1789

De innumeris generibus serierum maxime memorabilium, quibus omnium aequationum algebraicarum non solum radices ipsae sed etiam quaecumque earum potestates exprimi possunt

Leonhard Euler

DE INNVMERIS GENERIBVS SERIERVM MAXIME MEMORABILIVM, QVIBVS OMNIVM AEQVATIONVM ALGEBRAICARVM NON SOLVM RADICES IPSAE SED ETIAM QVAECVNQVE EARVM POTESTATES EXPRIMI POSSVNT.

Auctore

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§. 1.

Propofita aequatione quacunque algebraica ad hanc formam reducta:

$$\mathfrak{A}x^{\mathfrak{a}} + \mathfrak{B}x^{\mathfrak{b}} + \mathfrak{C}x^{\mathfrak{c}} + \mathfrak{D}x^{\mathfrak{d}} + \mathfrak{E}x^{\mathfrak{e}} + \text{etc.} = 0,$$

vbi exponentes $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$, etc. fint numeri quicunque, fiue integri, fiue fracti, fiue pofitiui, fiue etiam negatiui, in Tomo XV. Nouorum Commentariorum oftendi, non folum huius aequationis radicem x, fed etiam quamuis poteftatem x^n, per feriem infinitam fatis concinnam, cuius omnes termini fecundum certam legem formentur, pluribus modis exprimi poffe, quae eo magis omni attentione, atque adeo admiratione dignae videntur, quod earum ratio propemodum omnes vires Analyfeos fuperare videtur,

tur, propterea quod iſtae ſeries non ſolum per longas ambages ſint inuentae, ſed etiam in earum formatione plurimum inductioni ſit tributum. Quamobrem plurimum operae pretium fore videtur, vt omnes operationes, quibus hae ſeries formantur, dilucide exponantur et omnia momenta, quibus earum formatio innititur, ſummo ſtudio perpendantur. Tum enim demum ſperare licebit in veram originem iſtarum ſerierum penetrare atque omnia myſteria, quibus earum natura adhuc inuoluta deprehenditur, ad principia cognita reuocare.

§. 2. Prima operatio, qua aequationem quamcunque propoſitam ad hunc ſcopum praeparare oportet, in hoc conſiſtit, vt tota aequatio per quempiam ſuorum terminorum diuidatur, hocque modo ipſa aequatio ad huiusmodi formam reducatur:

$$1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta} + \frac{C}{x^\gamma} + \frac{D}{x^\delta} + \frac{E}{x^\varepsilon} + \text{etc.}$$

quod ergo totidem diuerſis modis fieri poterit, quot terminis ipſa aequatio fuerit compoſita; vbi iterum eſt monendum, exponentes α, β, γ, δ, ε, etc. quoscunque numeros denotare poſſe, ſiue poſitiuos, ſiue negatiuos, ſiue integros, ſiue fractos. Quin etiam perinde eſt quonam ordine iſti termini diſponantur. Tantum hic in limine notaſſe iuuabit, quoniam ipſi radici x vna dimenſio tribuitur, propter homogeneitatam omnium terminorum quantitatibus A, B, C reſpectiue tot dimenſiones aſſignari oportere, quot exponentes α, β, γ indicant. Ita littera A cenſenda eſt habere α dimenſiones; numerus vero dimenſionum litterae B eſt β, at litterae $C = \gamma$, etc., id quod ideo probe eſt obſeruandum, quod in ſerie, quae valorem radicis x referat, omnes eius termini vnicam dimenſionem conſtituere debent. In ſeriebus autem, quae radicis x poteſtatem

tatem

tatem quamcunque x^n exprimunt, euidens eſt in ſingulis terminis litteras A, B, C, D, etc. n dimenſiones complere debere.

§. 3. His expoſitis pro quacunque radicis poteſtate x^n tot diuerſae ſeries infinitae exhiberi poſſunt, quot litterae A, B, C, D, etc. in aequatione reperiuntur, propterea quod primus ſeriei cuiusque terminus ex quouis membro illius aequationis formari poteſt, quaſi reliqui non adeſſent. Ita ex membro $x^\alpha = 1$ ſequitur $x = A^{\frac{1}{\alpha}}$ et $x^n = A^{\frac{n}{\alpha}}$; ex membro autem ſecundo $\frac{B}{x^\beta} = 1$ prodiret $x^n = B^{\frac{n}{\beta}}$; ſimilique modo ex tertio membro prodiret $x^n = C^{\frac{n}{\gamma}}$; et ita porro. Vocemus igitur illud aequationis membrum, ex quo primum ſeriei terminum conſtituere libet, aequationis membrum principale, pro quo in poſterum aſſumamus perpetuo membrum primo loco poſitum $\frac{A}{x^\alpha}$, ita vt initium ſeriei inueſtigandae ſit $x^n = A^{\frac{n}{\alpha}}$.

§. 4. Conſtituto igitur primo termino ſeriei, quae valorem ipſius x^n exprimat, qui ergo eſt $x^n = A^{\frac{n}{\alpha}}$, manifeſtum eſt omnes terminos ſequentes ex reliquis litteris B, C, D, E, etc. tam ſingulis quam vtcunque inuicem combinatis, formari debere, vnde pro reliquis terminis infinitos ordines conſtitui conueniet, quorum primus contineat ſingulas litteras B, C, D, etc. ſolitarias, ſecundus ordo omnia producta ex binis harum litterarum, quae ergo ſunt primo earum quadrata B^2, C^2, D^2 tum vero producta ex binis diuerſis BC, BD, CD, etc. tertius vero ordo complectetur omnia producta ex tribus harum littera-

litterarum, siue iisdem, siue diuersis, quae ergo erunt B^3, C^3, D^3; B B C, B B D, B C C, B D D, B C D, etc. quartus vero ordo continebit, praeter potestates quartas harum litterarum, omnia producta ex quaternis earum inter se coniunctis. Ex quo intelligitur ordinem quintum inuoluere omnia producta ex quinis, ordinem sextum ex senis, et ita porro in infinitum. Hinc totum negotium huc redit, quomodo omnes terminos cuiusque ordinis formari oporteat, vt omnibus in vnam summam collectis et ad primum $A^{\frac{n}{\alpha}}$ additis, prodeat series infinita verum valorem potestatis x^n exprimens, quamobrem pro quolibet ordine regulas singulos terminos formandi exponamus, quemadmodum in dissertatione initio commemorata sunt traditae.

PRIMA REGVLA.

Pro formandis terminis primi ordinis.

