



1789

Innumera theoremata circa formulas integrales, quorum demonstratio vires analyseos superare videatur

Leonhard Euler

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INNVMERA
T H E O R E M A T A
 CIRCA FORMVLAS INTEGRALES
 QVORVM
 DEMONSTRATIO VIRES
A N A L Y S E O S
 SVPERARE VIDEATVR.

Auctore
 L. EYLERO.

Conuent. exhib. die 18 Mart. 1776.

Fundamentum horum Theorematum in eiusmodi formulis integralibus $\int V \partial x$ est constitutum, quarum valor a termino $x = 0$ vsque ad certum terminum definitum $x = k$ per expressionem finitam assignari queat. Quod si enim istum valorem littera P designemus, ita vt sit $\int V \partial x \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = k \end{array} \right] = P$, quoniam ipsa variabilis x in P non amplius inest, ea tanquam functio alius cuiuspiam quantitatis p , quae simul in functione V contineatur, spectari poterit; tum autem sub iisdem integrationis terminis innumerabiles aliae formulae integrales tam per differentiationem quam per integrationem, quemadmodum iam aliquoties fusius exposui, deriuari possunt, quae sunt:

A 2

Per

$$= (6) =$$

III.

$$\int \frac{x^p - 1 - p \log x - \frac{1}{2} p p (\log x)^2}{\Delta} \cdot \frac{\partial x}{x (\log x)^3} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] \\ = \int \partial p \int \partial p \int P \partial p.$$

IV.

$$\int \frac{x^p - 1 - p \log x - \frac{1}{2} p p (\log x)^2 - \frac{1}{6} p^3 (\log x)^3}{\Delta} \cdot \frac{\partial x}{x (\log x)^4} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] \\ \text{etc.} \quad = \int \partial p \int \partial p \int \partial p \int P \partial p.$$

Haecque Theoremata aequae subsistunt, sine p sit numerus positivus, sine negativus, sine etiam integer, sine fractus, dum ne sit $p - n > 0$, et integralia $\int P \partial p$, $\int \partial p \int P \partial p$, $\int \partial p \int \partial p \int P \partial p$, omniaque hinc deducta ita capiantur, ut evanescantposito $p = 0$.

ORDO SECVNDVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{x^p}{x^{-n} (1 + x^n)^2} \cdot \frac{\partial x}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\pi p}{n n \sin. \frac{p}{n} \pi}.$$

Ponamus hic iterum denominatorem $x^{-n} (1 + x^n)^2 = \Delta$, sitque $P = \frac{\pi p}{n n \sin. \frac{p}{n} \pi}$, ita ut P iterum sit functio ipsius p , ac primo per differentiationem hinc deducantur sequentia Theoremata:

I.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x \log x}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x (\log x)^2}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\partial \partial P}{\partial p^2}.$$

III.

===== (7) =====

$$\text{III.} \\ \int \frac{x^p}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\partial^3 P}{\partial p^3}.$$

$$\text{IV.} \\ \int \frac{x^p}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem inde sequentia Theoremata oriuntur:

$$\text{I.} \\ \int \frac{x^p - 1}{\Delta} \cdot \frac{\partial x}{x lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = fP \partial p.$$

$$\text{II.} \\ \int \frac{x^p - 1 - plx}{\Delta} \cdot \frac{\partial x}{x (lx)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = f \partial p f P \partial p.$$

$$\text{III.} \\ \int \frac{x^p - 1 - plx - \frac{1}{2} p p (lx)^2}{\Delta} \cdot \frac{\partial x}{x (lx)^3} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] \\ = f \partial p f \partial p f P \partial p.$$

$$\text{IV.} \\ \int \frac{x^p - 1 - plx - \frac{1}{2} p p (lx)^2 - \frac{1}{6} p^3 (lx)^3}{\Delta} \cdot \frac{\partial x}{x (lx)^4} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] \\ \text{etc.} \qquad \qquad \qquad = f \partial p f \partial p f \partial p f P \partial p.$$

vbi circa integrationes eadem sunt obseruanda, quae ante fuerant praecepta.

