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De innumeris curvis algebraicis, quarum longitudinem per arcus parabolicos metiri licet

Leonhard Euler

DE INNVMERIS CVRVIS ALGEBRAICIS, QVARVM LONGITVDINEM PER ARCVS PARABOLICOS METIRI LICET.

Auctore

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§. 1.

Non ita pridem ausus sum duo theoremata prorsus memorabilia in medium proferre, quorum altero statui: *nullam dari curuam algebraicam, cuius longitudo indefinita per quempiam logarithmum exprimi queat*; altero vero affirmaui: *praeter circulum nullas alias dari curuas algebraicas, quarum longitudo cuipiam arcui circulari esset aequalis.* Veritatem equidem horum theorematum grauissimis rationibus confirmare sum annisus; interim tamen fateri cogor, omnes has rationes a solida demonstratione, cuiusmodi in Geometria desiderari solet, adhuc plurimum abesse.

§. 2. Facile autem intelligitur totum hoc negotium felicissimo successu confectum iri, si sequens problema resoluere liceret: *Proposita formula differentiali quacunque $V\,dv$, vbi V sit*

functio

functio quaecunque data algebraica ipsius v, inuenire pro binis coordinatis x & y eiusmodi functiones algebraicas ipsius v, vt inde euadat $\sqrt{(\partial x^2 + d y^2)} = V \partial v$. Tum enim integrale $\int V \partial v$ vtique exprimeret longitudinem curuae cuiusdam algebraicae. Hic scilicet res eo rediret, vt ostenderetur, quibusnam casibus hoc problema vel nullam plane solutionem admitteret, quemadmodum euenire statuo casu $V \partial v = \frac{\partial v}{v}$; vel vnicam tantum solutionem, veluti casu $V \partial v = \frac{\partial v}{\sqrt{(1 - v v)}}$, siue etiam $V \partial v = \frac{\partial v}{1 + v v}$; vel denique, quibusnam casibus hoc problema innumerabiles solutiones recipere posset, quemadmodum ostensurus sum pro casu $V \partial v = \partial v \sqrt{(1 + v v)}$, quandoquidem eius integrale $\int \partial v \sqrt{(1 + v v)}$ exprimit arcum Parabolicum, cuius quippe coordinatae sunt v et $\frac{1}{2} v v$.

§. 3. Ante autem quam hoc problema particulare suscipiam, duplicem methodum aperiam, qua problema generale tractari conueniat. Ac primo quidem proposita aequatione

$$\sqrt{(\partial x^2 + \partial y^2)} = V \partial v$$

dispiciatur, num forte eiusmodi functionem ipsius v, quae sit U, explorare liceat, vt hae duae formulae: $\partial x = \frac{V \partial v \sqrt{(A + U)}}{\sqrt{(A + B)}}$ et $\partial y = \frac{V \partial v \sqrt{(B - U)}}{\sqrt{(A + B)}}$, fiant integrabiles; quoniam enim inde fit $\partial x^2 + \partial y^2 = V^2 \partial v^2$, quaestioni foret satisfactum. Vel etiam quaeratur eiusmodi angulus Φ, qui rationem algebraicam teneat ad variabilem v, ita vt ambae istae formulae $V \partial v \sin. \Phi$ et $V \partial v \cos. \Phi$ euadant integrabiles, quoniam hinc fieret

$$x = \int V \partial v \sin. \Phi \text{ et } y = \int V \partial v \cos. \Phi.$$

§. 4. Quando autem hoc tentamen nullo modo succedit, dispiciatur, vtrum formula proposita $V \partial v$ ad huiusmodi formam reduci queat: $\partial v \sqrt{(P^2 + Q^2)}$; tum enim statim haberetur solutio $x = \int P \partial v$ et $y = \int Q \partial v$, si modo hae formulae essent

essent integrabiles. At vero multo generalius solutionem tentare licebit, statuendo

$$\frac{\partial x}{\partial v} = \frac{P(\sqrt{A+U}) - Q\sqrt{(B-U)}}{\sqrt{(A+B)}},$$
$$\frac{\partial y}{\partial v} = \frac{P\sqrt{(B-U)} + Q\sqrt{(A+U)}}{\sqrt{(A+B)}},$$

vbi totum negotium eo redit, vt pro U eiusmodi functio ipsius v inuestigetur, qua istae duae formulae integrabiles reddantur. Vel etiam simplicius res redigi poterit ad inuentionem cuiuspiam anguli Φ, vt istae ambae formulae integrationem admittant:

$$\partial x = \partial v\,(P \text{ sin.}\, \Phi + Q \text{ cos.}\, \Phi),$$
$$\partial y = \partial v\,(P \text{ cos.}\, \Phi - Q \text{ sin.}\, \Phi),$$

siquidem hinc prodibit

$$\partial x^2 + \partial y^2 = \partial v^2\,(P^2 + Q^2).$$

Verum fatendum est, has regulas ita esse comparatas, vt si eas ad formulas determinatas applicare velimus, aqua nobis plerumque haereat.

