1789

De multiplicatione angulorum per factores expedienda

Leonhard Euler

DE MVLTIPLICATIONE ANGVLORVM PER FACTORES EXPEDIENDA.

Auctore

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§. 1.

Denotet ϕ angulum quemcunque propositum, sitque $n\phi$ eius multiplum, cuius tam sinum quam cosinum per factores exprimere oporteat. Ad hoc praestandum in subsidium vocetur formula imaginaria $u = \cos.\phi + \sqrt{-1}.\sin.\phi$, eritque $\frac{1}{u} = u^{-1} = \cos.\phi - \sqrt{-1}.\sin.\phi$, vnde ergo sequitur fore $u + u^{-1} = 2\cos.\phi$ et $u - u^{-1} = 2\sqrt{-1}.\sin.\phi$. Constat autem similem fore rationem omnium potestatum huius formulae, siquidem erit $u^n = \cos.n\phi + \sqrt{-1}\sin.n\phi$ et $u^{-n} = \cos.n\phi - \sqrt{-1}\sin.n\phi$, atque hinc colligimus

$$u^n + u^{-n} = 2\cos.n\phi \text{ et } u^n - u^{-n} = 2\sqrt{-1}\sin.n\phi.$$

Hinc igitur sequitur fore

$$\cos.n\phi = \tfrac{1}{2}(u^n + u^{-n}) \text{ et } \sin.n\phi = \frac{1}{2\sqrt{-1}}(u^n - u^{-n}).$$

Per huiusmodi ergo formulas etiam tangentem et cotangentem, item secantem et cosecantem anguli multipli $n\phi$ exprimere licebit, ita vt sufficiat eius tantum sinum et cosinum in factores resoluisse, quorum vtrumque hic seorsim euoluamus.

 I. Re-

I. Resolutio cosinus anguli multipli $n\phi$ in factores.

§. 2. Cum sit, vt modo vidimus,
$\cos. n\phi = \frac{1}{2}(u^n + u^{-n})$,
totum negotium huc redit, vt formula $u^n + u^{-n}$ in suos factores resoluatur. Facile autem perspicitur, singulos huius formulae factores talem formam esse habituros: $u - 2\cos.\omega + u^{-1}$, ac numerum talium factorum esse debere $= n$; quandoquidem si n huiusmodi factores in se inuicem multiplicentur, summa potestas positiua ipsius u vtique prodibit u^n, summa autem potestas negatiua $= u^{-n}$; ex quo necesse est, vt omnes potestates intermediae ipsius u se mutuo destruant. Ex hac igitur conditione singulos angulos ω definire oportebit, id quod vtique insignem laborem postularet; vnde hanc inuestigationem ex ipsa indole factorum deriuemus.

§. 3. Hic autem perspicuum est, si haec formula: $u - 2\cos.\omega + u^{-1}$, fuerit factor formae propositae $u^n + u^{-n}$, angulum ω ita comparatum esse debere, vt si statueretur iste factor $u - 2\cos.\omega + u^{-1} = 0$, tum etiam ipsa forma proposita $u^n + u^{-n}$ ad nihilum redigeretur. Neque vero hic putandum est, istam formulam reuera nihilo aequalem supponi, sed tantum hypothetice asseritur, si ista formula, esset $= 0$, etiamsi id fieri nequeat, tum etiam ipsam formam propositam in nihilum esse abituram; quemadmodum scilicet nouimus, formulae $1 - xx$ factorem esse $1 - x$, propterea quod posito $1 - x = 0$, siue $x = 1$, etiam fiat $1 - xx = 0$, etiamsi quantitas x fortasse nunquam vnitati fiat aequalis.

§. 4. Iam vero si esset $u - 2\cos.\omega + u^{-1} = 0$, inde foret $u = \cos.\omega + \sqrt{-1}\sin.\omega$, quamobrem hic valor, in formam $u^n + u^{-n}$ introductus, eam ad nihilum redigere deberet; tum autem foret, vt supra notauimus:

$$u^n = \text{cof.}\, n\,\omega + \sqrt{-1}\,\text{fin.}\, n\,\omega \text{ et}$$
$$u^{-n} = \text{cof.}\, n\,\omega - \sqrt{-1}\,\text{fin.}\, n\,\omega,$$

ficque forma propofita fiet $= 2\,\text{cof.}\, n\,\omega$, quae ergo quantitas neceffario euanefcere debet, quia aliter formula affumta factor formae propofitae effe non poffet.

§. 5. Hinc igitur patet angulum ω ita accipi debere, vt eius multipli $n\,\omega$ cofinus euanefcat; vnde manifeftum eft, fi ϱ defignet angulum rectum, tum ifti conditioni fatisfieri, fi ftatuatur vel $n\,\omega = \varrho$; vel $n\,\omega = 3\,\varrho$; vel $n\,\omega = 5\,\varrho$; vel $n\,\omega = 7\,\varrho$; etc. ex quo valores idonei pro angulo quaefito ω erunt fequentes: 1°. $\frac{\varrho}{n}$; 2°. $\frac{3\varrho}{n}$; 3°. $\frac{5\varrho}{n}$; 4°. $\frac{7\varrho}{n}$; etc. et in genere $\frac{i\varrho}{n}$, denotante i numerum imparem quemcunque; ita vt horum valorum numerus reuera fit infinitus, cum tamen pro noftro inftituto tantum n valoribus diuerfis indigeamus.

§. 6. Quanquam autem numerus valorum pro ω inuentorum fit infinitus, tamen inter eos innumerabiles dantur, qui pro cof. ω eundem valorem producunt. Cum enim omnibus iftis angulis: ψ, $4\varrho + \psi$, $8\varrho + \psi$, etc. idem cofinus conueniat, etiam omnes ifti anguli: $\frac{\varrho}{n}$, $\frac{(4n+1)\varrho}{n}$, $\frac{(8n+1)\varrho}{n}$, etc. communi gaudebunt cofinu, fimilique modo etiam omnes hi anguli: $\frac{3\varrho}{n}$, $\frac{(4n+3)\varrho}{n}$, $\frac{(8n+3)\varrho}{n}$, etc., tum vero etiam ifti: $\frac{5\varrho}{n}$, $\frac{(4n+5)\varrho}{n}$, $\frac{(8n+5)\varrho}{n}$, etc. communi cofinu erunt praediti; vnde ex quolibet ordine vnicum tantum valorem, eumque fimpliciffimum, fumi conueniet, donec eorum numerus fiat $= n$.

§. 7. Ex his igitur colligimus, omnes valores idoneos pro angulo ω affumendos ordine ita progredi:

$$\begin{array}{cccc} 1. & 2. & 3. & 4. \\ \frac{\varrho}{n}; & \frac{3\varrho}{n}; & \frac{5\varrho}{n}; & \frac{7\varrho}{n}; \text{ etc.} \end{array}$$

quorum numerus cum debeat esse $= n$, vltimus eorum erit $\frac{(2n-1)\varrho}{n}$. Si enim vlterius progredi vellemus, ad eiusmodi angulos perueniremus, quorum cosinus iam in praecedentibus occurrerent. Anguli enim proxime sequentis $\frac{(2n+1)\varrho}{n}$ cosinus foret ipsius vltimi $\frac{(2n-1)\varrho}{n}$ cosinui aequalis: in genere enim hi duo anguli ψ et $4\varrho - \psi$ communem habent cosinum; vnde si fuerit $\psi = \frac{(2n+1)\varrho}{n}$, erit $4\varrho - \psi = \frac{(2n-1)\varrho}{n}$, qui est terminus vltimus. Simili modo sequentium secundus $\frac{(2n+3)\varrho}{n}$, eundem habet cosinum quem angulus $\frac{(2n-3)\varrho}{n}$ habet, qui in nostro ordine est penultimus. Eodem modo sequentium tertius $\frac{(2n+5)\varrho}{n}$ cum nostro antepenultimo $\frac{(2n-5)\varrho}{n}$ communem habebit cosinum; quod idem de reliquis sequentibus est tenendum, quippe quorum omnium cosinus iam in nostro ordine occurrunt; vnde patet omnes valores idoneos anguli ω, quorum numerus est $= n$, contineri in hac serie:

$$\begin{array}{cccccccccc} 1. & 2. & 3. & 4. & . & . & . & . & . & n \\ \frac{\varrho}{n}; & \frac{3\varrho}{n}; & \frac{5\varrho}{n}; & \frac{7\varrho}{n}; & . & . & . & . & & \frac{(2n-1)\varrho}{n}. \end{array}$$

