1789

De innumeris curvis algebraicis, quarum longitudinem per arcus parabolicos metiri licet

Leonhard Euler

DE INNVMERIS CVRVIS ALGEBRAICIS, QVARVM LONGITVDINEM PER ARCVS ELLIPTICOS METIRI LICET.

Auctore

L. EVLERO.

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§. 1.

Pro Ellipsi, cuius singuli arcus nobis mensuram curuarum quaesitarum suppeditare debent, sit abscissa $= v$, applicata vero $= n\sqrt{(1 - vv)}$, vnde elementum arcus colligitur $= \frac{\partial v \sqrt{[1 + (nn - 1) vv]}}{\sqrt{(1 - vv)}}$; quamobrem sequens nobis propositum sit problema.

Problema.

Pro coordinatis x et y eiusmodi functiones algebraicas ipsius v inuestigare, vt fiat

$$\sqrt{(\partial x^2 + \partial y^2)} = \frac{\partial v \sqrt{[1 + (nn - 1) vv]}}{\sqrt{(1 - vv)}}.$$

Solutio.

§. 2. Vt formulae $\sqrt{(\partial x^2 + \partial y^2)}$ formam praescriptam conciliemus, quoniam denominator $\sqrt{(1 - vv)}$ duos habet factores $\sqrt{(1 + v)}$ et $\sqrt{(1 - v)}$, statuamus $\partial x = \frac{(p + q) \partial v}{\sqrt{2(1 + v)}}$

∂y

$\partial y = \frac{(p-q)\partial v}{\sqrt{2(1-v)}}$, hinc autem fiet

$$\sqrt{(\partial x^2 + \partial y^2)} = \partial v \sqrt{(pp + qq - 2pqv)},$$

vnde patet pro p et q eiusmodi quantitates quaeri debere, vt prodeat $pp + qq - 2pqv = 1 + (nn-1)vv$.

§. 3. Ante omnia autem hic euidens eſt, ſi modo pro litteris p et q functiones rationales integrae ipſius v aſſignari queant, ambas formulas pro ∂x et ∂y aſſumtas ſemper integrationem eſſe admiſſuras, propterea quod ambae iſtae formulae $\frac{v^i \partial v}{\sqrt{(1+v)}}$ et $\frac{v^i \partial v}{\sqrt{(1-v)}}$ ſemper ſunt integrabiles, ſi modo exponens i fuerit integer poſitiuus. Ad hoc igitur negotium abſoluendum ſequentes caſus euoluamus.

I. Caſus

quo $p = 1$ et $q = \alpha v$.

§. 4. Hic igitur erit $pp + qq = 1 + \alpha\alpha vv$ et $2pqv = 2\alpha vv$, quamobrem effici oportet

$$1 + \alpha\alpha vv - 2\alpha vv = 1 + (nn-1)vv$$

vnde patet ſumi debere $\alpha = 1 + n$, ita vt noſtra elementa hoc caſu fiant

$$\partial x = \frac{[1+(n+1)v]\partial v}{\sqrt{2(1+v)}} \text{ et } \partial y = \frac{[1-(n+1)v]\partial v}{\sqrt{2(1-v)}}$$

vbi integralibus ſumtis reperitur

$$x = \tfrac{1}{3}[1 - 2n + (n+1)v]\sqrt{2(1+v)} \text{ et}$$
$$y = \tfrac{1}{3}[2n - 1 + (n+1)v]\sqrt{2(1-v)}.$$

§. 5. Vt hinc quantitatem v eliminemus, addamus ambo quadrata, et obtinebimus

$$\frac{9(xx+yy)}{4} = (2n-1)^2 - 3(nn-1)vv,$$

ex

ex qua aequatione v facile per x et y determinatur; inde enim fit $vv = \frac{(2n-1)^2}{3(nn-1)} - \frac{3(xx+yy)}{4(nn-1)}$. Quo iam hunc valorem loco vv facilius ſubſtituere queamus, ſumamus productum noſtrarum formularum

$$\frac{9xy}{2} = [(n+1)^2 vv - (2n-1)^2] \sqrt{(1-vv)}$$

quae aequatio ſi quadretur, vbique tantum pares dimenſiones ipſius v occurrent, ac loco vv valore ſubſtituto aequatio inter x et y ad ſextum ordinem aſcendet.

II. Caſus

quo $p = 1 + \beta vv$ et $q = \alpha v$.