§. 5. His igitur praemissis problema, cuius solutionem pollicemur, aggrediamur.

Problema.

Inuenire innumerabiles curuas algebraicas, quarum longitudinem per arcus Parabolicos exprimere liceat, siue vt, positis binis coordinatis x *et* y, *fiat* $\sqrt{(\partial x^2 + \partial y^2)} = \partial v \sqrt{(1 + vv)}$, *simulque ipsae coordinatae* x *et* y *prodeant functiones algebraicae ipsius* v.

Solutio.

§. 6. Quodsi hanc formulam cum generali ante allegata comparemus, erit $P = 1$ et $Q = v$, vnde statim colligitur

tur $\partial x = \partial v$ et $\partial y = v\,\partial v$, hincque porro $x = v$ et $y = \frac{1}{2} v v$, ergo $y = \frac{1}{2} x x$, quae eſt aequatio pro ipſa Parabola. Problema autem noſtrum poſtulat, vt innumeras alias eiusmodi curvas inueſtigemus, quarum longitudo pari formula exprimatur; ſequamur igitur formulam priorem §. 4. traditam, vnde pro praeſenti caſu erit

$$\frac{\partial x}{\partial v} = \frac{\sqrt{(A + U)} - v\sqrt{(B - U)}}{\sqrt{(A + B)}} \text{ et } \frac{\partial y}{\partial v} = \frac{\sqrt{(B - U)} + v\sqrt{(A + U)}}{\sqrt{(A + B)}},$$

quae ambae formulae infinitis modis integrabiles reddi poſſunt. Primo ſcilicet ſi ſtatuamus $U = v$; deinde vero etiam ſi fuerit $U = \sqrt{v}$; porro quoque ſimili modo ſi ſumatur $U = \sqrt[3]{v}$, vel $U = \sqrt[4]{v}$, vel in genere $U = \sqrt[i]{v}$, ſi modo exponens i fuerit integer poſitiuus.

Euolutio caſus quo $U = v$.

§. 7. Hic igitur totum negotium redit ad integrationem talium duarum formularum:

$$\int \partial v \sqrt{(\alpha + \beta v)} \text{ et } \int v\,\partial v \sqrt{(\alpha + \beta v)}.$$

Statuamus igitur $\sqrt{(\alpha + \beta v)} = t$, eritque

$$v = \frac{tt - \alpha}{\beta} \text{ et } \partial v = \frac{2t\,\partial t}{\beta},$$

hinc ergo pro formula priore fiet

$$\partial v \sqrt{(\alpha + \beta v)} = \frac{2tt\,\partial t}{\beta},$$

pro altera vero formula erit

$$v\,\partial v \sqrt{(\alpha + \beta v)} = \frac{2tt\,\partial t}{\beta\beta}(tt - \alpha),$$

quocirca integrando eliciemus

$$\text{I. } \int \partial v \sqrt{(\alpha + \beta v)} = \frac{2t^3}{3\beta} = \frac{2}{3\beta}(\alpha + \beta v)^{\frac{3}{2}} \text{ et}$$

$$\text{II. } \int v\,\partial v \sqrt{(\alpha + \beta v)} = \frac{2t^5}{5\beta\beta} - \frac{2\alpha t^3}{3\beta\beta} = \frac{2t^3}{15\beta\beta}(3tt - 5\alpha).$$

ſiue

siue

$$\int v \partial v \sqrt{(\alpha + \beta v)} = \frac{2 (\alpha + \beta v)^{\frac{3}{2}}}{15 \beta \beta} (3 \beta v - 2 \alpha).$$

§. 8. Nunc igitur tantum superest, vt formulae supra exhibitae iuxta has regulas expediantur, quae ita se habebunt:

1°. $\int \partial v \sqrt{(A + v)} = \frac{2}{3} (A + v)^{\frac{3}{2}}$.