§. 8. Cum igitur, si pro ω accipiatur valor quicunque huius progressionis, ista formula $u - 2\,\text{cos}.\,\omega + u^{-1}$ certe sit factor formae $u^n + u^{-n}$, restituamus nunc valorem initio assumtum $u = \text{cos}.\,\Phi + \sqrt{-1}\,.\,\text{sin}.\,\Phi$, et quia hinc $u + u^{-1} = 2\,\text{cos}.\,\Phi$, huius formae $u^n + u^{-n}$ factor quicunque erit $2\,(\text{cos}.\,\Phi - \text{cos}.\,\omega)$, quare si loco ω successiue omnes eius valores scribamus, omnes factores obtinebimus, quorum numerus cum sit $= n$, forma nostra $u^n + u^{-n}$ aequabitur huic producto ex n factoribus constanti:

2^n

$$2^n (\text{cos}.\,\Phi - \text{cos}.\,\tfrac{\varrho}{n})(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{3\varrho}{n})(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{5\varrho}{n}) \times$$
$$\times (\text{cos}.\,\Phi - \text{cos}.\,\tfrac{7\varrho}{n}) \ldots (\text{cos}.\,\Phi - \text{cos}.\,\tfrac{(2n-1)\varrho}{n}).$$

§. 9. Cum igitur ſit $\text{cof}.\,n\Phi = \frac{1}{2}(u^n + u^{-n})$, ſi hoc productum ſubſtituamus, coſinus anguli $n\Phi$ ſequenti modo per productum ex n factoribus compoſitum exprimetur:

$$\text{cos}.\,n\Phi = 2^{n-1}(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{\varrho}{n})(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{3\varrho}{n}) \times$$
$$\times (\text{cos}.\,\Phi - \text{cos}.\,\tfrac{5\varrho}{n}) \ldots (\text{cos}.\,\Phi - \text{cos}.\,\tfrac{(2n-1)\varrho}{n}),$$

vnde ſequentes deducimus reſolutiones ſpeciales, dum loco n ordine numeros 1, 2, 3, 4, etc. aſſumimus:

$$\text{cos}.\,1\Phi = \text{cos}.\,\Phi - \text{cos}.\,\tfrac{\varrho}{1} = \text{cos}.\,\Phi.$$

$$\text{cos}.\,2\Phi = 2(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{1}{2}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{3}{2}\varrho).$$

$$\text{cos}.\,3\Phi = 4(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{1}{3}\varrho)(\text{cos}.\,\Phi - 0)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{5}{3}\varrho).$$

$$\text{cos}.\,4\Phi = 8(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{1}{4}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{3}{4}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{5}{4}\varrho) \times$$
$$\times (\text{cos}.\,\Phi - \text{cos}.\,\tfrac{7}{4}\varrho).$$

$$\text{cos}.\,5\Phi = 16(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{1}{5}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{3}{5}\varrho)(\text{cos}.\,\Phi - 0) \times$$
$$\times (\text{cos}.\,\Phi - \text{cos}.\,\tfrac{7}{5}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{9}{5}\varrho)$$

$$\text{cos}.\,6\Phi = 32(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{1}{6}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{1}{2}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{5}{6}\varrho) \times$$
$$\times (\text{cos}.\,\Phi - \text{cos}.\,\tfrac{7}{6}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{3}{2}\varrho)(\text{cos}.\,\Phi - \text{cos}.\,\tfrac{11}{6}\varrho).$$

etc. etc.

§. 10. Quodſi omnes iſtos factores attentius conſideremus, reperiemus binos factores ab extremis aeque diſtantes commode in vnum contrahi poſſe. Cum enim duorum angulorum, quorum ſumma eſt $180° = 2\varrho$, coſinus ſint aequales, ſed contrario ſigno affecti, ob angulum $\frac{(2n-1)\varrho}{n} = 2\varrho - \frac{\varrho}{n}$, eius coſinus erit $= -\text{cof}.\,\frac{\varrho}{n}$, ideoque vltimus factor $= \text{cof}.\,\Phi$

$+$ cof.

$+ \operatorname{cos}. \frac{\varrho}{n}$, qui ductus in primum praebet productum $\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \frac{\varrho}{n})^2$. Simili modo pro penultimo factore, ob $\frac{(2n-3)\varrho}{n} = 2\varrho - \frac{3\varrho}{n}$, huius anguli cosinus erit $= - \operatorname{cos}. \frac{3\varrho}{n}$, ideoque ipse factor penultimus $= \operatorname{cos}. \Phi + \operatorname{cos}. \frac{3\varrho}{n}$, qui ductus in secundum dabit productum $\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \frac{3\varrho}{n})^2$. Eodem modo factor tertius cum antepenultimo coniunctus dabit hoc productum $\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \frac{5\varrho}{n})^2$. Hoc igitur modo numerus factorum ad semissem reducetur, si numerus n fuerit par, sin autem fuerit impar, facta hac contractione solitarius relinquetur terminus medius, qui semper erit $\operatorname{cos}. \Phi$, ob $\operatorname{cos}. \varrho = 0$.

§. 11. Quodsi ergo hoc modo resolutionem in factores exhibere velimus, ea pro casibus simplicioribus ita se habebit:

$\operatorname{cos}. \Phi = \operatorname{cos}. \Phi.$

$\operatorname{cos}. 2\Phi = 2[\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{1}{2}\varrho)^2].$

$\operatorname{cos}. 3\Phi = 4[\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{1}{3}\varrho)^2] \operatorname{cos}. \Phi.$

$\operatorname{cos}. 4\Phi = 8[\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{1}{4}\varrho)^2][\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{3}{4}\varrho)^2].$

$\operatorname{cos}. 5\Phi = 16[\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{1}{5}\varrho)^2][\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{3}{5}\varrho)^2] \operatorname{cos}. \Phi.$

$\operatorname{cos}. 6\Phi = 32[\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{1}{6}\varrho)^2][\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{3}{6}\varrho)^2] \times$
$\times [\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{5}{6}\varrho)^2].$

$\operatorname{cos}. 7\Phi = 64[\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{1}{7}\varrho)^2][\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{3}{7}\varrho)^2] \times$
$\times [\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{5}{7}\varrho)^2] \operatorname{cos}. \Phi.$

$\operatorname{cos}. 8\Phi = 128[\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{1}{8}\varrho)^2][\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{3}{8}\varrho)^2] \times$
$\times [\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{5}{8}\varrho)^2][\operatorname{cos}. \Phi^2 - (\operatorname{cos}. \tfrac{7}{8}\varrho)^2].$

etc. etc.

§. 12. Iſtos autem factores in formam adhuc conciniorem transmutare licet. Cum enim in genere ſit

$$\text{coſ.}\,\psi^2 = \tfrac{1}{2} + \tfrac{1}{2}\,\text{coſ.}\,2\,\psi, \text{ erit}$$

$$\text{coſ.}\,\Phi^2 - \text{coſ.}\,\psi^2 = \tfrac{1}{2}\,\text{coſ.}\,2\,\Phi - \tfrac{1}{2}\,\text{coſ.}\,2\,\psi;$$

tum vero etiam conſtat eſſe in genere

$$\text{coſ.}\,2\,\Phi - \text{coſ.}\,2\,\psi = 2\,\text{ſin.}\,(\Phi + \psi)\,\text{ſin.}\,(\psi - \Phi),$$

conſequenter

$$\text{coſ.}\,\Phi^2 - \text{coſ.}\,\psi^2 = \text{ſin.}\,(\Phi + \psi)\,\text{ſin.}\,(\psi - \Phi).$$

§. 13. Quodſi ergo iſta reductione in ſuperioribus formulis vtamur, ſinguli factores ad ſimplices ſinus reuocabuntur; habebimus enim:

$$\text{coſ.}\,\Phi = \text{coſ.}\,\Phi.$$

$$\text{coſ.}\,2\,\Phi = 2\,\text{ſin.}\,(\tfrac{1}{2}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{1}{2}\varrho - \Phi)$$

$$\text{coſ.}\,3\,\Phi = 4\,\text{ſin.}\,(\tfrac{1}{3}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{1}{3}\varrho - \Phi)\,\text{coſ.}\,\Phi.$$

$$\text{coſ.}\,4\,\Phi = 8\,\text{ſin.}\,(\tfrac{1}{4}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{1}{4}\varrho - \Phi)\,\text{ſin.}\,(\tfrac{3}{4}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{3}{4}\varrho - \Phi).$$

$$\text{coſ.}\,5\,\Phi = 16\,\text{ſin.}\,(\tfrac{1}{5}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{1}{5}\varrho - \Phi)\,\text{ſin.}\,(\tfrac{3}{5}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{3}{5}\varrho - \Phi) \times \text{coſ.}\,\Phi$$

$$\text{coſ.}\,6\,\Phi = 32\,\text{ſin.}\,(\tfrac{1}{6}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{1}{6}\varrho - \Phi)\,\text{ſin.}\,(\tfrac{3}{6}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{3}{6}\varrho - \Phi) \times \text{ſin.}\,(\tfrac{5}{6}\varrho + \Phi)\,\text{ſin.}\,(\tfrac{5}{6}\varrho - \Phi).$$

etc. etc.

vnde in genere concludimus fore

$$\text{coſ.}\,n\,\Phi = 2^{n-1}\,\text{ſin.}\,(\tfrac{\varrho}{n} + \Phi)\,\text{ſin.}\,(\tfrac{\varrho}{n} - \Phi)\,\text{ſin.}\,(\tfrac{3\varrho}{n} + \Phi) \times \text{ſin.}\,(\tfrac{3\varrho}{n} - \Phi)\,\text{ſin.}\,(\tfrac{5\varrho}{n} + \Phi)\,\text{ſin.}\,(\tfrac{5\varrho}{n} - \Phi) \text{ etc.}$$

Vbi obſeruetur, ſi n fuerit numerus impar, tum vltimo factori accedere coſ. Φ. Caeterum hos factores eo vsque produci oportet, donec eorum numerus fiat $= n$.

