1789

Innumera theoremata circa formulas integrales, quorum demonstratio vires analyseos superare videatur

Leonhard Euler

INNVMERA
THEOREMATA
CIRCA FORMVLAS INTEGRALES
QVORVM
DEMONSTRATIO VIRES
ANALYSEOS
SVPERARE VIDEATVR.

Auctore
L. EVLERO.

Conuent. exhib. die 18 *Mart.* 1776.

Fundamentum horum Theorematum in eiusmodi formulis integralibus $\int V \partial x$ est constitutum, quarum valor a termino $x = 0$ vsque ad certum terminum definitum $x = k$ per expressionem finitam assignari queat. Quod si enim istum valorem littera P designemus, ita vt sit $\int V \partial x \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = k \end{bmatrix} = P$, quoniam ipsa variabilis x in P non amplius inest, ea tanquam functio alius cuiuspiam quantitatis p, quae simul in functione V contineatur, spectari poterit; tum autem sub iisdem integrationis terminis innumerabiles aliae formulae integrales tam per differentiationem quam per integrationem, quemadmodum iam aliquoties fusius exposui, deriuari possunt, quae sunt:

 Per

III.

$$\int \frac{x^p - 1 - p\,l\,x - \frac{1}{2}p\,p\,(l\,x)^2}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int \mathrm{P}\,\partial p.$$

IV.

$$\int \frac{x^p - 1 - p\,l\,x - \frac{1}{2}p\,p\,(l\,x)^2 - \frac{1}{6}p^3\,(l\,x)^3}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$

etc. $= \int \partial p \int \partial p \int \partial p \int \mathrm{P}\,\partial p.$

Haecque Theoremata aeque fubfiftunt, fiue p fit numerus pofitiuus, fiue negatiuus, fiue etiam integer, fiue fractus, dum ne fit $p - n > 0$, et integralia $\int \mathrm{P}\,\partial p$, $\int \partial p \int \mathrm{P}\,\partial p$, $\int \partial p \int \partial p \int \mathrm{P}\,\partial p$, omniaque hinc deducta ita capiantur, vt euanefcant pofito $p = 0$.

ORDO SECVNDVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{x^p}{x^{-n}\,(1 + x^n)^2}\,\frac{\partial x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi\,p}{n\,n\,\text{fin.}\,\frac{p}{n}\,\pi}.$$

Ponamus hic iterum denominatorem $x^{-n}\,(1 + x^n)^2 = \Delta$, fitque $\mathrm{P} = \frac{\pi\,p}{n\,n\,\text{fin.}\,\frac{p}{n}\,\pi}$, ita vt P iterum fit functio ipfius p, ac primo per differentiationem hinc deducentur fequentia Theoremata:

I.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\,l\,x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial \mathrm{P}}{\partial p}.$$

II.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\,(l\,x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial\,\partial \mathrm{P}}{\partial p^2}.$$

III.

III.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\,(l\,x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\,(l\,x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem inde ſequentia Theoremata oriuntur:

I.

$$\int \frac{x^p - 1}{\Delta} \cdot \frac{\partial x}{x\,l\,x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int P\,\partial p.$$

II.

$$\int \frac{x^p - 1 - p\,l\,x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int P\,\partial p.$$

III.

$$\int \frac{x^p - 1 - p\,l\,x - \frac{1}{2}\,p\,p\,(l\,x)^2}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\,\partial p.$$

IV.

$$\int \frac{x^p - 1 - p\,l\,x - \frac{1}{2}\,p\,p\,(l\,x)^2 - \frac{1}{6}\,p^3\,(l\,x)^3}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\,\partial p.$$

etc.

vbi circa integrationes eadem ſunt obſeruanda, quae ante fuerant praecepta.

ORDO

ORDO TERTIUS
Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + (f + \frac{1}{f}) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi}.$$

Ponamus hic iterum pro denominatore
$\Delta = x^n + (f + \frac{1}{f}) = x^{-n}$,
tum vero fit

$$P = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n(f^1 - f^{-1}) \sin. \frac{p}{n} \pi} = \frac{\pi (f^{1 + \frac{p}{n}} - f^{1 - \frac{p}{n}})}{n (ff - 1) \sin. \frac{p}{n} \pi}.$$

His positis ut ante per differentiationem sequentia Theoremata deducuntur:

I.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\, l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x (l x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x (l x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x (l x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem eliciuntur sequentia:

I.