2°. $\int \partial v \sqrt{(B - v)} = -\frac{2}{3} (B - v)^{\frac{3}{2}}$.

3°. $\int v \partial v \sqrt{(A + v)} = \frac{2}{15} (A + v)^{\frac{3}{2}} (3 v - 2 A)$.

4°. $\int v \partial v \sqrt{(B - v)} = -\frac{2}{15} (B - v)^{\frac{3}{2}} (3 v + 2 B)$.

His igitur valoribus substitutis ambae coordinatae x et y ita reperientur expressae:

$$x \sqrt{(A + B)} = \frac{2}{3} (A + v)^{\frac{3}{2}} + \frac{2}{15} (B - v)^{\frac{3}{2}} (3 v + 2 B) \text{ et}$$

$$y \sqrt{(A + B)} = -\frac{2}{3} (B - v)^{\frac{3}{2}} + \frac{2}{15} (A + v)^{\frac{3}{2}} (3 v - 2 A).$$

§. 9. Hac igitur ratione ambas coordinatas x et y per communem variabilem v algebraice expressas sumus consecuti, id quod ad curuam construendam sufficit; quandoquidem pro quolibet valore ipsius v quantitates vtriusque coordinatae assignare licet. Sin autem quantitatem v eliminare vellemus, in calculos molestissimos illaberemur, vix adeo extricabiles, atque aequatio inter x et y inde resultans ad plurimas dimensiones assurgeret, qui tamen labor nihil aliud esset praestaturus, nisi vt ordinem, ad quem has curuas referri oportet, assignare valeamus. Caeterum quia hic duae quantitates arbitrariae A et B sunt introductae, euidens est iam innumerabiles lineas curuas diuersas in hac sola solutione contineri.

§. 10.

§. 10. Quo formulas has ſatis complicatas exemplo illuſtremus, ponamus $A = 0$ et $B = 1$, ac perueniemus ad ſequentes formulas concinniores:

$$x = \tfrac{2}{3} v \sqrt{v} + \tfrac{2}{15} (1 - v)^{\frac{3}{2}} (3 v + 2) \text{ et}$$

$$y = - \tfrac{2}{3} (1 - v)^{\frac{3}{2}} + \tfrac{2}{5} v v \sqrt{v}.$$

Quodſi hic loco $\frac{15}{2} x$ et $\frac{15}{2} y$ ſcribamus X et Y, quandoquidem hoc modo natura curuae non mutatur, tum vero eliminemus terminum $(1 - v)^{\frac{3}{2}}$, peruenietur ad hanc aequationem:

$$5 X + Y (3 v + 2) = (25 + 6 v + 9 v v) v \sqrt{v},$$

quae aequatio denuo quadrari deberet ad rationalem efficiendam, tum vero littera v aſcenſura eſſet ad poteſtatem ſeptimam, vnde certe nemo determinationem huius litterae ſuſcipiet.

Euolutio caſus.

quo $U = \sqrt{v}$.

§. 11. Hic igitur occurrent binae ſequentes formulae integrandae:

$$\int \partial v \sqrt{(\alpha + \beta \sqrt{v})} \text{ et } \int v \partial v \sqrt{(\alpha + \beta \sqrt{v})},$$

quas mox patebit itidem eſſe integrabiles. Si enim ponatur $\sqrt{(\alpha + \beta \sqrt{v})} = t$, erit $\sqrt{v} = \frac{t t - \alpha}{\beta}$, conſequenter

$$v = \frac{t^4 - 2 \alpha t t + \alpha \alpha}{\beta \beta}, \text{ ergo } \partial v = \frac{4 t^3 \partial t - 4 \alpha t \partial t}{\beta \beta}.$$

Prior forma abibit in hanc: $\frac{4 t t \partial t}{\beta \beta} (t t - \alpha)$, cuius integrale eſt $\frac{4 t^5}{5 \beta \beta} - \frac{4 \alpha t^3}{3 \beta \beta}$, quamobrem habemus:

$$\int \partial v \sqrt{(\alpha + \beta \sqrt{v})} = \frac{4 (\alpha + \beta \sqrt{v})^{\frac{3}{2}}}{15 \beta \beta} (3 \beta \sqrt{v} - 2 \alpha).$$