§. 14. Si quis maluerit omnes iſtos factores etiam per coſinus exhibere, talis transformatio in promptu eſt. Cum enim ſit ſin. $\psi = \text{coſ.}\,(\varrho - \psi)$, noſtrae reſolutiones ſequenti modo procedent:

$$\text{coſ.}\,\phi = \text{coſ.}\,\phi.$$

$$\text{coſ.}\,2\phi = 2\,\text{coſ.}\,(\tfrac{1}{2}\varrho - \phi)\,\text{coſ.}\,(\tfrac{1}{2}\varrho + \phi).$$

$$\text{coſ.}\,3\phi = 4\,\text{coſ.}\,(\tfrac{2}{3}\varrho - \phi)\,\text{coſ.}\,(\tfrac{2}{3}\varrho + \phi)\,\text{coſ.}\,\phi.$$

$$\text{coſ.}\,4\phi = 8\,\text{coſ.}\,(\tfrac{3}{4}\varrho - \phi)\,\text{coſ.}\,(\tfrac{3}{4}\varrho + \phi)\,\text{coſ.}\,(\tfrac{1}{4}\varrho - \phi)\,\text{coſ.}\,(\tfrac{1}{4}\varrho + \phi)$$

$$\text{coſ.}\,5\phi = 16\,\text{coſ.}\,(\tfrac{4}{5}\varrho - \phi)\,\text{coſ.}\,(\tfrac{4}{5}\varrho + \phi)\,\text{coſ.}\,(\tfrac{2}{5}\varrho - \phi)\times$$
$$\times\,\text{coſ.}\,(\tfrac{2}{5}\varrho + \phi)\,\text{coſ.}\,\phi.$$

$$\text{coſ.}\,6\phi = 32\,\text{coſ.}\,(\tfrac{5}{6}\varrho - \phi)\,\text{coſ.}\,(\tfrac{5}{6}\varrho + \phi)\,\text{coſ.}\,(\tfrac{3}{6}\varrho - \phi)\,\text{coſ.}\,(\tfrac{3}{6}\varrho + \phi)\times$$
$$\times\,\text{coſ.}\,(\tfrac{1}{6}\varrho - \phi)\,\text{coſ.}\,(\tfrac{1}{6}\varrho + \phi).$$

$$\text{coſ.}\,7\phi = 64\,\text{coſ.}\,(\tfrac{6}{7}\varrho - \phi)\,\text{coſ.}\,(\tfrac{6}{7}\varrho + \phi)\,\text{coſ.}\,(\tfrac{4}{7}\varrho - \phi)\,\text{coſ.}\,(\tfrac{4}{7}\varrho + \phi)\times$$
$$\times\,\text{coſ.}\,(\tfrac{2}{7}\varrho - \phi)\,\text{coſ.}\,(\tfrac{2}{7}\varrho + \phi)\,\text{coſ.}\,\phi.$$

$$\text{coſ.}\,8\phi = 128\,\text{coſ.}\,(\tfrac{7}{8}\varrho - \phi)\,\text{coſ.}\,(\tfrac{7}{8}\varrho + \phi)\,\text{coſ.}\,(\tfrac{5}{8}\varrho - \phi)\,\text{coſ.}\,(\tfrac{5}{8}\varrho + \phi)\times$$
$$\times\,\text{coſ.}\,(\tfrac{3}{8}\varrho - \phi)\,\text{coſ.}\,(\tfrac{3}{8}\varrho + \phi)\,\text{coſ.}\,(\tfrac{1}{8}\varrho - \phi)\times$$
$$\times\,\text{coſ.}\,(\tfrac{1}{8}\varrho + \phi).$$

§. 15. Quo igitur ordo in his expreſſionibus clarius ob oculos ponatur, duos caſus diſtingui conueniet, quos hic ſeorſim referemus, prouti numerus n fuerit vel par vel impar.

Caſu quo $n = 2i$, erit:

$$\text{coſ.}\,2i\phi = 2^{2i-1}\,\text{coſ.}\,(\tfrac{\varrho}{2i} + \phi)\,\text{coſ.}\,(\tfrac{\varrho}{2i} - \phi)\,\text{coſ.}\,(\tfrac{3\varrho}{2i} + \phi)\times$$
$$\times\,\text{coſ.}\,(\tfrac{3\varrho}{2i} - \phi)\,\text{coſ.}\,(\tfrac{5\varrho}{2i} + \phi)\,\text{coſ.}\,(\tfrac{5\varrho}{2i} - \phi)\;\ldots\;\times$$
$$\times\,\text{coſ.}\,(\tfrac{(2i-1)\varrho}{2i} + \phi)\,\text{coſ.}\,(\tfrac{(2i-1)\varrho}{2i} - \phi).$$

Caſu

Casu quo $n = 2i + 1$, erit

$$\cos.(2i+1)\Phi = 2^{2i}\cos.\Phi\cos.\left(\frac{2\rho}{2i+1}+\Phi\right)\times$$
$$\times\cos.\left(\frac{2\rho}{2i+1}-\Phi\right)\cos.\left(\frac{4\rho}{2i+1}+\Phi\right)\cos.\left(\frac{4\rho}{2i+1}-\Phi\right)\times$$
$$\times\cos.\left(\frac{6\rho}{2i+1}+\Phi\right)\cos.\left(\frac{6\rho}{2i+1}-\Phi\right)\;.\;.\;.\;.\;\times$$
$$\times\cos.\left(\frac{2i\rho}{2i+1}+\Phi\right)\cos.\left(\frac{2i\rho}{2i+1}-\Phi\right).$$

II. Resolutio sinus anguli multipli $n\Phi$ in factores.

§. 16. Manentibus denominationibus initio stabilitis iam notauimus esse $\sin. n\Phi = \frac{1}{2\sqrt{-1}}(u^n - u^{-n})$; vnde nobis incumbit in factores huius formae $u^n - u^{-n}$ inquirere. Hic autem statim elucet, cunctos eius factores non habere posse formam supra assumtam $u - 2\cos.\omega + u^{-1}$, quandoquidem ex n huiusmodi factoribus maxima potestas negatiua proditura esset $+u^{-n}$, cum tamen hic sit $-u^{-n}$; facile autem patet formae nostrae propositae $u^n - u^{-n}$ vnum factorem certe esse $u - u^{-1}$, quandoquidem $uu - 1$ semper est factor formae $u^{2n} - 1$. Hoc igitur primo factore constituto, reliquorum, quorum ergo numerus erit $= 2n - 1$, forma tuto assumi potest supra vsurpata $u - 2\cos.\omega + u^{-1}$, vbi ergo angulum ω ita comparatum esse oportet, vt si poneretur $u - 2\cos.\omega + u^{-1} = 0$, id quod fit capiendo $u = \cos.\omega + \sqrt{-1}.\sin.\omega$, ipsa forma proposita $u^n - u^{-n}$ ad nihilum redigeretur.

§. 17. Supra autem iam obseruauimus, si fuerit $u = \cos.\omega + \sqrt{-1}\sin.\omega$, tum prodire

$$u^n - u^{-n} = 2\sqrt{-1}\sin.\omega,$$

 quem

quem ergo valorem nihilo aequalem esse oportet, vnde manente ϱ charactere anguli recti, quia omnium horum angulorum 2ϱ, 4ϱ, 6ϱ, 8ϱ, et in genere $2i\varrho$, sinus euanescunt, idonei anguli pro ω accipiendi hanc progressionem constituent: $\frac{2\varrho}{n}$; $\frac{4\varrho}{n}$; $\frac{6\varrho}{n}$; $\frac{8\varrho}{n}$; etc. quorum quidem numerus est infinitus; verum ob rationes iam ante allegatas hic tantum priores, numero $n-1$, accipi conueniet, ita vt vltimus valorum idoneorum futurus sit $\frac{2(n-1)\varrho}{n}$.