ORDO

ORDO TERTIVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + (f + \frac{1}{f}) + x^{-n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \cdot \text{fin. } \frac{p}{n} \pi}$$

Ponamus hic iterum pro denominatore

$$\Delta = x^n + (f + \frac{1}{f}) + x^{-n},$$

tum vero fit

$$P = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \cdot \text{fin. } \frac{p}{n} \pi} = \frac{\pi (f^{\frac{1}{n}} + \frac{p}{n} - f^{\frac{1}{n}} - \frac{p}{n})}{n (ff - 1) \cdot \text{fin. } \frac{p}{n} \pi}$$

His positis ut ante per differentiationem sequentia Theoremata deducuntur:

$$\text{I.} \\ \int \frac{x^p}{\Delta} \cdot \frac{\partial x / x}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\partial P}{\partial p}$$

$$\text{II.} \\ \int \frac{x^p}{\Delta} \cdot \frac{\partial x (1/x)^2}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\partial \partial P}{\partial p^2}$$

$$\text{III.} \\ \int \frac{x^p}{\Delta} \cdot \frac{\partial x (1/x)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\partial^3 P}{\partial p^3}$$

$$\text{IV.} \\ \int \frac{x^p}{\Delta} \cdot \frac{\partial x (1/x)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\partial^4 P}{\partial p^4} \\ \text{etc.}$$

Per integrationem autem eliciuntur sequentia:

I.

===== (9) =====

I.

$$\int \frac{x^p - 1}{\Delta} \cdot \frac{\partial x}{x l x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = f^p \partial p.$$

II.

$$\int \frac{x^p - 1 - p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = f \partial p f^p \partial p.$$

III.

$$\int \frac{x^p - 1 - p l x - \frac{1}{2} p p (l x)^2}{\Delta} \cdot \frac{\partial x}{x (l x)^3} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] \\ = f \partial p f \partial p f^p \partial p.$$

IV.

$$\int \frac{x^p - 1 - p l x - \frac{1}{2} p p (l x)^2 - \frac{1}{6} p^3 (l x)^3}{\Delta} \cdot \frac{\partial x}{x (l x)^4} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] \\ \text{etc.} \quad = f \partial p f \partial p f \partial p f^p \partial p.$$

Vbi denuo eadem sunt obseruanda, quae supra sunt praecepta.

ORDO QVARTVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2 \cos. \theta + x^{-n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\pi \sin. \frac{p}{n} \theta}{n \sin. \theta \sin. \frac{p}{n} \pi}.$$

Statuamus hic iterum $\Delta = x^n + 2 \cos. \theta + x^{-n}$, fitque

$$P = \frac{\pi \sin. \frac{p}{n} \theta}{n \sin. \theta \sin. \frac{p}{n} \pi},$$

ita vt P tanquam functio ipsius p spectari possit; vbi probe
Neua Acta Acad. Imp. Sc. T. V. B notan-

notandum est, hunc valorem integralem subsistere non posse, nisi sit $p < n$, ideoque fractio $\frac{p}{n}$ unitate minor, atque sub iisdem conditionibus per differentiationem sequentia hinc deducuntur Theoremata:

$$\text{I.} \quad \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \, l x}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial P}{\partial p}.$$

$$\text{II.} \quad \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^2}{x} \left[\begin{array}{l} \text{ab } = 0 \\ \text{ad } = 1 \end{array} \right] = \frac{\partial \partial P}{\partial p^2}.$$

$$\text{III.} \quad \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial^3 P}{\partial p^3}.$$

$$\text{IV.} \quad \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem colliguntur sequentia:

$$\text{I.} \quad \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x \, l x} \left[\begin{array}{l} \text{ad } x = 0 \\ \text{ad } x = 1 \end{array} \right] = f P \, \partial p.$$

$$\text{II.} \quad \int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x (l x)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = f \partial p \, f P \, \partial p.$$

III.