§. 6. Hic ergo erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta) vv + \beta\beta v^4 \text{ et}$$
$$2pqv = 1 + 2\alpha vv + 2\alpha\beta v^4,$$

vnde conditio adimplenda erit

$$1 + (\alpha\alpha + 2\beta - 2\alpha) vv$$
$$+ (\beta\beta - 2\alpha\beta) v^4 = 1 + (nn-1) vv.$$

Hic igitur ante omnia eſſe oportet $\beta\beta - 2\alpha\beta = 0$, ideoque $\beta = 2\alpha$, atque nunc ſupereſt vt fiat

$$\alpha\alpha + 2\beta - 2\alpha = \alpha\alpha + 2\alpha = nn - 1,$$

ſicque capi debet $\alpha = n - 1$ et $\beta = 2(n-1)$.

§. 7. Pro curua igitur definienda habebimus

$$p = 1 + 2(n-1) vv \text{ et } q = (n-1) v,$$

ſicque nunc erit

$$\partial x = \frac{1 + (n-1) v + 2(n-1) vv}{\sqrt{2(1+v)}} \partial v \text{ et}$$
$$\partial y = \frac{1 - (n-1) v + 2(n-1) vv}{\sqrt{2(1-v)}} \partial v$$

quarum integratio nulla amplius laborat difficultate, vnde hoc labore merito superſedemus.

III. Caſus
quo $p = 1 + \beta v v$ et $q = \alpha v + \gamma v^3$.

§. 8. Hic igitur erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta) v v + (\beta\beta + 2\alpha\gamma) v^4 + \gamma\gamma v^6 \text{ et}$$
$$p q = \alpha v + (\alpha\beta + \gamma) v^3 + \beta\gamma v^5,$$

vnde conficitur

$$pp + qq - 2pqv = 1 + (\alpha\alpha + 2\beta - 2\alpha) v v$$
$$+ (\beta\beta + 2\alpha\gamma - 2\alpha\beta - 2\gamma) v^4 + (\gamma\gamma - 2\beta\gamma) v^6,$$

quae quantitas aequari debet $1 + (nn - 1) v v$. Hic igitur primo poteſtas v^6 tolli debet, quod fit ponendo $\gamma\gamma - 2\beta\gamma = 0$, ideoque $\gamma = 2\beta$; deinde vero etiam poteſtatem quartam tolli oportet, vnde fit

$$\beta\beta + 2\alpha\gamma - 2\alpha\beta - 2\gamma = 0, \text{ ſiue}$$
$$\beta\beta + 2\alpha\beta - 4\beta = 0, \text{ ideoque}$$
$$\beta = 4 - 2\alpha \text{ et } \gamma = 8 - 4\alpha.$$

Iam vero coëfficiens ipſius vv erit

$$\alpha\alpha + 2\beta - 2\alpha = \alpha\alpha + 8 - 6\alpha$$

quem aequari oportet ipſi $nn - 1$, vnde colligitur $\alpha - 3 = n$, ſiue $\alpha = n + 3$, tum vero

$$\beta = -2(n+1) \text{ et } \gamma = -4(n+1).$$

§. 9. His igitur valoribus inuentis noſtrae formulae integrandae erunt

$$\partial x = \frac{1 + (n+3)v - 2(n+1)vv - 4(n+1)v^3}{\sqrt{2(1+v)}} \partial v \text{ et}$$
$$\partial y = \frac{1 - (n+3)v - 2(n+1)vv + 4(n+1)v^3}{\sqrt{2(1-v)}} \partial v,$$

qua-

quarum integrationi iterum non immorabimur. Vnicum tantum adhuc talem casum attingamus.

IV. Casus
quo $p = 1 + \beta vv + \delta v^4$ et $q = \alpha v + \gamma v^3$.

§. 10. Hic igitur erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta)vv + (\beta\beta + 2\delta + 2\alpha\gamma)v^4 + (2\beta\delta + \gamma\gamma)v^6 + \delta\delta v^8 \text{ et}$$

$$pq = \alpha v + (\alpha\beta + \gamma)v^3 + (\alpha\delta + \beta\gamma)v^5 + \gamma\delta v^7,$$

ex quibus conficitur formula

$$\begin{aligned} pp + qq - 2pqv = 1 &+ (\alpha\alpha + 2\beta - 2\alpha)vv \\ &+ (\beta\beta + 2\alpha\gamma + 2\delta - 2\alpha\beta - 2\gamma)v^4 \\ &+ (\gamma\gamma + 2\beta\delta - 2\alpha\delta - 2\beta\gamma)v^6 \\ &+ (\delta\delta - 2\gamma\delta)v^8, \end{aligned}$$

quae formula cum aequari debeat huic: $1 + (nn - 1)vv$, primo tollatur potestas octaua, vnde fit $\delta\delta - 2\gamma\delta = 0$, ideoque $\delta = 2\gamma$. Iam potestas sexta afficitur hac forma:

$$\gamma\gamma + 2\beta\delta - 2\alpha\delta - 2\beta\gamma = \gamma\gamma + 2\beta\gamma - 4\alpha\gamma,$$

quae nihilo aequata praebet $\gamma = 4\alpha - 2\beta$, hincque $\delta = 8\alpha - 4\beta$. Porro autem potestatis quartae coëfficiens est

$$\beta\beta + 2\alpha\gamma + 2\delta - 2\alpha\beta - 2\gamma =$$

$$\beta\beta - 6\alpha\beta - 4\beta + 8\alpha\alpha + 8\alpha = 0$$

quae aequatio diuisa per $\beta - 2\alpha$ praebet $\beta - 4\alpha - 4$, ita vt pro β geminos nanciscamur valores, alterum $\beta = 2\alpha$, alterum vero $\beta = 4\alpha + 4$, quorum vtrumque seorsim euoluamus.

§. 11. Sit igitur $\beta = 2\alpha$, eritque $\gamma = 0$ et $\delta = 0$, quo ergo casu res ad casum secundum reuoluitur. Sit igitur $\beta = 4\alpha + 4$, et nunc fiet $\gamma = -4(\alpha + 2)$ et $\delta = -8(\alpha + 2)$.

Verum hinc fiet poteſtatis vv coëfficiens $\alpha\alpha + 2\beta - 2\alpha$, ipſi $nn - 1$ aequandus, vnde fit $\alpha + 3 = n$, ſiue $\alpha = n - 3$, hincque porro fiet

$$\beta = 4(n-2);\ \gamma = -4(n-1) \text{ et } \delta = -8(n-1);$$

ex quibus ergo conficitur

$$p = 1 + 4(n-2)vv - 8(n-1)v^4 \text{ et}$$
$$q = (n-3)v - 4(n-1)v^3,$$

vnde tandem colligitur

$$x = \int \frac{(p+q)\partial v}{\sqrt{2(1+v)}} \text{ et } y = \int \frac{(p-q)\partial v}{\sqrt{2(1-v)}}$$

quem integrationis laborem ſuſcipere foret ſuperfluum.

Digreſſio
pro caſu $n = \pm 1$.

§. 12. Ex euolutione caſuum ſuperiorum manifeſtum eſt, curuas continuo ad altiores gradus aſcendere; hinc autem perpetuo excipi oportet caſum, quo foret $n = \pm 1$, quandoquidem arcus ellipticus $\int \frac{\partial v \sqrt{1 + (nn-1)vv}}{\sqrt{(1-vv)}}$ abiret in $\int \frac{\partial v}{\sqrt{(1-vv)}}$, hoc eſt in arcum circularem. Cum igitur praeter circulum nullae aliae dentur tales curuae, neceſſe eſt, vt curuae, ad quos caſus praecedentes nos deducunt, quando fuerit $n = \pm 1$, circulum exhibeant.

§. 13. Pro caſu autem primo, vbi integralia iam euoluimus, quando fit $n = \pm 1$ ideoque $nn - 1 = 0$, aequatio penultima abit in hanc: $\frac{9}{4}(xx + yy) = (2n - 1)^2$, hoc eſt aequabitur vel $= 1$ vel $= 9$, ita vt vtroque caſu curua manifeſto ſit circulus, cum tamen pro aliis omnibus valoribus ipſius n aequatio ad ſextum ordinem aſſurgere ſit obſeruata.

§. 14.

§. 14. Pro casu secundo faciamus primo $n = + 1$, eritque $\partial x = \frac{\partial v}{\sqrt{2(1+v)}}$ et $\partial y = \frac{\partial v}{\sqrt{2(1-v)}}$, vnde integrando fit $x = \sqrt{2(1+v)}$ et $y = -\sqrt{2(1-v)}$, ex quibus manifesto colligitur $xx + yy = 4$, quae vtique est aequatio ad circulum. Sin autem sumamus $n = -1$, reperitur

$$\partial x = \frac{1 - 2v - 4vv}{\sqrt{2(1+v)}} \partial v \text{ et}$$

$$\partial y = \frac{1 + 2v - 4vv}{\sqrt{2(1-v)}} \partial v,$$

hinc autem circulum enasci sequenti modo facillime ostendetur.