Pro

Pro altera autem formula habemus

$$v\,\partial v = \frac{(4t^7 - 12\alpha t^5 + 12\alpha\alpha t^3 - 4\alpha^3 t)\,\partial t}{\beta^4},$$

vnde colligitur

$$\int v\,\partial v\,\sqrt{(\alpha + \beta\sqrt{v})} = \frac{4t^9}{9\beta^4} - \frac{12\alpha t^7}{7\beta^4} + \frac{12\alpha\alpha t^5}{5\beta^4} - \frac{4\alpha^3 t^3}{3\beta^4}.$$

Haec autem formula iam nimis eſt complicata, quam vt operae pretium foret loco t eius valorem reſtituere; multo minus deinceps quisquam laborem eſſet ſuſcepturus, iſtas formulas integrales ad valores coordinatarum x et y transferendi.

§. 12. Hic igitur nobis ſufficiet oſtendiſſe, etiam hoc caſu curuas prodituras eſſe algebraicas, quod iam porro ſponte elucebit pro ſequentibus caſibus $U = \sqrt[3]{v}$; $U = \sqrt[4]{v}$; atque in genere $U = \sqrt[i]{v}$, quo caſu, poſito $\sqrt{(\alpha + \beta\sqrt[i]{v})} = t$, erit $\sqrt[i]{v} = \frac{tt - \alpha}{\beta}$, ideoque $v = (\frac{tt - \alpha}{\beta})^i$, ita vt v ſit functio rationalis integra ipſius t, dummodo exponens i fuerit poſitiuus et integer. Integratio igitur iſtarum formularum ſemper erit in poteſtate; quocirca etiam omnes iſti caſus perpetuo valores algebraicos pro coordinatis x et y ſuppeditabunt.

Alia Solutio
per angulos inſtituenda.

§. 13. Vtemur hic poſterioribus formulis §. 4. traditis, vbi ob $P = 1$ et $Q = v$ habebimus

$$\partial x = \partial v\,\text{ſin.}\,\Phi + v\,\partial v\,\text{coſ.}\,\Phi \text{ et}$$

$$\partial y = \partial v\,\text{coſ.}\,\Phi - v\,\partial v\,\text{ſin.}\,\Phi.$$

Hic ſcilicet requiritur, vt eiusmodi angulus Φ exploretur, quo iſtae formulae euadant integrabiles. Hoc facillime praeſtabitur, ſtatuendo $v = \text{ſin.}\,\theta$, vt ſit $\partial v = \partial\theta\,\text{coſ.}\,\theta$, quo facto erit

$$\partial x = \partial\theta \text{ cos. } \theta \text{ sin. } \Phi + \partial\theta \text{ sin. } \theta \text{ cos. } \theta \text{ cos. } \Phi \text{ et}$$
$$\partial y = \partial\theta \text{ cos. } \theta \text{ cos. } \Phi - \partial\theta \text{ sin. } \theta \text{ cos. } \theta \text{ sin. } \Phi.$$

Est vero $\text{sin. } \theta \text{ cos. } \theta = \frac{1}{2} \text{ sin. } 2\theta$,

$$\text{sin. } p \text{ cos. } q = \tfrac{1}{2} \text{ sin. } (p+q) + \tfrac{1}{2} \text{ sin. } (p-q),$$
$$\text{sin. } p \text{ sin. } q = \tfrac{1}{2} \text{ cos. } (p-q) - \tfrac{1}{2} \text{ cos. } (p+q),$$
$$\text{cos. } p \text{ cos. } q = \tfrac{1}{2} \text{ cos. } (p+q) - \tfrac{1}{2} \text{ cos. } (p+q).$$

His igitur reductionibus in subsidium vocatis reperiemus:

$$\frac{\partial x}{\partial\theta} = \tfrac{1}{2} \text{ sin. }(\Phi+\theta) + \tfrac{1}{2} \text{ sin. }(\Phi-\theta) + \tfrac{1}{4} \text{ sin. }(2\theta+\Phi) + \tfrac{1}{4} \text{ sin. }(2\theta-\Phi)$$
$$\frac{\partial y}{\partial\theta} = \tfrac{1}{2} \text{ cos. }(\Phi-\theta) + \tfrac{1}{2} \text{ cos. }(\Phi+\theta) - \tfrac{1}{4} \text{ cos. }(2\theta-\Phi) + \tfrac{1}{4} \text{ cos. }(2\theta+\Phi).$$

