1789

Uberior explicatio methodi singularis nuper expositae integralia alias maxime abscondita investigandi

Leonhard Euler

VBERIOR EXPLICATIO
METHODI SINGVLARIS
NVPER EXPOSITAE, INTEGRALIA ALIAS MAXIME ABSCONDITA INVESTIGANDI.

Auctore
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Methodus illa singularis, qua non ita pridem deductus sum sum ad integrationem formulae $\int \frac{x^a - x^b}{l\,x}\,\partial x$, cuius valorem a termino $x = 0$ vsque ad $x = 1$ extensum inueni esse $l\,\frac{a+1}{b+1}$ *), multo latius patet, ideoque accuratiorem euolutionem meretur, quandoquidem multo maiora incrementa scientiae analyticae polliceri videtur. Quo autem hoc feliciori successu et sine ambagibus praestari possit, necesse erit peculiarem signandi modum vsurpare, quem ergo ante omnia explicari conueniet.

EXPLICATIO CHARACTERVM in sequentibus adhibendorum.

I. Si V denotet functionem quamcunque binarum variabilium x et p, tum iste character: $\frac{\partial^\lambda}{x}$ V, mihi designabit eam

*) Iam in dissertatione praecedente annotauimus, Ill. Auctorem hanc integrationem exposuisse in Nouorum Commentariorum Tomo. XIX. pag. 70.

eam quantitatem, quae oritur, ſi functio V, ſolam x pro variabili ſumendo, toties ſucceſſiue differentietur, quot vnitates in indice λ continentur, ſimulque vbique differentiale ∂x reiiciatur. Eodem modo iſte character: $\frac{\partial^\lambda}{p}.V$, deſignabit eam quantitatem, quae per totidem differentiationes reſultat, dum ſola p vt variabilis tractatur. Hinc igitur iſta ſignandi ratio ſequenti modo ad formulas vſu receptas reducetur:

$$\frac{\partial}{x}.V = \left(\frac{\partial V}{\partial x}\right) \text{ et } \frac{\partial}{p}.V = \left(\frac{\partial V}{\partial p}\right),$$
$$\frac{\partial^2}{x}.V = \left(\frac{\partial\partial V}{\partial x^2}\right) \text{ et } \frac{\partial^2}{p}.V = \left(\frac{\partial\partial V}{\partial p^2}\right),$$
$$\frac{\partial^3}{x}.V = \left(\frac{\partial^3 V}{\partial x^3}\right) \text{ et } \frac{\partial^3}{p}.V = \left(\frac{\partial^3 V}{\partial p^3}\right).$$

II. Viciſſim autem integrando iſte character: $\frac{\int^\lambda}{x}.V$ deſignabit eam quantitatem, quae ex continua integratione λ vicibus repetita oritur, dum ſola x variabilis accipitur; et pariter hic character: $\frac{\int^\lambda}{p}.V$, eam quantitatem ſignificat, quae oritur per continuam integrationem λ vicibus repetitam, dum ſola p variabilis accipitur. Haec ergo ſequenti modo ad formas vſu receptas reuocabuntur:

$$\frac{\int}{x}.V = \int V\partial x \text{ et } \frac{\int}{p}.V = \int V\partial p,$$
$$\frac{\int^2}{x}.V = \int\partial x\int V\partial x \text{ et } \frac{\int^2}{p}.V = \int\partial p\int V\partial p,$$
$$\frac{\int^3}{x}.V = \int\partial x\int\partial x\int V\partial x \text{ et } \frac{\int^3}{p}.V = \int\partial p\int\partial p\int V\partial p.$$

III. At quoniam omnes quantitates per integrationem inuentae per ſe ſunt indeterminatae, in poſterum perpetuo omnia integralia ita capi ſtatuamus, vt euaneſcant poſito vel $x=0$ vel

vel $p = 0$; prius scilicet, si sola x vt variabilis fuerit tractata, posterius vero, si sola p fuerit variabilis.

IV. Hos iam characteres pro lubitu inter se coniungere licet, ac primo quidem haec formula: $\frac{\partial^{\mu}}{x} \cdot \frac{\partial^{\nu}}{p} \cdot V$, denotat, functionem V primo μ vicibus differentiari debere, sumta sola x variabili; tum vero quantitatem hinc oriundam denuo ν vicibus differentiari debere, sumta sola p variabili. Hinc istos characteres ad morem solitum reuocando erit

$$\frac{\partial}{x} \cdot \frac{\partial}{p} \cdot V = \left(\frac{\partial \partial V}{\partial x \partial p}\right), \quad \frac{\partial}{p} \cdot \frac{\partial}{x} \cdot V = \left(\frac{\partial \partial V}{\partial p \partial x}\right),$$
$$\frac{\partial^2}{x} \cdot \frac{\partial}{p} \cdot V = \left(\frac{\partial^3 V}{\partial x^2 \partial p}\right), \quad \frac{\partial}{x} \cdot \frac{\partial^2}{p} \cdot V = \left(\frac{\partial^3 V}{\partial x \partial p^2}\right),$$
$$\frac{\partial^3}{x} \cdot \frac{\partial^2}{p} \cdot V = \left(\frac{\partial^5 V}{\partial x^3 \partial p^2}\right), \quad \frac{\partial^2}{x} \cdot \frac{\partial^3}{p} \cdot V = \left(\frac{\partial^5 V}{\partial x^2 \partial p^3}\right),$$
$$\text{etc.} \qquad \text{etc.}$$

V. Ita formula $\frac{\partial^{\mu}}{x} \cdot \frac{\int^{\nu}}{p} \cdot V$ denotat, functionem V primo μ vicibus differentiari debere, sumta sola x variabili, tum vero quantitatem hinc oriundam ν vicibus integrari debere, sumta sola p variabili. Ita si fuerit $\mu = 2$ et $\nu = 1$, erit more solito $\frac{\partial^2}{x} \cdot \frac{\int^1}{p} \cdot V = \int \partial p \cdot \left(\frac{\partial \partial V}{\partial x^2}\right)$, vnde significatio aliorum huiusmodi characterum iam satis intelligi potest.

VI. Simili modo formula hoc charactere designata: $\frac{\int^{\mu}}{x} \cdot \frac{\partial^{\nu}}{p} \cdot V$, declarat, functionem V primo μ vicibus integrari debere, sumta sola x variabili; tum vero quantitatem hinc oriundam ν vicibus differentiari debere, sumta sola p variabili. Quae ergo significatio satis clare perspicitur, etsi more solito non tam commode indicari posset. Si enim esset $\mu = 2$ et $\nu = 2$, valor

lor huius formulae: $\frac{\int^2}{x} \cdot \frac{\partial^2}{p} \cdot V$, hoc modo repraesentari deberet: $\left(\frac{\partial\partial . \int \partial x \int V \partial x}{\partial p^2}\right)$.

VII. Denique iste character: $\frac{\int^\mu}{x} \cdot \frac{\int^\nu}{p} \cdot V$, significat, functionem V primo μ vicibus integrari debere, sumta sola x pro variabili; tum vero quantitatem resultantem denuo ν vicibus integrari debere, sumta sola p variabili. Vbi, quod in perpetuum est tenendum, priora integralia ita capi debent, vt euanescant posito $x = 0$, posteriora vero posito $p = 0$.

Hac characterum explicatione praemissa sequentia Theoremata probe notentur, quorum veritas ex iis, quae de indole functionum duarum variabilium sunt exposita, satis clare perspicitur.

Theorema I.

Si V fuerit functio quaecunque duarum variabilium x et p, sequens aequalitas semper locum habebit:

$$\frac{\partial^\mu}{x} \cdot \frac{\partial^\nu}{p} \cdot V = \frac{\partial^\nu}{p} \cdot \frac{\partial^\mu}{x} \cdot V.$$

Hinc ergo si ponamus

$$\frac{\partial^\mu}{x} \cdot V = Q \text{ et } \frac{\partial^\nu}{p} \cdot V = R, \text{ tum erit } \frac{\partial^\nu}{p} \cdot Q = \frac{\partial^\mu}{x} \cdot R.$$

Theorema II.

Si V fuerit functio quaecunque binarum variabilium x et p, tum sequens aequalitas semper locum habebit:

$$\frac{\int^\mu}{x} \cdot \frac{\partial^\nu}{p} \cdot V = \frac{\partial^\nu}{p} \cdot \frac{\int^\mu}{x} \cdot V.$$

Hinc

Hinc ſi ponamus

$$\frac{\int^\mu}{x} . V = Q \text{ et } \frac{\partial^\nu}{p} . V = R, \text{ erit } \frac{\partial^\nu}{p} . Q = \frac{\int^\mu}{x} . R.$$

Theorema III.

Si fuerit V functio quaecunque binarum variabilium x et p, tum ſequens aequalitas ſemper locum habebit:

$$\frac{\partial^\mu}{x} . \frac{\int^\nu}{p} . V = \frac{\int^\nu}{p} . \frac{\partial^\mu}{x} . V.$$

Hinc ſi ponamus $\frac{\partial^\mu}{x} . V = Q$ et $\frac{\int^\nu}{p} . V = R$, erit $\frac{\int^\nu}{p} . Q = \frac{\partial^\mu}{x} . R$.

Theorema IV.

Si fuerit V functio quaecunque binarum variabilium x et p, tum ſequens aequalitas ſemper locum habebit:

$$\frac{\int^\mu}{x} . \frac{\int^\nu}{p} . V = \frac{\int^\nu}{p} . \frac{\int^\mu}{x} . V.$$

Hinc ſi ponamus $\frac{\int^\mu}{x} . V = Q$ et $\frac{\int^\nu}{p} . V = R$, erit $\frac{\int^\nu}{p} . Q = \frac{\int^\mu}{x} R$.

Scholion.