===== (I I) =====

III.

$$\int \frac{x^p - x^{-p} - 2p/x}{\Delta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} ab \ x = c \\ ad \ x = 1 \end{bmatrix} = f \partial p f \partial p f P \partial p.$$

IV.

$$\int \frac{x^p + x^{-p} - 2 - \frac{2}{3} p p (lx)^2}{\Delta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} ab \ x = c \\ ad \ x = 1 \end{bmatrix} = f \partial p f \partial p f \partial p f P \partial p.$$

etc.

Quod si eadem integralia extendantur ab $x = 0$ ad $x = \infty$, eorum valores duplo euadent maiores.

O R D O Q V I N T V S

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^{-n}(1+x^n)^2} \begin{bmatrix} ab \ x = c \\ ad \ x = 1 \end{bmatrix} = \frac{\pi p}{n n \sin \frac{p}{n} \pi}.$$

Statuamus igitur hic pro denominatore $\Delta = x^{-n}(1+x^n)^2$, sitque $P = \frac{\pi p}{n n \sin \frac{p}{n} \pi}$, ita ut P spectari possit tanquam functione ipsius p , ubi perpetuo fractio $\frac{p}{n}$ unitate minor supponitur, quibus positis per differentiationem sequentia nascuntur Theoremata:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x / x}{x} \begin{bmatrix} ab \ x = c \\ ad \ x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} ab \ x = c \\ ad \ x = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

B 2

III.

$$\text{III.} \quad \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial^3 P}{\partial p^3}.$$

$$\text{IV.} \quad \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial^4 P}{\partial p^4}.$$

Per integrationem vero sequentia deducuntur:

$$\text{I.} \quad \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x l x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = f P \partial p.$$

$$\text{II.} \quad \int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x (l x)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = f \partial p f P \partial p.$$

$$\text{III.} \quad \int \frac{x^p - x^{-p} - 2 p (l x)}{\Delta} \cdot \frac{\partial x}{x (l x)^3} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f P \partial p.$$

$$\text{IV.} \quad \int \frac{x^p + x^{-p} - 2 - p p (l x)^2}{\Delta} \cdot \frac{\partial x}{x (l x)^4} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f \partial p f P \partial p$$

etc.

At si haec integralia ab $x = 0$ ad $x = \infty$ capiantur, eorum valores euadent duplo maiores.

ORDO SEXTVS

Theorematum ex forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + (f + \frac{1}{f}) + x^{-n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi}$$

Statuamus

$$\Delta = x^n + (f + \frac{1}{f}) + x^{-n} = \frac{1}{x^n} (x^n + f) (x^n + \frac{1}{f}), \text{ et fit}$$

$$P = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi},$$

vbi iterum fractio $\frac{p}{n}$ vnitatem minor supponitur. His obseruatis per differentiationem colligimus:

$$\text{I.} \quad \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial P}{\partial p}$$

$$\text{II.} \quad \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial^2 P}{\partial p^2}$$

$$\text{III.} \quad \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial^3 P}{\partial p^3}$$

$$\text{IV.} \quad \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem sequentia Theoremata nascuntur:

(14)

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x l x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = f P \partial p.$$

II.

$$\int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x (l x)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = f \partial p f P \partial p.$$

III.

$$\int \frac{x^p - x^{-p} - 2 p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^3} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f P \partial p.$$

IV.

$$\int \frac{x^p + x^{-p} - 2 - p p (l x)^2}{\Delta} \cdot \frac{\partial x}{x (l x)^4} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f \partial p f P \partial p.$$

Quòd si haec integralia ab $x = 0$ ad $x = \infty$ extendantur, eorum valores erunt duplo maiores. Ceterum hic perspicuum est, quantitatem f esse debere positivam, quia alias potestates $f^{\pm \frac{p}{n}}$ fieri possent imaginariae.