I.

$$\int \frac{x^p - 1}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{x^p - 1 - p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{x^p - 1 - p\, l\, x - \frac{1}{2}\, p\, p\, (l\, x)^2}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{x^p - 1 - p\, l\, x - \frac{1}{2}\, p\, p\, (l\, x)^2 - \frac{1}{6}\, p^3\, (l\, x)^3}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix}$$

etc.
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

Vbi denuo eadem ſunt obſeruanda, quae ſupra ſunt praecepta.

ORDO QVARTVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2 \operatorname{coſ.} \theta + x^{-n}} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \frac{\pi \operatorname{ſin.} \frac{p}{n} \theta}{n \operatorname{ſin.} \theta \operatorname{ſin.} \frac{p}{n} \pi}.$$

Statuamus hic iterum $\Delta = x^n + 2 \operatorname{coſ.} \theta + x^{-n}$, ſitque

$$P = \frac{\pi \operatorname{ſin.} \frac{p}{n} \theta}{n \operatorname{ſin.} \theta \operatorname{ſin.} \frac{p}{n} \pi},$$

ita vt P tanquam functio ipſius p ſpectari poſſit; vbi probe notandum eſt, hunc valorem integralem ſubſiſtere non poſſe, niſi ſit $p < n$, ideoque fractio $\frac{p}{n}$ vnitate minor, atque ſub iisdem conditionibus per differentiationem ſequentia hinc deducuntur Theoremata:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \, l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^2}{x} \begin{bmatrix} \text{ab } = 0 \\ \text{ad } = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem colliguntur ſequentia:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x \, l x} \begin{bmatrix} \text{ad } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P \, \partial p.$$

II.

$$\int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x \, (l x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P \, \partial p.$$

III.

III.

$$\int \frac{x^p - x^{-p} - 2 p\, l x}{\Delta} \cdot \frac{\partial x}{x (l x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \mathrm{P}\, \partial p.$$

IV.

$$\int \frac{x^p + x^{-p} - 2 - \tfrac{2}{2} p p (l x)^2}{\Delta} \cdot \frac{\partial x}{x (l x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int \mathrm{P}\, \partial p.$$

etc.

Quod si eadem integralia extendantur ab $x = 0$ ad $x = \infty$, eorum valores duplo euadent maiores.

ORDO QVINTVS
Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^{-n} (1 + x^n)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi p}{n n \sin. \frac{p}{n} \pi}.$$

Statuamus igitur hic pro denominatore $\Delta = x^{-n} (1 + x^n)^2$, sitque $\mathrm{P} = \frac{\pi p}{n n \sin. \frac{p}{n} \pi}$, ita vt P spectari possit tanquam functio ipsius p, vbi perpetuo fractio $\frac{p}{n}$ vnitate minor supponitur, quibus positis per differentiationem sequentia nascuntur Theoremata:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x\, l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial \mathrm{P}}{\partial p}.$$

II.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial \partial \mathrm{P}}{\partial p^2}.$$

III.

III.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x\,(l\,x)^3}{x} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x\,(l\,x)^4}{x} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

Per integrationem vero ſequentia deducuntur:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x\,l\,x} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix} = \int P\,\partial p.$$

II.

$$\int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^2} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix} = \int \partial p \int P\,\partial p.$$

III.

$$\int \frac{x^p - x^{-p} - 2\,p\,(l\,x)}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^3} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\,\partial p.$$

IV.

$$\int \frac{x^p + x^{-p} - 2 - p\,p\,(l\,x)^2}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^4} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\,\partial p$$

etc.

At ſi haec integralia ab $x = 0$ ad $x = \infty$ capiantur, eorum valores euadent duplo maiores.