Hae aequalitates per ſe ita ſunt manifeſtae, vt quouis caſu euolutae euadant identicae. Ita ſi ſumatur $V = x^m p^n$, ex theoremate primo ſumto, $\mu = 2$ et $\nu = 1$, reperietur

$$Q = \frac{\partial^2}{x} . V = m(m-1) x^{m-2} p^n \text{ et}$$
$$R = \frac{\partial}{p} . V = n p^{n-1} x^m.$$

Hinc vero elicitur

 $\frac{\partial}{p} . Q$

$$\frac{\partial}{p} \cdot Q = m\,n\,(m-1)\,x^{m-2}\,p^{n-1} \text{ et}$$
$$\frac{\partial^2}{x} \cdot R\; m\,n\,(m-1)\,x^{m-2}\,p^{n-1},$$

qui duo valores manifefto congruunt. Ex fecundo autem theoremate $\mu = 2$ et $\nu = 1$ fiet

$$Q = \frac{\int}{x} V = \frac{p^n\,x^{m+2}}{(m+1)\,(m+2)} \text{ et } R = \frac{\partial}{p} V = n\,p^{n-1}\,x^m.$$

Hinc ergo erit

$$\frac{\partial}{p} Q = \frac{n\,p^{n-1}\,x^{m+2}}{(m+1)\,(m+2)} \text{ et } \frac{\int^2}{x} \cdot R = \frac{n\,p^{n-1}\,x^{m+2}}{(m+1)\,(m+2)}.$$

Ex tertio theoremate, manente $\mu = 2$ et $\nu = 1$, erit

$$Q = \frac{\partial^2}{x} \cdot V = m\,(m-1)\,x^{m-2}\,p^n \text{ et } R = \frac{\int}{p} V = \frac{p^{n+1}\,x^m}{n+1}.$$

Hinc igitur erit

$$\frac{\int}{p} Q = \frac{m\,(m-1)\,x^{m-2}\,p^{n+1}}{n+1} \text{ et}$$
$$\frac{\partial^2}{x} R = \frac{m\,(m-1)\,x^{m-2}\,p^{n+1}}{n+1}.$$

Ex quarto denique theoremate erit

$$Q = \frac{\int^2}{p} V = \frac{x^{m+2}\,p^n}{(m+1)\,(m+2)} \text{ et } R = \frac{\int}{p} V = \frac{x^m\,p^{n+1}}{n+1}.$$

Hinc ergo colligitur:

$$\frac{\int}{p} Q = \frac{x^{m+2}\,p^{n+1}}{(n+1)(m+1)(m+2)} \text{ et } \frac{\int^2}{x} R = \frac{x^{m+2}\,p^{n+1}}{(n+1)(m+1)(m+2)}.$$

Ob has igitur aequalitates adeo identicas nullae conclufiones hinc deduci poffe videbuntur. Verum longe aliter fe res habere deprehenditur, fi poft omnes operationes inftitutas ipfi x determinatus valor, veluti $x = 1$, tribui debeat, quemadmodum in in quatuor Problematibus ſequentibus oſtendemus, quae ſe ad quatuor Theoremata praecedentia referunt.

Problema I.

Si V fuerit functio quaecunque binarum variabilium x et p, et omnes operationes in Theoremate primo indicatae abſoluantur, tum vero ſtatuatur $x = 1$, exhibere aequalitatem, ad quam hoc Theorema perducit.

Solutio.

Quoniam in noſtro primo Theoremate poſuimus $\frac{\partial^\mu}{x} V = Q$, deinde vero haec quantitas, ſola p variabili ſumta, differentiari debet, ita vt iam x pro conſtanti habeatur, ſtatim loco x vnitas ſcribi poterit, quo facto abeat Q in M, ita vt nunc M futura ſit functio ſolius p. Manente igitur $R = \frac{\partial^\nu}{p} . V$ conſequemur hanc aequationem: $\frac{\partial^\nu}{p} M = \frac{\partial^\mu}{x} R$, vbi plerumque eueniet, vt quantitas M multo promptius differentiari queat quam functio Q, vnde aequalitas inuenta plerumque non adeo erit obuia, id quod ſequentibus exemplis illuſtraſſe iuuabit, in quibus omnibus aſſumemus $V = x^{n+p}$, ita vt eius valor poſito $x = 1$ abeat in 1.

Exemplum I, quo $\mu = 1$ et $\nu = 1$.

Hic ergo erit

$$Q = \tfrac{\partial}{x} . x^{n+p} = (n+p)\, x^{n+p-1},$$

vnde ergo poſito $x = 1$ fit $M = n + p$; quare cum ſit

$$R = \tfrac{\partial}{p} . x^{n+p} = x^{n+p}\, l\, x,$$

nancisnancifcimur hanc aequationem: $1 = \frac{\partial}{x} . x^{n+p} l x$. Vnde patet, fi poft differentiationem ponatur $x = 1$, fore more exprimendi folito $\frac{1}{\partial x} \partial . x^{n+p} l x = 1$, id quod non amplius tam eft obvium: eft enim

$$\partial . x^{n+p} l x = (n+p) x^{n+p-1} \partial x l x + x^{n+p-1} \partial x,$$

quae expreffio per ∂x diuifa, pofitoque $x = 1$, abit in 1.

Exemplum II, quo $\mu = 2$ et $\nu = 1$.

Hic igitur erit

$$Q = \frac{\partial^2}{x} x^{n+p} = (n+p)(n+p-1) x^{n+p-2},$$

pofito ergo $x = 1$, erit $M = (n+p)(n+p-1)$. Quare cum fit $R = x^{n+p} l x$, erit

$$\frac{\partial}{p} (n+p)(n+p-1) = \frac{\partial^2}{x} x^{n+p} l x,$$

quamobrem per folitum exprimendi modum habebimus:

$$\frac{\partial \partial\, x^{n+p} l x}{\partial x^2} = 2(n+p) - 1,$$

poftquam fcilicet gemina differentiatione abfoluta ponitur $x = 1$.

Exemplum III, quo $\mu = 1$ et $\nu = 2$.

Hic igitur erit

$$Q = \frac{\partial}{x} . x^{n+p} = (n+p) x^{n+p-1},$$

vnde pofito $x = 1$ fit $M = n + p$. Quare cum fit

$$R = \frac{\partial^2}{p} . x^{n+p} = x^{n+p} (l x)^2, \text{ erit}$$

$$\frac{\partial^2}{p} (n+p) = \frac{\partial}{x} x^{n+p} (l x)^2,$$

fiue folito exprimendi more $\frac{\partial . x^{n+p} (l x)^2}{\partial x} = 0$, poftquam fcilicet differentiatione abfoluta ponitur $x = 1$.

Ex-

Exemplum IV, quo $\mu = 2$ et $\nu = 2$.

Cum igitur hoc casu sit

$$Q = \frac{\partial^2}{x} x^{n+p} = (n+p)(n+p-1)\, x^{n+p-2},$$

ideoque

$$M = (n+p)(n+p-1) \text{ et } R = \frac{\partial^2}{p} x^{n+p} = x^{n+p}(l\,x)^2,$$

erit

$$\frac{\partial\partial\,.\,x^{n+p}(l\,x)^2}{\partial x^2} = \frac{\partial\partial M}{\partial p^2} = 2.$$

Corollarium.

Ex his exemplis iam abunde fit perspicuum, si exponentes μ et ν fuerint quicunque, tum posito $x = 1$ fore

$$M = (n+p)(n+p-1) \ldots\ldots (n+p-\mu+1),$$

ideoque functionem ipsius p tantum. Quare cum sit $R = x^{n+p}(l\,x)^\nu$, erit more solito $\frac{\partial^\mu x^{n+p}(l\,x)^\nu}{\partial x^\mu} = \frac{\partial^\nu . M}{\partial p^\nu}$, quando scilicet omnibus operationibus peractis statuitur $x = 1$.

Scholion.

Quemadmodum hic assumsimus $V = x^{n+p}$, ita eadem opera expedire licet hanc formam latius patentem: $V = x^p X$, denotante X functionem quamcunque ipsius x tantum, ita vt altera quantitas p non ingrediatur. Ponamus igitur sumto $x = 1$ fieri $X = A$, $\frac{\partial X}{\partial x} = A'$, $\frac{\partial\partial X}{\partial x^2} = A''$, etc. atque cum fiat

$$Q = \frac{\partial}{x} V = p\, x^{p-1} X + x^p \frac{\partial X}{\partial x},$$

erit hoc casu $M = p\,A + A'$. Deinde vero habebimus

$$\frac{\partial^2}{x} V = p(p-1)\, x^{p-2} X + 2p\, x^{p-1} \frac{\partial X}{\partial x} + x^p \frac{\partial\partial X}{\partial x^2} = Q;$$

 hinc

hinc ergo colligitur $M = p(p-1)A + 2pA' + A''$. Prodit porro

$$\frac{\partial^3}{x} V = p(p-1)(p-2)\, x^{p-3} X + 3p(p-1)\, x^{p-2} \frac{\partial X}{\partial x} + 3p\, x^{p-1} \cdot \frac{\partial \partial X}{\partial x^2} + x^p \frac{\partial^3 X}{\partial x^3},$$

hinc ergo erit

$$M = p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A'''.$$

Hinc iam patet, ex formula $\frac{\partial^4}{x} V$ oriturum eſſe valorem

$$M = p(p-1(p-2)(p-3)A + 4p(p-1)(p-2)A' + 6p(p-1)A'' + 4pA''' + A'''',$$

vnde lex progreſſionis ſatis eſt manifeſta. At vero pro altera littera R habebimus:

caſu $\nu = 1$, $R = x^p X\, l x$,

caſu $\nu = 2$, $R = x^p X (l x)^2$,

caſu $\nu = 3$, $R = x^p X (l x)^3$,

atque adeo in genere caſu $\nu = \nu$, erit $R = x^p X (l x)^\nu$. Ex his igitur formulis nanciſcemur valores differentialium omnium ordinum formulae $x^p X (l x)^\nu$, poſtquam factis omnibus operationibus poſitum fuerit $x = 1$:

$$1^\circ.\ \frac{1}{\partial x} \cdot \partial .\, x^p X (l x)^\nu = \frac{\partial^\nu (pA + A')}{\partial p^\nu},$$

qui valor ſemper erit $= 0$, excepto caſu $\nu = 1$, quo prodit $= A$.

$$2^\circ.\ \frac{1}{\partial x^2} \partial\partial .\, x^p X (l x)^\nu = \frac{\partial^\nu (p(p-1)A + 2pA' + A'')}{\partial p^\nu},$$

qui valor ſemper eſt o quando $\nu > 2$.

$$3^\circ.\ \frac{1}{\partial x^3} \partial^3 .\, x^p X (l x)^\nu = \frac{\partial^\nu (p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^\nu},$$

qui valor ſemper euaneſcit, exceptis caſibus quibus $\nu = < 3$.

