1789

Analysis facilis et plana ad eas series maxime abstrusas perducens, quibus omnium aequationum algebraicarum non solum radices ipsae sed etiam quaevis earum potestates exprimi possunt

Leonhard Euler

ANALYSIS FACILIS ET PLANA AD EAS SERIES MAXIME ABSTRVSAS PERDVCENS, QVIBVS OMNIVM AEQVATIONVM ALGEBRAICARVM NON SOLVM RADICES IPSAE, SED ETIAM QVAEVIS EARVM POTESTATES EXPRIMI POSSVNT.

Auctore
L. EVLERO.

Conuent. exhib. die 15 *April.* 1776.

Problema.

Propofita aequatione algebraica tribus terminis conftante, quam femper hac forma repraefentare licet: $1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta}$, inuenire feriem, quae exprimat valorem ipfius x^n.

Solutio.

§. 1. Haec aequatio, ponendo $x = A^{\frac{1}{\alpha}} Z$, femper ad hanc formam fimpliciorem reuocari poteft: $1 = \frac{1}{Z^\alpha} + \frac{B}{A^{\frac{\beta}{\alpha}} Z^\beta}$, vnde ponendo $B = A^{\frac{\beta}{\alpha}} C$, erit $1 = \frac{1}{Z^\alpha} + \frac{C}{Z^\beta}$, quae per Z^n mul-

multiplicata praebet $Z^n = Z^{n-\alpha} + C Z^{n-\beta}$; hinc igitur quaeri oportet valorem poteſtatis Z^n, quandoquidem hinc erit $x^n = A^{\frac{n}{\alpha}} Z^n$. Manifeſtum autem eſt valorem ipſius Z^n exprimi debere per ſeriem, in quam exponens n ingrediatur, quam ergo ſpectare licebit tanquam functionem ipſius n, et quia haec ſeries ex infinitis terminis conſtabit, eam ita repraeſentemus:

$$Z^n = f^\circ : n + f' : n + f'' : n + f''' : n + f'''' : n + \text{etc.}$$

quae ergo forma ita debet eſſe comparata, vt poſito $n = 0$ fiat $Z^n = 1$; vnde patet ſtatui debere $f^\circ : n = 1$, reliquos vero terminos factorem habere debere n, vt euaneſcant poſito $n = 0$, prodeatque $x^\circ = 1$.

§. 2. Conſtituta hac ſerie, ſi loco n ſcribamus $n - \alpha$, habebimus:

$$Z^{n-\alpha} = f^\circ : (n-\alpha) + f' : (n-\alpha) + f'' : (n-\alpha) + f''' : (n-\alpha) + \text{etc.}$$

ſimilique modo erit

$$Z^{n-\beta} = f^\circ : (n-\beta) + f' : (n-\beta) + f'' : (n-\beta) + \text{etc.}$$

vbi iterum notetur eſſe $f^\circ : (n - \alpha) = 1$ et $f^\circ : (n - \beta) = 1$. Cum iam noſtra aequatio ſit $Z^n - Z^{n-\alpha} = C Z^{n-\beta}$, ſcribamus loco poteſtatum ipſius Z ſeries aſſumtas ſequenti modo:

$$\begin{array}{rcl} + Z^n &=& + f^\circ : n \quad + f' : n \quad + f'' : n \quad + f''' : n + \text{etc.} \\ - Z^{n-\alpha} &=& - f^\circ : (n-\alpha) - f' : (n-\alpha) - f'' : (n-\alpha) - f''' : (n-\alpha) - \text{etc.} \\ \hline = C Z^{n-\beta} &=& C f^\circ : (n-\beta) + C f' : (n-\beta) + C f'' : (n-\beta) + C f''' : (n-\beta) + \text{etc.} \end{array}$$

nunc functiones iſtae indefinitae ita determinentur, vt fiat

$$\text{I. } f' : n - f' : (n - \alpha) = C f^\circ : (n - \beta) = C,$$

$$\text{II. } f'' : n - f'' : (n - \alpha) = C f' : (n - \beta);$$

III.

$$\text{III. } f''':n - f''':(n-\alpha) = C f'':(n-\beta);$$
$$\text{IV. } f'''':n - f'''':(n-\alpha) = C f''':(n-\beta).$$
etc. etc.

§. 3. Ope harum aequationum ergo primo quaeri debet natura functionis $f':n$, vt primae aequationi satisfiat; qua inuenta innotescet functio $f':(n-\beta)$, ex eaque per secundam aequationem quaeri debet indoles functionis $f'':n$, vnde innotescet functio $f'':(n-\beta)$, hincque porro simili modo ex aequatione tertia deducetur indoles functionis $f''':n$, et ita porro, donec lex pateat, qua singulae hae functiones vlterius progrediuntur: vnde patet resolutionem omnium harum aequationum reuocari ad hanc quaestionem, qua proposita functione ipsius n quaeritur alia functio, veluti $\Phi:n$, vt fiat $\Phi:n - \Phi:(n-\alpha) = N$, quem in finem sequentia Lemmata euoluamus.

Lemma I.

§. 4. Si fuerit $\Phi:n = \Delta n$, erit $\Phi:(n-\alpha) = \Delta(n-\alpha)$, ideoque $\Phi:n - \Phi:(n-\alpha) = \Delta\alpha$; vnde viciſſim, ſi ponatur $\Delta\alpha = k$, vt fieri debeat $\Phi:n - \Phi:(n-\alpha) = k$, reperietur $\Phi:n = \frac{kn}{\alpha}$. Quare cum ex prima aequatione esse debeat $f':n - f':(n-\alpha) = C$, necesse est vt sit $f':n = \frac{Cn}{\alpha}$, vnde pro secunda aequatione fiet $f':(n-\beta) = \frac{C}{\alpha}(n-\beta)$.

Lemma II.