ORDO

ORDO SEXTVS
Theorematum ex forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + (f + \frac{1}{f}) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi}.$$

Statuamus

$$\Delta = x^n + (f + \tfrac{1}{f}) + x^{-n} = \frac{1}{x^n} (x^n + f)(x^n + \tfrac{1}{f}), \text{ et fit}$$

$$P = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi},$$

vbi iterum fractio $\frac{p}{n}$ vnitate minor ſupponitur. His obſeruatis per differentiationem colligimus:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \, l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem ſequentia Theoremata naſcuntur:

I.

$$\int \frac{x^{p} - x^{-p}}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{x^{p} + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^{2}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{x^{p} - x^{-p} - 2\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^{3}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$

$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{x^{p} + x^{-p} - 2 - p\, p\, (l\, x)^{2}}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^{4}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$

$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

Quod ſi haec integralia ab $x = 0$ ad $x = \infty$ extendantur, eorum valores erunt duplo maiores. Ceterum hic perſpicuum eſt, quantitatem f eſſe debere poſitiuam, quia alias poteſtates $f^{\pm \frac{p}{n}}$ fieri poſſent imaginariae.

ORDO SEPTIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\cos.\, p\, l\, x}{x^{n} + 2 \cos.\, \theta + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2\, n \sin.\, \theta} \left(\frac{e^{\frac{p}{n}\theta} - e^{-\frac{p}{n}\theta}}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}} \right).$$

Statua-

Statuamus hic iterum pro denominatore

$$\Delta = x^n + 2\,\text{cof.}\,\theta + x^{-n},$$

fitque

$$P = \frac{\pi}{2n\,\text{fin.}\,\theta} \cdot \frac{e^{\frac{p}{n}\theta} - e^{-\frac{p}{n}\theta}}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}},$$

quae ergo quantitas iterum vt functio ipfius p fpectari poteft; vbi autem non amplius neceffe eft vt fractio $\frac{p}{n}$ fit vnitate minor. Hinc igitur per differentiationem fequentia deriuantur Theoremata:

I.

$$\int \frac{\text{fin.}\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,l\,x}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = -\frac{\partial P}{\partial p}.$$

II.

$$\int \frac{\text{cof.}\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,(l\,x)^2}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = -\frac{\partial\partial P}{\partial p^2}.$$

III.

$$\int \frac{\text{fin.}\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,(l\,x)^3}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = +\frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{\text{cof.}\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,(l\,x)^4}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = +\frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem vero

I.

$$\int \frac{\text{fin.}\,p\,l\,x}{\Delta} \cdot \frac{\partial x}{x\,l\,x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \int P\,\partial p.$$

II.

II.

$$\int \frac{1 - \text{cos}.\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{p\, l\, x - \text{sin}.\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ab } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{\frac{1}{2} p\, p\, (l\,x)^2 - 1 + \text{cos}.\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

V.

$$\int \frac{\frac{1}{6} p^3\, (l\,x)^3 - p\, l\, x + \text{sin}.\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^5} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

etc.

Haec igitur integralia, si ab $x = 0$ ad $x = \infty$ extendantur, iterum duplo fiunt maiora.

ORDO OCTAVUS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{cos}.\, p\, l\, x}{x^{-n}\,(x^n + 1)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{n\, n} \cdot \frac{p}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}}.$$

Statuamus hic pro denominatore $\Delta = x^{-n}\,(x^n + 1)^2$, sitque

P =

$$P = \frac{\pi}{n\,n} \cdot \frac{p}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}},$$

atque per differentiationem hinc deducentur ſequentia Theoremata:

I.

$$\int \frac{\text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, l\, x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\partial P}{\partial p}.$$

II.

$$\int \frac{\text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{\text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = +\frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{\text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = +\frac{\partial^4 P}{\partial p^4}.$$

Per integrationem vero elicitur

I.

$$\int \frac{\text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{1 - \text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{p\, l\, x - \text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{\frac{1}{2}pp(lx)^2 - 1 + \text{cof.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

V.

$$\int \frac{\frac{1}{6}p^3(lx)^3 - p\, l\, x + \text{fin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x(lx)^5} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

etc.