In his formulis notaſſe iuuabit eſſe

Pro

Pro prima:

$$\frac{\partial (pA + A')}{\partial p} = A.$$

Pro secunda:

$$\frac{\partial (p(p-1)A + 2pA' + A'')}{\partial p} = (2p - 1)A + 2A' \text{ et}$$

$$\frac{\partial\partial (p(p-1)A + 2pA' + A'')}{p^2} = 2A,$$

sequentia autem sunt o.

Pro tertia:

$$\frac{\partial (p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p} =$$
$$(3pp - 6p + 2)A + 3(2p-1)A' + 3A'';$$

$$\frac{\partial\partial (p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^2} =$$
$$(6p - 6)A + 6A' \text{ et}$$

$$\frac{\partial^3 (p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^3} = 6A,$$

sequentia omnia euanescunt.

Problema II.

Si V fuerit functio quaecunque binarum variabilium x et p, et omnes operationes in Theoremate secundo indicatae absoluantur, tum vero statuatur $x = 1$; exhibere aequalitatem ad quam hoc Theorema perducit.

Solutio.

Quoniam in nostro secundo Theoremate posuimus $\frac{\int^{\mu}}{x} V = Q$, deinde vero haec quantitas, sumta sola p variabili, ν vicibus differentiari debet, posita x constante, iam ante has differentiationes ponere licet $x = 1$. Hoc ergo facto abeat Q in M, sicque habebitur $\frac{\partial^{\nu}}{p}. Q = \frac{\partial^{\nu} M}{\partial p^{\nu}}$, quod iam est membrum primum aequalitatis quaesitae more solito expressum, quandoquidem M est sola functio ipsius p. Pro altero membro

 cum

cum fit $R = \frac{\partial^\nu}{p} . V$, erit hoc alterum membrum $\frac{\int^\mu}{x} R$. Quamobrem fi poft omnes has μ integrationes peractas (quae autem fingula integralia femper ita funt capienda, vt euanefcant pofito $x = 0$), ftatuatur: $x = 1$, femper erit $\frac{\int^\mu}{x} R = \frac{\partial^\nu M}{\partial p^\nu}$, de quo valore certi fumus, etiamfi forte integratio abfolui nequeat, quamobrem hanc veritatem exemplis illuftremus, in quibus affumemus $V = x^{n+p}$.

Exemplum I, quo $\mu = 1$ et $\nu = 1$.

Hoc ergo cafu erit

$$Q = \int x^{n+p} \partial x = \frac{x^{n+p+1}}{n+p+1},$$

vnde fit $M = \frac{1}{n+p+1}$. Deinde vero erit

$$R = \frac{\partial}{p} x^{n+p} = x^{n+p} \, l \, x,$$

ex quibus aequatio noftra fiet

$$\int x^{n+p} \partial x \, l \, x = \partial \frac{1}{n+p+1} = \frac{-1}{(n+p-1)^2}.$$

Exemplum II, quo $\mu = 1$ et $\nu = 2$.

Hoc ergo cafu erit

$$Q = \frac{\int}{x} x^{n+p} = \frac{x^{n+p+1}}{n+p+1},$$

ideoque M vt ante $\frac{1}{n+p+1}$. Deinde vero erit

$$R = \frac{\partial^2}{p} x^{n+p} = x^{n+p} (l \, x)^2,$$

quocirca pofito $x = 1$ habebitur ifta aequatio:

$$\int x^{n+p} \partial x \, (l \, x)^2 = \frac{\partial \partial}{p} \frac{1}{n+p+1} = \frac{+2}{(n+p+1)^3}.$$

Ex-

Exemplum III, quo $\mu = 1$ et $\nu = 3$.

Hoc igitur caſu erit

$$Q = \frac{x^{n+p+1}}{n+p+1} \text{ et } M = \frac{1}{n+p+1}.$$

Tum vero erit $R = x^{n+p} (l\,x)^3$, vnde naſcitur haec aequalitas:

$$\int x^{n+p}\,\partial x\,(l\,x)^3 = + \frac{\partial^3}{\partial p^3} \cdot \frac{1}{n+p+1} = \frac{-6}{(n+p+1)^4}.$$

Exemplum IV, quo $\mu = 1$ et $\nu = \nu$.

Hic ex praecedentibus ſatis liquet, aequationem hinc reſultantem fore

$$\int x^{n+p}\,\partial x\,(l\,x)^\nu = \pm \frac{1.\,2.\,3.\,\ldots\,.\,\nu}{(n+p+1)^{\nu+1}},$$

vbi ſignum ſuperius valet ſi ν ſit numerus par, inferius vero ſi impar, quae reductio eo magis eſt notatu digna, quod alias per plures ambages ad eam perueniri ſolet.

Exemplum V, quo $\mu = 2$ et $\nu = 1$.

Hoc ergo caſu erit

$$Q = \frac{\int^2}{x} x^{n+p} = \frac{x^{n+p+2}}{(n+p+1)\,(n+p+2)},$$

quamobrem habebitur $M = \frac{1}{(n+p+1)(n+p+2)}$, qui valor reducitur ad hunc:

$$M = \frac{1}{n+p+1} - \frac{1}{n+p+2};$$

tum vero erit $R = x^{n+p}\,l\,x$, vnde ſequens aequalitas deducitur:

$$\int \partial x \int x^{n+p}\,\partial x\,l\,x = \frac{-1}{(n+p+1)^2} + \frac{1}{(n+p+2)^2},$$

quae aequalitas more ſolito indagata iam ſatis moleſtos calculos poſtulat.

Exemplum VI, quo $\mu = 2$ et $\nu = 2$.

Hic ergo erit vt ante

$$M = \frac{1}{(n+p+1)(n+p+2)} = \frac{1}{n+p+1} - \frac{1}{n+p+2}$$

et $R = x^{n+p}(l\,x)^2$, vti in Exemplo II. vnde ſtatim colligitur iſta aequatio:

$$\int \partial x \int x^{n+p}\,\partial x\,(l\,x)^2 = \frac{\partial\partial M}{\partial p^2} = \frac{2}{(n+p+1)^3} - \frac{2}{(n+p+2)^3}.$$

Exemplum VII, quo $\mu = 2$ et $\nu = \nu$.

Hic ergo erit

$$M = \frac{1}{n+p+1} - \frac{1}{n+p+2} \text{ et } R = x^{n+p}(l\,x)^\nu,$$

vnde reſultat aequatio:

$$\int \partial x \int x^{n+p}\,\partial x\,(l\,x)^\nu = \pm \frac{1 \ldots \nu}{(n+p+1)^{\nu+1}} \mp \frac{1 \ldots \nu}{(n+p+2)^{\nu+1}},$$

vbi iterum ſigna ſuperiora valent ſi ν numerus par, inferiora vero ſi impar.

Exemplum VIII, quo $\mu = 3$ et $\nu = \nu$.

Pro hoc caſu ob

$$Q = \frac{\int^3}{x} x^{n+p} = \frac{n+p+3}{(n+p+1)(n+p+2)(n+p+3)},$$

poſito $x = 1$ fit

$$M = \frac{1}{(n+p+1)(n+p+2)(n+p+2)},$$

quae fractio reſoluatur in ſuas ſimplices, fietque

$$M = \frac{1}{2(n+p+1)} - \frac{1}{n+p+2} + \frac{1}{2(n+p+3)},$$

vnde facile patet, ſequentem prodituram eſſe aequalitatem:

$$\int \partial x \int \partial x \int x^{n+p}(l\,x)^\nu = \frac{3\cdot4\cdot5 \ldots \nu}{(n+p+1)^{\nu+1}} \mp \frac{2 \ldots \nu}{(n+p+2)^{\nu+1}} \pm \frac{3\cdot4 \ldots \nu}{(n+p+2)^{\nu+1}}$$

$$= \pm 3\cdot4\cdot5 \ldots \nu \left(\frac{1}{(n+p+1)^{\nu+1}} - \frac{2}{(n+p+2)^{\nu+1}} + \frac{1}{(n+p+3)^{\nu+1}} \right),$$

vbi

vbi ratio ſigni ambigui eſt eadem vt ante. Facile autem intelligitur, ſi quis formulam illam integralem euoluere voluerit, eum in calculos valde moleſtos eſſe delapſurum.

Scholion.

Superfluum foret indici μ maiores valores tribuere, ſiquidem euolutio ſimili modo expediri poſſet. Praecipuum autem negotium conſiſtit in reſolutione fractionis M in ſuas fractiones ſimplices, id quod neceſſe eſt, vt deinceps facilius omnes differentiationes, atque adeo ſecundum indicem indefinitum ν inſtitui queant. Hic autem labor ſubſidio ſequentis Propoſitionis promptiſſime abſolui poterit.

Propoſitio.