§. 5. Si fuerit $\Phi:n = \Delta n(n+\alpha-v)$, erit
$$\Phi:(n-\alpha) = \Delta(n-\alpha)(n-v), \text{ vnde colligitur}$$
$$\Phi:n - \Phi:(n-\alpha) = 2\Delta\alpha(n-\tfrac{1}{2}v).$$
Quodſi ergo prodire debeat
$$\Phi:n - \Phi:(n-\alpha) = k(n-\lambda),$$

ob $\Delta = \frac{k}{2\alpha}$ et $v = 2\lambda$, erit

$$\phi : n = \frac{kn}{2\alpha}(n + \alpha - 2\lambda).$$

Quare cum aequatio ſecunda iam ſit

$$f'' : n - f'' : (n - \alpha) = C f' : (n - \beta) = \frac{cc}{\alpha}(n - \beta),$$

ob $k = \frac{cc}{\alpha}$ et $\lambda = \beta$ erit

$$f'' : n = \frac{cc}{2\alpha\alpha}\, n\,(n + \alpha - 2\beta),$$

vnde pro tertia aequatione fiet

$$f'' : (n - \beta) = \frac{cc}{2\alpha\alpha}(n - \beta)(n + \alpha - 3\beta).$$

Lemma III.

§. 6. Si fuerit

$$\phi : n = \Delta\, n\,(n + \alpha - v)(n + 2\alpha - v), \text{ erit}$$

$$\phi : (n - \alpha) = \Delta\,(n - \alpha)(n - v)(n + \alpha - v),$$

hinc ergo fit

$$\phi : n - \phi : (n - \alpha) = 3\Delta\,\alpha\,(n + \alpha - v)(n - \tfrac{1}{3}v),$$

vnde viciſſim, poſito $3\Delta\alpha = k$ et $\frac{1}{3}v = \lambda$, vt prodeat

$k(n + \alpha - 3\lambda)(n - \lambda)$, ſumi debet

$$\phi : n = \frac{kn}{3\alpha}(n + \alpha - 3\lambda)(n + 2\alpha - 3\lambda).$$

Quia nunc pro noſtra aequatione tertia fieri debet

$$f''' : n - f''' : (n - \alpha) = \frac{c^3}{2\alpha\alpha}(n - \beta)(n + \alpha - 3\beta),$$

facta applicatione fiet $k = \frac{c^3}{2\alpha\alpha}$ et $\lambda = \beta$, hincque concluditur fore

$$f''' : n = \frac{c^3}{6\alpha^3} \cdot n\,(n + \alpha - 3\beta)(n + 2\alpha - 3\beta),$$

vnde pro aequatione ſequente habebimus:

$$f''' : (n - \beta) = \frac{c^3}{6\alpha^3}(n - \beta)(n + \alpha + 4\beta)(n + 2\alpha - 4\beta).$$

Lem-

Lemma IV.

§. 7. Si fuerit

$$\Phi : n = \Delta\, n\, (n+\alpha-v)\,(n+2\alpha-v)\,(n+3\alpha-v), \text{ erit}$$

$$\Phi : (n-\alpha) = \Delta\,(n-\alpha)\,(n-v)\,(n+\alpha-v)\,(n+2\alpha-v),$$

hincque

$$\Phi : n - \Phi : (n-\alpha) = 4\Delta\,\alpha\,(n+\alpha-v)\,(n+2\alpha-v)\,(n-\tfrac{1}{4}v).$$

Quare si debeat esse

$$\Phi : n - \Phi : (n-\alpha) = k\,(n-\lambda)\,(n+\alpha-4\lambda)\,(n+2\alpha-4\lambda),$$

sumi debet $\Delta = \frac{k}{4\alpha}$ et $v = 4\lambda$, hincque fiet

$$\Phi : n = \frac{kn}{4\alpha}(n+\alpha-4\lambda)\,(n+2\alpha-4\lambda)\,(n+3\alpha-4\lambda).$$

Quare cum aequatio nostra quarta sit

$$f'''' : n - f'''' : (n-\alpha) = \frac{c^4}{6\alpha^3}(n-\beta)\,(n+\alpha-4\beta)\,(n+2\alpha-4\beta),$$

facta applicatione fiet $k = \frac{c^4}{6\alpha^3}$ et $\lambda = \beta$, hincque concluditur fore

$$f'''' : n = \frac{c^4}{24\alpha^4}\,n\,(n+\alpha-4\beta)\,(n+2\alpha-4\beta)\,(n+3\alpha-4\beta),$$

vnde pro quinta aequatione nanciscemur:

$$f'''' : (n-\beta) = \frac{c^4}{24\alpha^4}(n-\beta)\,(n+\alpha-5\beta)\,(n+2\alpha-5\beta)\,(n+3\alpha-5\beta).$$

Lemma V.

§. 8. Si fuerit

$$\Phi : n = \Delta\, n\,(n+\alpha-v)\,(n+2\alpha-v)\,(n+3\alpha-v)\,(n+4\alpha-v),$$

erit

$$\Phi : (n-\alpha) = \Delta\,(n-\alpha)\,(n-v)\,(n+\alpha-v)\,(n+2\alpha-v)\,(n+3\alpha-v),$$

hincque

$$\Phi : n - \Phi : (n-\alpha) = 5\,\Delta\,\alpha\,(n+\alpha-v)\,(n+2\alpha-v)\,(n+3\alpha-v)\,(n-\tfrac{1}{5}v).$$

Quare si debeat esse

$$\phi:n-\phi:(n-\alpha)=k(n-\lambda)\,(n+\alpha-5\lambda)\,(n+2\alpha-5\lambda)\,(n+3\alpha-5\lambda),$$

ſumi debet $\Delta = \frac{k}{5\alpha}$ et $5\lambda = v$, tum vero erit

$$\phi:n=\frac{k}{5\alpha}n\,(n+\alpha-5\lambda)\,(n+2\alpha-5\lambda)\,(n+3\alpha-5\lambda)\,(n+4\alpha-5\lambda).$$

Aequatio autem quinta cum ita ſe habeat:

$$f''''':n-f''''':(n-\alpha)=\frac{c^5}{24\alpha^4}(n-\beta)\,(n+\alpha-5\beta)\,(n+2\alpha-5\beta)\,(n+3\alpha-5\beta),$$

hic ſumi debet $k = \frac{c^5}{24\alpha^4}$ et $\lambda = \beta$, vnde concluditur

$$f''''':n=\frac{c^5}{120\alpha^5}n\,(n+\alpha-5\beta)\,(n+2\alpha-5\beta)\,(n+3\alpha-5\beta)\,(n+4\alpha-5\beta).$$

Hinc iam ſine vlteriore calculo concludere licet fore

$$f^{VI}:n=\frac{c^6}{720\alpha^6}n\,(n+\alpha-6\beta)\,(n+2\alpha-6\beta)\,(n+3\alpha-6\beta)\times$$
$$\times(n+4\alpha-6\beta)\,(n+5\alpha-6\beta) \text{ et}$$

$$f^{VII}:n=\frac{c^7}{5040\alpha^7}n\,(n+\alpha-7\beta)\,(n+2\alpha-7\beta)\,(n+3\alpha-7\beta)\times$$
$$\times(n+4\alpha-7\beta)\,(n+5\alpha-7\beta)\,(n+6\alpha-7\beta).$$

Concluſio finalis.