§. 5. Hic ante omnia patet cum qualibet litterarum B, C, D, E, etc. eiusmodi potestatem ipsius A coniungi debere, vt dimensionum numerus prodeat $= n$. Quoniam igitur littera B censetur habere β dimensiones, litterae vero A numerus dimensionum est α, si pro B statuatur ista potestas A^λ, tum producti $A^\lambda B$ numerus dimensionum erit $\lambda\alpha + \beta = n$, vnde colligitur: $\lambda = \frac{n-\beta}{\alpha}$, sicque ex littera B nascitur $A^{\frac{n-\beta}{\alpha}} B$. Similique modo ex littera C orietur $A^{\frac{n-\gamma}{\alpha}} C$. Tum vero ex littera D orietur $A^{\frac{n-\delta}{\alpha}} D$, et ita porro. Praeterea vero his singulis productis praefigi opportet communem multiplicatorem $\frac{n}{\alpha}$; quamobrem termini seriei quaesitae ex singulis litteris B, C, D, E oriundi ita se habebunt:

$$\frac{n}{\alpha} A^{\frac{n-\beta}{\alpha}} B + \frac{n}{\alpha} A^{\frac{n-\gamma}{\alpha}} C + \frac{n}{\alpha} A^{\frac{n-\delta}{\alpha}} D + \frac{n}{\alpha} A^{\frac{n-\varepsilon}{\alpha}} E + \text{etc.}$$

ficque habentur omnes termini primi ordinis pro ferie quam inueftigamus.

SECVNDA REGVLA.

Pro formandis terminis fecundi ordinis.

§. 6. Quoniam hic occurrunt termini dupplicis formae, fcilicet vel ipfa quadrata B^2, C^2, D^2, vel producta ex binis diverfis B C, B D, B E, C D, etc. multiplicatores cuique formae iungendos feorfim euoluamus. Ac primo quidem pro forma B^2, quae habet 2β dimenfiones, iungi debet haec poteftas $A^{\frac{n-2\beta}{\alpha}}$; deinde vero ex iis, quae in differtatione iam allegata demonftraui, patet infuper adiungi debere multiplicatorem ex duobus factoribus conftantem $\frac{n}{\alpha} \cdot \frac{n+\alpha-2\beta}{2\alpha}$, vnde termini feriei quaefitae, qui ex quadratis B^2, C^2, D^2 oriuntur, erunt fequentes:

Ex forma	nafcitur terminus
B B	$\frac{n}{\alpha} \cdot \frac{n+\alpha-2\beta}{2\alpha} A^{\frac{n-2\beta}{\alpha}} BB$
C C	$\frac{n}{\alpha} \cdot \frac{n+\alpha-2\gamma}{2\alpha} A^{\frac{n-2\gamma}{\alpha}} CC$
D D	$\frac{n}{\alpha} \cdot \frac{n+\alpha-2\delta}{2\alpha} A^{\frac{n-2\delta}{\alpha}} DD$
E E	$\frac{n}{\alpha} \cdot \frac{n+\alpha-2\varepsilon}{2\alpha} A^{\frac{n-2\varepsilon}{\alpha}} EE$
etc.	etc.

§. 7. Pro formis autem binas litteras diuerfas inuolventibus fufficiet confideraffe formam B C, quae primo recipit coëfficientem numericum numero permutationum harum litterarum refpondentem, qui eft 2, vt hinc habeatur 2 B C. Deinde quia numerus dimenfionum eft $\beta + \gamma$, poteftas ipfius A iungengenda erit $A^{\frac{n-\beta-\gamma}{\alpha}}$; ac denique multiplicator insuper adiiciendus erit $\frac{n}{\alpha} \cdot \frac{n+\alpha-\beta-\gamma}{2\alpha}$. His igitur coniunctis ex singulis nostris formis huius ordinis orientur sequentes termini pro serie quaesita:

Ex forma	nascitur terminus
B C	$2 \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\beta-\gamma}{2\alpha} A^{\frac{n-\beta-\gamma}{\alpha}} BC$
B D	$2 \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\beta-\delta}{2\alpha} A^{\frac{n-\beta-\delta}{\alpha}} BD$
B E	$2 \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\beta-\varepsilon}{2\alpha} A^{\frac{n-\beta-\varepsilon}{\alpha}} BE$
C D	$2 \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\gamma-\delta}{2\alpha} A^{\frac{n-\gamma-\delta}{\alpha}} CD$
C E	$2 \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\gamma-\varepsilon}{2\alpha} A^{\frac{n-\gamma-\varepsilon}{\alpha}} CE$
D E	$2 \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\delta-\varepsilon}{2\alpha} A^{\frac{n-\delta-\varepsilon}{\alpha}} DE$
etc.	etc.

TERTIA REGVLA.

Pro formandis terminis tertii ordinis.

§. 8. In hoc ordine primo occurrit B^3, cuius index naturalis ex permutatione ortus est $= 1$; deinde potestas ipsius A, quia dimensionum numerus ipsius B^3 est 3β, erit $A^{\frac{n-3\beta}{\alpha}}$. At vero multiplicator insuper adiungendus nunc constabit ex tribus factoribus, eritque $\frac{n}{\alpha} \cdot \frac{n+\alpha-3\beta}{2\alpha} \cdot \frac{n+2\alpha-3\beta}{3\alpha}$, quibus coniunctis termini ex singulis cubis oriundi se habebunt sequenti modo:

Ex

Ex forma	nascitur terminus
B^3	$\frac{n}{\alpha}\cdot\frac{n+\alpha-3\beta}{2\alpha}\cdot\frac{n+2\alpha-3\beta}{3\alpha}\,A^{\frac{n-3\beta}{\alpha}}B^3$
C^3	$\frac{n}{\alpha}\cdot\frac{n+\alpha-3\gamma}{2\alpha}\cdot\frac{n+2\alpha-3\gamma}{3\alpha}\,A^{\frac{n-3\gamma}{\alpha}}C^3$
D^3	$\frac{n}{\alpha}\cdot\frac{n+\alpha-3\delta}{2\alpha}\cdot\frac{n+2\alpha-3\delta}{3\alpha}\,A^{\frac{n-3\delta}{\alpha}}D^3$
etc.	etc.