Si X fuerit functio quaecunque ipſius x, ac poſt integrationes ſtatui debeat $x = 1$, tum ſemper iſta formula integralis complicata $\frac{\int^{\mu}}{x} . X$ reduci poteſt ad iſtam formulam integralem ſimplicem more ſolito expreſſam: $\frac{\int X \partial x (1-x)^{\mu-1}}{1. 2. 3. \ldots (\mu-1)}$.
Hinc enim ſtatim patet, pro noſtro caſu quo $X = x^{n+p}$, quantitatem M ſequenti modo expreſſum iri

$$M = \frac{1}{1.2.\ldots(\mu-1)} \left(\frac{1}{n+p+1} - \frac{(\mu-1)}{n+p+2} + \frac{(\mu-1)(\mu-2)}{1.2.\ldots(n+p+3)} - \frac{(\mu-1)(\mu-2)(\mu-3)}{1.2.3.\ldots(n+p+4)} \text{ etc.} \right),$$

vnde iam facile differentialia omnium ordinum ipſius M derivari poſſunt. Ceterum hic adhuc obſeruaſſe iuuabit, loco functionis illius V vix alium valorem accipi poſſe praeter x^{n+p}, propterea quod hoc ſolo caſu omnia $\frac{\int^{\mu}}{x}$ V actu expedire licet,

id

id quod ad nostrum institutum imprimis requiritur, quia alioquin nullae aequationes memorabiles inde deduci possent.

Problema III.

Si V fuerit functio quaecunque binarum variabilium x et p, et omnes operationes in Theoremate tertio indicatae actu absoluantur, tum vero statuatur $x = 1$, exhibere aequalitatem, ad quam hoc Theorema perducit.

Solutio.

Quoniam in nostro tertio Theoremate posuimus

$$\frac{\partial^\mu}{x} V = Q \text{ et } \frac{\int^\nu}{p} V = R,$$

hinc deduximus sequentem aequalitatem: $\frac{\int^\nu}{p} Q = \frac{\partial^\mu}{x} R$, vbi in valore pro Q inuento loco x vnitas scribi debet, vnde resultet quantitas M, quae iam tantum erit functio ipsius p, ita vt nunc aequalitas nostra euadat $\frac{\partial^\nu}{p} M = \frac{\partial^\mu}{x} R$. Quod si iam loco V hanc accipiamus functionem: x^{n+p}, pro variis valoribus indicis μ littera M sequentes sortietur valores:

1°. Si $\mu = 1$ erit $M = n + p$,
2°. Si $\mu = 2$ erit $M = (n+p)(n+p-1)$,
4°. Si $\mu = 3$ erit $M = (n+p)(n+p-1)(n+p-2)$,
etc.

hincque in genere

$$M = (n+p)(n+p-1) \ldots\ldots (n+p-\mu+1).$$

Pro littera autem R ex valoribus simplicioribus indicis ν colligetur:

si

1°. Si $\nu = 1$, valor $R = \frac{x^{n+p}}{l x} + C$,

quae constans C cum ita debeat accipi, vt integrale euanescat posito $p = 0$, erit hac correctione adhibita $R = \frac{x^{n+p}}{l x} - \frac{x^n}{l x}$, quae formula ducta in ∂p et denuo integrata, adiectaque debita constante praebet:

2°. Si $\nu = 2$. --- $R = \frac{x^{n+p} - x^n}{(l x)^2} - \frac{p x^n}{l x}$,

3°. Si $\nu = 3$. --- $R = \frac{x^{n+p} - x^n}{(l x)^3} - \frac{p x^n}{(l x)^2} - \frac{p p x^n}{2 l x}$,

4°. Si $\nu = 4$. --- $R = \frac{x^{n+p} - x^n}{(l x)^4} - \frac{p x^n}{(l x)^3} - \frac{p p x^n}{2 (l x)^2} - \frac{p^3 x^n}{6 l x}$,

vnde concluditur in genere esse proditurum:

$$R = \frac{x^{n+p}}{(l x)^\nu} - x^n \left(\frac{1}{(l x)^\nu} + \frac{p}{(l x)^{\nu-1}} + \frac{p p}{1 . 2 (l x)^{\nu-2}} + \dots \frac{+ p^{\nu-1}}{1 . 2 . 3 .. (\nu-1) l x} \right).$$

His igitur valoribus euolutis sequentia exempla euoluamus.

Exemplum I, quo $\mu = 1$ et $\nu = 1$.

Hoc ergo casu erit $M = n + p$ et $R = \frac{x^{n+p} - x^n}{l x}$,

vnde oritur haec aequalitas:

$$\frac{1}{\partial x} \cdot \frac{\partial . (x^{n+p} - x^n)}{l x} = \int_p (n + p) = n p + \frac{p p}{2},$$

more solito expressa. Hic scilicet forma $\frac{x^{n+p} - x^n}{l x}$ per solam variabilem x differentiata et per ∂x diuisa, si loco x scribatur

batur 1, producet hunc valorem: $np + \frac{1}{2}pp$, id quod neutiquam tam facile perſpicitur. Si enim illa quantitas differentietur, omiſſo elemento ∂x, peruenitur ad iſtam expreſſionem:

$$\frac{(n+p)\, x^{n+p-1} - n\, x^{n-1}}{l\, x} - \frac{(x^{n+p-1} - x^{n-1})}{(l\, x)^2},$$

vbi iam poni oportet $x = 1$; tum autem vtrumque membrum euadit infinitum, quamobrem has duas fractiones ante omnia ad eundem denominatorem reduci conuenit, vt habeatur iſta fractio: $\frac{(n+p)\, x^{n+p-1}\, l x - n\, x^{n-1}\, l x - x^{n+p-1} + x^{n-1}}{(l\, x)^2}$, cuius tam numerator quam denominator euaneſcunt facto $x = 1$. Quamobrem ſecundum regulam cognitam loco tam numeratoris quam denominatoris eorum differentialia ſcribantur, ac pro numeratore reperietur:

$$(n+p)(n+p-1)\, x^{n+p-2}\, lx + (n+p)\, x^{n+p-2} - n(n-1) x^{n-2}\, lx - n x^{n-2}$$
$$- (n+p-1)\, x^{n+p-2} + (n-1)\, x^{n-2};$$

denominator vero erit $\frac{2\, l\, x}{x}$, ita vt iam tota fractio fit

$$\frac{(n+p)(n+p-1) x^{n+p-1} lx + (n+p) x^{n+p-1} - n(n-1) x^{n-1} lx - (n+p-1) x^{n+p-1} - x^{n-1}}{2\, l\, x}$$

vbi denuo, poſito $x = 1$, tam numerator quam denominator euaneſcunt; quamobrem eorum loco iterum differentialia ſubſtituamus, quo facto prodibit fractio, cuius numerator erit

$$(n+p-1)^2\, x^{n+p-2}\, [(n+p)\, l\, x - 1] + 2\, (n+p)\, (n+p-1)\, x^{n+p-2}$$
$$- n\, (n-1)^2\, x^{n-2}\, l\, x - (n\, n - 1)\, x^{n-2},$$

denominator vero erit $\frac{2}{x}$. Hic iam facto $x = 1$ numerator dabit

$$2\,(n+p)\,(n+p-1) - (n+p-1)^2 - (nn-1) = 2\,np + pp,$$

deno-

denominator vero 2, vnde valor quaesitus resultat $n p + \frac{1}{2} p p$, prorsus vti supra inuenimus. Hinc igitur abunde patet egregius vsus nostrae reductionis. Quin etiam casus adhuc simplicior, quo $\mu = 0$, haud exiguam moram creat.

Exemplum II, quo $\mu = 0$ et $\nu = 1$.

Hic erit $M = 1$, ob $Q = x^{n+p}$, manente $R = \frac{x^{n+p} - x^n}{l x}$, tum erit $\frac{\int}{p}. M = p$, vnde aequatio more solito expressa fiet $\frac{x^{n+p} - x^n}{l x} = p$. Posito autem $x = 1$ in parte sinistra tam numerator quam denominator euanescunt, vnde eorum differentialibus substitutis ista fractio euadet

$$\frac{(n+p) x^{n+p-1} - n x^{n-1}}{1 : x},$$

quae fractio posito $x = 1$ praebet p.

Exemplum III, quo $\mu = 0$ et $\nu = 2$.

Hic ergo erit $M = 1$, ideoque $\frac{\int^2}{p} . M = \frac{1}{2} p p$, cui ergo ipsa quantitas R aequabitur; sicque orietur haec aequatio:

$$\frac{x^{n+p} - x^n}{(l x)^2} - \frac{p x^n}{l x} = \frac{1}{2} p p,$$

cuius veritas neutiquam in oculos incurrit; quamobrem quantitas R ad vnicam fractionem reducatur, quae erit $\frac{x^{n+p} - x^n - p x^n l x}{(l x)^2}$, quae fractio, si loco numeratoris et denominatoris eorum differentialia substituantur, abit in sequentem:

$$\frac{(n+p)\,x^{n+p} - n\,x^n - n\,p\,x^n\,l\,x - p\,x^n}{2\,l\,x};$$

haec vero fractio eadem operatione instituta reducitur ad hanc:

$$\frac{(n+p)^2\,x^{n+p} - n\,n\,x^n - n\,n\,p\,x^n\,l\,x - 2\,n\,p\,x^n}{2}$$

quae expressio posito $x = 1$ manifesto abit in $\frac{1}{2}\,p\,p$.

Exemplum IV, quo $\mu = 0$ et $\nu = \nu$.

Hic ergo erit $M = 1$, ideoque $\frac{\int^{\nu}}{p} M = \frac{p^{\nu}}{1.\,2.\,3\,\ldots\,\nu}$.

Porro vero vidimus esse

$$R = \frac{x^{n+p}}{(l\,x)^{\nu}} - x^n \left(\frac{1}{(l\,x)^{\nu}} + \frac{p}{(l\,x)^{\nu-1}} + \ldots + \frac{p^{\nu-1}}{1.\,2.\,3\ldots(\nu-1)\,l\,x} \right),$$

atque haec expressio R ita est comparata, vt posito $x = 1$ eius valor futurus sit $\frac{p^{\nu}}{1.\,2.\,3\,\ldots\,\nu}$.