§. 18. Hic quidem videtur, primo loco statui debuisse angulum $\frac{0\varrho}{n}$; verum hinc nasceretur factor $u - 2 + u^{-1}$, qui inuoluit quadratum, dum est $\frac{1}{u}(u-1)^2$, eiusque radix $u-1$, quae his casibus tantum accipi debet, iam in nostro factore primo $u - u^{-1}$ continetur. Praeterea vero in isto factore primo etiam continetur $u+1$, qui autem in valore nostrum ordinem proxime sequente, qui esset $\frac{2n\varrho}{n} = 2\varrho = \omega$, contineretur, siquidem hinc resultaret factor

$$u + 2 + u^{-1} = \frac{1}{u}(u+1)^2.$$

Caeterum vero patet, si angulos illos pro ω datos vltra terminum praescriptum continuare vellemus, eorum cosinus iam in antecedentibus contineri. Veluti si sumeremus $\omega = \frac{2(n+1)\varrho}{n}$, foret $4\varrho - \omega = \frac{2(n-1)\varrho}{n}$, qui in assignatis valoribus iam continetur.

§. 19. Restituamus nunc loco u valorem assumtum $\cos.\Phi - \sqrt{-1}\,\sin.\Phi$, ac pro primo factore habebimus $u - u^{-1} = 2\sqrt{-1}\,\sin.\Phi$. Deinde vero pro quolibet factore reliquorum fiet

$$u - 2\cos.\omega + u^{-1} = 2(\cos.\Phi - \cos.\omega),$$

quamobrem si loco ω successiue valores debitos scribamus,

forma

forma propofita $u^n - u^{-n}$ fequenti producto aequabitur;

$$2^n \sqrt{-1} \,\text{fin.}\, \Phi \,(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2\rho}{n})(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{4\rho}{n}) \times$$
$$\times (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{6\rho}{n})(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{8\rho}{n}),$$

quorum factorum vltimus erit $\text{cof.}\, \Phi - \text{cof.}\, (\frac{2(n-1)\rho}{n})$.

§. 20. Cum igitur fit $\text{fin.}\, n\Phi = \frac{1}{2\sqrt{-1}}(u^n - u^{-n})$, hoc valore, quem modo inuenimus, fubftituto, reperiemus:

$$\text{fin.}\, n\Phi = 2^{n-1} \,\text{fin.}\, \Phi \,(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2\rho}{n})(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{4\rho}{n}) \times$$
$$\times (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{6\rho}{n}) \;.\;.\;.\; (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2(n-1)\rho}{n}),$$

vnde fequentes cafus fpeciales deriuaffe iuuabit:

$\text{fin.}\, \Phi = \text{fin.}\, \Phi.$

$\text{fin.}\, 2\Phi = 2\,\text{fin.}\, \Phi \,\text{cof.}\, \Phi.$

$\text{fin.}\, 3\Phi = 4\,\text{fin.}\, \Phi (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2}{3}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{4}{3}\rho).$

$\text{fin.}\, 4\Phi = 8\,\text{fin.}\, \Phi (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2}{4}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{4}{4}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{6}{4}\rho).$

$\text{fin.}\, 5\Phi = 16\,\text{fin.}\, \Phi (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2}{5}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{4}{5}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{6}{5}\rho) \times$
$\times (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{8}{5}\rho).$

$\text{fin.}\, 6\Phi = 32\,\text{fin.}\, \Phi (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2}{6}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{4}{6}\rho)(\text{cf.}\, \Phi - \text{cf.}\, \tfrac{6}{6}\rho) \times$
$\times (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{8}{6}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{10}{6}\rho).$

$\text{fin.}\, 7\Phi = 64\,\text{fin.}\, \Phi (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2}{7}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{4}{7}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{6}{7}\rho) \times$
$\times (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{8}{7}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{10}{7}\rho) \times$
$\times (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{12}{7}\rho).$

$\text{fin.}\, 8\Phi = 128\,\text{fin.}\, \Phi (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{2}{8}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{4}{8}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{6}{8}\rho) \times$
$\times (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{8}{8}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{10}{8}\rho) \times$
$\times (\text{cof.}\, \Phi - \text{cof.}\, \tfrac{12}{8}\rho)(\text{cof.}\, \Phi - \text{cof.}\, \tfrac{14}{8}\rho).$

etc. etc.

§. 21. Sepofito iam primo factore fin. Φ, in reliquis, quorum numerus eſt $n - 1$, idem vſu venit, quod ante, vt ſcilicet illi factores ab extremis aequidiſtantes in vnum contrahi queant. Cum enim vltimus factor ſit coſ. $\frac{2(n-1)\varrho}{n}$, ob angulum $\frac{2(n-1)\varrho}{n} = 2\varrho - \frac{2\varrho}{n}$, eius coſinus erit $=$ — coſ. $\frac{2\varrho}{n}$, ita vt iſte factor ſit coſ. Φ + coſ. $\frac{2\varrho}{n}$, qui ergo per primum multiplicatus dat productum coſ. $\Phi^2 - ($coſ. $\frac{2\varrho}{n})^2$, Simili modo factores ſecundus et penultimus contrahentur in hoc productum: coſ. $\Phi^2 - ($coſ. $\frac{4\varrho}{n})^2$, hocque modo factorum numerus ad dimidium reuocabitur, ſiquidem $n - 1$ fuerit numerus par, ideoque n impar; caſibus autem, quibus $n - 1$ eſt impar, ideoque n par, inſuper accedet factor medius, qui ſemper eſt coſ. Φ; vnde hoc productum ita ſe habebit:

$$\text{ſin.}\, n\Phi = 2^{n-1}\,\text{ſin.}\,\Phi\,[\text{coſ.}\,\Phi^2 - (\text{coſ.}\,\tfrac{2\varrho}{n})^2]\,[\text{coſ.}\,\Phi^2 - (\text{coſ.}\,\tfrac{4\varrho}{n})^2] \times$$
$$\times\,[\text{coſ.}\,\Phi^2 - \text{coſ.}\,\tfrac{6\varrho}{n})^2]\,[\text{coſ.}\,\Phi^2 - \text{coſ.}\,\tfrac{8\varrho}{n})^2]\ \text{etc.}$$

§. 22. Supra autem vidimus eſſe

$$\text{coſ.}\,\Phi^2 - \text{coſ.}\,\psi^2 = \tfrac{1}{2}\,\text{coſ.}\,2\Phi - \tfrac{1}{2}\,\text{coſ.}\,2\psi\,,$$

quae formula denuo reſoluitur in hos duos factores:

$$\text{ſin.}\,(\Phi + \psi)\,\text{ſin.}\,(\psi - \Phi),$$

ſicque hoc modo omnium factorum numerus erit $n - 1$. Hac igitur reſolutione adhibita reperietur:

$$\text{ſin.}\,n\Phi = 2^{n-1}\,\text{ſin.}\,\Phi\,\text{ſin.}\,(\tfrac{2\varrho}{n} + \Phi)\,\text{ſin.}\,(\tfrac{2\varrho}{n} - \Phi)\,\text{ſin.}\,(\tfrac{4\varrho}{n} + \Phi) \times$$
$$\times\,\text{ſin.}\,(\tfrac{4\varrho}{n} - \Phi)\,\text{ſin.}\,(\tfrac{6\varrho}{n} + \Phi)\,\text{ſin.}\,(\tfrac{6\varrho}{n} - \Phi)\ \text{etc.}$$

ſi modo obſeruetur caſibus, quibus n eſt numerus par, adiungendum eſſe factorem coſ. Φ.

§. 23. Quo indoles huius expreſſionis clarius perſpiciatur, ſequentes caſus ſpeciales euoluamus:

$$\sin.\Phi = \sin.\Phi.$$

$$\sin.2\Phi = 2\sin.\Phi\cos.\Phi.$$

$$\sin.3\Phi = 4\sin.\Phi\sin.(\tfrac{2}{3}\rho+\Phi)\sin.(\tfrac{2}{3}\rho-\Phi).$$

$$\sin.4\Phi = 8\sin.\Phi\sin.(\tfrac{2}{4}\rho+\Phi)\sin.(\tfrac{2}{4}\rho-\Phi)\cos.\Phi.$$

$$\sin.5\Phi = 16\sin.\Phi\sin.(\tfrac{2}{5}\rho+\Phi)\sin.(\tfrac{2}{5}\rho-\Phi)\sin.(\tfrac{4}{5}\rho+\Phi)\times \\ \times\sin.(\tfrac{4}{5}\rho-\Phi).$$

$$\sin.6\Phi = 32\sin.\Phi\sin.(\tfrac{2}{6}\rho+\Phi)\sin.(\tfrac{2}{6}\rho-\Phi)\sin.(\tfrac{4}{6}\rho+\Phi)\sin.(\tfrac{4}{6}\rho-\Phi)\times \\ \times\cos.\Phi.$$

$$\sin.7\Phi = 64\sin.\Phi\sin.(\tfrac{2}{7}\rho+\Phi)\sin.(\tfrac{2}{7}\rho-\Phi)\sin.(\tfrac{4}{7}\rho+\Phi)\sin.(\tfrac{4}{7}\rho-\Phi)\times \\ \times\sin.(\tfrac{6}{7}\rho+\Phi)\sin.(\tfrac{6}{7}\rho-\Phi).$$

$$\sin.8\Phi = 128\sin.\Phi\sin.(\tfrac{2}{8}\rho+\Phi)\sin.(\tfrac{2}{8}\rho-\Phi)\sin.(\tfrac{4}{8}\rho+\Phi)\sin.(\tfrac{4}{8}\rho-\Phi)\times \\ \times\sin.(\tfrac{6}{8}\rho+\Phi)\sin.(\tfrac{6}{8}\rho-\Phi)\cos.\Phi.$$

etc. etc.