§. 9. Secunda forma in hoc ordine occurrens est BBC, cui conuenit index naturalis $= 3$, et quia numerus dimensionum est $2\beta+\gamma$, potestas ipsius A iungenda erit $A^{\frac{n-2\beta-\gamma}{\alpha}}$; tum vero multiplicator insuper adiiciendus erit

$$\frac{n}{\alpha}\cdot\frac{n+\alpha-2\beta-\gamma}{2\alpha}\cdot\frac{n+2\alpha-2\beta-\gamma}{3\alpha}$$

vnde pro singulis formis formabuntur sequentes termini:

Ex forma	nascitur terminus
B B C	$3\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-2\beta-\gamma}{2\alpha}\cdot\frac{n+2\alpha-2\beta-\gamma}{3\alpha}\,A^{\frac{n-2\beta-\gamma}{\alpha}}BBC$
B C C	$3\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-\beta-2\gamma}{2\alpha}\cdot\frac{n+2\alpha-\beta-2\gamma}{3\alpha}\,A^{\frac{n-\beta-2\gamma}{\alpha}}BCC$
B B D	$3\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-2\beta-\delta}{2\alpha}\cdot\frac{n+2\alpha-2\beta-\delta}{3\alpha}\,A^{\frac{n-2\beta-\delta}{\alpha}}BBD$
B D D	$3\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-\beta-2\delta}{2\alpha}\cdot\frac{n+2\alpha-\beta-2\delta}{3\alpha}\,A^{\frac{n-\beta-2\delta}{\alpha}}BDD$
etc.	etc.

In hoc ordine superest terminus B C D ternas litteras dispares inuoluens, cuius index numericus ex permutatione natus est 6; et quia numerus dimensionum hic est $\beta+\gamma+\delta$, potestas ipsius A iungenda erit $A^{\frac{n-\beta-\gamma-\delta}{\alpha}}$; at vero multiplicator insuper iungendus erit

$$\frac{n}{\alpha}\cdot\frac{n+\alpha-\beta-\gamma-\delta}{2\alpha}\cdot\frac{n+2\alpha-\beta-\gamma-\delta}{3\alpha},$$

vnde

vnde ex forma B C D nafcitur ifte terminus pro ferie quaefita:

$$6.\ \frac{n}{\alpha}\cdot\frac{n+\alpha-\beta-\gamma-\delta}{2\alpha}\cdot\frac{n+2\alpha-\beta-\gamma-\delta}{3\alpha}A^{\frac{n-\beta-\gamma-\delta}{\alpha}}BCD;$$

vnde facile patet quomodo termini ex reliquis formis huius naturae B C E, C D E, etc. fint formandi, quos fuperfluum foret hic feorfim apponere, quandoquidem omnes has litteras inter fe permutare licet, quamobrem etiam in fequentibus vnicam formam ex quolibet ordine euoluiffe fufficiet.

QVARTA REGVLA.

Pro formandis terminis quarti ordinis.

§. 10. Ex hactenus allatis facile intelligitur, quomodo pro fingulis formis, quae in hoc ordine occurrunt, termini inde refultantes formari debeant.

I. Ex forma B^4 nafcitur ifte terminus:

$$1.\ \frac{n}{\alpha}\cdot\frac{n+\alpha-4\beta}{2\alpha}\cdot\frac{n+2\alpha-4\beta}{3\alpha}\cdot\frac{n+3\alpha-4\beta}{4\alpha}A^{\frac{n-4\beta}{\alpha}}B^4.$$

II. Ex forma $B^3\,C$ nafcitur terminus

$$4.\ \frac{n}{\alpha}\cdot\frac{n+\alpha-3\beta-\gamma}{2\alpha}\cdot\frac{n+2\alpha-3\beta-\gamma}{3\alpha}\cdot\frac{n+3\alpha-3\beta-\gamma}{4\alpha}A^{\frac{n-3\beta-\gamma}{\alpha}}B^3C.$$

III. Ex forma $B^2\,C^2$ nafcitur terminus

$$6.\ \frac{n}{\alpha}\cdot\frac{n+\alpha-2\beta-2\gamma}{2\alpha}\cdot\frac{n+2\alpha-2\beta-2\gamma}{3\alpha}\cdot\frac{n+3\alpha-2\beta-2\gamma}{4\alpha}A^{\frac{n-2\beta-2\gamma}{\alpha}}B^2C^2.$$

IV. Ex forma $B^2\,C\,D$ nafcitur terminus

$$12.\ \frac{n}{\alpha}\cdot\frac{n+\alpha-2\beta-\gamma-\delta}{2\alpha}\cdot\frac{n+2\alpha-2\beta-\gamma-\delta}{3\alpha}\cdot\frac{n+3\alpha-2\beta-\gamma-\delta}{4\alpha}A^{\frac{n-2\beta-\gamma-\delta}{\alpha}}B^2CD.$$

V. Ex forma B C D E nafcitur terminus

$$24.\ \frac{n}{\alpha}\cdot\frac{n+\alpha-\beta-\gamma-\delta-\varepsilon}{2\alpha}\cdot\frac{n+2\alpha-\beta-\gamma-\delta-\varepsilon}{3\alpha}\cdot\frac{n+3\alpha-\beta-\gamma-\delta-\varepsilon}{4\alpha}A^{\frac{n-\beta-\gamma-\delta-\varepsilon}{\alpha}}BCDE.$$

QVINTA REGVLA.

Pro formandis terminis quinti ordinis.

§. 11. In hoc ordine ſequentes occurrunt caſus ſeorſim euoluendi:

I. Ex forma B^5 naſcitur terminus

$$\frac{n}{\alpha}\cdot\frac{n+\alpha-5\beta}{2\alpha}\cdot\frac{n+2\alpha-5\beta}{3\alpha}\cdot\frac{n+3\alpha-5\beta}{4\alpha}\cdot\frac{n+4\alpha-5\beta}{5\alpha}\,A^{\frac{n-5\beta}{\alpha}}\,B^5.$$

II. Ex forma $B^4 C$, poſito brev. gr. $4\beta+\gamma=\theta$, naſcitur terminus

$$5\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-\theta}{2\alpha}\cdot\frac{n+2\alpha-\theta}{3\alpha}\cdot\frac{n+3\alpha-\theta}{4\alpha}\cdot\frac{n+4\alpha-\theta}{5\alpha}\,A^{\frac{n-\theta}{\alpha}}\,B^4\,C.$$

III. Ex forma $B^3 C^2$, poſito brev. gr. $3\beta+2\gamma=\theta$, naſcitur terminus

$$10\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-\theta}{2\alpha}\cdot\frac{n+2\alpha-\theta}{3\alpha}\cdot\frac{n+3\alpha-\theta}{4\alpha}\cdot\frac{n+4\alpha-\theta}{5\alpha}\,A^{\frac{n-\theta}{\alpha}}\,B^3\,C^2.$$