Exemplum V, quo $\mu = 1$ et $\nu = \nu$.

Hic ergo erit $M = n + p$ ideoque

$$\frac{\int^{\nu}}{p} M = \frac{n\,(\nu+1)\,p^{\nu} + p^{\nu+1}}{1.\,2.\,3\,\ldots\,(\nu+1)}.$$

Quod si iam ponatur

$$R = \frac{x^{n+p}}{(l\,x)^{\nu}} - x^n \left(\frac{1}{(l\,x)^{\nu}} + \frac{p}{(l\,x)^{\nu-1}} + \frac{p\,p}{1.\,2\,(l\,x)^{\nu-2}} + \ldots + \frac{p^{\nu-1}}{1.\,2\ldots(\nu-1)\,l\,x} \right)$$

quae expressio vt functio solius x spectetur, tum posito $x = 1$ erit more solito $\left(\frac{\partial R}{\partial x}\right) = \frac{p^{\nu}\,[n\,(\nu+1) + p]}{1.\,2.\,3\,\ldots\,(\nu+1)}$. Vbi facile intelligitur, differentiale ipsius R formulam producere multo magis

magis complicatam, cuius omnibus terminis ad communem denominatorem reductis, qui erit $(l\,x)^{\nu+1}$, si per regulam vulgarem istius fractionis valorem casu $x = 1$ explorare vellemus, tum tam numerator quam denominator $\nu + 1$ vicibus differentiari deberent, antequam eius verus valor definiri posset, quem tamen nunc certe nouimus fore $\frac{p^\nu [n(\nu + 1) + p]}{1.\,2.\,3\,\ldots\,(\nu + 1)}$.

Exemplum VI, quo $\mu = 2$ et $\nu = \nu$.

Hic ergo erit

$$M = (n+p)(n+p-1) = n(n-1) + (2n-1)p + pp$$

ideoque

$$\frac{\int^\nu}{p} M = \frac{n(n-1)p^\nu}{1.\,2.\,3\,\ldots\,\nu} + \frac{(2n-1)p^{\nu+1}}{1.\,2.\,3\ldots(\nu+1)} + \frac{p^{\nu+2}}{1.\,2.\,3\ldots(\nu+2)},$$

tum igitur, si vt ante fuerit

$$R = \frac{x^{n+p}}{(lx)^\nu} - x^n\left(\frac{1}{(l\,x)^\nu} + \frac{p}{(lx)^{\nu-1}} + \frac{pp}{1.\,2\,(l\,x)^{\nu-2}} + \ldots + \frac{p^{\nu-1}}{1.\,2\ldots(\nu-1)\,l\,x}\right)$$

casu $x = 1$ erit

$$\left(\frac{\partial\partial R}{\partial x^2}\right) = \frac{n(n-1)p^\nu}{1.\,2.\,3\,\ldots\,\nu} + \frac{(2n-1)p^{\nu+1}}{1.\,2.\,3\,..(\nu+1)} + \frac{p^{\nu+2}}{1.\,2.\,3\,..(\nu+2)}$$

quam veritatem more consueto euoluere nemo certe fusceperit. Atque ex his iam facile apparet, quomodo has conclusiones pro maioribus valoribus indicis μ formari oporteat.

Problema IV.

Si V fuerit functio quaecunque binarum variabilium x et p, et omnes operationes in Theoremate quarto indicatae absoluantur, tum vero statuatur $x = 1$, exhibere aequalitatem ad quam hoc Theorema perducit.

Solutio.

Quoniam in noſtro Theoremate quarto poſuimus $Q = \frac{\int^{\mu}}{x} . V$, qui valor poſito $x = 1$ abeat in M, ita vt M futura ſit ſola functio ipſius p, tum vero $R = \frac{\int^{\nu}}{p} . V$, vi noſtri Theorematis ſemper erit $\frac{\int^{\mu}}{x} R = \frac{\int^{\nu}}{x} . M$, ſiquidem omnes integrationes ita abſoluantur, vt ſingula integralia euaneſcant, poſito ſiue $x = 0$, ſiue $p = 0$, omnibus autem operationibus peractis ſtatuatur $x = 1$. Quod ſi iam pro V accipiamus hanc functionem: x^{n+p}, primo valores litterae M pro variis indicibus μ ſequenti modo ſe habebunt:

1°. Si $\mu = 0$ erit $M = 1$;
2°. Si $\mu = 1$ erit $M = \frac{1}{n+p+1}$;
3°. Si $\mu = 2$ erit $M = \frac{1}{(n+p+1)(n+p+2)}$;
4°. Si $\mu = 3$ erit $M = \frac{1}{(n+p+1)(n+p+2)(n+p+3)}$.

Hi autem valores ipſius M ope propoſitionis ſupra allegatae, qua erat

$$M = \frac{1}{1.2.3\ldots(\mu-1)}\left(\frac{1}{n+p+1} - \frac{\mu-1}{n+p+2} + \frac{(\mu-1)(\mu-2)}{1.2(n+p+3)} - \frac{(\mu-1)(\mu-2)(\mu-3)}{1.2.3(n+p+4)} \text{ etc.}\right)$$

ſequenti modo pro variis valoribus indicis μ ſe habebunt:

Si $\mu = 0$ valor $M = 1$;
Si $\mu = 1$ . . . $M = \frac{1}{n+p+1}$;
Si $\mu = 2$ . . . $M = \frac{1}{n+p+1} - \frac{1}{n+p+2}$;
Si $\mu = 3$ . . . $M = \frac{1}{2}\left(\frac{1}{n+p+1} - \frac{2}{n+p+2} + \frac{1}{n+p+3}\right)$
Si $\mu = 4$ . . . $M = \frac{1}{6}\left(\frac{1}{n+p+1} - \frac{3}{n+p+2} + \frac{3}{n+p+3} - \frac{1}{n+p+4}\right)$
Si $\mu = 5$ . . . $M = \frac{1}{24}\left(\frac{1}{n+p+1} - \frac{4}{n+p+2} + \frac{6}{n+p+3} - \frac{4}{n+p+4} + \frac{1}{n+p+5}\right)$.
etc. etc.

Dein-

Deinde pro littera R, si indici ν successiue tribuantur valores 0, 1, 2, 3, 4, etc. reperietur:

1°. Si $\nu = 0$ fore $R = x^{n+p}$;

2°. Si $\nu = 1$ ... $R = \frac{x^{n+p} - x^n}{l x}$;

3°. Si $\nu = 2$ ... $R = \frac{x^{n+p} - x^n}{(l x)^2} - \frac{p x^n}{l x}$;

4°. Si $\nu = 3$ ... $R = \frac{x^{n+p} - x^n}{(l x)^3} - \frac{p x^n}{(l x)^2} - \frac{p p x^n}{2 l x}$;

5°. Si $\nu = 4$ ... $R = \frac{x^{n+p} - x^n}{(l x)^4} - \frac{p x^n}{(l x)^3} - \frac{p p x^n}{2 (l x)^2} - \frac{p^3 x^n}{6 l x}$.

Hinc igitur sequentia Exempla euoluamus.

Exemplum I, quo $\mu = 0$ et $\nu = 0$.

Hoc casu erit $M = 1$ et $R = x^{n+p}$, vnde facto $x = 1$ erit vtique $x^{n+p} = 1$.

Exemplum II, quo $\mu = 0$ et $\nu = 1$.

Hoc ergo casu erit $M = 1$ et $R = \frac{x^{n+p} - x^n}{l x}$, vnde posito $x = 1$ fiet $\frac{x^{n+p} - x^n}{l x} = p$.

Exemplum III, quo $\mu = 0$ et $\nu = 2$.

Hoc ergo casu adhuc est

$$M = 1 \text{ et } R = \frac{x^{n+p} - x^n}{(l x)^2} - \frac{p x^n}{l x}.$$

Hinc ergo posito $x = 1$ prodibit ista aequalitas:

$$\frac{x^{n+p}}{(l x)^2} - x^n \left(\frac{1}{(l x)^2} + \frac{p}{l x} \right) = \frac{p p}{2}.$$

Exem-

Exemplum IV, quo $\mu = 0$ et $\nu = 3$.

Hic ergo, manente $M = 1$, erit

$$R = \frac{x^{n+p}}{(lx)^3} - x^n \left(\frac{1}{(lx)^3} + \frac{p}{(lx)^2} + \frac{pp}{2lx} \right);$$

quare posito $x = 1$ habebitur ista aequatio:

$$\frac{x^{n+p}}{(lx)^3} - x^n \left(\frac{1}{(lx)^3} + \frac{p}{(lx)^2} + \frac{pp}{2(lx)} \right) = \frac{p^3}{6}.$$

Haec autem exempla iam in praecedente problemate occurrunt, quia signa $\int^0$ et ∂^0 aequiualent.

Exemplum V, quo $\mu = 1$ et $\nu = 1$.

Hoc casu erit $M = \frac{1}{n+p+1}$ et $R = \frac{x^{n+p} - x^n}{lx}$, vnde cum fiat $\int R \partial x = \int M \partial p$, erit

$$\int \frac{x^{n+p} - x^n}{lx} \partial x = l \frac{n+p+1}{n+1},$$

quod est illud ipsum Theorema, quod non ita pridem inueneram et Geometris proposueram.

Exemplum VI, quo $\mu = 2$ et $\nu = 1$.

Hoc casu erit $M = \frac{1}{n+p+1} - \frac{1}{n+p+2}$, manente $R = \frac{x^{n+p} - x^n}{lx}$. Hinc igitur posito $x = 1$ oritur ista aequatio:

$$\int \partial x \int \frac{x^{n+p} - x^n}{lx} \partial x = l \frac{n+p+1}{n+1} - l \frac{n+p+2}{n+2};$$

haec autem veritas haud difficulter ex praecedenti exemplo deduci potest. Cum enim in genere sit $\int \partial x \int R \partial x = x \int R \partial x - \int R x \partial x$, ideoque casu $x = 1$

$\int \partial x$

$$\int \partial x \int R \partial x = \int R \partial x - \int R x \partial x,$$

ob $R = \frac{x^{n+p} - x^n}{l x}$ erit ex exemplo praecedente $\int R \partial x = l \frac{n+p+1}{n+1}$, atque indidem, loco n fcribendo $n+1$, erit $\int R x \partial x = l \frac{n+p+2}{n+2}$, ficque ipfe valor inuentus prodit.