§. 24. Vt nunc has formulas rurſus generales reddamus, duos caſus conſtitui conuenit, prouti numerus n fuerit vel par vel impar.

I. Sit $n = 2i$, erit

$$\sin.2i\Phi = 2^{2i-1}\sin.\Phi\cos.\Phi\sin.(\tfrac{\rho}{i}+\Phi)\sin.(\tfrac{\rho}{i}-\Phi)\times \\ \times\sin.(\tfrac{2\rho}{i}+\Phi)\sin.(\tfrac{2\rho}{i}-\Phi)\sin.(\tfrac{3\rho}{i}+\Phi)\sin.(\tfrac{3\rho}{i}-\Phi\times \\ \times\ .\ .\ .\ .\ \sin.(\tfrac{(i-1)\rho}{i}+\Phi)\sin.\tfrac{(i-1)\rho}{i}-\Phi).$$

II. Sit

II. Sit $n = 2i + 1$, erit

$$\sin.(2i+1)\phi = 2^{2i}\sin.\phi\,\sin.\left(\tfrac{2\varrho}{2i+1}+\phi\right)\sin.\left(\tfrac{2\varrho}{2i+1}-\phi\right) \times \sin.\left(\tfrac{4\varrho}{2i+1}+\phi\right)\sin.\left(\tfrac{4\varrho}{2i+1}-\phi\right)\sin.\left(\tfrac{6\varrho}{2i+1}+\phi\right) \times \sin.\left(\tfrac{6\varrho}{2i+1}-\phi\right)\ldots\sin.\left(\tfrac{2i\varrho}{2i+1}+\phi\right)\sin.\left(\tfrac{2i\varrho}{2i+1}-\phi\right).$$

§. 25. Quoniam hic omnes factores praeter $\cos.\phi$ sunt sinus, eos, si lubet, in cosinus conuertere possumus, ope formulae $\sin.\psi = \cos.(\varrho - \psi)$, ac pro binis casibus principalibus sequentes expressiones reperiemus:

Si $n = 2i$, erit

$$\sin.2i\phi = 2^{2i-1}\sin.\phi\cos.\phi\cos.\left(\tfrac{\varrho}{i}+\phi\right)\cos.\left(\tfrac{\varrho}{i}-\phi\right) \times \cos.\left(\tfrac{2\varrho}{i}+\phi\right)\cos.\left(\tfrac{2\varrho}{i}-\phi\right)\cos.\left(\tfrac{3\varrho}{i}+\phi\right) \times \cos.\left(\tfrac{3\varrho}{i}-\phi\right)\cos.\left(\tfrac{4\varrho}{i}+\phi\right)\cos.\left(\tfrac{4\varrho}{i}-\phi\right) \times \ldots \cos.\left(\tfrac{(i-1)\varrho}{i}+\phi\right)\cos.\left(\tfrac{(i-1)\varrho}{i}-\phi\right).$$

Si $n = 2i + 1$, erit

$$\sin.(2i+1)\phi = 2^{2i}\sin.\phi\cos.\left(\tfrac{\varrho}{2i+1}+\phi\right)\cos.\left(\tfrac{\varrho}{2i+1}-\phi\right) \times \cos.\left(\tfrac{3\varrho}{2i+1}+\phi\right)\cos.\left(\tfrac{3\varrho}{2i+1}-\phi\right)\cos.\left(\tfrac{5\varrho}{2i+1}+\phi\right) \times \cos.\left(\tfrac{5\varrho}{2i+1}-\phi\right)\ldots\cos.\left(\tfrac{(2i-1)\varrho}{2i+1}+\phi\right) \times \cos.\left(\tfrac{(2i-1)\varrho}{2i+1}-\phi\right).$$

§. 26. Cum igitur duplici modo anguli multipli $n\phi$ tam cosinum quam sinum per factores secundum sinus siue cosinus procedentes exhibuerimus, hic istas expressiones inuentas coniunctim ante oculos exponamus:

I. Si

I. Si fuerit $n = 2i$, erit

Pro cofinu:

$$1^\circ.\ \text{cof.}\ 2i\phi = 2^{2i-1}\ \text{fin.}\ (\tfrac{\varrho}{2i}+\phi)\ \text{fin.}\ (\tfrac{\varrho}{2i}-\phi)\ \text{fin.}\ (\tfrac{3\varrho}{2i}+\phi)\ \times$$
$$\times\ \text{fin.}\ (\tfrac{3\varrho}{2i}-\phi)\ \text{fin.}\ (\tfrac{5\varrho}{2i}+\phi)\ \text{fin.}\ (\tfrac{5\varrho}{2i}-\phi)\ \times$$
$$\times\ \ldots\ \text{fin.}\ (\tfrac{(2i-1)\varrho}{2i}+\phi)\ \text{fin.}\ (\tfrac{(2i-1)\varrho}{2i}-\phi).$$

$$2^\circ.\ \text{cof.}\ 2i\phi = 2^{2i-1}\ \text{cof.}\ (\tfrac{\varrho}{2i}+\phi)\ \text{cof.}\ (\tfrac{\varrho}{2i}-\phi)\ \text{cof.}\ (\tfrac{3\varrho}{2i}+\phi)\ \times$$
$$\times\ \text{cof.}\ (\tfrac{3\varrho}{2i}-\phi)\ \text{cof.}\ (\tfrac{5\varrho}{2i}+\phi)\ \text{cof.}\ (\tfrac{5\varrho}{2i}-\phi)\ \times$$
$$\times\ \ldots\ \text{cof.}\ (\tfrac{(2i-1)\varrho}{2i}+\phi)\ \text{cof.}\ (\tfrac{(2i-1)\varrho}{2i}-\phi).$$

Pro finu:

$$1^\circ.\ \text{fin.}\ 2i\phi = 2^{2i-1}\ \text{fin.}\phi\ \text{cof.}\phi\ \text{fin.}\ (\tfrac{\varrho}{i}+\phi)\ \text{fin.}\ (\tfrac{\varrho}{i}-\phi)\ \text{fin.}\ (\tfrac{2\varrho}{i}+\phi)\ \times$$
$$\times\ \text{fin.}\ (\tfrac{2\varrho}{i}-\phi)\ \text{fin.}\ (\tfrac{3\varrho}{i}+\phi)\ \text{fin.}\ (\tfrac{3\varrho}{i}-\phi)\ \times$$
$$\times\ \ldots\ \text{fin.}\ (\tfrac{(i-1)\varrho}{i}+\phi)\ \text{fin.}\ (\tfrac{(i-1)\varrho}{i}-\phi).$$

$$2^\circ.\ \text{fin.}\ 2i\phi = 2^{2i-1}\ \text{fin.}\ \phi\ \text{cof.}\ \phi\ \text{cof.}\ (\tfrac{\varrho}{i}+\phi)\ \text{cof.}\ (\tfrac{\varrho}{i}-\phi)\ \times$$
$$\times\ \text{cof.}(\tfrac{2\varrho}{i}+\phi)\ \text{cof.}(\tfrac{2\varrho}{i}-\phi)\ \text{cof.}(\tfrac{3\varrho}{i}+\phi)\ \text{cof.}\ (\tfrac{3\varrho}{i}-\phi)\ \times$$
$$\times\ \ldots\ \text{cof.}\ (\tfrac{(i-1)\varrho}{i}+\phi)\ \text{cof.}\ (\tfrac{(i-1)\varrho}{i}-\phi).$$

II. Si fuerit $n = 2i+1$, erit

Pro cofinu:

$$1^\circ.\ \text{cof.}\ (2i+1)\phi = 2^{2i}\ \text{cof.}\ \phi\ \text{fin.}\ (\tfrac{\varrho}{2i+1}+\phi)\ \text{fin.}\ (\tfrac{\varrho}{2i+1}-\phi)\ \times$$
$$\times\ \text{fin.}\ (\tfrac{3\varrho}{2i+1}+\phi)\ \text{fin.}\ (\tfrac{3\varrho}{2i+1}-\phi)\ \text{fin.}\ (\tfrac{\varrho}{2i+1}+\phi)\ \times$$
$$\times\ \text{fin.}(\tfrac{5\varrho}{2i+1}-\phi)\ldots\text{fin.}(\tfrac{(2i-1)\varrho}{2i+1}+\phi)\ \text{fin.}(\tfrac{(2i-1)\varrho}{2i+1}-\phi).$$