IV. Ex forma $B^3 C D$, poſito brev. gr. $3\beta+\gamma+\delta=\theta$, naſcitur terminns

$$20\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-\theta}{2\alpha}\cdot\frac{n+2\alpha-\theta}{3\alpha}\cdot\frac{n+3\alpha-\theta}{4\alpha}\cdot\frac{n+4\alpha-\theta}{5\alpha}\,A^{\frac{n-\theta}{\alpha}}\,B^3\,C\,D.$$

V. Ex forma $B^2 C^2 D$, poſito brev. gr. $2\beta+2\gamma+\delta=\theta$, naſcitur terminus,

$$30\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-\theta}{2\alpha}\cdot\frac{n+2\alpha-\theta}{3\alpha}\cdot\frac{n+3\alpha-\theta}{4\alpha}\cdot\frac{n+4\alpha-\theta}{5\alpha}\,A^{\frac{n-\theta}{\alpha}}\,B^2\,C^2\,D.$$

VI. Ex forma $B^2 C D E$, poſito brev. gr. $2\beta+\gamma+\delta+\varepsilon=\theta$, naſcitur

$$60\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-\theta}{2\alpha}\cdot\frac{n+2\alpha-\theta}{3\alpha}\cdot\frac{n+3\alpha-\theta}{4\alpha}\cdot\frac{n+4\alpha-\theta}{5\alpha}\,A^{\frac{n-\theta}{\alpha}}\,B^2\,C\,D\,E.$$

VII. Ex forma $B\,C\,D\,E\,F$, poſito brev. gr. $\beta+\gamma+\delta+\varepsilon+\zeta=\theta$, naſcitur.

$$120\cdot\frac{n}{\alpha}\cdot\frac{n+\alpha-\theta}{2\alpha}\cdot\frac{n+2\alpha-\theta}{3\alpha}\cdot\frac{n+3\alpha-\theta}{4\alpha}\cdot\frac{n+4\alpha-\theta}{5\alpha}\,A^{\frac{n-\theta}{\alpha}}\,B\,C\,D\,E\,F.$$

Hinc

Hinc iam lex progreſſionis ita manifeſto perſpicitur, vt ſuperfluum foret vlteriores ordines euoluere.

REGVLA GENERALIS.
Pro terminis formae B^i inueniendis.

§. 12. Quoniam igitur hic terminus ad ordinem i pertinet, per ſe patet eius exponentem i neceſſario eſſe debere numerum integrum poſitiuum, dum reliqui exponentes α, β, γ, δ omnes plane numeros tam poſitiuos quam negatiuos, ſiue integros, ſiue fractos admittant. Deinde quia hic terminus eſt ſimplex poteſtas, eius index numericus erit $= 1$. Porro quia litterae A tribuuntur β dimenſiones, hic numerus dimenſionum erit $i\beta = \theta$, hincque poteſtas ipſius A iungenda erit $A^{\frac{n-\theta}{\alpha}}$. Denique multiplicator ab exponente n pendens continebit i factores, qui ſunt

$$\frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \ldots\ldots\ldots \frac{n+(i-1)\alpha-\theta}{i\alpha}.$$

His ergo coniunctis terminus ex forma B^i oriundus erit

$$1 \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \frac{n+3\alpha-\theta}{4\alpha} \cdot \ldots\ldots \frac{n+(i-1)\alpha-\theta}{i\alpha} A^{\frac{n-\theta}{\alpha}} B^i$$

quae formula facile ad terminos formae vel C^i, vel D^i, vel E^i, vel etc. transfertur.

REGVLA GENERALIS.
Pro terminis formae B^b C^c inueniendis.

§. 13. Hic vt ante ſponte intelligitur exponentes b et c non niſi numeros integros poſitiuos deſignare poſſe, vnde iſte terminus pertinebit ad ordinem $b+c$. Ponamus autem $b+c = i$. Primo igitur huius termini index numericus definiri debet, quem indicemus littera N, et ex Theoria Permuta-

 tio-

tionum constat fore $N = \frac{1.2.3 \ldots \ldots i}{1.2.3 \ldots b \times 1.2.3 \ldots c}$. Deinde quia litterae B dimensiones β, litterae vero C dimensiones γ tribuuntur, numerus dimensionum erit $b\beta + c\gamma = \theta$, hincque potestas ipsius A adiungenda erit $A^{\frac{n-\theta}{\alpha}}$. Tertio vero multiplicator ab exponente n pendens erit vt ante

$$\frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \ldots \ldots \cdot \frac{n+(i-1)\alpha-\theta}{i\alpha},$$

nisi quod hic ipsi θ debitus valor est assignandus, quamobrem his coniunctis reperietur terminus ex forma proposita $B^b C^c$ oriundus iste:

$$N. \frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \frac{n+3\alpha-\theta}{4\alpha} \ldots \frac{n+(i-1)\alpha-\theta}{i\alpha} A^{\frac{n-\theta}{\alpha}} B^b C^c.$$

REGVLA GENERALIS.

Pro terminis formae B^b. C^c. D^d formandis.

§. 14. Ponatur hic index ordinis, ad quem haec forma est referenda, $b + c + d = i$; numerus vero dimensionum in hoc termino contentarum statuatur vt hactenus $b\beta + c\gamma + d\delta = \theta$, vnde tam potestas ipsius A quam multiplicator definientur vt ante; quare cum index numericus sit

$$N = \frac{1.2.3.4 \ldots \ldots i}{1.2 \ldots b \times 1.2 \ldots c \times 1.2 \ldots d},$$

his coniungendis terminus formae propositae respondens erit

$$N. \frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \frac{n+3\alpha-\theta}{4\alpha} \ldots \frac{n+(i-1)\alpha-\theta}{i\alpha} A^{\frac{n-\theta}{\alpha}} B^b. C^c. D^d.$$

FORMATIO SERIEI

Valorem ipsius x^n exprimentis.

§. 15. Postquam secundum praecepta exposita omnes termini cuiusque ordinis in infinitum fuerint inuenti, nil aliud opus

opus eſt, niſi vt omnes iſti termini in vnam ſummam colligantur, quae ad primum terminum addita praebebit valorem poteſtatis x^n. Hinc ergo ſi ſumma omnium terminorum primi ordinis ponatur $= \mathfrak{A}$, ſecundi ordinis $= \mathfrak{B}$, tertii ordinis $= \mathfrak{C}$, etc. erit

$$x^n = A^{\frac{n}{\alpha}} + \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \text{etc.},$$

hicque valor propterea ſatisfaciet aequationi propoſitae

$$1 = \frac{A}{x^{\alpha}} + \frac{B}{x^{\beta}} + \frac{C}{x^{\gamma}} + \frac{D}{x^{\delta}} + \text{etc.},$$

quae cum habere queat plures radices, ſeu valores ipſius x, hic in genere vix definire licet, quemnam valorem expreſſio inuenta ſit exhibitura: plerumque quidem omnium radicum maxima quandoque etiam minima hic locum habere deprehenditur.