Exemplum VII, quo $\mu = 3$ et $\nu = 1$.

Hoc ergo cafu erit

$$M = \frac{1}{2}\left(\frac{1}{n+p+1} - \frac{2}{n+p+2} + \frac{1}{n+p+3}\right)$$

hincque

$$\int M \partial p = \frac{1}{2} l \frac{n+p+1}{n+1} - \frac{2}{2} l \frac{n+p+2}{n+2} + \frac{1}{2} l \frac{n+p+3}{n+3};$$

at pro R habetur adhuc valor praecedens $R = \frac{x^{n+p} - x^n}{l x}$. Quare cum per propofitionem fupra allatam fit

$$\int \partial x \int \partial x \int R \partial x = \int \frac{R \partial x (1-x)^2}{1.2},$$

habebimus per fimplex fignum fummatorium

$$\int \frac{(1-x)^2 (x^{n+p} - x^n)}{l x} \partial x = l \frac{n+p+1}{n+1} - 2 l \frac{n+p+2}{n+2} + l \frac{n+p+3}{n+3}.$$

Exemplum VIII, quo $\mu = 4$ et $\nu = 1$.

Hoc cafu erit

$$M = \frac{1}{6}\left(\frac{1}{n+p+1} - \frac{3}{n+p+2} + \frac{3}{n+p+3} - \frac{1}{n+p+4}\right)$$

hincque

$$\int M \partial p = \frac{1}{6} l \frac{n+p+1}{n+1} - \frac{3}{6} l \frac{n+p+2}{n+2} + \frac{3}{6} l \frac{n+p+3}{n+3} - \frac{1}{6} l \frac{n+p+4}{n+4}.$$

Deinde cum vt ante fit $R = \frac{x^{n+p} - x^n}{l x}$, ob

$$\int \partial x \int \partial x \int \partial x \int R \partial x = \frac{1}{6} \int R \partial x (1-x)^3, \text{ erit}$$

$$\int \frac{(1-x)^3 (x^{n+p}-x^n)}{l x} \partial x = l\frac{n+p+1}{n+1} - 3\, l\frac{n+p+2}{n+2} + 3\, l\frac{n+p+3}{n+3} - l\frac{n+p+4}{n+4}.$$

Superfluum autem foret indici μ maiores valores tribuere, cum facta euolutione formulae $(1-x)^{\mu-1}$ ex exemplo V^{to} iidem valores essent produturi.

Exemplum IX, quo $\mu = 1$ et $\nu = 2$.

Hoc ergo casu erit $M = \frac{1}{n+p+1}$, hincque

$$\int M \partial p = l\frac{n+p+1}{n+1} \text{ et } \int \partial p \int M \partial p = (n+p+1)\, l\frac{n+p+1}{n+1} - p.$$

Facilius autem hic valor reperitur ope reductionis generalis

$$\int \partial p \int M \partial p = p \int M \partial p - \int M p \partial p;$$

namque ob $M = \frac{1}{n+p+1}$ erit $\int M \partial p = l\frac{n+p+1}{n+1}$, deinde vero ob $M p = \frac{p}{n+p-1} = 1 - \frac{n+1}{n+p+1}$, erit

$$\int M p \partial p = p - (n+1)\, l\frac{n+p+1}{n+1},$$

vnde colligitur

$$\int \partial p \int M \partial p = p\, l\frac{n+p+1}{n+1} + (n+1)\, l\frac{n+p+1}{n+1} - p,$$

vt ante. Tum vero erit

$$R = \frac{x^{n+p}}{(l x)^2} - x^n \left(\frac{1}{(l x)^2} + \frac{p}{l x}\right);$$

hinc cum sit $\int R \partial x = \int \partial p \int M \partial p$, erit

$$\int \frac{x^{n+p}}{(l x)^2} \partial x - \int x^n \left(\frac{1}{(l x)^2} + \frac{p}{l x}\right) \partial x = (n+p+1)\, l\frac{n+p+1}{n+1} - p.$$

Exemplum X, quo $\mu = 2$ et $\nu = 2$.

Hoc ergo casu erit $M = \frac{1}{n+p+1} - \frac{1}{n+p+2}$ hincque

$$\int M \partial p = l\frac{n+p+1}{n+1} - l\frac{n+p+2}{n+2},$$

et

et ob ſuperiorem reductionem hinc fit

$$M p = \frac{p}{n+p+1} - \frac{p}{n+p+2} = -\frac{n+1}{n+p+1} + \frac{n+2}{n+p+2},$$

ideoque

$$\int M p \partial p = -(n+1) l \frac{n+p+1}{n+1} + (n+2) l \frac{n+p+2}{n+2},$$

ita vt iam ſit

$$\int \partial p \int M \partial p = p l \frac{n+p+1}{n+1} - p l \frac{n+p+2}{n+2}$$
$$+ (n+1) l \frac{n+p+1}{n+1} - (n+2) l \frac{n+p+2}{n+2};$$

quare cum ſit $\int \partial x \int R \partial x = \int \partial p \int M \partial p$, ob

$$\int \partial x \int R \partial x = \int R \partial x - \int R x \partial x,$$

aequatio hinc oriunda fiet

$$\int \frac{(1-x) x^{n+p} \partial x}{(l x)^2} - \int (1-x) x^n \left(\frac{1}{(l x)^2} + \frac{p}{l x}\right) \partial x$$
$$= (n+p+1) l \frac{n+p+1}{n+1} - (n+p+2) l \frac{n+p+2}{n+2}.$$

Exemplum XI, quo $\mu = 3$ et $\nu = 2$.

Hoc ergo caſu eſt

$$M = \tfrac{1}{2}\left(\frac{1}{n+p+1} - \frac{2}{n+p+2} + \frac{1}{n+p+3}\right), \text{ hinc}$$

$$\int M \partial p = \tfrac{1}{2} l \frac{n+p+1}{n+1} - \tfrac{2}{2} l \frac{n+p+2}{n+2} + \tfrac{1}{2} l \frac{n+p+3}{n+3};$$

tum vero

$$M p = -\frac{\frac{1}{2}(n+1)}{n+p+1} + \frac{\frac{2}{2}(n+2)}{n+p+2} - \frac{\frac{1}{2}(n+3)}{n+p+3},$$

ideoque

$$\int M p \partial p = -\tfrac{1}{2}(n+1) l \frac{n+p+1}{n+1} + \tfrac{2}{2}(n+2) l \frac{n+p+2}{n+2}$$
$$- \tfrac{1}{2}(n+3) l \frac{n+p+3}{n+3};$$

conſequenter

$$\int \partial p \int M \partial p = \begin{cases} + \frac{1}{2}(n+p+1)\, l\frac{n+p+1}{n+1} \\ - \frac{2}{2}(n+p+2)\, l\frac{n+p+2}{n+2} \\ + \frac{1}{2}(n+p+3)\, l\frac{n+p+3}{n+3} \end{cases}.$$

Deinde vero manente R vt ante, quoniam fumto $x = 1$ in genere eft

$$\int \partial x \int \partial x \int R\, \partial x = \tfrac{1}{2} \int R\, \partial x\, (1 - x)^2,$$

hinc refultabit fequens aequatio:

$$\int \frac{(1-x)^2\, x^{n+p}\, \partial x}{(l\,x)^2} - \int (1-x)^2\, x^n \left(\frac{1}{(l\,x)^2} + \frac{p}{l\,x}\right) \partial x$$

$$= \begin{cases} + (n+p+1)\, l\frac{n+p+1}{n+1} \\ - (n+p+2)\, l\frac{n+p+2}{n+1} \\ + (n+p+3)\, l\frac{n+p+2}{n+3} \end{cases}.$$

Exemplum XII, quo $\mu = 1$ et $\nu = 3$.

Hoc igitur cafu erit $M = \frac{1}{n+p+1}$, et quia in genere eft

$$\int \partial p \int \partial p \int M\, \partial p = \tfrac{1}{2}\, p\, p \int M\, \partial p - \tfrac{2}{2}\, p \int M\, p\, \partial p + \tfrac{1}{2} \int M\, p\, p\, \partial p,$$

habebimus: $\int M\, \partial p = l\frac{n+p+1}{n+1}$,

$$\int M\, p\, \partial p = p - (n+1)\, l\frac{n+p+1}{n+1} \text{ et}$$

$$\int M\, p\, p\, \partial p = \tfrac{1}{2}\, p\, p - (n+1)\, p + (n+1)^2\, l\frac{n+p+}{n+1},$$

ex his colligitur

$$\int \partial p \int \partial p \int M \partial p = \tfrac{1}{2}(n+p+1)^2\, l\frac{n+p+1}{n+1} + \tfrac{3}{4}\, p\, p - \tfrac{1}{2}(n+1)\, p.$$

Deinde erit hic

$$R = \frac{x^{n+p}}{(l\,x)^3} - x^n \left(\frac{1}{(l\,x)^3} + \frac{p}{(l\,x)^2} + \frac{p\,p}{2\,l\,x}\right).$$

Hinc

Hinc igitur resultat sequens aequatio:

$$\int \frac{x^{n+p}\,\partial x}{(l\,x)} - \int x^n \left(\frac{1}{(l\,x)^3} + \frac{p}{(l\,x)^2} + \frac{pp}{2\,l\,x}\right)\partial x =$$

$$\tfrac{1}{2}(n+p+1)^2\, l\,\frac{n+p+1}{n+1} + \tfrac{3}{4}pp - \tfrac{1}{2}(n+1)p.$$

Exemplum XIII, quo $\mu = 2$ et $\nu = 3$.