§. 14. Iam vero euidens est, singulas has partes integrationem esse admissuras, si modo anguli Φ et θ rationem inter se teneant rationalem. Sit igitur $\Phi = \lambda\theta$, existente λ numero quocunque, siue integro, siue fracto, siue positiuo, siue negatiuo, quin etiam generalius statui poterit $\Phi = \lambda\theta + \alpha$, quo facto habebimus

$$\frac{\partial x}{\partial\theta} = \tfrac{1}{2} \text{ sin. } [(\lambda+1)\theta+\alpha] + \tfrac{1}{2} \text{ sin. } [(\lambda-1)\theta+\alpha]$$
$$+ \tfrac{1}{4} \text{ sin. } [(\lambda+2)\theta+\alpha] - \tfrac{1}{4} \text{ sin } [(\lambda-2)\theta+\alpha].$$
$$\frac{\partial y}{\partial\theta} = \tfrac{1}{2} \text{ cos. } [(\lambda-1)\theta+\alpha] + \tfrac{1}{2} \text{ cos. } [(\lambda+1)\theta+\alpha]$$
$$- \tfrac{1}{4} \text{ cos. } [(\lambda-2)\theta+\alpha] + \tfrac{1}{4} \text{ cos. } [(\lambda+2)\theta+\alpha],$$

tum autem integratio nobis praebebit istas expressiones:

$$x = -\frac{\text{cos. } [(\lambda+1)\theta+\alpha]}{2(\lambda+1)} - \frac{\text{cos. } [(\lambda-1)\theta+\alpha]}{2(\lambda-1)} - \frac{\text{cos. } [(\lambda+2)\theta+\alpha]}{4(\lambda+2)}$$
$$+ \frac{\text{cos. } [(\lambda-2)\theta+\alpha]}{4(\lambda-2)},$$
$$y = +\frac{\text{sin. } [(\lambda-1)\theta+\alpha]}{2(\lambda-1)} + \frac{\text{sin. } [(\lambda+1)\theta+\alpha]}{2(\lambda+1)} - \frac{\text{sin. } [(\lambda-2)\theta+\alpha]}{4(\lambda-2)}$$
$$+ \frac{\text{sin. } [(\lambda+2)\theta+\alpha]}{4(\lambda+2)},$$

quae formulae semper ergo erunt algebraicae, nisi fuerit vel $\lambda = \pm 1$, vel $\lambda = \pm 2$.

§. 15. Consideremus casum quo $\lambda = \frac{1}{2}$ et $\alpha = 0$, ac reperietur

$$x = \text{cos.}\, \tfrac{1}{2}\theta - \tfrac{1}{3}\, \text{cos.}\, \tfrac{3}{2}\theta - \tfrac{1}{10}\, \text{cos.}\, \tfrac{5}{2}\theta \quad \text{et}$$
$$y = \text{sin.}\, \tfrac{1}{2}\theta + \tfrac{1}{6}\, \text{sin.}\, \tfrac{3}{2}\theta + \tfrac{1}{10}\, \text{sin.}\, \tfrac{5}{2}\theta.$$

Porro cum sit

$$\text{sin.}\, \tfrac{1}{2}\theta = \text{sin.}\, \tfrac{3}{2}\theta\, \text{cos.}\, \theta - \text{cos.}\, \tfrac{3}{2}\theta\, \text{sin.}\, \theta,$$
$$\text{cos.}\, \tfrac{1}{2}\theta = \text{cos.}\, \tfrac{3}{2}\theta\, \text{cos.}\, \theta + \text{sin.}\, \tfrac{3}{2}\theta\, \text{sin.}\, \theta,$$
$$\text{sin.}\, \tfrac{5}{2}\theta = \text{sin.}\, \tfrac{3}{2}\theta\, \text{cos.}\, \theta + \text{cos.}\, \tfrac{3}{2}\theta\, \text{sin.}\, \theta,$$
$$\text{cos.}\, \tfrac{5}{2}\theta = \text{cos.}\, \tfrac{3}{2}\theta\, \text{cos.}\, \theta - \text{sin.}\, \tfrac{3}{2}\theta\, \text{sin.}\, \theta,$$

his valoribus substitutis habebimus

$$x = \tfrac{9}{10}\, \text{cos.}\, \tfrac{3}{2}\theta\, \text{cos.}\, \theta + \tfrac{11}{10}\, \text{sin.}\, \tfrac{3}{2}\theta\, \text{sin.}\, \theta - \tfrac{1}{3}\, \text{cos.}\, \tfrac{3}{2}\theta \quad \text{et}$$
$$y = \tfrac{11}{10}\, \text{sin.}\, \tfrac{3}{2}\theta\, \text{cos.}\, \theta - \tfrac{9}{10}\, \text{cos.}\, \tfrac{3}{2}\, \text{sin.}\, \theta + \tfrac{1}{6}\, \text{sin.}\, \tfrac{3}{2}\theta.$$

Interim tamen et hic calculo satis taedioso foret opus, si hinc aequationem inter x et y elicere vellemus.