§. 16. Si praeter terminum principalem $\frac{A}{x^{\alpha}}$ aequatio vnicum contineat terminum $\frac{B}{x^{\beta}}$, in omnibus ordinibus vnicus tantum habebitur terminus, vnde pro hoc caſu ſeries inuenta ſe ita habebit:

$$x^n = A^{\frac{n}{\alpha}} + \frac{n}{\alpha} A^{\frac{n-\beta}{\alpha}} B + \frac{n}{\alpha} \cdot \frac{n+\alpha-2\beta}{2\alpha} A^{\frac{n-2\beta}{\alpha}} BB$$

$$+ \frac{n}{\alpha} \cdot \frac{n+\alpha-3\beta}{2\alpha} \cdot \frac{n+2\alpha-3\beta}{3\alpha} A^{\frac{n-3\beta}{\alpha}} B^3$$

$$+ \frac{n}{\alpha} \cdot \frac{n+\alpha-4\beta}{2\alpha} \cdot \frac{n+2\alpha-4\beta}{3\alpha} \cdot \frac{n+3\alpha-4\beta}{4\alpha} A^{\frac{n-4\beta}{\alpha}} B^4$$

$$+ \frac{n}{\alpha} \cdot \frac{n+\alpha-5\beta}{2\alpha} \cdot \frac{n+2\alpha-5\beta}{3\alpha} \cdot \frac{n+3\alpha-5\beta}{4\alpha} \cdot \frac{n+4\alpha-5\beta}{5\alpha} A^{\frac{n-5\beta}{\alpha}} B^5$$

etc.

quae ergo feries conuenit aequationi $1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta}$, ita vt radix poteftatis n ex hac ferie extracta exhibeat quampiam radicem iftius aequationis; et quoniam pro n numerum quemcunque, fiue pofitiuum, fiue negatiuum, fiue integrum, fiue fractum accipere licet, hinc innumerabiles diuerfae formae pro eadem aequationis radice x exhiberi poffunt.

§. 17. Sin autem aequatio praeter terminum principalem $\frac{A}{x^\alpha}$ duos pluresue complectatur terminos, tum etiam in ferie inuenta numerus terminorum cuiusque ordinis continuo magis augebitur. Scilicet fi numerus litterarum B, C, D, etc. fuerit $= k$, qui ergo numerum terminorum in primo ordine contentorum indicabit, tum numerus terminorum fecundi ordinis erit $= \frac{k(k+1)}{1.2}$, tertii vero ordinis $= \frac{k(k+1)(k+2)}{1.2.3}$, quarti ordinis $= \frac{k(k+1)(k+2)(k+3)}{1.2.3.4}$, atque adeo in genere numerus terminorum ordinis i erit $= \frac{k(k+1)(k+2)\ldots\ldots(k+i-1)}{1.2.3\ldots\ldots i}$.

§. 18. Quoniam igitur in enumeratione terminorum cuiusque ordinis probe cauendum eft, ne vlla forma praetermittatur, hoc obtinebitur fi generatim pro ordine i euoluatur ifta poteftas: $(B + C + D + E + \text{etc.})^i$, tum enim non folum omnes formae ad hunc ordinem pertinentes prodibunt, fed etiam quaelibet forma iam adiunctum habebit fuum indicem numericum, qui ipfi iuxta regulam permutationum conuenit, et quem fupra defignauimus littera N, ita vt non opus fit eum feorfim inueftigare.

PRAEPARATIO.

ad demonſtrationem ſerierum inuentarum.

§. 19. Quanquam iſtas ſeries iam in Tomo XV. Nouorum Commentariorum exhibui, tamen iam obſeruaui, eas methodo maxime indirecta ac per plures ambages eſſe inuentas, praeterquam quod plerumque etiam ſoli inductioni innituntur, quamobrem ob ſummam dignitatem harum ſerierum maxime eſt optandum, vt earum veritas per demonſtrationem ſolidam extra omne dubium collocetur; inprimis autem deſideratur methodus directa, qua ad easdem ſeries, per operationes iam vſu receptas, peruenire liceat. Quoniam autem poſteriori requiſito nondum ſatisfacere valui, ſaltem veritatem harum ſerierum, quam firma demonſtratione corroborare mihi licuit, hic ſum expoſiturus.

§. 20. Docebo igitur ſeries, quarum conſtructionem hic tradidi, apprime ſatisfacere ipſi aequationi propoſitae:

$$1 = \frac{A}{x^{\alpha}} + \frac{B}{x^{\beta}} + \frac{C}{x^{\gamma}} + \frac{D}{x^{\delta}} + \text{etc.},$$

quae per x^n multiplicata induit hanc formam:

$$x^n = A\,x^{n-\alpha} + B\,x^{n-\beta} + C\,x^{n-\gamma} + D\,x^{n-\delta} + \text{etc.}$$

quamobrem oſtendi oportet ſeriem, quam ſupra pro exprimenda poteſtate x^n inuenire docui, reuera aequalem eſſe ſummae ſerierum, quae ſecundum eandem legem pro poteſtatibus $x^{n-\alpha}$, $x^{n-\beta}$, $x^{n-\gamma}$, etc. formantur, ſi ſcilicet reſpectiue per litteras A, B, C, D, etc. multiplicentur. Euidens autem eſt has poſteriores ſeries ex noſtra ſerie generali erui poſſe, ſi loco exponentis n ſucceſſiue ſcribantur numeri $n-\alpha$, $n-\beta$, $n-\gamma$, $n-\delta$, etc. Hoc autem ſequenti modo pro qualibet terminorum forma oſtendere licebit, vbi quidem ſufficiet

ficiet veritatem ternarum poſtremarum regularum generalium demonſtraſſe.

DEMONSTRATIO
primae regulae generalis pro terminis formae B^i.