§. 9. His igitur colligendis ſi aequatio propoſita fuerit $1 = \frac{1}{Z^\alpha} + \frac{C}{Z^\beta}$, tum pro poteſtate quacunque ipſius Z ſequens reſultat ſeries:

$$Z^n = 1 + \frac{c}{\alpha}n + \frac{cc}{2\alpha\alpha}n\,(n+\alpha-2\beta) + \frac{c^3}{6\alpha^3}n\,(n+\alpha-3\beta)\,(n+2\alpha-3\beta)$$
$$+\frac{c^4}{24\alpha^4}n\,(n+\alpha-4\beta)\,(n+2\alpha-4\beta)\,(n+3\alpha-4\beta)$$
$$+\frac{c^5}{120\alpha^5}n\,(n+\alpha-5\beta)\,(n+2\alpha-5\beta)\,(n+3\beta-5\beta)\times$$
$$\times(n+4\alpha-5\beta)+\text{ etc.}$$

Scholion.

§. 10. Haec ſeries, quam eruimus, eo magis eſt notatu digna, quod nulla alia via patet eam inueniendi. Quin etiam

iam Analysis nostra ita est comparata, vt veritas solutionis non solum ad omnes exponentes integros n, sed etiam ad quosuis valores fractos, atque adeo negatiuos extenditur. Praeterea vero etiam ex nostra serie generali logarithmus Hyperbolicus ipsius Z exprimi potest. Cum enim semper, casu $n = 0$, sit

$$\frac{Z^n - 1}{n} = l\,Z,$$

erit nostro casu

$$l\,Z = \tfrac{c}{\alpha} + \tfrac{cc}{2\alpha^2}(\alpha - 2\beta) + \tfrac{c^3}{6\alpha^3}(\alpha - 3\beta)(2\alpha - 3\beta)$$
$$+ \tfrac{c^4}{24\alpha^4}(\alpha - 4\beta)(2\alpha - 4\beta)(3\alpha - 4\beta)$$
$$+ \tfrac{c^5}{120\alpha^5}(\alpha - 5\beta)(2\alpha - 5\beta)(3\alpha - 5\beta(4\alpha - 5\beta) + \text{etc.}$$

Vnde si tota haec series designetur littera Δ, vt sit $l\,Z = \Delta$, erit $Z = e^{\Delta}$, ideoque $Z^n = e^{n\Delta}$, quae ergo quantitas aequalis erit seriei supra inuentae pro Z^n. At vero ista expressio $e^{n\Delta}$ in seriem euoluta praebet

$$Z^n = 1 + n\,\Delta + \tfrac{1}{2}nn\,\Delta^2 + \tfrac{1}{6}n^3\,\Delta^3 + \tfrac{1}{24}n^4\,\Delta^4 + \tfrac{1}{120}n^5\,\Delta^5 + \text{etc.}$$

quae ergo series seriei supra inuentae necessario erit aequalis, id quod etiam comprobabitur, dum saltem priores termini euoluentur. Cum enim sit

$$\Delta = \tfrac{c}{\alpha} + \tfrac{cc}{2\alpha^2}(\alpha - 2\beta) + \text{etc. erit}$$
$$\Delta^2 = \tfrac{cc}{\alpha^2} + \tfrac{c^3}{\alpha^3}(\alpha - 2\beta)$$
$$\Delta^3 = \tfrac{c^3}{\alpha^3},$$

vnde deducimus

$$Z^n = 1 + \tfrac{c}{\alpha}n + \tfrac{cc}{2\alpha\alpha}n(\alpha - 2\beta) + \tfrac{c^3}{6\alpha^3}n(\alpha - 3\beta)(2\alpha - 3\beta) + \text{etc.}$$
$$\tfrac{cc}{2\alpha\alpha}nn \qquad + \tfrac{c^3}{2\alpha^3}nn(\alpha - 2\beta) + \text{etc.}$$
$$+ \tfrac{c^3}{6\alpha^3}n^3 + \text{etc.}$$

siue

$Z^n =$

$$Z^n = 1 + \frac{c}{\alpha} n + \frac{cc}{2\alpha\alpha} n(n + \alpha - 2\beta)$$
$$+ \frac{c^3}{6\alpha^3}\left(n(\alpha - 3\beta)(2\alpha - 3\beta) + nn(n + 3\alpha - 6\beta)\right),$$

quod cum ſerie pro Z^n ſupra inuenta perfecte congruit.

Theorema generale.

§. 11. Quodſi ergo propoſita fuerit aequatio initio commemorata: $1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta}$, quoniam poſuimus $Z = \frac{x}{A^{\frac{1}{\alpha}}}$ et $C = \frac{B}{A^{\frac{\beta}{\alpha}}}$, erit

$$\frac{x}{A^{\frac{n}{\alpha}}} = 1 + \frac{B}{A^{\frac{\beta}{\alpha}}} \cdot \frac{n}{\alpha} + \frac{B^2}{A^{\frac{2\beta}{\alpha}}} \frac{n}{\alpha} \cdot \frac{(n + \alpha - 2\beta)}{2\alpha}$$
$$+ \frac{B^3}{A^{\frac{3\beta}{\alpha}}} \cdot \frac{n}{\alpha} \cdot \frac{(n + \alpha - 3\beta)}{2\alpha} \frac{(n + 2\alpha - 3\beta)}{3\alpha}$$
$$+ \frac{B^4}{A^{\frac{4\beta}{\alpha}}} \cdot \frac{n}{\alpha} \cdot \frac{(n + \alpha - 4\beta)}{2\alpha} \frac{(n + 2\alpha - 4\beta)}{3\alpha} \frac{(n + 3\alpha - 4\beta)}{4\alpha} \text{ etc.}$$