ORDO NONVS
Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{cof.}\, p\, l\, x}{x^n + (f + \frac{1}{f}) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{2\pi\, \text{fin.}\, \frac{p}{n} . lf}{n(f - \frac{1}{f})(e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi})}.$$

Statuatur $\Delta = x^n + (f + \frac{1}{f}) + x^{-n}$ fitque

$$P = \frac{2\pi\, \text{fin.}\, \frac{p}{n} . lf}{n(f - \frac{1}{f})(e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi})}$$

atque hinc per differentiationem sequentia prodeunt Theoremata:

I.

$$\int \frac{\text{fin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, l\, x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\partial P}{\partial p}.$$

II.

$$\int \frac{\text{cof.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\partial\partial P}{\partial p^2}.$$

III.

III.

$$\int \frac{\text{fin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = + \frac{\partial^3 P}{\partial x^3}.$$

IV.

$$\int \frac{\text{cof.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = + \frac{\partial^4 P}{\partial x^4}.$$

etc.

Per integrationem vero

I.

$$\int \frac{\text{fin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{1 - \text{cof.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{p\, l\, x - \text{fin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{\frac{1}{2} pp (l\, x)^2 - 1 + \text{cof.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

etc.

Hic manifeſtum eſt quantitatem f negatiuam accipi non poſſe, quia alias iam ipſa functio P fieret imaginaria.

Adiungamus his theoremata ſimpliciora, quae ex hactenus allatis naſcuntur, dum angulus θ ſumitur rectus, vt ſit cof. $\theta = 0$ et ſin. $\theta = 1$. Hinc ergo ſequentes ordines adiiciamus

ORDO DECIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{2\, n \operatorname{cof.} \frac{\pi p}{2n}}.$$

Haec forma ſcilicet nata eſt ex prima, ſumendo $\theta = \frac{\pi}{2}$, vnde poſito $\Delta = x^n + x^{-n}$ et $P = \frac{\pi}{2\, n \operatorname{cof.} \frac{\pi p}{2n}}$ naſcuntur eaedem formulae, quae in ordine primo ſunt allatae. Hic autem imprimis notari meretur, quod integrale $\int P\, \partial p$ per logarithmos exhiberi poteſt: erit enim

$$\int P\, \partial p = \int \frac{\pi\, \partial p}{2\, n \operatorname{cof.} \frac{\pi p}{2n}} = l \operatorname{tang.} \left(45^\circ + \frac{\pi p}{4n}\right)$$

quod integrale ita eſt ſumtum, vt euaneſcat facto $p = 0$.

ORDO VNDECIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2\, n \operatorname{cof.} \frac{\pi p}{2n}}.$$

Hic ſcilicet ordo natus eſt ex quarto, ponendo $\theta = \frac{\pi}{2}$; quamobrem ſtatuamus $\Delta = x^n + x^{-n}$ et $P = \frac{\pi}{2\, n \operatorname{cof.} \frac{\pi p}{2n}}$, eademque theoremata inde naſcuntur, quae ſupra pro ordine quar-

quarto ſunt allata, vbi ergo iterum commode vſu venit vt ſit
$\int P \partial p = l \text{ tang. } (45^\circ + \frac{\pi p}{4n})$.

ORDO XII.

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{coſ. } p\, l\, x}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\pi}{2n} \cdot \frac{1}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Quod ſi ergo ſtatuamus

$$\Delta = x^n + x^{-n} \text{ et } P = \frac{\pi}{2n\,(e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}})},$$

eadem plane Theoremata hinc oriuntur, quae ſupra pro caſu ſeptimo ſunt allata. Hic autem iterum notaſſe iuuabit integrale $\int P \partial p$ reuera exhiberi poſſe. Cum enim ſit

$$\int P \partial p = \int \frac{\pi \partial p}{2n\,(e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}})},$$

ponatur $\frac{\pi p}{2n} = z$, eritque

$$\int P \partial p = \int \frac{\partial z}{e^z + e^{-z}} = \int \frac{e^z \partial z}{e^{2z} + 1}.$$

Sit porro $e^z = v$, erit $\partial v = e^z \partial z$, hincque fiet

$$\int P \partial p = \int \frac{\partial v}{1 + vv} = \text{A tang. } v;$$

quare retro ſubſtituendo habebimus

$$\int P \partial p = \text{A tang. } e^{\frac{\pi p}{2n}}.$$

Denique adhuc referamus formulas illas integrales, in quarum denominatore erat $1 - x^{2n}$, quas quidem iam olim breuiter tetigi, nunc autem vberius euoluam.