§. 21. Supra vidimus terminum ex hac forma oriundum pro poteſtate x^n eſſe

$$1 \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \frac{n+3\alpha-\theta}{4\alpha} \cdots \frac{n+(i-1)\alpha-\theta}{i\alpha} A^{\frac{n-\theta}{\alpha}} \cdot B^i,$$

exiſtente $\theta = i\beta$, qui ergo aequalis eſſe debet terminis eiusdem formae, qui ex membris aequationis primo et ſecundo $A x^{n-\alpha} + B x^{n-\beta}$ oriuntur, quandoquidem in reliquis membris nulli termini formae B^i locum inueniunt, propterea quod horum membrorum termini omnes vel litteram C, vel D, vel E neceſſario inuoluunt. Ex primi igitur membri poteſtate $x^{n-\alpha}$ ſumi debet terminus formae B^i, quippe qui ductus in A naturam non mutat; at vero ex ſecundi membri poteſtate $x^{n-\beta}$, tantum capi debet terminus formae B^{i-1}, quippe qui per B multiplicatus praebet formam propoſitam B^i. Sicque hos binos terminos iunctos aequales fieri oportet ipſi termino propoſito.

§. 22. Quaeramus igitur primo terminum formae B^i, in poteſtate $x^{n-\alpha}$ occurrentem, quem ergo ex ipſa formula propoſita deducemus, loco n ſcribendo $n-\alpha$, qui idcirco erit

$$\frac{n-\alpha}{\alpha} \cdot \frac{n-\theta}{2\alpha} \cdot \frac{n+\alpha-\theta}{3\alpha} \cdot \frac{n+2\alpha-\theta}{4\alpha} \cdots \frac{n+(i-2)\alpha-\theta}{i\alpha} A^{\frac{n-\alpha-\theta}{\alpha}} B^i,$$

exiſtente $\theta = i\beta$. Deinde vero terminus formae B^{i-1} poteſtati $x^{n-\beta}$ ex ipſa formula propoſita eruetur, ſi primo loco i ſcribatur $i-1$, porro loco θ nunc ſcribi debet $\theta-\beta$, tertio

tio vero loco n ſcribendum eſt $n - \beta$, quo facto terminus reſultans erit

$$\frac{n-\beta}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \ldots \cdot \frac{n+(i-2)\alpha-\theta}{(i-1)\alpha} A^{\frac{n-\theta}{\alpha}} B^{i-1},$$

quamobrem nunc priorem terminum ducamus in A, poſteriorem vero in B, et quia ambo hi termini communem habebunt factorem

$$\frac{n+\alpha-\theta \cdot n+2\alpha-\theta \cdot n+3\alpha-\theta \cdot \ldots \cdot n+(i-2)\alpha-\theta}{\alpha \cdot 2\alpha \cdot 3\alpha \cdot 4\alpha \cdot \ldots \cdot (i-1)\alpha} A^{\frac{n-\theta}{\alpha}} B^{i} = \Delta,$$

eorum ſumma erit

$$\Delta \left(\frac{(n-\alpha)\cdot(n-\theta)}{i\alpha} + n - \beta\right) = \frac{n[n+(i-1)\alpha-\theta]}{i\alpha} \cdot \Delta,$$

ob $i\beta = \theta$. Quamobrem reſtituto pro Δ valore aſſumto et factoribus in debitum ordinem diſpoſitis ſumma iſtorum terminorum erit:

$$\frac{n(n+\alpha-\theta)(n+2\alpha-\theta)(n+3\alpha-\theta) \ldots [n+(i-2)\alpha-\theta]\,[n+(i-1)\alpha-\theta]}{\alpha \cdot 2\alpha \cdot 3\alpha \cdot 4\alpha \cdot \ldots \cdot (i-1)\alpha \cdot i\alpha} A^{\frac{n-\theta}{\alpha}} \cdot B^{i},$$

qui eſt ipſe terminus poteſtati x^n reſpondens, ideoque huius regulae veritas eſt demonſtrata.

DEMONSTRATIO

ſecundae regulae generalis, pro terminis formae $B^b C^c$.

§. 23. Poſtquam hic poſuimus $b+c=i$ et $b\beta+c\gamma=\theta$, tum vero $N = \frac{1.2.3.\ldots\ldots i}{1.2.\ldots b \times 1.2.\ldots c}$, ipſe terminus huius formae in poteſtatem x^n ingrediens inuentus eſt

$$N \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \ldots \cdot + \frac{n+(i-1)\alpha-\theta}{i\alpha} \cdot A^{\frac{n-\theta}{\alpha}} B^b C^c.$$

Haec igitur expreſſio aggregatum eſſe demonſtrandum eſt 1°. ex termino formae $B^b C^c$, poteſtati $x^{n-\alpha}$ reſpondente, ducto in A; 2°. ex termino formae $B^{b-1} C^c$, poteſtati $x^{n-\beta}$ reſpondente, ducto

ducto in B; 3°. ex termino formae $B^b\,C^{c-1}$, poteſtati $x^{n-\gamma}$ reſpondente, ducto in C. Euidens enim eſt in reliquis membris, quae per D, E, F, etc. multiplicata ſunt, formam propoſitam occurrere non poſſe, quandoquidem aequatio, cui ſatisfieri oportet, eſt haec:

$$x^n = A\,x^{n-\alpha} + B\,x^{n-\beta} + C\,x^{n-\gamma} + D\,x^{n-\delta} + \text{etc.}$$

§. 24. Pro inueniendo autem termino formae $B^b\,C^c$, in poteſtate $x^{n-\alpha}$ occurrente, perſpicuum eſt in ipſa formula propoſita tantum loco n ſcribi debere $n-\alpha$, litteras autem i, θ et N eosdem valores retinere, vnde iſte terminus, poteſtati $x^{n-\alpha}$ conueniens, erit

$$\text{I. } N \cdot \frac{n-\alpha}{\alpha} \cdot \frac{n-\theta}{2\alpha} \cdot \frac{n+\alpha-\theta}{3\alpha} \cdot \frac{n+2\alpha-\theta}{4\alpha} \ldots \frac{n+(i-2)\alpha-\theta}{i\alpha}\, A^{\frac{n-\alpha-\theta}{\alpha}}\, B^b\, C^c.$$

Deinde terminus formae $B^{b-1}\,C^c$, in poteſtate $x^{n-\beta}$ occurrens, deriuabitur ex ipſa formula propoſita, ſi primo loco n ſcribatur $n-\beta$; tum vero loco i hic ſcribi debet $b-1+c=i-1$; deinde loco θ ſcribi debet $(b-1)\beta + c\gamma = \theta - \beta$; denique pro indice N ſcribendum erit $\frac{1.2.3.\ldots.(i-1)}{1.2.\ldots b \times 1.2.\ldots c}$; ſicque loco N ſcribi oportebit $N \cdot \frac{b}{i}$, vnde ipſe iſte terminus erit

$$\text{II. } \frac{b}{i} N \cdot \frac{n-\beta}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \ldots\ldots \frac{n+(i-2)\alpha-\theta}{(i-1)\alpha}\, A^{\frac{n-\theta}{\alpha}} \cdot B^{b-1}\, C^c.$$