2°. $\cos.(2i+1)\Phi = 2^{2i}\cos.\Phi\cos.(\frac{2\varrho}{2i+1}+\Phi)\cos.(\frac{2\varrho}{2i+1}-\Phi)\times$
$\times\cos.(\frac{4\varrho}{2i+1}+\Phi)\cos.(\frac{4\varrho}{2i+1}-\Phi)\cos.(\frac{6\varrho}{2i+1}+\Phi)\times$
$\times\cos.(\frac{6\varrho}{2i+1}-\Phi)\ldots\cos.(\frac{2i\varrho}{2i+1}+\Phi)\cos.(\frac{2i\varrho}{2i+1}-)\Phi.$

Pro finu:

1°. $\sin.(2i+1)\Phi = 2^{2i}\sin.\Phi\sin.(\frac{2\varrho}{2i+1}+\Phi)\sin.(\frac{2\varrho}{2i+1}-\Phi)\times$
$\times\sin.(\frac{4\varrho}{2i+1}+\Phi)\sin.(\frac{4\varrho}{2i+1}-\Phi)\sin.(\frac{6\varrho}{2i+1}+\Phi)\times$
$\times\sin.(\frac{6\varrho}{2i+1}-\Phi)\ldots\sin.(\frac{2i\varrho}{2i+1}+\Phi)\sin.(\frac{2i\varrho}{2i+1}-\Phi).$

2°. $\sin.(2i+1)\Phi = 2^{2i}\sin.\Phi\cos.(\frac{\varrho}{2i+1}+\Phi)\cos.(\frac{\varrho}{2i+1}-\Phi)\times$
$\times\cos.(\frac{3\varrho}{2i+1}+\Phi)\cos.(\frac{3\varrho}{2i+1}-\Phi\cos.(\frac{5\varrho}{2i+1}+\Phi)\times$
$\times\cos.(\frac{5\varrho}{2i+1}-\Phi)\ldots\cos.(\frac{(2i-1)\varrho}{2i+1}+\Phi)\cos.\frac{(2i-1)\varrho}{2i+1}-\Phi).$

§. 27. Hinc patet, cafu quo $n = 2i$, tam cofinum quam finum per aequales angulos exprimi, vnde fi binae formae pro cofinu $2i\Phi$ inuentae per fe inuicem diuidantur, nafcetur ifta aequatio:

$$1 = \tang.(\frac{\varrho}{2i}+\Phi)\tang.(\frac{\varrho}{2i}-\Phi)\tang.(\frac{3\varrho}{2i}+\Phi)\times$$
$$\times\tang.(\frac{3\varrho}{2i}-\Phi)\tang.(\frac{5\varrho}{2i}+\Phi)\tang.(\frac{5\varrho}{2i}-\Phi)\times$$
$$\times\ldots\ldots\tang.(\frac{(2i-1)\varrho}{2i}+\Phi)\tang.(\frac{(2i-1)\varrho}{2i}-\Phi).$$

Hinc igitur fequentes cafus fpeciales deducuntur:

Si $i=1$, erit

$$1 = \tang.(\tfrac{1}{2}\varrho+\Phi)\tang.(\tfrac{1}{2}\varrho-\Phi).$$

Si $i=2$, erit

$$1 = \tang.(\tfrac{1}{4}\varrho+\Phi)\tang.(\tfrac{1}{4}\varrho-\Phi)\tang.(\tfrac{3}{4}\varrho+\Phi)\tang.(\tfrac{3}{4}\varrho-\Phi).$$

Si $i=3$, erit

$$1 = \text{tang.}\left(\tfrac{1}{6}\varrho + \Phi\right)\text{tang.}\left(\tfrac{1}{6}\varrho - \Phi\right)\text{tang.}\left(\tfrac{3}{6}\varrho + \Phi\right) \times$$
$$\times\,\text{tang.}\left(\tfrac{3}{6}\varrho - \Phi\right)\text{tang.}\left(\tfrac{5}{6}\varrho + \Phi\right)\text{tang.}\left(\tfrac{5}{6}\varrho - \Phi\right).$$

etc. etc.

Quae aequalitas quidem ſponte in oculos incurrit, quoniam productum ex primo factore in vltimum eſt $= 1$, ſimilique modo productum ex ſecundo in penultimum $= 1$, et ita porro, ſiquidem bini tales anguli iunctim ſumpti conficiunt angulum rectum ϱ.

§. 28. Diuidamus ſimili modo binos valores pro ſinu $2i\Phi$ inuentos, priorem per poſteriorem, ac reſultabit iſta aequalitas:

$$1 = \text{tang.}\left(\tfrac{\varrho}{i} + \Phi\right)\text{tang.}\left(\tfrac{\varrho}{i} - \Phi\right)\text{tang.}\left(\tfrac{2\varrho}{i} + \Phi\right)\text{tang.}\left(\tfrac{2\varrho}{i} - \Phi\right) \times$$
$$\times\,\text{tang.}\left(\tfrac{3\varrho}{i} + \Phi\right)\text{tang.}\left(\tfrac{3\varrho}{i} - \Phi\right) \;.\;.\;.\;.\;.\; \times$$
$$\times\,\text{tang.}\left(\tfrac{(i-1)\varrho}{i} + \Phi\right)\text{tang.}\left(\tfrac{(i-1)\varrho}{i} - \Phi\right),$$

vnde ſequuntur hi caſus ſpeciales:

Si $i = 1$, erit

$$1 = \text{tang.}(\varrho + \Phi)\,\text{tang.}(\varrho - \Phi).$$

Si $i = 2$, erit

$$1 = \text{tang.}\left(\tfrac{1}{2}\varrho + \Phi\right)\text{tang.}\left(\tfrac{1}{2}\varrho - \Phi\right).$$

Si $i = 3$, erit

$$1 = \text{tang.}\left(\tfrac{1}{3}\varrho + \Phi\right)\text{tang.}\left(\tfrac{1}{3}\varrho - \Phi\right)\text{tang.}\left(\tfrac{2}{3}\varrho + \Phi\right)\text{tang.}\left(\tfrac{2}{3}\varrho - \Phi\right).$$

etc. etc.

Vbi iterum ratio per ſe eſt manifeſta, cum in genere ſit $\text{tang.}\,\psi\,\text{tang.}(\varrho - \psi) = 1$, ob $\text{tang.}(\varrho - \psi) = \text{cot.}\,\psi$.

III. Refolutio tangentis anguli $n\Phi$ in factores.

§. 29. Ifta refolutio peculiari Analyfi non eget, propterea quod tang. $n\Phi = \frac{fin.\ n\Phi}{cof.\ n\Phi}$, quamobrem tantum opus eft, vt finguli factores ipfius fin. $n\Phi$ diuidantur per fingulos factores ipfius cof. $n\Phi$. Cum igitur tam pro finu quam cofinu duplicem dederimus folutionem, ea combinatione vti conueniet, quae fimpliciffimam folutionem fuppeditat. Hic autem iterum duos cafus a fe inuicem diftinxiffe iuuabit, prouti numerus n fuerit vel par vel impar.

I. Cafus quo $n = 2i$.

§. 30. Hic primo cafus fpeciales euoluamus, tribuendo numero n valores 1, 2, 3 etc. eritque

$$\text{tang. } 2\Phi = \frac{\text{fin. } \Phi \text{ cof. } \Phi}{\text{cof. } (\frac{1}{2}\varrho - \Phi) \text{ cof. } (\frac{1}{2}\varrho + \Phi)},$$

$$\text{tang. } 4\Phi = \frac{\text{fin. } \Phi \text{ cof. } \Phi \text{ cof. } (\frac{3}{4}\varrho + \Phi) \text{ cof. } (\frac{3}{4}\varrho - \Phi)}{\text{cof.} (\frac{1}{4}\varrho + \Phi) \text{ cof.} (\frac{1}{4}\varrho - \Phi) \text{ cof.} (\frac{3}{4}\varrho + \Phi) \text{ cof.} (\frac{3}{4}\varrho - \Phi)}.$$

Cum igitur hic nulla contractio locum inueniat, has expreffiones vlterius continuare fuperfluum foret; at quando n eft numerus impar, ob contractionem formulae prodeunt notatu maxime dignae

II. Cafus quo $n = 2i + 1$.

§. 31. Loco i ergo fumamus fucceffiue numeros 0, 1, 2, 3, etc., ac primo quidem:

Si $i = 0$ et $n = 1$, erit

$$\text{tang. } \Phi = \frac{fin.\ \Phi}{cof.\ \Phi}.$$

Si

Si $i = 1$ et $n = 3$, erit

$$\text{tang. } 3\,\phi = \text{tang. } \phi \text{ tang. } (\tfrac{2}{3}\varrho + \phi) \text{ tang. } (\tfrac{2}{3}\varrho - \phi).$$

Si $i = 2$ et $n = 5$, erit

$$\text{tang. } 5\,\phi = \text{tang. } \phi \text{ tang. } (\tfrac{2}{5}\varrho + \phi) \text{ tang. } (\tfrac{2}{5}\varrho - \phi) \times$$
$$\times \text{ tang. } (\tfrac{4}{5}\varrho + \phi) \text{ tang. } (\tfrac{4}{5}\varrho - \phi).$$

et generaliter, pro quocunque numero impari $2n + 1$, erit

$$\text{tang. } (2i+1)\phi = \text{tang. } \phi \text{ tang. } (\tfrac{2\varrho}{2i+1} + \phi) \text{ tang. } (\tfrac{2\varrho}{2i+1} - \phi) \times$$
$$\times \text{ tang. } (\tfrac{4\varrho}{2i+1} + \phi) \text{ tang. } (\tfrac{4\varrho}{2i+1} - \phi) \times$$
$$\times \text{ tang. } (\tfrac{6\varrho}{2i+1} + \phi) \text{ tang. } (\tfrac{6\varrho}{2i+1} - \phi) \times$$
$$\times \;.\;.\;.\;.\; \text{tang. } (\tfrac{2i\varrho}{2i+1} + \phi) \text{ tang. } (\tfrac{2i\varrho}{2i+1} - \phi).$$

§. 32. Hic igitur egregia theoremata Trigonometrica deducimus. Scilicet ex angulo $3\,\phi$ habemus:

$$\text{tang. } 3\,\phi = \text{tang. } \phi \text{ tang. } (60^\circ + \phi) \text{ tang. } (60^\circ - \phi),$$

ſeu quia tang. $60 + \omega = \text{cot. } (30 - \omega)$, erit

$$\text{tang. } 3\,\phi = \text{tang. } \phi \text{ cot. } (30^\circ - \phi) \text{ cot. } (30^\circ + \phi),$$

vnde colligitur

$$\text{tang. } 3\,\phi \text{ tang. } (30^\circ - \phi) \text{ tang. } (30^\circ + \phi) = \text{tang. } \phi.$$

Veluti ſi fuerit $\phi = 20^\circ$, habebimus ſequentes relationes:

$$\text{tang. } 60^\circ = \text{tang. } 20^\circ \times \text{tang. } 80^\circ \times \text{tang. } 40^\circ, \text{ ideoque}$$
$$\text{tang. } 60^\circ \times \text{tang. } 10^\circ = \text{tang. } 20^\circ \times \text{tang. } 40^\circ,$$

hinc ſequentem proportionem deducimus

$$\text{tang. } 10^\circ : \text{tang. } 20^\circ = \text{tang. } 40^\circ : \text{tang. } 60^\circ,$$

ideoque logarithmos ſumendo erit

$$l\,\text{tang. } 10^\circ + l\,\text{tang. } 60^\circ = l\,\text{tang. } 20^\circ + l\,\text{tang. } 40^\circ.$$

Ex tabulis autem est

l tang. $10^\circ = 9{,}2463188$	l tang. $20^\circ = 9{,}5610659$
l tang. $60 = 10{,}2385606$	l tang. $40 = 9{,}9238135$
l tg. $10^\circ + l$ tg. $60^\circ = 9{,}4848794$	l tg. $20^\circ + l$ tg. $40^\circ = 9{,}4848794.$

§. 33. Simili modo formula pro tang. $5\,\Phi$ inuenta praebet

$$\text{tang. } 5\,\Phi = \text{tang. } \Phi \text{ tang. } (36^\circ + \Phi) \text{ tang. } (36^\circ - \Phi) \times$$
$$\times \text{ tang. } (72^\circ + \Phi) \text{ tang. } (72^\circ - \Phi), \text{ siue}$$

$$\text{tang. } 5\,\Phi \text{ tang. } (18^\circ + \Phi) \text{ tang. } (18^\circ - \Phi) = \text{tang. } \Phi \times$$
$$\times \text{ tang. } (36^\circ + \Phi) \text{ tang. } (36^\circ - \Phi),$$

hincque in logarithmis:

$$l \text{ tang. } 5\,\Phi + l \text{ tang. } (18^\circ + \Phi) + l \text{ tang. } (18^\circ - \Phi)$$
$$= l \text{ tang. } \Phi + l \text{ tang. } (36^\circ + \Phi) + l \text{ tang. } (36^\circ - \Phi).$$

Sit exempli gratia $\Phi = 10^\circ$, eritque

$$l \text{ tang. } 50 + l \text{ tang. } 28^\circ + l \text{ tang. } 8^\circ$$
$$= l \text{ tang. } 10^\circ + l \text{ tang. } 46^\circ + l \text{ tang. } 26^\circ,$$

veluti in sequente schematismo videre licet:

l tang. $50^\circ = 10{,}0761865$	l tang. $10^\circ = 9{,}2463188$
l tang. $28 = 9{,}7256744$	l tang. $46 = 10{,}0151628$
l tang. $8 = 9{,}1478025$	l tang. $26 = 9{,}6881818$
l tg. $50^\circ + l$ tg. $28^\circ + l$ tg. $8^\circ = 8{,}9496634$	Summa $= 8{,}9496634.$

§. 34. Quoniam igitur hactenus tam sinus et cosinus quam tangentes angulorum multiplorum per producta expressimus, istas formulas commode per logarithmos euoluere licebit, qui deinceps per differentiationem tolli poterunt, dum scilicet angulus Φ tanquam variabilis spectatur.

Euolu-

Euolutio

formularum pro $\cos. n\Phi$ inuentarum, per logarithmos et differentiationem.

§. 35. Cum supra inuenerimus

$$\cos. 2\Phi = 2\cos.(\tfrac{1}{2}\varrho + \Phi)\cos.(\tfrac{1}{2}\varrho - \Phi),$$

erit sumtis logarithmis

$$l\cos. 2\Phi = l\,2 + l\cos.(\tfrac{1}{2}\varrho + \Phi) + l\cos.(\tfrac{1}{2}\varrho - \Phi),$$

quae aequatio quia vera est pro angulo quocunque Φ, spectetur Φ vt quantitas variabilis, ac differentiatio nobis praebebit hanc aequationem:

$$-\frac{2\partial\Phi\,\sin. 2\Phi}{\cos. 2\Phi} = -\frac{\partial\Phi\,\sin.(\tfrac{1}{2}\varrho + \Phi)}{\cos.(\tfrac{1}{2}\varrho + \Phi)} + \frac{\partial\Phi\,\sin.(\tfrac{1}{2}\varrho - \Phi)}{\cos.(\tfrac{1}{2}\varrho - \Phi)},$$

quae per $-\partial\Phi$ diuisa per tangentes dabit

$$2\,\tang. 2\Phi = \tang.(45° + \Phi) - \tang.(45° - \Phi),$$

id quod exemplo illustrasse iuuabit. Sit igitur $\Phi = 17°, 30'$, eritque

$$2\,\tang. 35° = \tang.(62°, 30') - \tang.(27°, 30').$$

Est vero ex tabulis

	$\tang. 62°, 30' = 1,9209821$
	$\tang. 27, 30 = 0,5205671$
$2\,\tang. 35° = 1,4004150$	Differentia $= 1,4004150$

§. 36. Deinde quia supra inuenimus

$$\cos. 3\Phi = 4\cos.(60° + \Phi)\cos.(60° - \Phi)\cos.\Phi,$$

erit per logarithmos

$$l\cos. 3\Phi = l\,4 + l\cos.\Phi + l\cos.(60° + \Phi)\; l\cos.(60° - \Phi).$$

Hinc

Hinc differentiando erit

$$3 \tan g. 3\phi = \tan g. \phi + \tan g. (60^\circ + \phi) - \tan g. (60 - \phi).$$

Exemplum. Sumatur $\phi = 25^\circ$, eritque

$$3 \tan g. 75^\circ = \tan g. 25^\circ + \tan g. 85^\circ - \tan g. 35^\circ,$$

cuius veritas ex ſubiecto calculo videre licet:

	tang. 25° = 0, 4663077
	tang. 85, = 11, 4300520
	Summa = 11, 8963597
	tang. 35° = 0, 7002075
3 tang. 75° = 11, 1961524	11, 1961522.