§. 15. Hunc in finem statuatur $v = \text{cos}.\, 2\Phi$, eritque $\partial v = -2\partial\Phi \,\text{sin}.\, 2\Phi$ et $\sqrt{2(1+v)} = 2\,\text{cos}.\,\Phi$, similique modo $\sqrt{2(1-v)} = 2\,\text{sin}.\,\Phi$. Ergo pro priore formula erit $\frac{\partial v}{\sqrt{2(1+v)}} = -2\partial\Phi\,\text{sin}.\,\Phi$, alter vero factor fiet

$$= 1 - 2\,\text{cos}.\,2\Phi - 4\,\text{cos}.\,2\Phi^2$$

$$= -1 - 2\,\text{cos}.\,2\Phi - 2\,\text{cos}.\,4\Phi$$

quamobrem habebimus

$$\partial x = 2\partial\Phi\,\text{sin}.\,\Phi\,(1 + 2\,\text{cos}.\,2\Phi + 2\,\text{cos}.\,4\Phi).$$

Constat autem esse

$$2\,\text{sin}.\,\Phi\,\text{cos}.\,2\Phi = \text{sin}.\,3\Phi - \text{sin}.\,\Phi \text{ et}$$

$$2\,\text{sin}.\,\Phi\,\text{cos}.\,4\Phi = \text{sin}.\,5\Phi - \text{sin}.\,3\Phi,$$

quibus substitutis obtinebitur $\partial x = 2\partial\Phi\,\text{sin}.\,5\Phi$, cuius integrale est $x = -\frac{2}{5}\,\text{cos}.\,5\Phi$. Simili modo pro altera formula prodit $\frac{\partial v}{\sqrt{2(1-v)}} = -2\partial\Phi\,\text{cos}.\,\Phi$, alter vero factor erit

$$1 + 2\,\text{cos}.\,2\Phi - 4\,\text{cos}.\,2\Phi^2 = -1 + 2\,\text{cos}.\,2\Phi - 2\,\text{cos}.\,4\Phi$$

sicque fiet

$$\partial y = 2\partial\Phi\,\text{cos}.\,\Phi\,(1 - 2\,\text{cos}.\,2\Phi + 2\,\text{cos}.\,4\Phi).$$

Constat autem esse

$$2\,\text{cos}.\,\Phi\,\text{cos}.\,2\Phi = \text{cos}.\,3\Phi + \text{cos}.\,\Phi \text{ et}$$

$$2\,\text{cos}.\,\Phi\,\text{cos}.\,4\Phi = \text{cos}.\,5\Phi + \text{cos}.\,3\Phi,$$

K 3 quo-

quocirca proueniet $\partial y = 2\,\partial\Phi\,\text{cof.}\,5\,\Phi$, ideoque $y = \frac{2}{5}\,\text{fin.}\,5\,\Phi$, confequenter hic cafus nobis fuppeditat $xx + yy = \frac{4}{25}$, quae iterum manifefto eft aequatio ad circulum.

§. 16. Tractemus fimili modo cafum tertium, ponendo primo $n = +1$, et habebimus

$$\partial x = \frac{1 + 4v - 4vv - 8v^3}{\sqrt{2(1+v)}}\,\partial v \text{ et}$$
$$\partial y = \frac{1 - 4v - 4vv + 8v^3}{\sqrt{2(1-v)}}\,\partial v.$$

Statuamus iterum $v = \text{cof.}\,2\,\Phi$ fietque

$$\partial x = -2\,\partial\Phi\,\text{fin.}\,\Phi\,(1 + 4\,\text{cof.}\,2\,\Phi - 4\,\text{cof.}\,2\,\Phi^2 - 8\,\text{cof.}\,2\,\Phi^3) \text{ et}$$
$$\partial y = -2\,\partial\Phi\,\text{cof.}\,\Phi\,(1 - 4\,\text{cof.}\,2\,\Phi - 4\,\text{cof.}\,2\,\Phi^2 + 8\,\text{cof.}\,2\,\Phi^3),$$

vbi notetur effe

$$4\,\text{cof.}\,2\,\Phi^2 = 2 + 2\,\text{cof.}\,4\,\Phi \text{ et}$$
$$8\,\text{cof.}\,2\,\Phi^3 = 6\,\text{cof.}\,2\,\Phi + 2\,\text{cof.}\,6\,\Phi,$$

vnde habebimus

$$\partial x = 2\,\partial\Phi\,\text{fin.}\,\Phi\,(1 + 2\,\text{cof.}\,2\,\Phi + 2\,\text{cof.}\,4\,\Phi + 2\,\text{cof.}\,6\,\Phi) \text{ et}$$
$$\partial y = 2\,\partial\Phi\,\text{cof.}\,\Phi\,(1 - 2\,\text{cof.}\,2\,\Phi + 2\,\text{cof.}\,4\,\Phi - 2\,\text{cof.}\,6\,\Phi).$$

Quodfi iam has reductiones vlterius profequamur, nancifcemur tandem

$$\partial x = 2\,\partial\Phi\,\text{fin.}\,7\,\Phi, \text{ ideoque } x = -\tfrac{2}{7}\,\text{cof.}\,7\,\Phi,$$

eodemque modo

$$\partial y = 2\,\partial\Phi\,\text{cof.}\,7\,\Phi, \text{ ergo } y = \tfrac{2}{7}\,\text{fin.}\,7\,\Phi,$$

vnde iterum colligitur $xx + yy = \frac{4}{49}$, ideoque pro circulo.