ſiue

$$X^n = A^{\frac{n}{\alpha}} + A^{\frac{n-\beta}{\alpha}} B \frac{n}{\alpha} + A^{\frac{n-2\beta}{\alpha}} B^2 \frac{n}{\alpha} \frac{(n + \alpha - 2\beta)}{2\alpha}$$
$$+ A^{\frac{n-3\beta}{\alpha}} B^3 \frac{n}{\alpha} \frac{(n + \alpha - 3\beta)}{2\alpha} \frac{(n + 2\alpha - 3\beta)}{3\alpha}$$
$$+ A^{\frac{n-4\beta}{\alpha}} B^4 \cdot \frac{n}{\alpha} \frac{(n + \alpha - 4\beta)}{2\alpha} \frac{(n + 2\alpha - 4\beta)}{3\alpha} \frac{(n + 3\alpha - 4\beta)}{4\alpha}$$
$$+ A^{\frac{n-5\beta}{\alpha}} B^5 \cdot \frac{n}{\alpha} \frac{(n + \alpha - 5\beta)}{2\alpha} \frac{(n + 2\alpha - 5\beta)}{3\alpha} \times$$
$$\times \frac{(n + 3\alpha - 5\beta)}{4\alpha} \frac{(n + 4\alpha - 5\beta)}{5\alpha} + \text{ etc.}$$

Vicis-

Viciſſim igitur propoſita hac ſerie, eius ſumma erit $= x^n$, exiſtente x radice huius aequationis: $1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta}$, id quod aliquot exemplis illuſtrare liceat.

Exemplum 1.

§. 12. Statuamus $\alpha = 1$ et $\beta = 1$, vt propoſita ſit iſta aequatio: $1 = \frac{A}{x} + \frac{B}{x}$, vnde ſit $x = A + B$, conſequenter $x^n = (A + B)^n$, ſeries autem inuenta hoc caſu nobis dat

$$(A + B)^n = A^n + \tfrac{n}{1} A^{n-1} B + \tfrac{n(n-1)}{1.\ 2.} A^{n-2} B^2$$
$$+ \tfrac{n(n-1)(n-2)}{1.\ 2.\ 3.} A^{n-3} B^3 + \text{etc.}$$

quae eſt ipſa euolutio Binomii Newtoniana, quam nunc patet veram eſſe, quicunque numerus pro exponente n accipiatur, ſiue integer, ſiue fractus, ſiue poſitiuus, ſiue negatiuus, ſiue etiam ſurdus; cum in Algebra communi, vbi haec euolutio eſt tractata, exponens n neceſſario ſit integer poſitiuus.

Exemplum 2.

§. 13. Ponamus, vt ante, $\alpha = 1$, at ſumatur $\beta = 0$, ita vt ſit $1 = \frac{A}{x} + B$, vnde ſit $x = \frac{A}{1 - B}$, conſequenter

$$x^n = \frac{A^n}{(1 - B)^n} = A^n (1 - B)^{-n},$$

ſeries autem, ad quam ſumus perducti, hoc caſu erit

$$A^n (1 - B)^n = A^n + \tfrac{n}{1} A^n B + \tfrac{n(n+1)}{1.\ 2.} A^n B^2 + \text{etc.}$$

ſiue

$$(1 - B)^n = 1 + \tfrac{n}{1} B + \tfrac{n(n+1)}{1.\ 2.} B^2 + \tfrac{n(n+1)(n+2)}{1.\ 2.\ 3.} B^3 + \text{etc.}$$

quae eſt demonſtratio eiusdem theorematis Newtoniani pro exponentibus negatiuis.

Exem-

Exemplum 3.

§. 14. Sumamus $A = 2a$ et $B = b$, fitque porro $\alpha = 1$ et $\beta = 2$, vt noftra aequatio fiat $1 = \frac{2a}{x} + \frac{b}{xx}$, fiue $xx = 2ax + b$, vnde fit $x = a + \sqrt{(aa + b)}$, quo valore fubftituto feries ante inuenta praebebit

$$(a + \sqrt{aa+b})^n = 2^n a^n + \frac{n}{1} 2^{n-2} a^{n-2} b + \frac{n(n-3)}{1.\ 2.} 2^{n-4} a^{n-4} bb$$
$$+ \frac{n(n-5)(n-4)}{1.\ 2.\ 3.} 2^{n-6} a^{n-6} b^3 + \frac{n(n-7)(n-6)(n-5)}{1.\ 2.\ 3.\ 4.} 2^{n-8} a^{n-8} b^4$$
$$+ \text{etc.}$$

cuius veritas pro cafu, quo $n = 1$, ex euolutione vulgari confirmari poteft. Sumto enim $n = 1$ erit

$$a + \sqrt{aa+b} = 2a + \frac{b}{2a} - \frac{bb}{2^3 a^3} + \frac{3b^3}{2^4 a^5} - \frac{5b^4}{2^7 a^7} - \frac{7b^5}{2^8 a^9} - \text{etc.}$$

nouimus autem ex refolutione vulgari effe

$$\sqrt{(aa + b)} = a + \frac{b}{2a} - \frac{1.\ 1.}{2.\ 4.} \frac{bb}{a^3} + \frac{1.\ 1.\ 3.}{2.\ 4.\ 6.} \frac{b^3}{a^5} - \frac{1.\ 1.\ 3.\ 5}{2.\ 4.\ 6.\ 8} \frac{b^4}{a^7} + \text{etc.}$$

cui fi addatur a, ipfa illa feries prodit.

Exemplum 4.