Denique terminus $B^b\,C^{c-1}$, in poteſtate $x^{n-\gamma}$ occurrens, deriuabitur ex ipſa formula propoſita, ſi primo loco n ſcribatur $n-\gamma$; tum vero loco i hic ſcribi debet $b+c-1=i-1$; tertio loco θ ſcribi debet $b\beta + (c-1)\gamma = \theta - \gamma$; denique pro indice N ſcribendum erit $\frac{1.2.3.\ldots.(i-1)}{1.2.3.\ldots b \times 1.2.3.\ldots (c-1)}$, ſicque pro N ſcribi oportebit $N \cdot \frac{c}{i}$, vnde ipſe iſte terminus erit

$$\text{III. } \frac{c}{i} N \cdot \frac{n-\gamma}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \ldots \frac{n+(i-2)\alpha-\theta}{(i-1)\alpha}\, A^{\frac{n-\theta}{\alpha}}\, B^b\, C^{c-1}.$$

Quodſi ergo harum formularum prima ducatur in A, ſecunda

in

in B et tertia in C, oſtendendum eſt earum ſummam ipſi formulae propoſitae fore aequalem.

§. 25. Tum autem euidens eſt poſtremos factores litterales inter ſe prodituros eſſe aequales, ſcilicet $A^{\frac{n-\theta}{\alpha}} B^b C^c$; priores autem factores hunc communem habent factorem:

$$\frac{N.(n+\alpha-\theta)(n+2\alpha-\theta)\ldots\ldots\ldots\ldots[n+(i-2)\alpha-\theta]}{\alpha.2\alpha.3\alpha.4\alpha\ldots\ldots\ldots\ldots\ldots(i-1)\alpha},$$

qui ſi deſignetur per Δ, ſumma harum trium formularum erit

$$\Delta\left(\frac{(n-\alpha)(n-\theta)}{i\alpha}+\frac{b(n-\beta)}{i}+\frac{c(n-\gamma)}{i}\right),$$

quae expreſſio, facta euolutione, abit in hanc:

$$\frac{\Delta}{i\alpha}(nn-\alpha n-\theta n+\alpha b n+\alpha c n+\alpha\theta-\alpha b\beta-\alpha c\gamma).$$

Cum autem ſit $b\beta+c\gamma=\theta$ et $b+c=i$, iſta forma reducetur ad hanc:

$$\frac{\Delta}{i\alpha}(nn-\alpha n-\theta n+\alpha i n)=\frac{\Delta}{i\alpha}n[n+(i-1)\alpha-\theta],$$

quamobrem iſtud aggregatum reperietur hoc modo expreſſum:

$$N.\frac{n}{\alpha}.\frac{n+\alpha-\theta}{2\alpha}.\frac{n+2\alpha-\theta}{3\alpha}\ldots\ldots\frac{n+(i-1)\alpha-\theta}{i\alpha}A^{\frac{n-\theta}{\alpha}}.B^b.C^c,$$

quae cum congruat cum ipſa forma propoſita, declarat veritatem noſtrae regulae.

DEMONSTRATIO
tertiae regulae generalis, pro terminis formae $B^b.C^c.D^d$.

§. 26. Poſuimus hic primo $b+c+d=i$, ſecundo $b\beta+c\gamma+d\delta=\theta$ et tertio $N=\frac{1.2.3.4\ldots\ldots i}{1.2\ldots b\times 1.2\ldots c\times 1.2\ldots d}$, hincque terminum iſtius formae pro poteſtate x^n inuenimus:

$$N \cdot \frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \ldots \cdot \frac{n+(i-1)\alpha-\theta}{i\alpha} A^{\frac{n-\theta}{\alpha}} \cdot B^b \cdot C^c \cdot D^d.$$

Nunc igitur demonſtrandum eſt hanc formulam compoſitam eſſe ex quatuor ſequentibus partibus: 1°. ex termino formae eiusdem $B^b \cdot C^c \cdot D^d$, poteſtati $x^{n-\alpha}$ reſpondente, ducto in A; 2°. ex termino formae $B^{b-1} \cdot C^c \cdot D^d$, poteſtati $x^{n-\beta}$ reſpondente, ducto in B; 3°. ex termino formae $B^b \cdot C^{c-1} \cdot D^d$, pro poteſtate $x^{n-\gamma}$, ducto in C; 4°. ex termino formae $B^b \cdot C^c \cdot D^{d-1}$, pro poteſtate $x^{n-\delta}$, ducto in D.

§. 27. Prima igitur pars formabitur ex noſtra formula generali, ſi modo loco n ſcribatur $n-\alpha$, vnde facta multiplicatione per A iſta pars erit

$$\text{I. } N \cdot \frac{n-\alpha}{\alpha} \cdot \frac{n-\theta}{2\alpha} \cdot \frac{n+\alpha-\theta}{3\alpha} \cdot \ldots \cdot \frac{n+(i-2)\alpha-\theta}{i\alpha} \cdot A^{\frac{n-\theta}{\alpha}} \cdot B^b \cdot C^c \cdot D^d.$$

Pro parte autem ſecunda eruenda primo loco n ſcribi debet $n-\beta$, ſecundo loco i ſcribi debet $i-1$, tertio loco θ ſcribendum eſt $\theta-\beta$, denique loco indicis N habebimus

$$\frac{1 \cdot 2 \cdot 3 \ldots \ldots \ldots \ldots i-1}{1 \cdot 2 \ldots (b-1) \times 1 \cdot 2 \ldots c \times 1 \cdot 2 \ldots d},$$

ita vt nunc loco N ſcribi oporteat $\frac{b}{i} \cdot N$, quibus obſeruatis tota pars ſecunda ita ſe habebit:

$$\text{II. } \frac{b}{i} \cdot N \cdot \frac{n-\beta}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \ldots \cdot \frac{n+(i-2)\alpha-\theta}{(i-1)\alpha} A^{\frac{n-\theta}{\alpha}} \cdot B^b \cdot C^c \cdot D^d.$$

Tertia forma ſimili modo formabitur ex ipſa propoſita, ſcribendo primo $n-\gamma$ loco n, ſecundo $i-1$ loco i et tertio $\theta-\gamma$ loco θ, at vero quarto pro indice N habebimus $\frac{c}{i}$ N, quibus obſeruatis iſta pars tertia tota erit

$$\text{III. } \frac{c}{i} \cdot \frac{n-\gamma}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot \ldots \cdot \frac{n+(i-2)\alpha-\theta}{(i-1)\alpha} A^{\frac{n-\theta}{\alpha}} \cdot B^b \cdot C^c \cdot D^d.$$