§. 17. Si pro eodem cafu tertio ponatur $n = -1$, fiet

$$\partial x = \frac{1 + 2v}{\sqrt{2(1+v)}}\,\partial v \text{ et } \partial y = \frac{1 - 2v}{\sqrt{2(1-v)}}\,\partial v.$$

Statuamus igitur $v = \text{cof.}\,2\,\Phi$, eritque $\sqrt{2(1+v)} = 2\,\text{cof.}\,\Phi$ et $\sqrt{2(1-v)} = 2\,\text{fin.}\,\Phi$ et $\partial v = -2\,\partial\Phi\,\text{fin.}\,2\,\Phi$; quamobrem

obrem habebitur

$$\partial x = -\frac{(1+2\,cof.\,2\,\Phi)}{2\,cof.\,\Phi}\partial\Phi\,\text{fin.}\,2\,\Phi = -2\,\partial\Phi\,\text{fin.}\,\Phi\,(1+2\,\text{cof.}\,2\,\Phi) \text{ et}$$

$$\partial y = -\frac{(1-2\,cof.\,2\,\Phi)}{2\,fin.\,\Phi}\partial\Phi\,\text{fin.}\,2\,\Phi = -2\,\partial\Phi\,\text{cof.}\,\Phi\,(1-2\,\text{cof.}\,2\,\Phi),$$

quae formulae porro reducuntur ad has:

$$\partial x = -2\,\partial\Phi\,\text{fin.}\,3\,\Phi \text{ et } \partial y = -2\,\partial\Phi\,\text{cof.}\,3\,\Phi;$$

hincque integrando fiet

$$x = \tfrac{2}{3}\,\text{cof.}\,3\,\Phi \text{ et } y = -\tfrac{2}{3}\,\text{fin.}\,3\,\Phi,$$

vnde colligitur $xx + yy = \frac{4}{9}$, quae eſt aequatio pro circulo, cuius radius $= \frac{2}{3}$.

§. 18. Quodſi quis ſimili modo caſum quartum euoluere voluerit, ponendo ſiue $n = +1$ ſiue $n = -1$, itidem reperiet curuas ſatisfacientes pariter ad circulum reduci. Hinc igitur anſam arripimus problema noſtrum alio modo reſoluendi, dum ſcilicet in formulam, qua arcus curuae exprimi debet, ſtatim ſinum coſinumue cuiuspiam anguli introducemus.

Alia problematis ſolutio ex calculo angulorum petita.

§. 19. Cum elementum arcus curuarum quaeſitarum debeat eſſe

$$\partial s = \frac{\partial v \sqrt{[1+(nn-1)\,vv]}}{\sqrt{(1-vv)}},$$

ponamus ſtatim $v = \text{fin.}\,\Phi$, vt fiat $\frac{\partial v}{\sqrt{(1-vv)}} = \partial\Phi$, eritque

$$\partial s = \partial\Phi\sqrt{(\text{cof.}\,\Phi^2 + nn\,\text{fin.}\,\Phi^2)},$$

vnde ſtatim manifeſtum eſt capi poſſe

$$\partial x = \partial\Phi\,\text{cof.}\,\Phi \text{ et } \partial y = n\,\partial\Phi\,\text{fin.}\,\Phi,$$

vnde fit $x = \text{fin.}\,\Phi$ et $y = -n\,\text{cof.}\,\Phi$, ideoque $nx = n\,\text{fin.}\,\Phi$, vnde colligitur $nnxx + yy = nn$, quae eſt ipſa aequatio pro Ellipſi,

Ellipsi, cuius arcus mensuram reliquarum curuarum constituere debent.