§. 15. Sumamus $\alpha = 2$ et $\beta = 1$, vt fit $1 = \frac{A}{xx} + \frac{B}{x}$, fiue $xx = A + Bx$, ideoque $x = \frac{B + \sqrt{(BB + 4A)}}{2}$, loco A autem fcribamus aa et $2b$ loco B, vt fit $x = b + \sqrt{(bb + aa)}$, quocirca feries inuenta nobis dabit

$$(b + \sqrt{bb + aa})^n = a^n + \frac{n}{2} a^{n-1} . 2b + \frac{n}{2} . \frac{n}{4} a^{n-2} . 4bb$$
$$+ \frac{n(n-1)(n+1)}{2.\ 4.\ 6.} a^{n-3} . 8b^3 + \frac{n}{2} . \frac{(n-2)}{4} \frac{n}{6} \frac{(n+2)}{8} a^{n-4} . 16 b^4 + \text{etc.}$$

quae reducitur ad hanc formam:

$$(b + \sqrt{bb + aa})^n = a^n + \frac{n}{1} a^{n-1} b + \frac{n}{1} . \frac{n}{2} a^{n-2} bb$$
$$+ \frac{n}{1} . \frac{(n-1)(n+1)}{2.\ 3.} a^{n-3} b^3 + \frac{n}{1} \frac{(n-2)}{2} \frac{n}{3} . \frac{(n+2)}{4} a^{n-4} b^4 \text{ etc.}$$

haec autem forma vlterius reducetur ad hanc:

$(b +$

$$(b+\sqrt{bb+aa})^n = a^n + \tfrac{n}{1}\, a^{n-1}\, b + \tfrac{nn}{1.2}\, a^{n-2}\, bb$$
$$+ \tfrac{n(nn-1)}{1.\,2.\,3.}\, a^{n-3}\, b^3 + \tfrac{nn(nn-4)}{1.\,.\,.\,.\,4}\, a^{n-4}\, b^4 + \tfrac{n(nn-1)(nn-9)}{1.\,.\,.\,.\,.\,.\,5}\, a^{n-5}\, b^5$$
$$+ \tfrac{nn(nn-4)(nn-16)}{1.\,.\,.\,.\,.\,.\,6}\, a^{n-6}\, b^6 + \tfrac{n(nn-1)(nn-9)(nn-25)}{1.\,.\,.\,.\,.\,.\,.\,.\,7}\, a^{n-7}\, b^7$$
$$+ \text{etc.}$$

Ita si sumamus $n = 1$, habebimus

$$b+\sqrt{bb+aa} = a + b + \tfrac{1}{1.2}\,\tfrac{bb}{a} - \tfrac{3}{1\ldots4}\,\tfrac{b^4}{a^3} + \tfrac{3.15}{1\ldots6}\,\tfrac{b^6}{a^5} - \text{etc.}$$

nouimus autem esse

$$\sqrt{bb+aa} = a + \tfrac{1}{2}\,\tfrac{bb}{a} - \tfrac{1.1}{2.4}\,\tfrac{b^4}{a^3} + \tfrac{1.1.3}{2.4.6}\,\tfrac{b^6}{a^5} - \text{etc.}$$

cui si addatur b, ipsa illa series prodit.

Exemplum 5.

§. 16. Sumamus $\alpha = 1$ et $\beta = -1$, vt nostra aequatio fit $1 = \frac{A}{x} + Bx$, vnde fit $x = \frac{1+\sqrt{1-4AB}}{2B}$, hinc ergo prodit

$$\left(\tfrac{1\pm\sqrt{(1-4AB)}}{2B}\right)^n = A^n + \tfrac{n}{1}\, A^{n+1}\, B + \tfrac{n(n+3)}{1.\,2.}\, A^{n+2}\, B^2$$
$$+ \tfrac{n}{1}\,\tfrac{(n+4)(n+5)}{2.\,3.}\, A^{n+3}\, B^3 + \tfrac{n}{1}\,\tfrac{(n+5)(n+6)(n+7)}{2.\,3.\,4.}\, A^{n+4}\, B^4$$
$$+ \tfrac{n(n+6)(n+7)(n+8)(n+9)}{1.\,2.\,3.\,4.\,5}\, A^{n+5}\, B^5 + \text{etc.}$$

Hinc ergo si sumamus $n = 1$, erit

$$\tfrac{1\pm\sqrt{(1-4AB)}}{2B} = A + A^2 B + \tfrac{4}{2}\, A^3 B^2 + \tfrac{5.6}{2.3}\, A^4 B^3$$
$$+ \tfrac{6.7.8}{2.3.4}\, A^5 B^4 + \tfrac{7.8.9.10}{2.3.4.5}\, A^6 B^5 + \text{etc.}$$

Est vero

$$\sqrt{(1-4AB)} = 1 - 2AB - 2A^2B^2 - 4A^3B^3$$
$$- 2.5\, A^4 B^4 - \text{etc.}$$

quae series ab vnitate subtracta et per $2B$ diuisa praebet seriem modo inuentam.

 Scho-

Scholion.

§. 17. Series autem generalis, quam ſupra elicuimus, primum ab acutiſſimo *Lamberto* ex principiis maxime diuerſis eſt inuenta, quam idcirco *Lambertinam* appellare liceat, propterea quod inter egregia huius viri inuenta merito eſt referenda. Methodus autem, qua hic vſi ſumus, ad aequationes multo generaliores extendi poteſt, quando ſcilicet aequatio propoſita quatuor pluresue terminos continet; id quod pro caſu quatuor terminorum oſtendiſſe operae erit pretium.

Problema generalius.

Si propoſita fuerit aequatio algebraica huius formae:

$$1 - \frac{1}{Z^\alpha} = \frac{B}{Z^\beta} + \frac{C}{Z^\gamma},$$

inuenire ſeriem, quae valorem poteſtatis cuiuscunque ipſius Z, puta Z^n exprimat.

Solutio.

§. 18. Multiplicetur aequatio propoſita per Z^n, vt habeatur $Z^n - Z^{n-\alpha} = B\,Z^{n-\beta} + C\,Z^{n-\gamma}$; et poteſtatem quaeſitam Z^n vt ante tanquam functionem ipſius n ſpectare licebit, quae per partes continuo procedentes ita repraeſentetur, vt ſit

$$Z^n = f^\circ : n + f' : n + f'' : n + f''' : n + f'''' : n + \text{etc.}$$

vbi cum ſumto $n = 0$ fieri debeat $Z^n = 1$, ſit perpetuo $f^\circ : n = 1$, tum vero vt reliquae partes caſu $n = 0$ euaneſcant, ſingulas factorem n habere neceſſe eſt. Hinc ergo erit

$$Z^{n-\alpha} = f^\circ : (n-\alpha) + f' : (n-\alpha) + f'' : (n-\alpha) + \text{etc.}$$
$$Z^{n-\beta} = f^\circ : (n-\beta) + f' : (n-\beta) + f'' : (n-\beta) + \text{etc.}$$
$$Z^{n-\gamma} = f^\circ : (n-\gamma) + f' : (n-\gamma) + f'' : (n-\gamma) + \text{etc.}$$