ORDO SEPTIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\cos. p l x}{x^n + 2 \cos. \theta + x^{-n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\pi}{2 n \sin. \theta} \left(\frac{e^{\frac{p}{n}} - e^{-\frac{p}{n} \theta}}{e^{\frac{p}{n} \pi} - e^{-\frac{p}{n} \pi}} \right).$$

Statua-

Statuamus hic iterum pro denominatore

$$\Delta = x^n + 2 \cos. \theta + x^{-n},$$

fitque

$$P = \frac{\pi}{2n \sin. \theta} \cdot \frac{e^{p\theta} - e^{-p\theta}}{e^{n\pi} - e^{-n\pi}},$$

quae ergo quantitas iterum vt functio ipsius p spectari potest; vbi autem non amplius necesse est vt fractio $\frac{p}{n}$ sit vnitatem minor. Hinc igitur per differentiationem sequentia deriuntur Theoremata:

$$\text{I.} \quad \int \frac{\sin. p l x}{\Delta} \cdot \frac{\partial x l x}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = - \frac{\partial P}{\partial p}.$$

$$\text{II.} \quad \int \frac{\cos. p l x}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = - \frac{\partial \partial P}{\partial p^2}.$$

$$\text{III.} \quad \int \frac{\sin. p l x}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = + \frac{\partial^3 P}{\partial p^3}.$$

$$\text{IV.} \quad \int \frac{\cos. p l x}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = + \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem vero

$$\text{I.} \quad \int \frac{\sin. p l x}{\Delta} \cdot \frac{\partial x}{x l x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int P \partial p.$$

II.

(16)

$$\text{II.} \quad \int \frac{1 - \text{cof. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = f \partial p f P \partial p.$$

$$\text{III.} \quad \int \frac{p l x - \text{fin. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^3} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f P \partial p.$$

$$\text{IV.} \quad \int \frac{\frac{1}{2} p p (l x)^2 - 1 + \text{cof. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^4} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f \partial p f P \partial p.$$

$$\text{V.} \quad \int \frac{\frac{1}{5} p^3 (l x)^3 - p l x + \text{fin. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^5} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f \partial p f \partial p f P \partial p.$$

etc.

Hæc igitur integralia, si ab $x = 0$ ad $x = \infty$ extendantur, iterum duplo fiunt maiora.

ORDO OCTAVVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{cof. } p l x}{x^{-n} (x^n + 1)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\pi}{n n} \cdot \frac{p}{e^n - e^{-n}} = \frac{p \pi}{n}.$$

Statuamus hic pro denominatore $\Delta = x^{-n} (x^n + 1)^2$,
fitque

P =

$$P = \frac{\pi}{nn} \cdot \frac{p}{e^{\frac{p}{n}} \pi - e^{-\frac{p}{n}} \pi},$$

atque per differentiationem hinc deducuntur sequentia Theoremata :

$$\text{I.} \quad \int \frac{\sin. plx}{\Delta} \cdot \frac{\partial x lx}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = - \frac{\partial P}{\partial p}.$$

$$\text{II.} \quad \int \frac{\cos. plx}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = - \frac{\partial \partial P}{\partial p^2}.$$

$$\text{III.} \quad \int \frac{\sin. plx}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = + \frac{\partial^3 P}{\partial p^3}.$$

$$\text{IV.} \quad \int \frac{\cos. plx}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = + \frac{\partial^4 P}{\partial p^4}.$$

Per integrationem vero elicitur

$$\text{I.} \quad \int \frac{\sin. plx}{\Delta} \cdot \frac{\partial x}{x lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int P \partial p.$$

$$\text{II.} \quad \int \frac{1 - \cos. plx}{\Delta} \cdot \frac{\partial x}{x (lx)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \partial p \int P \partial p.$$

$$\text{III.} \quad \int \frac{plx - \sin. plx}{\Delta} \cdot \frac{\partial x}{x (lx)^3} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \partial p \int \partial p \int P \partial p.$$

IV.

$$\int \frac{\frac{1}{2} p p (l x)^2 - 1 + \text{cof. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^4} \left[\begin{array}{l} \text{ab } x = c \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f \partial p f P \partial p.$$

V.