§. 20. Ex hac autem solutione infinitas alias curuas quaesito pariter satisfacientes deriuare possumus, ponendo

$$\partial x = \partial \phi \operatorname{cos.} \phi \operatorname{cos.} \omega - n \partial \phi \operatorname{sin.} \phi \operatorname{sin.} \omega \text{ et}$$
$$\partial y = \partial \phi \operatorname{cos.} \phi \operatorname{sin.} \omega + n \partial \phi \operatorname{sin.} \phi \operatorname{cos.} \omega,$$

sic enim euadet

$$\partial x + \partial y^2 = \partial \phi^2 \operatorname{cos.} \phi^2 + n n \partial \phi^2 \operatorname{sin.} \phi^2 = \partial s^2.$$

Tantum igitur superest vt istae duae formulae differentiales integrabiles reddantur, quod manifesto in genere eueniet, sumendo $\omega = \lambda \phi$, tum enim per reductiones satis cognitas nanciscemur:

$$\frac{2 \partial x}{\partial \phi} = (n+1) \operatorname{cos.} (\lambda + 1) \phi - (n-1) \operatorname{cos.} (\lambda - 1) \phi \text{ et}$$
$$\frac{2 \partial y}{\partial \phi} = (n+1) \operatorname{sin.} (\lambda + 1) \phi - (n-1) \operatorname{sin.} (\lambda - 1) \phi,$$

atque hinc integrando impetrabimus:

$$2x = \frac{n+1}{\lambda+1} \operatorname{sin.} (\lambda + 1) \phi - \frac{(n-1)}{(\lambda-1)} \operatorname{sin.} (\lambda - 1) \phi,$$
$$2y = \frac{n+1}{\lambda+1} \operatorname{cos.} (\lambda + 1) \phi - \frac{n-1}{\lambda-1} \operatorname{cos.} (\lambda - 1) \phi,$$

quae ergo ambae formulae semper sunt algebraicae, solo casu excepto vbi $\lambda = \pm 1$. Caeterum quando $n = \pm 1$, curuae resultantes manifesto abeunt in circulum, quicunque valor ipsi λ tribuatur.

§. 21. Haec solutio non solum est admodum succincta, sed etiam multo latius patet quam praecedens, quandoquidem praecedentes casus ex hac solutione deducuntur, sumendo $\lambda = \pm \frac{1}{2}$; vel $\lambda = \pm \frac{3}{2}$; vel $\lambda = \pm \frac{5}{2}$; vel $\lambda = \pm \frac{7}{2}$. Quatenus igitur hic pro λ numeros integros accipere licet, vel

vel etiam quascunque alias fractiones, eatenus haec solutio longe alias suppeditat lineas curuas, quae ex priori solutione nullo modo deduci possunt. Euoluamus igitur aliquot exempla.

Exemplum 1.

§. 22. Quia pro λ vnitatem assumere non licet, ponamus statim $\lambda = 2$, atque habebimus

$$x = + \frac{n+1}{6} \text{ sin. } 3\,\Phi - \frac{(n-1)}{2} \text{ sin. } \Phi \text{ et}$$

$$y = - \frac{(n+1)}{6} \text{ cos. } 3\,\Phi + \frac{(n-1)}{2} \text{ cos. } \Phi,$$

hinc iam colligimus

$$x\,x + y\,y = \frac{(n+1)^2}{36} + \frac{(n-1)^2}{4} - \frac{(n\,n-1)}{6} \text{ cos. } 2\,\Phi,$$

ex qua aequatione angulus Φ haud difficulter per x et y determinatur, qui deinceps in alterutra substitutus praebebit aequationem inter x et y.

Exempum 2.

Sumamus etiam $\lambda = \frac{1}{2}$, erit

$$x = + \frac{(n+1)}{3} \text{ sin. } \tfrac{3}{2}\,\Phi + (n-1) \text{ sin. } \tfrac{1}{2}\,\Phi \text{ et}$$

$$y = - \frac{(n+1)}{3} \text{ cos. } \tfrac{3}{2}\,\Phi + (n-1) \text{ cos. } \tfrac{1}{2}\,\Phi.$$

Facile autem patet hoc exemplum cum casu supra §. 4. tractato congruere.

Scholion.

§. 23. Haec igitur solutio praecedentem maxime supereminet, cum non solum infinities plures curuas in se complectatur, sed etiam valores pro coordinatis x et y inuenti tam simpliciter exprimantur, vt duobus tantum terminis constent, cum sit

$$2x = \frac{n+1}{\lambda+1}\,\text{fin.}\,(\lambda+1)\,\Phi - \frac{(n-1)}{\lambda-1}\,\text{fin.}\,(\lambda-1)\,\Phi \text{ et}$$

$$-2y = \frac{n+1}{\lambda+1}\,\text{cof.}\,(\lambda+1)\,\Phi - \frac{(n-1)}{\lambda-1}\,\text{cof.}\,(\lambda-1)\,\Phi.$$

Ex qua forma coordinatarum colligitur iſtas curuas omnes affines eſſe Epicycloidibus, et generari poſſe ex prouolutione circuli ſuper circulo, dum ſcilicet ſtylus deſcribens non in ipſa peripheria circuli mobilis aſſumitur. Interim tamen ne haec quidem ſolutio pro generali haberi poteſt: namque innumerabiles alias curuas ſatisfacientes aſſignare licet, quae ne in hac ſolutione continentur, quam inuentionem hic ſubiungamus.