Iam

Iam iſtae ſeries loco harum poteſtatum in noſtra aequatione ſubſtituantur, et cum partes in membro ſiniſtro ſponte ſe tollant, reliquae partes ſiniſtrae partibus antecedentibus in dextro membro aequari debebunt, vnde ſequentes aequationes reſultabunt.

$$\text{I. } f' : n - f' : (n-\alpha) = \mathrm{B} f^{\circ} : (n-\beta) + \mathrm{C} f^{\circ} : (n-\gamma) = \mathrm{B} + \mathrm{C}$$
$$\text{II. } f'' : n - f'' : (n-\alpha) = \mathrm{B} f' : (n-\beta) + \mathrm{C} f' : (n-\gamma)$$
$$\text{III. } f''' : n - f''' : (n-\alpha) = \mathrm{B} f'' : (n-\beta) + \mathrm{C} f'' : (n-\gamma)$$
$$\text{IV. } f'''' : n - f'''' : (n-\alpha) = \mathrm{B} f''' : (n-\beta) + \mathrm{C} f''' : (n-\gamma)$$

etc. etc.

§. 19. In ſubſidium iam vocemus lemmata ſupra allata, ex quibus conſtat fore vt ſequitur:

I. Si fuerit $\phi : n - \phi : (n-\alpha) = k$, erit $\phi : n = \frac{kn}{\alpha}$,

II. Si fuerit $\phi : n - \phi : (n-\alpha) = k\,(n-\lambda)$, erit
$$\phi : n = \frac{kn}{2\alpha}\,(n+\alpha-2\lambda).$$

III. Si fuerit $\phi : n - \phi : (n-\alpha) = k\,(n-\lambda)\,(n+\alpha-3\lambda)$, erit
$$\phi : n = \frac{kn}{3\alpha}\,(n+\alpha-3\lambda)\,(n+2\alpha-3\lambda).$$

IV. Si fuerit $\phi : n - \phi : (n-\alpha) = k\,(n-\lambda)\,(n+\alpha-4\lambda)\,(n+2\alpha-4\lambda)$, erit
$$\phi : n = \frac{kn}{4\alpha}\,(n+\alpha-4\lambda)\,(n+2\alpha-4\lambda)\,(n+3\alpha-4\lambda).$$

V. Si fuerit $\phi : n - \phi : (n-\alpha) = k(n-\lambda)(n+\alpha-5\lambda)(n+2\alpha-5\lambda)(n+3\alpha-5\lambda)$, erit
$$\phi : n = \frac{kn}{5\alpha}(n+\alpha-5\lambda)(n+2\alpha-5\lambda)(n+3\alpha-5\lambda)(n+4\alpha-5\lambda).$$

et ita porro.

§. 20. Harum lemmatum ope ex prima aequatione, vbi pro lemmate primo eſt $k = B + C$, elicimus $f' : n = \frac{Bn}{\alpha} + \frac{Cn}{\alpha}$, hinc igitur pro ſecunda aequatione erit

$$B f' : (n - \beta) = B^2 \frac{(n - \beta)}{\alpha} + B\,C \frac{(n - \beta)}{\alpha} \text{ et}$$
$$C f' : (n - \gamma) = C^2 \frac{(n - \gamma)}{\alpha} + B\,C \frac{(n - \gamma)}{\alpha},$$

$$\text{Summa} = \frac{BB\,(n - \beta)}{\alpha} + \frac{2BC}{\alpha}\left(n - \frac{\beta - \gamma}{2}\right) + \frac{CC}{\alpha}(n - \gamma),$$

quae formula quia ex tribus conſtat partibus, ſingulas cum lemmate ſecundo conferri oportet, ac pro prima parte erit $k = \frac{BB}{\alpha}$ et $\lambda = \beta$, pro ſecunda parte eſt $k = \frac{2BC}{\alpha}$ et $\lambda = \frac{\beta + \gamma}{2}$, pro tertia parte eſt $k = \frac{CC}{\alpha}$ et $\lambda = \gamma$, vnde ex omnibus ſimul ſumtis colligitur:

$$f'' : n = \frac{BB}{2\alpha^2}(n + \alpha - 2\beta) + \frac{2B.C}{2\alpha^2} n (n + \alpha - \beta - \gamma)$$
$$+ \frac{CC}{2\alpha\alpha} n (n + \alpha - 2\gamma).$$

§. 21. Progrediamur iam ad aequationem tertiam, ac pro eius membro dextro habebimus:

$$B f'' : (n - \beta) = \frac{B^3}{2\alpha^2}(n - \beta)(n + \alpha - 3\beta)$$
$$+ \frac{2B.B.C}{2\alpha^2}(n - \beta)(n + \alpha - 2\beta - \gamma) + \frac{B\,C^2}{2\alpha^2}(n - \beta)(n + \alpha - \beta - 2\gamma),$$
$$C f'' : (n - \gamma) = \frac{C^3}{2\alpha^2}(n - \gamma)(n + \alpha - 3\gamma)$$
$$+ \frac{B.B.C}{2\alpha^2}(n - \gamma)(n + \alpha - \beta - 2\gamma)$$
$$+ \frac{2BCC}{2\alpha^2}(n - \gamma)(n + \alpha - \beta - 2\gamma)$$

$$\text{Summa} = \frac{B^3}{2\alpha^2}(n - \beta)(n + \alpha - 3\beta)$$
$$+ \frac{3BBC}{2\alpha^2}(n + \alpha - 2\beta - \gamma)\left(n - \frac{2\beta - \gamma}{3}\right)$$
$$+ \frac{3BCC}{2\alpha^2}(n + \alpha - \beta - 2\gamma)\left(n - \frac{\beta - 2\gamma}{3}\right)$$
$$+ \frac{C^3}{2\alpha^2}(n - \gamma)(n + \alpha - 3\gamma),$$