Quarta forma orietur ex principali, ſcribendo primo $n-\delta$ lo-

co

co n, ſecundo $i-1$ loco i, tertio $\theta-\delta$ loco θ et quarto $\frac{d}{i}$ N loco N, vnde fiet quarta pars

$$\text{IV.}\ \frac{d}{i}.\text{N}.\frac{n-\delta}{\alpha}.\frac{n+\alpha-\theta}{2\alpha}.\frac{n+2\alpha-\theta}{3\alpha}\ldots\frac{n+(i-2)\alpha-\theta}{(i-1)\alpha}.\text{A}^{\frac{n-\theta}{\alpha}}.\text{B}^{b}.\text{C}^{c}.\text{D}^{d}.$$

§. 28. Quoniam quatuor iſtae partes communem habent factorem hunc:

$$\Delta = \frac{\text{N}.(n+\alpha-\theta)(n+2\alpha-\theta)(n+3\gamma-\theta)\ldots n+(i-2)\alpha-\theta}{\alpha.2\alpha.3\alpha.4\alpha\ldots(i-1)\alpha}\text{A}^{\frac{n-\theta}{\alpha}}.\text{B}^{b}.\text{C}^{c}.\text{D}^{d},$$

ſumma omnium harum partium erit

$$\Delta\left(\frac{(n-\alpha)(n-\theta)}{i\alpha}+\frac{b(n-\beta)+c(n-\gamma)+d(n-\delta)}{i}\right),$$

quae expreſſio euoluta, ob $b+c+d=i$ et $b\beta+c\gamma+d\delta=\theta$, reducitur ad hanc formam: $\frac{\Delta}{i\alpha}[n(n+(i-1)\alpha-\theta]$, conſequenter tota ſumma quatuor harum partium erit

$$\text{N}.\frac{n}{\alpha}.\frac{n+\alpha-\theta}{2\alpha}.\frac{n+2\alpha-\theta}{3\alpha}\ldots\frac{n+(i-1)\alpha-\theta}{i\alpha}.\text{A}^{\frac{n-\theta}{\alpha}}.\text{B}^{b}.\text{C}^{c}.\text{D}^{d},$$

quae eſt ipſa forma demonſtranda. Vnde iam manifeſtum eſt omnes ſeries hac methodo reſultantes veritati eſſe conſentaneas.

REGVLA FACILIS ET SVCCINCTA

omnes ſeries huius generis expedite formandi

§. 29. Propoſita aequatione quacunque huius formae:

$$1=\frac{\text{A}}{x^{\alpha}}+\frac{\text{B}}{x^{\beta}}+\frac{\text{C}}{x^{\gamma}}+\frac{\text{D}}{x^{\delta}}+\text{etc.}$$

ponatur breuitatis gratia

$$\text{A}^{-\frac{\beta}{\alpha}}\text{B}+\text{A}^{-\frac{\gamma}{\alpha}}\text{C}+\text{A}^{-\frac{\delta}{\alpha}}\text{D}+\text{etc.}=\text{V},$$

 hinc-

hincque ſemper habebitur:

$$\frac{x^n}{A^{\frac{n}{\alpha}}} = 1 + \frac{n}{\alpha} V + \frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot V^2 + \frac{n}{\alpha} \cdot \frac{n+\alpha-\theta}{2\alpha} \cdot \frac{n+2\alpha-\theta}{3\alpha} \cdot V^3 + \text{etc.}$$

ſi modo notetur omnes poteſtates ipſius V actu euolui debere, litteram vero θ non habere valorem fixum, ſed pro quouis termino euoluto ipſi peculiarem valorem tribui debere. Scilicet pro termino, qui continet productum $B^b . C^c . D^d . E^e$. etc. iſti litterae θ tribui oportet hunc valorem:

$$\theta = b\beta + c\gamma + d\delta + e\varepsilon + \text{etc},$$

hocque in ſingulis terminis obſeruato orientur eaedem ſeries, quas per regulas ſupra datas formare docuimus.

§. 30. Quanquam nulla adhuc methodus directa patet ad iſtam expreſſionem perueniendi, tamen ſequenti modo eius ratio quadantenus explicari poteſt. Cum ſit

$$1 - \frac{B}{x^\beta} - \frac{C}{x^\gamma} - \frac{D}{x^\delta} - \text{etc.} = \frac{A}{x^\alpha},$$

in membro ſiniſtro loco x vbique ſcribatur eius valor ex prima parte oriundus, ſcilicet $A^{\frac{1}{\alpha}}$, fietque

$$x^\alpha = \frac{A}{1 - A^{-\frac{\beta}{\alpha}} . B - A^{-\frac{\gamma}{\alpha}} . C - A^{-\frac{\delta}{\alpha}} . D - \text{etc.}},$$

vnde ſi br. gr. ponatur

$$V = A^{-\frac{\beta}{\alpha}} B + A^{-\frac{\gamma}{\alpha}} C + A^{-\frac{\delta}{\alpha}} . D + \text{etc.}$$

erit $x^\alpha = \frac{A}{1-V}$, hincque ſumtis poteſtatibus exponentis $\frac{n}{\alpha}$, erit

$x^n = \frac{A^{\frac{n}{\alpha}}}{(1-V)^{\frac{n}{\alpha}}}$, quae expreſſio ſi more ſolito in ſeriem con-

ver-

vertatur, prodibit

$$\frac{x^n}{A^{\frac{n}{\alpha}}} = (1 - V)^{-\frac{n}{\alpha}} = 1 + \frac{n}{\alpha} V + \frac{n}{\alpha} \frac{(n + \alpha)}{2\alpha} V^2$$

$$+ \frac{n}{\alpha} \cdot \frac{n + \alpha}{2\alpha} \cdot \frac{n + 2\alpha}{3\alpha} V^3 + \text{etc.}$$

hacc ergo expreſſio introductione litterae θ, modo ante expoſito definiendae, transformabitur in ipſam formam modo traditam:

$$\frac{x^n}{A^{\frac{n}{\alpha}}} = 1 + \frac{n}{\alpha} V + \frac{n}{\alpha} \cdot \frac{n + \alpha - \theta}{2\alpha} \cdot V^2 + \frac{n}{\alpha} \cdot \frac{n + \alpha - \theta}{2\alpha} \cdot \frac{n + 2\alpha - \theta}{3\alpha} \cdot V^3 + \text{etc.}$$

haecque ſpeculatio aliis forte occaſionem praebere poterit formationem huius ſeriei a priori inueſtigandi.

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