Adhuc alia ſolutio

problematis propoſiti.

§. 24. Maneat vt ante $v = \text{fin.}\,\Phi$, et cum hinc fiat

$$\partial s = \partial\Phi\,\sqrt{(\text{cof.}\,\Phi^2 + nn\,\text{fin.}\,\Phi^2)},$$

ſcribamus $1 - \text{fin.}\,\Phi^2$ loco $\text{cof.}\,\Phi^2$, eritque

$$\partial s = \partial\Phi\,\sqrt{(1 + (nn-1)\,\text{fin.}\,\Phi^2)}.$$

Faciamus autem breuitatis gratia $nn - 1 = mm$, atque huic conditioni ſtatim ſatisfieret, ſumendo $\partial x = \partial\Phi$ et $\partial y = m\,\partial\Phi\,\text{fin.}\,\Phi$; hinc autem ob $x = \Phi$ prodiret curua tranſcendens, quod tamen non impedit, quo minus infinitae curuae algebraicae hinc deduci queant. Statuamus enim

$$\partial x = \partial\Phi\,\text{cof.}\,\lambda\Phi - m\,\partial\Phi\,\text{fin.}\,\Phi\,\text{fin.}\,\lambda\Phi \text{ et}$$

$$\partial y = \partial\Phi\,\text{fin.}\,\lambda\Phi + m\,\partial\Phi\,\text{fin.}\,\Phi\,\text{cof.}\,\lambda\Phi,$$

atque hinc prodit

$$\partial x^2 + \partial y^2 = \partial\Phi^2\,(1 + mm\,\text{fin.}\,\Phi^2).$$

Nunc igitur membra poſteriora more ſolito euoluantur et obtinebitur

$$\frac{2\partial x}{\partial \phi} = 2\,\mathrm{cof.}\,\lambda\,\phi - m\,\mathrm{cof.}\,(\lambda - 1)\,\phi + m\,\mathrm{cof.}\,(\lambda + 1)\,\phi \text{ et}$$
$$\frac{2\partial y}{\partial \phi} = 2\,\mathrm{fin.}\,\lambda\,\phi + m\,\mathrm{fin.}\,(\lambda + 1)\,\phi - m\,\mathrm{fin.}\,(\lambda - 1)\,\phi,$$

vnde fumtis integralibus erit

$$2x = \frac{2}{\lambda}\,\mathrm{fin.}\,\lambda\,\phi - \frac{m}{\lambda - 1}\,\mathrm{fin.}\,(\lambda - 1)\,\phi + \frac{m}{\lambda + 1}\,\mathrm{fin.}\,(\lambda + 1)\,\phi \text{ et}$$
$$-2y = \frac{2}{\lambda}\,\mathrm{cof.}\,\lambda\,\phi - \frac{m}{\lambda - 1}\,\mathrm{cof.}\,(\lambda - 1)\,\phi + \frac{m}{\lambda + 1}\,\mathrm{cof.}\,(\lambda + 1)\,\phi,$$

quae ergo formulae etiam funt algebraicae, folo cafu $\lambda = \pm 1$ excepto. Perfpicuum autem eft, hos valores penitus effe diuerfos a praecedentibus, propterea quod terna membra inuoluunt.

§. 25. Praeterea vero hic manifefto affumfimus effe $nn > 1$, ita vt haec folutio extendi nequeat ad cafus quibus $nn < 1$, cum prior folutio pro omnibus valoribus numeri n valeat; interim tamen etiam haec folutio adaptari poteft ad cafus quibus $nn < 1$, ita vt fit

$$\partial s = \partial \phi \sqrt{(1 - (1 - nn)\,\mathrm{fin.}\,\phi)},$$

quae expreffio, pofito $\mathrm{fin.}\,\phi^2 = 1 - \mathrm{cof.}\,\phi^2$, abit in hanc:

$$\partial s = \partial \phi \sqrt{(nn + (1 - nn)\,\mathrm{cof.}\,\phi^2)},$$

et pofito breuitatis gratia $1 - nn = kk$, fiet

$$\partial s = \partial \phi \sqrt{(nn + kk\,\mathrm{cof.}\,\phi^2)},$$

vbi notetur effe $nn + kk = 1$.