Cum hoc casu sit $M = \frac{1}{n+p+1} - \frac{1}{n+p+2}$, ob

$$\int\partial p\int\partial p\int M\,\partial p = \tfrac{1}{2}pp\int M\,\partial p - \tfrac{2}{2}p\int Mp\,\partial p + \tfrac{1}{2}\int M\,pp\,\partial p,$$

quaeratur

$$\int M\,\partial p = l\,\tfrac{n+p+1}{n+1} - l\,\tfrac{n+p+2}{n+2}.$$

Porro ob $M\,p = -\frac{n+1}{n+p+1} + \frac{n+2}{n+p+2}$, erit

$$\int M\,p\,\partial p = -(n+1)\,l\,\tfrac{n+p+1}{n+1} + (n+2)\,l\,\tfrac{n+p+2}{n+2} \text{ et}$$

$$\int M\,p\,p\,\partial p = -(n+1)\,p + (n+1)^2\,l\,\tfrac{n+p+1}{n+1}$$
$$+ (n+2)\,p - (n+2)^2\,l\,\tfrac{n+p+2}{n+2},$$

vnde fit

$$\int p\,\partial\int p\,\partial\int M\,\partial p = \tfrac{1}{2}(n+p+1)^2\,l\,\tfrac{n+p+1}{n+1}$$
$$- \tfrac{1}{2}(n+p+2)^2\,l\,\tfrac{n+p+2}{n+2} + \tfrac{1}{2}p.$$

Deinde manente R vt supra erit $\int\partial x\int R\,\partial x = \int R\,\partial x\,(1-x)$, vnde colligimus:

$$\int\frac{(1-x)\,x^{n+p}\,\partial x}{(l\,x)^3} - \int(1-x)\,x^n\left(\frac{1}{(l\,x)^3} + \frac{p}{(l\,x)^2} + \frac{pp}{2\,l\,x}\right)\partial x$$
$$= \tfrac{1}{2}(n+p+1)^2\,l\,\tfrac{n+p+1}{n+1} - \tfrac{1}{2}(n+p+2)^2\,l\,\tfrac{n+p+2}{n+2} + \tfrac{1}{2}p.$$

Scholion.

Ad illuſtranda haec problemata loco V alia functione determinata, praeter $V = x^{n+p}$, vti non licuit, propterea quod alia huiusmodi forma non conſtat, cuius omnium ordinum integralia ex variabilitate ipſius x oriunda re ipſa exhiberi eorumque valores caſu $x = 1$ dari queant. Hic enim ob nullum plane vſum memorabilem reiici conuenit tales formas: $V = X + P$ et $V = XP$, vbi X ſignificaret functionem ipſius X tantum, P vero ipſius p tantum. Sin autem in vnica integratione ex ſola variabili x nata acquieſcere velimus, praeter formulam hactenus tractatam x^{n+p} etiam duae ſequentes in vſum vocari poſſunt:

$$V = \frac{x^{n+p-1} + x^{n-p-1}}{1 + x^{2n}} \text{ et } V = \frac{x^{n+p-1} - x^{n-p-1}}{1 - x^{2n}},$$

quandoquidem oſtendi, vtroque caſu valorem integralis $\frac{\int}{x} V$ ſiue $\int V \partial x$, caſu quo ponitur $x = 1$, admodum commode per functionem ſolius p exprimi poſſe, poſtquam ſcilicet integrale ita fuerit ſumtum, vt euaneſcat poſito $x = 0$. Iam dudum enim demonſtraui (*) ſub his conditionibus fore

$$\text{I. } \int \frac{x^{n+p-1} + x^{n-p-1}}{1 + x^{2n}} \partial x = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}.$$

$$\text{II. } \int \frac{x^{n+p-1} - x^{n-p-1}}{1 - x^{2n}} \partial x = -\frac{\pi}{2n} \text{tang.} \frac{\pi p}{2n}.$$

Quamobrem operae pretium erit bina problemata II et IV. etiam per has formulas illuſtrare. Ex vtroque ſcilicet problemate, ſumto indice $\mu = 1$, primo deduximus $Q = \frac{\int}{x} V$, tum vero poſito $x = 1$ fecimus $Q = M$, vnde caſu formulae prioris

(*) Videatur Diſſertatio Ill. Euleri: *De valore formulae integralis*

$$\int \frac{z^{m-1} \pm z^{n-m-1} \partial z}{1 \pm z^{2n}},$$

caſu quo poſt integrationem ponitur $z = 1$. Nouor. Comment. T. XIX.

ris perpetuo erit $M = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$, casu posterioris formulae $M = -\frac{\pi}{2n} \text{tang.} \frac{\pi p}{2n}$. Pro altera autem littera R in problemate secundo erat $R = \frac{\partial^\nu}{p} V$, vnde pro formula prima casu $\nu = 1$ erit $R = \frac{(x^{n+p-1} - x^{n-p-1})}{1 + x^{2n}} l x$, et pro posteriore

$$R = \frac{(x^{n+p-1} + x^{n-p-1})}{1 - x^{2n}} l x.$$

Deinde vero sumto $\nu = 2$, erit pro priore formula:

$$R = \frac{x^{n+p-1} + x^{n-p-1})}{1 + x^{2n}} (l x)^2$$ et pro posteriore

$$R = \frac{x^{n+p-1} - x^{n-p-1}}{1 - x^{2n}} (l x)^2.$$

Simili modo sumto $\nu = 3$ erit pro priore formula:

$$R = \frac{x^{n+p-1} - x^{n-p-1}}{1 + x^{2n}} (l x)^3;$$ pro posteriore vero

$$R = \frac{x^{n+p-1} + x^{n-p-1}}{1 - x^{2n}} (l x)^3.$$

Atque adeo in genere pro omni indice ν erit pro priore forma:

$$R = \frac{x^{n+p-1} \pm x^{n-p-1}}{1 + x^{2n}} (l x)^\nu,$$ pro posteriore vero

$$R = \frac{x^{n+p-1} \mp x^{n-p-1}}{1 - x^{2n}} (l x)^\nu.$$

Vbi signa superiora valent si ν numerus par, inferiora vero si impar.

Pro quarto autem problemate, vbi quantitas R per integrationes definiri debet, cum sit $R = \frac{\int^\nu}{p} V$, reperimus, sumto to $\nu = 1$, pro priore formula

$$R = \frac{x^{n+p-1} - x^{n-p-1}}{(1 + x^{2n})\, l x},$$

pro posteriore vero formula reperitur

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2 x^{n-1}}{(1 - x^{2n})\, l x}.$$

Sumto autem $\nu = 2$ habebimus pro formula priore

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2 x^{n-1}}{(1 + x^{2n})\, (l x)^2},$$ pro posteriore vero

$$R = \frac{x^{n+p-1} - x^{n-p-1}}{(1 - x^{2n})\, (l x)^2} - \frac{2 x^{n-1} p}{(1 - x^{2n})\, l x}$$ siue

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2 x^{n-1} . p\, l x}{(1 - x^{2n})\, (l x)^2}.$$

Deinde vero sumto $\nu = 3$, erit pro priore formula

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2 p\, x^{n-1}\, l x}{(1 + x^{2n})\, (l x)^3}$$

et pro posteriore formula:

$$R = \frac{x^{n+p-1} + x^{n-p-1} - p^2 x^{n-1} (l x)^2}{(1 + x^{2n})\, (l x)^3}.$$

Sumatur porro $\nu = 4$, ac reperiemus pro formula priore

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2 x^{n-1} - p\, p\, x^{n-1} (l x)^2}{(1 + x^{2n})\, (l x)^4},$$

pro posteriore vero

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2 p\, x^{n-1}\, l x - \frac{1}{3} p^3 x^{n-1} (l x)^3}{(1 - x^{2n})\, (l x)^4}.$$

Sumatur porro $\nu = 5$ ac habebimus pro priore formula:

R =

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2px^{n-1}\,lx - \frac{1}{3}p^3 x^{n-1}(lx)^3}{(1+x^{2n})(lx)^5} \text{ et}$$

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1} - ppx^{n-1}(lx)^2 - \frac{1}{12}p^4 x^{n-1}(lx)^4}{(1-x^{2n})(lx)^5}.$$

Sit $\nu = 6$, eritque

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1}(1 + \frac{1}{2}p(lx)^2 + \frac{1}{24}p^4(lx)^4)}{(1+x^{2n})(lx)^6},$$

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2x^{n-1}(p\,lx + \frac{1}{6}p^3(lx)^3 + \frac{1}{120}p^5(lx)^5)}{(1-x^{2n})(lx)^6},$$

et hinc lex iam ſatis elucet, qua ſequentes valores progrediuntur.

CONSIDERATIO AEQVATIONIS

$$\int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x} = \frac{\pi}{2n} \text{ ſec. } \frac{\pi p}{2n}.$$

Quod ſi hic breuitatis gratia ponamus $M = \frac{\pi}{2n}$ ſec. $\frac{\pi p}{2n}$, primo, caſu $x = 1$, ex problemate ſecundo deriuantur ſequentes aequalitates:

$$\text{I. } \int \frac{x^{n+p} - x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}\, lx = \frac{\partial M}{\partial p},$$

$$\text{II. } \int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^2 = \frac{\partial\partial M}{\partial p^2},$$

$$\text{III. } \int \frac{x^{n+p} - x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^3 = \frac{\partial^3 M}{\partial p^3},$$

$$\text{IV. } \int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^4 = \frac{\partial^4 M}{\partial p^4}.$$

etc.