$$\int \frac{\frac{1}{2} p^3 (l x)^3 - p l x + \text{fin. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^5} \left[\begin{array}{l} \text{ab } x = c \\ \text{ad } x = 1 \end{array} \right] \\ = f \partial p f \partial p f \partial p f \partial p f P \partial p. \\ \text{etc.}$$

ORDO NONVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{cof. } p l x}{x^n + (f + \frac{1}{f}) + x^{-n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{2 \pi \text{fin. } \frac{p}{n} \cdot l f}{n (f - \frac{1}{f}) (e^{\frac{p}{n} \pi} - e^{-\frac{p}{n} \pi})}$$

Statuatur $\Delta = x^n + (f + \frac{1}{f}) + x^{-n}$ sitque

$$P = \frac{2 \pi \text{fin. } \frac{p}{n} \cdot l f}{n (f - \frac{1}{f}) (e^{\frac{p}{n} \pi} - e^{-\frac{p}{n} \pi})}$$

atque hinc per differentiationem sequentia prodeunt Theoremata:

I.

$$\int \frac{\text{fin. } p l x}{\Delta} \cdot \frac{\partial x l x}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = - \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{\text{cof. } p l x}{\Delta} \cdot \frac{\partial x (l x)^2}{x} \left[\begin{array}{l} \text{ab } x = c \\ \text{ad } x = 1 \end{array} \right] = - \frac{\partial \partial P}{\partial p^2}.$$

III.

==== (19) ====

III.

$$\int \frac{\text{fin. } p l x}{\Delta} \cdot \frac{\partial x (l x)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = + \frac{\partial^3 P}{\partial x^3}.$$

IV.

$$\int \frac{\text{cof. } p l x}{\Delta} \cdot \frac{\partial x (l x)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = + \frac{\partial^4 P}{\partial x^4}.$$

etc.

Per integrationem vero

I.

$$\int \frac{\text{fin. } p l x}{\Delta} \cdot \frac{\partial x}{x l x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int P \partial p.$$

II.

$$\int \frac{1 - \text{cof. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^2} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \partial p \int P \partial p.$$

III.

$$\int \frac{p l x - \text{fin. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^3} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \partial p \int \partial p \int P \partial p.$$

IV.

$$\int \frac{\frac{1}{2} p p (l x)^2 - 1 + \text{cof. } p l x}{\Delta} \cdot \frac{\partial x}{x (l x)^4} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] \\ = \int \partial p \int \partial p \int \partial p \int P \partial p.$$

etc.

Hic manifestum est quantitatem f negativam accipi non posse, quia alias iam ipsa functio P feret imaginaria.

Adiungamus his theoremata simpliciora, quae ex hactenus allatis nascuntur, dum angulus θ sumitur rectus, ut fit $\text{cof. } \theta = 0$ et $\text{fin. } \theta = 1$. Hinc ergo sequentes ordines adiiciamus

ORDO DECIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + x^{-n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\pi}{2n \text{ cof. } \frac{\pi p}{2n}}$$

Haec forma scilicet nata est ex prima, sumendo $\theta = \frac{\pi}{2}$, unde posito $\Delta = x^n + x^{-n}$ et $P = \frac{\pi}{2n \text{ cof. } \frac{\pi p}{2n}}$ nascuntur eadem formulae, quae in ordine primo sunt allatae. Hic autem imprimis notari meretur, quod integrale $\int P \partial p$ per logarithmos exhiberi potest: erit enim

$$\int P \partial p = \int \frac{\pi \partial p}{2n \text{ cof. } \frac{\pi p}{2n}} = l \text{ tang. } (45^\circ + \frac{\pi p}{4n})$$

quod integrale ita est sumtum, ut euanescat facto $p = c$.

ORDO VNDECIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + x^{-n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\pi}{2n \text{ cof. } \frac{\pi p}{2n}}$$

Hic scilicet ordo natus est ex quarto, ponendo $\theta = \frac{\pi}{2}$; quamobrem statuamus $\Delta = x^n + x^{-n}$ et $P = \frac{\pi}{2n \text{ cof. } \frac{\pi p}{2n}}$, eademque theoremata inde nascuntur, quae supra pro ordine quar-

quarto sunt allata, vbi ergo iterum commode vsu venit vt fit
 $\int P \partial p = l \text{ tang. } (45^\circ + \frac{\pi p}{2n}).$

O R D O XII.