§. 26. Huic ergo formulae ftatim fatisfiet ponendo

$$\partial x = n\,\partial \phi \text{ et } \partial y = k\,\partial \phi\,\mathrm{cof.}\,\phi,$$

vnde autem curua refultaret tranfcendens, quare vt curuas algebraicas eruamus, ftatuamus vt ante

 ∂x

$$\partial x = n \partial \phi \text{ fin. } \lambda \phi + k \partial \phi \text{ cof. } \phi \text{ cof. } \lambda \phi \text{ et}$$
$$\partial y = n \partial \phi \text{ cof. } \lambda \phi - k \partial \phi \text{ cof. } \phi \text{ fin. } \lambda \phi,$$
vnde ambo valores prodibunt algebraici, dum ne fit $\lambda = \pm 1$.

§. 27. Reductione igitur folita in vfum vocata nanciscemur has formulas:

$$\frac{2 \partial x}{\partial \phi} = 2n \text{ fin. } \lambda \phi + k \text{ cof. } (\lambda + 1) \phi + k \text{ cof. } (\lambda - 1) \phi \text{ et}$$
$$\frac{2 \partial y}{\partial \phi} = 2n \text{ cof. } \lambda \phi - k \text{ fin. } (\lambda + 1) \phi - k \text{ fin. } (\lambda - 1) \phi,$$

vnde integrando deducimus:

$$2x = -\frac{2n}{\lambda} \text{ cof. } \lambda \phi + \frac{k}{\lambda + 1} \text{ fin. } (\lambda + 1) \phi + \frac{k}{\lambda - 1} \text{ fin. } (\lambda - 1) \phi \text{ et}$$
$$2y = +\frac{2n}{\lambda} \text{ fin. } \lambda \phi + \frac{k}{\lambda + 1} \text{ cof. } (\lambda + 1) \phi + \frac{k}{\lambda - 1} \text{ cof. } (\lambda - 1) \phi,$$

quae curuae itidem maxime difcrepant a praecedente folutione.

Scholion.

§. 28. Quamuis autem hae folutiones infinities infinitas fuppeditent lineas curuas algebraicas problemati noftro fatisfacientes, tamen vix affirmari poffe videtur, in his formulis omnes plane folutiones contineri: tam parum enim adhuc iftud argumentum eft elaboratum, vt vix quicquam certi in hoc negotio ftatui poffe videatur; fed potius quaeftio generalis, qua curuae algebraicae defiderantur, quarum longitudo per datam formulam integralem $\int V \partial v$ exprimatur, vbi V denotet functionem quamcunque ipfius v, tantopere etiamnunc tenebris obuoluta deprehenditur, vt folutionem pauciffimis tantum cafibus euoluere liceat, quemadmodum nobis folutio fucceffit pro arcubus Parabolicis et Ellipticis. Si enim talis quaeftio circa arcus Hyperbolicos proponatur, fateri cogor, nullo adhuc modo me vel vnicam faltem curuam algebraicam eruere potuiffe, cuius finguli arcus per formulam:

$\int$

$$\int \frac{\partial v}{vv} \sqrt{(1 + v^4)},$$

exprimerentur. Si enim v denotet abſciſſam Hyperbolae aequilaterae inter aſymptotos, applicata erit $y = \frac{1}{v}$, ideoque $\partial y = -\frac{\partial v}{vv}$, vnde elementum arcus reſultat

$$\partial s = \frac{\partial v}{vv} \sqrt{(1 + v^4)}.$$

Sin autem aequationem generalem pro Hyperbola aſſumere velimus, qua eſt $y = n \sqrt{(1 + vv)}$, elementum arcus inde naſcitur $\partial s = \frac{\partial v \sqrt{(1 + (nn+1) vv)}}{\sqrt{(1 + vv)}}$, quae formula ita comparata eſt, vt omnia artificia, quae quidem mihi detegere licuit, penitus fruſtretur. Quin etiam hic nullo modo calculus angulorum cum vllo ſucceſſu in ſubſidium vocari poteſt. Neutiquam autem etiamnunc aſſeuerare auſim, praeter Hyperbolam nullas alias dari curuas algebraicas, quarum longitudinem per arcus Hyperbolicos metiri liceat, quemadmodum hoc de circulo audacter pronunciare non dubitaui. Hac igitur ſpeculatione ampliſſimus campus aperitur, in quo Geometrae non ſine inſigni fructu et Analyſeos vlterioris perfectione elaborare poterunt.