ORDO XIII.

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p - x^{-p}}{x^n - x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2n} \text{ tang. } \frac{\pi p}{2n}.$$

Hic igitur iterum ſtatuamus

$\Delta = x^n - x^{-n}$ et $P = \frac{\pi}{2n} \text{ tang. } \frac{\pi p}{2n}$,

atque per differentiationem nanciſcemur ſequentia Theoremata.

I.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x\, l\, x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x\, (l\, x)^2}{x} \begin{bmatrix} \text{ab } = 0 \\ \text{ad } = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x\, (l\, x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x\, (l\, x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Integratio autem ſequentia ſuppeditat:

I.

$$\int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ad } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

II.

$$\int \frac{x^p - x^{-p} - 2\,p\,l\,x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \mathrm{P}\,\partial p.$$

III.

$$\int \frac{x^p + x^{-p} - 2 - \frac{2}{2}\,p\,p\,(l\,x)^2}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \mathrm{P}\,\partial p.$$

IV.

$$\int \frac{x^p - x^{-p} - 2\,p\,l\,x - \frac{2}{6}\,p^3\,(l\,x)^3}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int \mathrm{P}\,\partial p.$$

V.

$$\int \frac{x^p + x^{-p} - 2 - \frac{2}{2}\,p\,p\,(l\,x)^2 - \frac{2}{24}\,p^4\,(l\,x)^4}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^5} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int \partial p \int \mathrm{P}\,\partial p.$$

etc.

vbi iterum notetur formulam integralem $\int \mathrm{P}\,\partial p$ actu exhiberi posse; erit enim

$$\int \mathrm{P}\,\partial p = \int \tfrac{\pi\,\partial p}{2\,n}\,\text{tang.}\,\tfrac{\pi p}{2 n} = -\,l\,\text{cos.}\,\tfrac{\pi p}{2 n} = +\,l\,\text{sec.}\,\tfrac{\pi p}{2 n}.$$

Hic probe notandum est, fractionem $\frac{p}{n}$ semper esse debere vnitate minorem.

ORDO

ORDO XIV.

Theorematum ex hac forma generali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{sin.}\, p\, l\, x}{x^{-n} - x^{+n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{4n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{+\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Statuatur igitur vt hactenus $\Delta = x^{-n} - x^{n}$ et

$$P = \frac{\pi}{4n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}},$$

atque differentiatio nobis praebebit sequentia Theoremata:

I.

$$\int \frac{\text{cos.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, l\, x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{\text{sin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = - \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{\text{cos.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = - \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{\text{sin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = + \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per

Per integrationem autem impetramus ſequentia:

I.

$$\int \frac{1 - \text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{p\, l\, x - \text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{\frac{1}{2} p\, p\, (l\, x)^2 - 1 + \text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{\frac{1}{6} p^3\, (l\, x)^3 - p\, l\, x + \text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

etc.

Vbi iterum commode euenit vt $\int P\, \partial p$ exhiberi poſſit, ſiquidem habemus

$$\int P\, \partial p = \int \frac{\pi\, \partial p}{4\, n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Ponatur enim $\frac{p\pi}{2n} = \phi$, eritque

$$\int P\,\partial p = \int \tfrac{1}{2}\,\partial\phi \,.\, \frac{e^{-\phi} - e^{+\phi}}{e^{\phi} + e^{-\phi}},$$

vbi denominatoris differentiale eſt $e^{\phi}\,\partial\phi - e^{-\phi}\,\partial\phi$, vnde concluditur

$$\int P\,\partial p = -l\sqrt{(e^{\phi} + e^{-\phi})} + C$$

quae conſtans C ita aſſumi debet, vt integrale euaneſcat poſito $\phi = 0$, vnde fit

$$\int P\,\partial p = \tfrac{1}{2}\, l\, \frac{2}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Hic autem perinde eſt, vtrum fractio $\frac{p}{n}$ maior ſit minorue vnitate.

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