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{cos. } p l x}{x^n + x^{-n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = - \frac{\pi}{2n} \cdot \frac{1}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Quod si ergo statuamus

$$\Delta = x^n + x^{-n} \text{ et } P = \frac{\pi}{2n (e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}})},$$

eadem plane Theoremata hinc oriuntur, quae supra pro casu septimo sunt allata. Hic autem iterum notasse iuuabit integrale $\int P \partial p$ reuera exhiberi posse. Cum enim fit

$$\int P \partial p = \int \frac{\pi \partial p}{2n (e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}})},$$

ponatur $\frac{\pi p}{2n} = z$, eritque

$$\int P \partial p = \int \frac{\partial z}{e^z + e^{-z}} = \int \frac{e^z \partial z}{e^{2z} + 1}.$$

Sit porro $e^z = v$, erit $\partial v = e^z \partial z$, hincque fiet

$$\int P \partial p = \int \frac{\partial v}{1 + v^2} = A \text{ tang. } v;$$

quare retro substituendo habebimus

$$\int P \partial p = A \text{ tang. } e^{\frac{\pi p}{2n}}.$$

Denique adhuc referamus formulas illas integrales, in quarum denominatore erat $1 - x^{2n}$, quas quidem iam olim breuiter tetigi, nunc autem vberius euoluam.

O R D O XIII.

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p - x^{-p}}{x^n - x^{-n}} \left[\begin{array}{l} ab \ x \equiv 0 \\ ad \ x \equiv 1 \end{array} \right] = \frac{\pi}{2n} \text{tang.} \frac{\pi p}{2n}.$$

Hic igitur iterum statuamus

$$\Delta = x^n - x^{-n} \text{ et } P = \frac{\pi}{2n} \text{tang.} \frac{\pi p}{2n},$$

atque per differentiationem nanciscemur sequentia Theoremata.

$$\text{I.} \\ \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \ l x}{x} \left[\begin{array}{l} ab \ x \equiv 0 \\ ad \ x \equiv 1 \end{array} \right] = \frac{\partial P}{\partial p}.$$

$$\text{II.} \\ \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \ (l x)^2}{x} \left[\begin{array}{l} ab \equiv 0 \\ ad \equiv 1 \end{array} \right] = \frac{\partial^2 P}{\partial p^2}.$$

$$\text{III.} \\ \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \ (l x)^3}{x} \left[\begin{array}{l} ab \ x \equiv 0 \\ ad \ x \equiv 1 \end{array} \right] = \frac{\partial^3 P}{\partial p^3}.$$

$$\text{IV.} \\ \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \ (l x)^4}{x} \left[\begin{array}{l} ab \ x \equiv 0 \\ ad \ x \equiv 1 \end{array} \right] = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Integratio autem sequentia suppeditat:

$$\text{I.} \\ \int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x \ l x} \left[\begin{array}{l} ad \ x \equiv 0 \\ ad \ x \equiv 1 \end{array} \right] = \int P \ \partial p.$$

II.

II.

$$\int \frac{x^p - x^{-p} - 2plx}{\Delta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} ab \ x = 0 \\ ad \ x = 1 \end{bmatrix} \\ = f \partial p f P \partial p.$$

III.

$$\int \frac{x^p + x^{-p} - 2 - \frac{2}{3} p p (lx)^2}{\Delta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} ab \ x = 0 \\ ad \ x = 1 \end{bmatrix} \\ = f \partial p f \partial p f P \partial p.$$

IV.

$$\int \frac{x^p - x^{-p} - 2plx - \frac{2}{5} p^3 (lx)^3}{\Delta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} ab \ x = 0 \\ ad \ x = 1 \end{bmatrix} \\ = f \partial p f \partial p f \partial p f P \partial p.$$

V.