quae

quae quia quatuor conſtat partibus, cum lemmate III. conferendis, pro prima parte erit $k = \frac{B^3}{2\alpha^2}$ et $\lambda = \beta$, pro ſecunda parte erit $k = \frac{3BBC}{2\alpha^2}$ et $\lambda = \frac{2\beta+\gamma}{3}$; pro tertia vero parte eſt $k = \frac{3BCC}{2\alpha^2}$ et $\lambda = \frac{\beta+2\gamma}{3}$, denique pro quarta parte eſt $k = \frac{C^3}{2\alpha^2}$ et $\lambda = \gamma$, quibus obſeruatis functio quaeſita f''' itidem ex quatuor partibus conſtabit, quae ſunt:

$$f''' : n = \left\{\begin{array}{l} + \frac{B^3}{6\alpha^3} n (n + \alpha - 3\beta)(n + 2\alpha - 3\beta) \\ + \frac{3BBC}{6\alpha^3} n (n + \alpha - 2\beta - \gamma)(n + 2\alpha - 2\beta - \gamma) \\ + \frac{3BCC}{6\alpha^3} n (n + \alpha - \beta - 2\gamma)(n + 2\alpha - \beta - 2\gamma) \\ + \frac{C^3}{6\alpha^3} n (n + \alpha - 3\gamma)(n + 2\alpha - 3\gamma). \end{array}\right\}.$$

§. 22. Tractemus ſimili modo aequationem quartam, atque ex valore $f''' : n$ inuento habebimus:

$$Bf''' : (n-\beta) = \left\{\begin{array}{l} + \frac{B^4}{6\alpha^3} (n - \beta)(n + \alpha - 4\beta)(n + 2\alpha - 4\beta) \\ + \frac{3B^3C}{6\alpha^3} (n-\beta)(n+\alpha-3\beta-\gamma)(n+2\alpha-3\beta-\gamma) \\ + \frac{3B^2C^2}{6\alpha^3} (n-\beta)(n+\alpha-2\beta-2\gamma)(n+2\alpha-2\beta-2\gamma) \\ + \frac{BC^3}{6\alpha^3} (n-\beta)(n+\alpha-\beta-3\gamma)(n+2\alpha-\beta-3\gamma) \end{array}\right\}.$$

$$Cf''' : (n-\gamma) = \left\{\begin{array}{l} + \frac{3BC^3}{6\alpha^3} (n-\gamma)(n+\alpha-\beta-3\gamma)(n+2\alpha-\beta-3\gamma) \\ + \frac{B^3C}{6\alpha^3} (n-\gamma)(n+\alpha-3\beta-\gamma)(n+2\alpha-3\beta-\gamma) \\ + \frac{3B^2C^2}{6\alpha^3} (n-\gamma)(n+\alpha-2\beta-2\gamma)(n+2\alpha-2\beta-2\gamma) \\ + \frac{C^4}{6\alpha^3} (n-\gamma)(n+\alpha-4\gamma)(n+2\alpha-4\gamma) \end{array}\right\}.$$

His iam terminis collectis valor formulae

$$Bf''' : (n - \beta) + Cf''' : (n - \gamma),$$

conſtabit ſequentibus quinque partibus:

$$\frac{B^4}{6\alpha^3}(n-\beta)(n+\alpha-4\beta)$$
$$+\frac{B^3C}{6\alpha^3}\left(n-\frac{3\beta-\gamma}{4}\right)(n+\alpha-3\beta-\gamma)(n+2\alpha-3\beta-\gamma)$$
$$+\frac{6B^2C^2}{6\alpha^3}\left(n-\frac{\beta-\gamma}{2}\right)(n+\alpha-2\beta-2\gamma)(n+2\alpha-2\beta-2\gamma)$$
$$+\frac{4BC^3}{6\alpha^3}\left(n-\frac{\beta-3\gamma}{4}\right)(n+\alpha-\beta-3\gamma)(n+2\alpha-\beta-3\gamma)$$
$$+\frac{C^4}{6\alpha^3}(n-\gamma)(n+\alpha-4\gamma)(n+2\alpha-4\gamma)$$

§. 23. Quoniam igitur functio quaesita $f^{IV}:n$ ex quinque partibus componitur, singulas cum lemmate quarto comparari oportebit, ac pro parte prima erit $k=\frac{B^4}{6\alpha^3}$ et $\lambda=\beta$; pro parte secunda est $k=\frac{4B^3C}{6\alpha^3}$ et $\lambda=\frac{3\beta+\gamma}{4}$; pro parte tertia $k=\frac{6B^2C^2}{6\alpha^3}$ et $\lambda=\frac{\beta+\gamma}{2}$; pro parte quarta est $k=\frac{4BC^3}{6\alpha^3}$ et $\lambda=\frac{\beta+3\gamma}{4}$; denique pro parte quinta erit $k=\frac{C^4}{6\alpha^3}$ et $\lambda=\gamma$; vnde collectis omnibus terminis reperietur

$$f^{IV}:n=\left\{\begin{array}{l}\frac{B^4}{24\alpha^4}n(n+\alpha-4\beta)(n+2\alpha-4\beta)(n+3\alpha-4\beta)\\+\frac{4B^3C}{24\alpha^4}n(n+\alpha-3\beta-\gamma)(n+2\alpha-3\beta-\gamma)(n+3\alpha-3\beta-\gamma)\\+\frac{6B^2C^2}{24\alpha^4}n(n+\alpha-2\beta-2\gamma)(n+2\alpha-2\beta-2\gamma\cdot(n+3\alpha-2\beta-2\gamma)\\+\frac{4BC^3}{24\alpha^4}n(n+\alpha-\beta-3\gamma)(n+2\alpha-\beta-3\gamma)(n+3\alpha-\beta-3\gamma)\\+\frac{C^4}{24\alpha^4}n(n+\alpha-4\gamma)(n+2\alpha-4\gamma)(n+3\alpha-4\gamma).\end{array}\right\}$$