 At

At vero ex problemate quarto prodeunt ſequentes aequalitates:

$$\text{I. } \int \frac{x^{n+p} - x^{n-p}}{1 + x^{2n}} \cdot \frac{\partial x}{x\, l x} = \int M \partial p,$$

$$\text{II. } \int \frac{x^{n+p} + x^{n-p} - 2x^n}{1 + x^{2n}} \cdot \frac{\partial p}{x (l x)^2} = \int \partial p \int M \partial p,$$

$$\text{III. } \int \frac{x^{n+p} - x^{n-p} - 2x^n \cdot p l x}{1 + x^{2n}} \cdot \frac{\partial x}{x (l x)^3} = \int \partial p \int \partial p \int M \partial p,$$

$$\text{IV. } \int \frac{x^{n+p} + x^{n-p} - 2x^n (1 + \frac{1}{2} p p (l x)^2)}{1 + x^{2n}} \cdot \frac{\partial x}{x (l x)^4}$$
$$= \int \partial p \int \partial p \int \partial p \int M \partial p,$$

$$\text{V. } \int \frac{x^{n+p} - x^{n-p} - 2x^n (p\, l x + \frac{1}{6} p^3 (l x)^3}{1 + x^{2n}} \cdot \frac{\partial x}{x (l x)^5}$$
$$= \int \partial p \int \partial p \int \partial p \int \partial p \int M \partial p$$

$$\text{VI. } \int \frac{x^{n+p} + x^{n-p} - 2x^n (1 + \frac{1}{2} p^2 (l x)^2 + \frac{1}{24} p^4 (l x)^4)}{1 + x^{2n}}$$
$$= \int \partial p \int \partial p \int \partial p \int \partial p \int \partial p \int M \partial p.$$

etc.

CONSIDERATIO AEQVATIONIS

$$\int \frac{x^{n+p} - x^{n-p}}{1 - x^{2n}} \cdot \frac{\partial x}{x} = -\frac{\pi}{2n} \operatorname{tang.} \frac{\pi p}{2n}.$$

Ponamus hic diſtinctionis gratia $N = -\frac{\pi}{2n} \operatorname{tang.} \frac{\pi p}{2n}$, atque ex problemate ſecundo naſcuntur ſequentes aequalitates:

$$\text{I. } \int \frac{x^{n+p} + x^{n-p}}{1 - x^{2n}} \cdot \frac{\partial x}{x} \cdot l x = + \frac{\partial N}{\partial p};$$

$$\text{II. } \int \frac{x^{n+p} - x^{n-p}}{1 - x^{2n}} \cdot \frac{\partial x}{x} (l x)^2 = \frac{\partial \partial N}{\partial p^2};$$

$$\text{III. } \int \frac{x^{n+p} + x^{n-p}}{1 - x^{2n}} \cdot \frac{\partial x}{x} (l x)^3 = \frac{\partial^3 N}{\partial p^3};$$

$$\text{IV. } \int \frac{x^{n+p} - x^{n-p}}{1 - x^{2n}} \cdot \frac{\partial x}{x} (l x)^4 = \frac{\partial^4 N}{\partial p^4};$$

etc.

Verum ex theoremate quarto ſequentes reſultant aequalitates:

$$\text{I. } \int \frac{x^{n+p} + x^{n-p} - 2 x^n}{1 - x^{2n}} \cdot \frac{\partial x}{x l x} = \int N \partial p;$$

$$\text{II. } \int \frac{x^{n+p} - x^{n-p} - 2 x^n . p l x}{1 - x^{2n}} \cdot \frac{\partial x}{x (l x)^2} = \int \partial p \int N \partial p;$$

$$\text{III. } \int \frac{x^{n+p} + x^{n-p} - 2 x^n (1 + \frac{1}{2} p p (l x)^2)}{1 - x^{2n}} \cdot \frac{\partial x}{x (l x)^3} = \int \partial p \int \partial p \int N \partial p;$$

$$\text{IV. } \int \frac{x^{n+p} - x^{n-p} - 2 x^n (p l x + \frac{1}{6} p^3 (l x)^3)}{1 - x^{2n}} \cdot \frac{\partial x}{x (l x)^4} = \int \partial p \int \partial p \int \partial p \int N \partial p;$$

$$\text{V. } \int \frac{x^{n+p} + x^{n-p} - 2 x^n (1 + \frac{1}{2} p p (l x)^2 + \frac{1}{24} p^4 (l x)^4)}{1 - x^{2n}} \cdot \frac{\partial x}{x (l x)^5} = \int \partial p \int \partial p \int \partial p \int \partial p \int N \partial p.$$

In his ſcilicet formulis quantitates M et N ſpectantur vt fractiones ipſius p, atque ex eius variabilitate tam differentiantur quam integrantur.

Ex his igitur abunde intelligitur, omnia quae ſuper hoc argumento a me non ita pridem ſunt prolata, tanquam caſus valde particulares in praeſenti tractatione contineri.

Scholion.

Formulae autem iftae fequenti modo fuccinctius exhiberi poffunt, ad quas intelligendas notetur in formulis ad finiftram pofitis valores integralium effe extendendas ab $x = 0$ ad $x = 1$, in formulis autem ad dextram pofitis quantitatem p fpectari vt variabilem et integralia ita capi, vt euanefcant pofito $p = 0$; tum vero loco $\frac{\pi}{2}$ hic litteram ϱ fcribi, ita vt ϱ fit character anguli recti. His igitur praenotatis ex integrali priori:

$$\int \frac{x^p + x^{-p}}{x^n + x^{-n}} \frac{\partial x}{x} = \frac{\varrho}{n} \text{ fec. } \frac{p\varrho}{n},$$

per differentiationem fequentia deducuntur:

$$\text{I. } \int \frac{x^p - x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x} l x = \frac{\varrho}{n \partial p} \partial \, . \text{ fec. } \frac{p\varrho}{n},$$

$$\text{II. } \int \frac{x^p + x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x} (l x)^2 = \frac{\varrho}{n \partial p^2} \partial \partial \, . \text{ fec. } \frac{p\varrho}{n},$$

$$\text{III. } \int \frac{x^p - x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x} (l x)^3 = \frac{\varrho}{n \partial p^3} \partial^3 \, . \text{ fec. } \frac{p\varrho}{n},$$

$$\text{IV. } \int \frac{x^p + x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x} (l x)^4 = \frac{\varrho}{n \partial p^4} \partial^4 \, . \text{ fec. } \frac{p\varrho}{n},$$

per integrationem vero fequentes aequalitates oriuntur:

$$\text{I. } \int \frac{x^p - x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x \, l x} = \frac{\varrho}{n} \int \partial p \text{ fec. } \frac{p\varrho}{n},$$

$$\text{II. } \int \frac{x^p + x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x (l x)^2} = \frac{\varrho}{n} \int \partial p \int \partial p \text{ fec. } \frac{p\varrho}{\varrho},$$

$$\text{III. } \int \frac{x^p - x^{-p} - 2 p \, l x}{x^n + x^{-n}} \cdot \frac{\partial x}{x (l x)^3} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \text{ fec. } \frac{p\varrho}{n},$$

IV.

IV. $\int \frac{x^p + x^{-p} - 2(1 + \frac{1}{2}p^2(lx)^2}{x^n + x^{-n}} \cdot \frac{\partial x}{x(lx)^4}$

$$= \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \text{ fec. } \frac{p\varrho}{n},$$

V. $\int \frac{x^p - x^{-p} - 2(p\, l x + \frac{1}{6}p^3(lx)^3)}{x^n + x^{-n}} \cdot \frac{\partial x}{x(lx)^5}$

$$= \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \int \partial p \text{ fec. } \frac{p\varrho}{n}.$$

etc.

Ex altera autem formula integrali principali:

$$\int \frac{x^p - x^{-p}}{x^n - x^{-n}} = \frac{\varrho}{n} \text{ tang. } \frac{p\varrho}{n},$$

per differentiationem nafcuntur fequentes aequationes:

I. $\int \frac{x^p + x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x} l x = \frac{\varrho}{n} \partial \,.\, \text{tang. } \frac{p\varrho}{n},$

II. $\int \frac{x^p - x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x} (lx)^2 = \frac{\varrho}{n} \partial\partial \,.\, \text{tang. } \frac{p\varrho}{n},$

III. $\int \frac{x^n + x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x} (lx)^3 = \frac{\varrho}{n} \partial^3 \,.\, \text{tang. } \frac{p\varrho}{n},$

IV. $\int \frac{x^p - x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x} (lx)^4 = \frac{\varrho}{n} \partial^4 \,.\, \text{tang. } \frac{p\varrho}{n},$

etc.

per differentiationem vero colliguntur fequentes:

I. $\int \frac{x^p + x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)} = \frac{\varrho}{n} \int \partial p \text{ tang. } \frac{p\varrho}{n},$

II. $\int \frac{x^p - x^{-p} - 2p\, l x}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^2} = \frac{\varrho}{n} \int \partial p \int \partial p \text{ tang. } \frac{p\varrho}{n},$

$$\text{III.}\ \int \frac{x^p + x^{-p} - 2\left(1 + \tfrac{1}{2}p^2 (l\,x)^2\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x\,(l\,x)^3}$$

$$= \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \ \text{tang.}\, \frac{p\,\varrho}{n},$$

$$\text{IV.}\ \int \frac{x^p - x^{-p} - 2\left(p\,l\,x + \tfrac{1}{6}p^3 (l\,x)^3\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x\,(l\,x)^4}$$

$$= \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \ \text{tang.}\, \frac{p\,\varrho}{n},$$

$$\text{V.}\ \int \frac{x^p + x^{-p} - 2\left(1 + \tfrac{1}{2}p^2 (l\,x)^2 + \tfrac{1}{24}p^4 (l\,x)^4\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x(l\,x)^5}$$

$$= \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \int \partial p \ \text{tang.}\, \frac{p\,\varrho}{n}.$$

Denique circa omnes has integrationes notari operae erit pretium, ſi integralia ad ſiniſtram poſita a termino $x = 0$ vsque ad $x = \infty$ extendantur, tum eorum valores duplo fieri maiores.

ANALY-