$$\int \frac{x^p + x^{-p} - 2 - \frac{2}{3} p p (lx)^2 - \frac{2}{5} p^3 (lx)^4}{\Delta} \cdot \frac{\partial x}{x(lx)^5} \begin{bmatrix} ab \ x = 0 \\ ad \ x = 1 \end{bmatrix} \\ = f \partial p f \partial p f \partial p f \partial p f P \partial p.$$

etc.

vbi iterum notetur formulam integram $f P \partial p$ actu exhiberi posse; erit enim

$$f P \partial p = f \frac{\pi \partial p}{2n} \text{ tang. } \frac{\pi p}{2n} = -l \text{ cof. } \frac{\pi p}{2n} = +l \text{ sec. } \frac{\pi p}{2n}.$$

Hic probe notandum est, fractionem $\frac{p}{n}$ semper esse debere unitate minorem.

ORDO

O R D O X I V .

Theorematum ex hac forma generali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{fin. } p l x}{x^{-n} - x^{+n}} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\pi}{4n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{+\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}} .$$

Statuatur igitur vt haectenus $\Delta = x^{-n} - x^n$ et

$$P = \frac{\pi}{4n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{2n} + e^{-2n}} ,$$

atque differentiatio nobis praebebit sequentia Theoremata:

I.

$$\int \frac{\text{cof. } p l x}{\Delta} \cdot \frac{\partial x l x}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\partial P}{\partial p} .$$

II.

$$\int \frac{\text{fin. } p l x}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = - \frac{\partial \partial P}{\partial p^2} .$$

III.

$$\int \frac{\text{cof. } p l x}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = - \frac{\partial^3 P}{\partial p^3} .$$

IV.

$$\int \frac{\text{fin. } p l x}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = + \frac{\partial^4 P}{\partial p^4} .$$

etc.

Per

Per integrationem autem impetramus sequentia:

I.

$$\int \frac{1 - \cos. plx}{\Delta} \cdot \frac{\partial x}{x lx} \left[\begin{array}{l} ab \ x = 0 \\ ad \ x = 1 \end{array} \right] = fP \partial p.$$

II.

$$\int \frac{plx - \sin. plx}{\Delta} \cdot \frac{\partial x}{x (lx)^2} \left[\begin{array}{l} ab \ x = 0 \\ ad \ x = 1 \end{array} \right] = f\partial p fP \partial p.$$

III.

$$\int \frac{\frac{1}{2} p p (lx)^2 - 1 + \cos. plx}{\Delta} \cdot \frac{\partial x}{x (lx)^3} \left[\begin{array}{l} ab \ x = 0 \\ ad \ x = 1 \end{array} \right] \\ = f\partial p f\partial p fP \partial p.$$

IV.

$$\int \frac{\frac{1}{2} p^3 (lx)^3 - plx + \sin. plx}{\Delta} \cdot \frac{\partial x}{x (lx)^4} \left[\begin{array}{l} ab \ x = 0 \\ ad \ x = 1 \end{array} \right] \\ = f\partial p f\partial p f\partial p fP \partial p.$$

etc.

Vbi iterum commode euenit vt $fP \partial p$ exhiberi possit, siquidem habemus

$$fP \partial p = \int \frac{\pi \partial p}{4n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Ponatur enim $\frac{p\pi}{2n} = \Phi$, eritque

$$\int P \partial p = \int \frac{1}{2} \partial \Phi \cdot \frac{e^{-\Phi} - e^{+\Phi}}{e^{\Phi} + e^{-\Phi}},$$

vbi denominatoris differentiale est $e^{\Phi} \partial \Phi - e^{-\Phi} \partial \Phi$, vnde concluditur

$$\int P \partial p = -l \sqrt{(e^{\Phi} + e^{-\Phi})} + C$$

quae constans C ita assumi debet, vt integrale euanescat posito $\Phi = 0$, vnde fit

$$\int P \partial p = \frac{1}{2} \int \frac{2}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Hic autem perinde est, vtrum fractio $\frac{p}{n}$ maior sit minorue vnitae.