Aliud exemplum. Sumamus $\phi = 29^\circ$, eritque

$$3 \tan g. 87^\circ = \tan g. 29^\circ + \tan g. 89^\circ - \tan g. 31^\circ,$$

veluti ex ſubiecto calculo videre licet

tang. 87° = 19, 081137	tang. 29° = 0, 554309
3 tang. 87° = 57, 243411	tang. 89 = 57, 289962
	Summa = 57, 844271
	tang. 31° = 0, 600860
	different. = 57, 243411.

§. 37. Sumamus etiam exemplum quo anguli occurrunt maiores recto, ſitque $\phi = 58^\circ$, eſſeque oportet:

$$3 \tan g. 174^\circ = \tan g. 58^\circ + \tan g. 118^\circ - \tan g. 2^\circ,$$

quae aequatio reducitur ad ſequentem:

$$3 \tan g. 6^\circ = \tan g. 62^\circ + \tan g. 2^\circ - \tan g. 58^\circ,$$

et calculus ita ſe habebit:

tang.

tang. $6^\circ = 0,1051042$	tang. $62^\circ = 1,8807265$
3 tang. $6^\circ = 0,3153126$	tang. $2^\circ = 0,0349208$
	Summa $= 1,9156473$
	tang. $38^\circ = 1,6003345$
	different. $= 0,3153128$.

§. 38. Sit nunc in genere $n = 2i$, et quia inuenimus

$$\operatorname{cof.} 2i\Phi = 2^{2i-1} \operatorname{cof.}(\tfrac{\rho}{2i} + \Phi) \operatorname{cof.}(\tfrac{\rho}{2i} - \Phi) \operatorname{cof.}(\tfrac{3\rho}{2i} + \Phi) \times$$
$$\times \operatorname{cof.}(\tfrac{3\rho}{2i} - \Phi) \operatorname{cof.}(\tfrac{5\rho}{2i} + \Phi) \ldots \operatorname{cof.}(\tfrac{(2i-1)\rho}{2i} + \Phi) \times$$
$$\times \operatorname{cof.}(\tfrac{(2i-1)\rho}{2i} - \Phi),$$

hinc per logarithmos erit

$$l \operatorname{cof.} 2i\Phi = l\, 2^{2i-1} + l \operatorname{cof.}(\tfrac{\rho}{2i} + \Phi) + l \operatorname{cof.}(\tfrac{\rho}{2i} - \Phi)$$
$$+ l \operatorname{cof.}(\tfrac{3\rho}{2i} + \Phi) + \operatorname{cof.} l (\tfrac{3\rho}{2i} - \Phi) \ldots$$
$$+ l \operatorname{cof.}(\tfrac{(2i-1)\rho}{2i} + \Phi) + l \operatorname{cof.}(\tfrac{(2i-1)\rho}{2i} - \Phi),$$

vnde ergo per differentiationem nancifcimur

$$2i \operatorname{tang.} 2i\Phi = \operatorname{tang.}(\tfrac{\rho}{2i} + \Phi) - \operatorname{tang.}(\tfrac{\rho}{2i} - \Phi)$$
$$+ \operatorname{tang.}(\tfrac{3\rho}{2i} + \Phi) - \operatorname{tang.}(\tfrac{3\rho}{2i} - \Phi) \text{ etc. } \ldots$$
$$+ \operatorname{tang.}(\tfrac{(2i-1)\rho}{2i} + \Phi) - \operatorname{tang.}(\tfrac{(2i-1)\rho}{2i} - \Phi).$$

§. 39. Hinc ergo pro i fumendis numeris minoribus habebimus:

$$4 \operatorname{tang.} 4\Phi = \operatorname{tang.}(22\tfrac{1}{2}^\circ + \Phi) \operatorname{tang.}(22\tfrac{1}{2}^\circ - \Phi)$$
$$+ \operatorname{tang.}(67\tfrac{1}{2}^\circ + \Phi) - \operatorname{tang.}(67\tfrac{1}{2}^\circ - \Phi);$$

$$6 \operatorname{tang.} 6\Phi = \operatorname{tang.}(15^\circ + \Phi) - \operatorname{tang.}(15^\circ - \Phi)$$
$$+ \operatorname{tang.}(45^\circ + \Phi) - \operatorname{tang.}(45^\circ - \Phi)$$
$$+ \operatorname{tang.}(75^\circ + \Phi) - \operatorname{tang.}(75^\circ - \Phi);$$

$$8 \text{ tang. } 8\Phi = \text{tang. } (11\tfrac{1}{4}^\circ + \Phi) - \text{tang. } (11\tfrac{1}{4}^\circ - \Phi)$$
$$+ \text{tang. } (33\tfrac{3}{4}^\circ + \Phi) - \text{tang. } (33\tfrac{3}{4}^\circ - \Phi)$$
$$+ \text{tang. } (56\tfrac{1}{4}^\circ + \Phi) - \text{tang. } (56\tfrac{1}{4}^\circ - \Phi)$$
$$+ \text{tang. } (78\tfrac{3}{4}^\circ + \Phi) - \text{tang. } (78\tfrac{3}{4}^\circ - \Phi).$$

§. 40. Sumamus nunc etiam $n = 2i + 1$, quo cafu vidimus effe

$$\text{cof.} (2i + 1) \Phi = 2^{2i} \text{cof.} \Phi \text{ cof.} (\tfrac{2\varrho}{2i+1} + \Phi) \text{ cof.} (\tfrac{2\varrho}{2i+1} - \Phi) \times$$
$$\times \text{cof.} (\tfrac{4\varrho}{2i+1} + \Phi) \text{ cof.} (\tfrac{4\varrho}{2i+1} - \Phi) \text{ cof.} (\tfrac{6\varrho}{2i+1} + \Phi) \times$$
$$\times \; . \; . \; . \; . \; . \; \text{cof.} (\tfrac{2i\varrho}{2i+1} + \Phi) \text{ cof.} (\tfrac{2i\varrho}{2i+1} - \Phi),$$

hinc fumtis logarithmis erit

$$l \text{ cof.} (2i + 1) \Phi = l\, 2^{2i} + l \text{ cof.} \Phi + l \text{ cof.} (\tfrac{2\varrho}{2i+1} + \Phi)$$
$$+ l \text{ cof.} (\tfrac{2\varrho}{2i+1} - \Phi) \; . \; . \; . \; . + l \text{ cof.} (\tfrac{2i\varrho}{2i+1} + \Phi)$$
$$+ l \text{ cof.} (\tfrac{2i\varrho}{2i+1} - \Phi),$$

vnde differentiando nancifcimur

$$(2i + 1) \text{ tang. } (2i + 1) \Phi = \text{tang. } \Phi + \text{tang. } (\tfrac{2\varrho}{2i+1} + \Phi)$$
$$- \text{tang. } (\tfrac{2\varrho}{2i+1} - \Phi) + \text{tang. } (\tfrac{4\varrho}{2i+1} + \Phi)$$
$$- \text{tang. } (\tfrac{4\varrho}{2i+1} - \Phi) \; . \; . \; . \; . \; .$$
$$+ \text{tang. } (\tfrac{2i\varrho}{2i+1} + \Phi) - \text{tang. } (\tfrac{2i\varrho}{2i+1} - \Phi).$$

§. 41. Cafum quo $2i + 1 = 3$ iam euoluimus, fit igitur $2i + 1 = 5$, eritque

$$5 \text{ tang. } 5\Phi = \text{tang. } \Phi + \text{tang. } (36^\circ + \Phi) - \text{tang. } (36^\circ - \Phi)$$
$$+ \text{tang. } (72^\circ + \Phi) - \text{tang. } (72^\circ - \Phi).$$

Sit

Sit nunc $2i + 1 = 7$, eritque

$$7 \text{ tang. } 7\Phi = \text{tang. } \Phi + \text{tang. } (25\tfrac{5}{7}{}^{\circ} + \Phi)$$
$$- \text{tang. } (25\tfrac{5}{7}{}^{\circ} - \Phi) + \text{tang. } (51\tfrac{1}{7}{}^{\circ} + \Phi)$$
$$- \text{tang. } (51\tfrac{1}{7}{}^{\circ} - \Phi) + \text{tang. } (76\tfrac{5}{7}{}^{\circ} + \Phi)$$
$$- \text{tang. } (76\tfrac{5}{7}{}^{\circ} - \Phi).$$

Posito $2i + 1 = 9$ habebimus

$$9 \text{ tang. } 9\Phi = \text{tang. } \Phi + \text{tang. } (20^{\circ} + \Phi) - \text{tang. } (20^{\circ} - \Phi)$$
$$+ \text{tang. } (40^{\circ} + \Phi) - \text{tang. } (40^{\circ} - \Phi)$$
$$+ \text{tang. } (60^{\circ} + \Phi) - \text{tang. } (60^{\bullet} - \Phi)$$
$$+ \text{tang. } (80^{\circ} + \Phi) - \text{tang. } (80^{\circ} - \Phi).$$