§. 24. Superfluum foret hos calculos vlterius prosequi; quandoquidem ex allatis iam tuto concludere licet, sequentem functionem $f^{V}:n$ hunc habituram esse valorem:

$$f^{\mathrm{v}}:n=\left\{\begin{array}{l}
+\frac{B^5}{120\,\alpha^5}\,n\,(n+\alpha-5\beta)\,(n+2\alpha-5\beta)\,(n+3\alpha-5\beta)\,(n+4\alpha-5\beta)\\
+\frac{5\,B^4\,C}{120\,\alpha^5}\,n\,(n+\alpha-4\beta-\gamma)\,(n+2\alpha-4\beta-\gamma)\,(n+3\alpha-4\beta-\gamma)\,\times\\
\qquad\qquad\qquad\qquad\qquad\qquad\times\,(n+4\alpha-4\beta-\gamma)\\
+\frac{10\,B^3\,C^2}{120\,\alpha^5}\,n\,(n+\alpha-3\beta-2\gamma)(n+2\alpha-3\beta-2\gamma)(n+3\alpha-3\beta-2\gamma)\,\times\\
\qquad\qquad\qquad\qquad\qquad\qquad\times\,(n+4\alpha-3\beta-2\gamma)\\
+\frac{10\,B^2\,C^3}{120\,\alpha^5}\,n\,(n+\alpha-2\beta-3\gamma)\,(n+2\alpha-2\beta-\gamma)\,(n+3\alpha-2\beta-3\gamma)\,\times\\
\qquad\qquad\qquad\qquad\qquad\qquad\times\,(n+4\alpha-2\beta-3\gamma)\\
+\frac{5\,B\,C^4}{120\,\alpha^5}\,n\,(n+\alpha-\beta-4\gamma)\,(n+2\alpha-\beta-4\gamma)\,(n+3\alpha-\beta 4-\gamma)\,\times\\
\qquad\qquad\qquad\qquad\qquad\qquad\times\,(n+4\alpha-\beta-4\gamma)\\
+\frac{C^5}{120\,\alpha^5}\,n\,(n+\alpha-5\beta)\,(n+2\alpha-4\beta)\,(n+3\alpha-5\beta)\,(n+4\alpha-5\beta)
\end{array}\right\}$$

vnde formatio omnium ſequentium functionum ſatis dilucide perſpicitur.

§. 25. Quodſi iam omnes iſti valores, quos pro functionibus $f':n$; $f'':n$; $f''':n$; etc. elicuimus, in vnam ſummam colligantur, et ob $f^{\circ}:n=1$ vnitas praefigatur, obtinebitur ſeries deſiderata, quae ſcilicet valorem poteſtatis Z^n exprimit, neque ergo opus eſt omnes iſtas functiones hic denuo collectas referre.

Corollarium.

Hae ergo ſeries infinitis conſtant terminis, in quibus omnes poſſibiles binarum litterarum B et C combinationes occurrunt. Quin etiam pro combinatione quacunquae, quae ſit $B^b\,C^c$, in genere terminus eam inuoluens aſſignari poterit. Primo enim iſta forma multiplicetur per numerum omnium combinationum, qui poſito breuitatis gratia $b+c=i$, ſi indicetur littera N, erit vti conſtat $N=\frac{1.2.3.4\ldots.i}{1.2.3\ldots.b.1.2\ldots.c}$, deinde ſi ponamus $b\beta+c\gamma=\theta$, erit terminus huic formae reſpondens:

N B^b

$$\frac{N\,B^b\,C^c}{1.\,2.\,3.\,\ldots\ldots\, i\,\alpha^i}\, n\,(n+\alpha-\theta)\,(n+2\alpha-\theta)\times$$
$$\times\,(n+3\alpha-\theta)\ldots\ldots[n+(i-1)\,\alpha-\theta].$$

Veluti si forma proposita fuerit $B^3\,C^2$, erit $i=5$, hincque $N=\frac{1.\,2.\,3.\,4.\,5}{1.\,2.\,3.\,1.\,2}=10$, deinde vero erit $\theta=3\beta+2\gamma$, sicque ipse terminus huius formae erit

$$\frac{10\,B^3\,C^2}{120\,\alpha^5}\, n\,(n+\alpha-3\beta-2\gamma)\,(n+2\alpha-3\beta-2\gamma)\times$$
$$\times\,(n+3\alpha-3\beta-2\gamma)\,(n+4\alpha-3\beta-2\gamma),$$

prorsus vt supra est exhibitus.

Scholion.

§. 26. Hinc iam abunde patet, si aequatio proposita pluribus adhuc constet terminis, habeatque hanc formam:

$$1-\frac{1}{Z^\alpha}=\frac{B}{Z^\beta}+\frac{C}{Z^\gamma}+\frac{D}{Z^\delta}+\frac{E}{Z^\varepsilon}+\text{etc.}$$

tum ope eiusdem methodi seriem infinitam inuestigari posse, quae valorem potestatis Z^n exprimat; ista enim series incipiens ab vnitate infinitos inuoluet terminos ex omnibus plane combinationibus litterarum B, C, D, E, etc. formatos. Si enim in genere proponatur haec combinatio: $B^b.\,C^c.\,D^d.\,E^e.$ etc. statuatur primo $b+c+d+e=i$, et quaeratur numerus N, vt sit

$$N=\frac{1.\,2.\,3.\,4.\,\ldots\ldots\ldots\, i}{1.\,2.\,3.\ldots b.\,1.\,2.\,3.\ldots c.\,1.\,2.\,3.\ldots d.\,1.\,2.\,3.\ldots e.},$$

sitque breuitatis gratia $1.\,2.\,3.\,4.\ldots\ldots i=I$, factor prior huius termini erit $\frac{N\,B^b.\,C^c.\,D^d.\,E^e.}{I.\,\alpha^i}$, praeterea vero adiungendi sunt factores exponentem n inuoluentes, pro quibus inueniendis statuatur:

$$b\,\beta+c\,\gamma+d\,\delta+e\,\varepsilon=\theta,$$

erunt-

eruntque isti factores numero i isti:

$$n\,(n+\alpha-\theta)\,(n+2\alpha-\theta)\,(n+3\alpha-\theta)\ldots\ldots[n+(i-1)\,\alpha-\theta],$$

ita vt totus terminus sit

$$\frac{N\,.\,B^b\,.\,C^c\,.\,D^d\,.\,E^e}{1\,.\,\alpha^i}\,n\,(n+\alpha-\theta)\,(n+2\,\alpha-\theta)\times$$
$$\times\,(n+3\,\alpha-\theta)\,\ldots\ldots\,[n+(i-1)\,\alpha-\theta].$$

Quamobrem super indole omnium harum serierum maxime memorabilium, quas olim in Tomo XV. Nouorum Commentariorum fusius, sed sufficiente demonstratione descripsi, nihil amplius desiderari posse videtur.