§. 16. Euidens est hinc pariter innumerabiles inueniri lineas curuas problemati satisfacientes, quoniam litteras α et λ in infinitum variare licet. Vtrum autem omnes istae solutiones a praecedentibus sint diuersae nec ne, quaestio est altioris indaginis: in priori enim methodo variae solutiones deductae sunt ex variis formulis radicalibus, $\sqrt{v}$, $\sqrt[3]{v}$, $\sqrt[4]{v}$, dum in posteriori petitae sunt ex multiplicatione seu diuisione angulorum. Nulla autem affinitas inter has diuersas determinationes intercedere videtur; atque adeo vix vllum est dubium, quin in linearum ordinibus inferioribus nullae plane dentur eiusmodi curuae, quarum arcus per arcus parabolicos exprimere liceat.

Adhuc alia Solutio eiusdem Problematis.

§. 17. Ponamus hic ſtatim $v = \text{ſin.}\,\theta$, vt formula noſtra adimplenda ſit

$$\sqrt{(\partial x^2 + \partial y^2)} = \partial\theta\,\text{coſ.}\,\theta\,\sqrt{(1 + \text{ſin.}\,\theta^2)}$$
$$= \partial\theta\,\text{coſ.}\,\theta\,\sqrt{(\text{coſ.}\,\theta^2 + 2\,\text{ſin.}\,\theta^2)}.$$

Faciamus $P = \text{coſ.}\,\theta$ et $Q = \text{ſin.}\,\theta\sqrt{2} = n\,\text{ſin.}\,\theta$, exiſtente $\partial v = \partial\theta\,\text{coſ.}\,\theta$, et nunc ex §. 4. habebimus

$$\frac{\partial x}{\partial\theta\,\text{coſ.}\,\theta} = \text{coſ.}\,\theta\,\text{ſin.}\,\Phi + n\,\text{ſin.}\,\theta\,\text{coſ.}\,\Phi \text{ et}$$
$$\frac{\partial y}{\partial\theta\,\text{coſ.}\,\theta} = \text{coſ.}\,\theta\,\text{coſ.}\,\Phi - n\,\text{ſin.}\,\theta\,\text{ſin.}\,\Phi,$$

quae aequationes in $\text{coſ.}\,\theta$ ductae, ob $\text{coſ.}\,\theta^2 = \frac{1}{2} + \frac{1}{2}\,\text{coſ.}\,2\theta$ et $\text{ſin.}\,\theta\,\text{coſ.}\,\theta = \frac{1}{2}\,\text{ſin.}\,2\theta$, abeunt in iſtas:

$$\frac{2\partial x}{\partial\theta} = \text{ſin.}\,\Phi + \text{coſ.}\,2\theta\,\text{ſin.}\,\Phi + n\,\text{ſin.}\,2\theta\,\text{coſ.}\,\Phi \text{ et}$$
$$\frac{2\partial y}{\partial\theta} = \text{coſ.}\,\Phi + \text{coſ.}\,2\theta\,\text{coſ.}\,\Phi - n\,\text{ſin.}\,2\theta\,\text{ſin.}\,\Phi,$$

hae autem porro ob

$$\text{coſ.}\,2\theta\,\text{ſin.}\,\Phi = \tfrac{1}{2}\,\text{ſin.}\,(\Phi + 2\theta) + \tfrac{1}{2}\,\text{ſin.}\,(\Phi - 2\theta) \text{ et}$$
$$\text{ſin.}\,2\theta\,\text{coſ.}\,\Phi = \tfrac{1}{2}\,\text{ſin.}\,(\Phi + 2\theta) - \tfrac{1}{2}\,\text{ſin.}\,(\Phi - 2\theta),$$
$$\text{coſ.}\,2\theta\,\text{coſ.}\,\Phi = \tfrac{1}{2}\,\text{coſ.}\,(\Phi + 2\theta) + \tfrac{1}{2}\,\text{coſ.}\,(\Phi - 2\theta) \text{ et}$$
$$\text{ſin.}\,2\theta\,\text{ſin.}\,\Phi = \tfrac{1}{2}\,\text{coſ.}\,(\Phi - 2\theta) - \tfrac{1}{2}\,\text{coſ.}\,(\Phi + 2\theta),$$

transformabuntur in ſequentes:

$$\frac{4\partial x}{\partial\theta} = 2\,\text{ſin.}\,\Phi + (n+1)\,\text{ſin.}\,(\Phi + 2\theta) - (n-1)\,\text{ſin.}\,(\Phi - 2\theta) \text{ et}$$
$$\frac{4\partial y}{\partial\theta} = 2\,\text{coſ.}\,\Phi + (n+1)\,\text{coſ.}\,(\Phi + 2\theta) - (n-1)\,\text{coſ.}\,(\Phi - 2\theta).$$

§. 18.

§. 18. Nunc ambae istae formulae sponte rectificabiles reddentur, si modo statuatur $\Phi = \alpha + \lambda\theta$, tum enim pro coordinatis curuae quaesitae habebimus

$$4x = -\frac{2}{\lambda}\operatorname{cos.}(\alpha+\lambda\theta) + \frac{n+1}{\lambda+2}\operatorname{cos.}(\alpha+(\lambda+2)\theta) + \frac{n-1}{\lambda-2}\operatorname{cos.}(\alpha+(\lambda-2)\theta).$$

$$4y = \frac{2}{\lambda}\operatorname{fin.}(\alpha+\lambda\theta) + \frac{n+1}{\lambda+2}\operatorname{fin.}(\alpha+(\lambda+2)\theta) - \frac{n-1}{\lambda-2}\operatorname{fin.}(\alpha+(\lambda-2)\theta).$$

Hae igitur ambae formulae erunt algebraicae, dummodo ne fit vel $\lambda = 2$, vel $\lambda = -2$, reliquis casibus omnibus, quibus λ est numerus rationalis, siue integer, siue fractus, curua prodibit algebraica.

§. 19. Hic ergo sine dubio casus elicietur simplicissimus, si capiatur $\lambda = 1$ et $\alpha = 0$, tum enim habebimus

$$4x = -(n+1)\operatorname{cos.}\theta - \frac{(n+1)}{3}\operatorname{cos.}3\theta \text{ et}$$

$$4y = -(n-3)\operatorname{fin.}\theta + \frac{(n+1)}{3}\operatorname{fin.}3\theta,$$

vbi litteram n scripsimus loco $\sqrt{2}$.

§. 20. Ad has formulas tractandas ponamus tang. $\theta = t$, fietque

$$\operatorname{fin.}\theta = \frac{t}{\sqrt{(1+tt)}} \text{ et } \operatorname{cof.}\theta = \frac{1}{\sqrt{(1+tt)}},$$

tum vero erit tang. $3\theta = \frac{3t-t^3}{1-3tt}$, vnde fit

$$\operatorname{fin.}3\theta = \frac{3t-t^3}{(1+tt)^{\frac{3}{2}}} \text{ et } \operatorname{cof.}3\theta = \frac{1-3tt}{(1+tt)^{\frac{3}{2}}},$$

quibus valoribus substitutis reperiemus

$$-4x = \frac{+(n+1)}{\sqrt{(1+tt)}} + \frac{(n+1)(1-3tt)}{3(1+tt)^{\frac{3}{2}}} = \frac{4(n+1)}{3(1+tt)^{\frac{3}{2}}},$$

ideoque

$$x = \frac{-(n+1)}{(1+tt)^{\frac{3}{2}}} \text{ et}$$

$$y = \frac{t}{(1+tt)^{\frac{3}{2}}}(3-(n+2)tt).$$

Diuidatur posterior aequatio per priorem et prodibit

$$\frac{(n+1)y}{x} = (n+2)t^3 - 3t,$$

Hinc autem satis liquet, si vellemus quantitatem t eliminare, aequationem inter x et y ad plurimas dimensiones esse adscensuram. Sufficiat igitur tres formulas generales exhibuisse, quarum singulae innumerabiles curuas algebraicas suppeditare possunt, ita vt in omnibus longitudo arcus curuae $\int\sqrt{(\partial x^2 + \partial y^2)}$, aequetur arcui parabolico: $\int \partial v \sqrt{(1+vv)}$.

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