1789

Evolutio formulae integralis $\int \partial x \cdot (1/(1-x) + 1/\log(x))$ a termino $x=0$ ad $x=1$ extensae

Leonhard Euler

EVOLVTIO FORMVLAE INTEGRALIS

$$\int \partial x \left(\frac{1}{1-x} + \frac{1}{lx} \right)$$

A TERMINO $x = 0$ VSQVE AD $x = 1$ EXTENSAE.

Auctore
L. EVLERO.

Conuent. exhib. die 29 *Febr.* 1776.

§. 1.

Ista formula integralis eo magis est notatu digna, quod eius valorem ostendi conuenire cum eo, quem praebet ista expressio: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} - ln$, si numerus n sumatur infinite magnus, et quem per approximationem olim inveni esse $= 0,5772156649015325$, cuius valorem nullo adhuc modo ad mensuras transcendentes iam cognitas redigere potui; vnde haud inutile erit resolutionem huius formulae propositae pluribus modis tentare. Ac primo quidem, quoniam duabus constat partibus $\int \frac{\partial x}{1-x}$ et $\int \frac{\partial x}{lx}$, manifestum est prioris partis valorem $-l(1-x)$, posito $x = 1$, fore $-l0$ ideoque $= \infty$; tum vero etiam facile perspicitur, posterioris partis valorem quoque esse infinitum, sed signo contrario affectum, ita vt haud difficulter intelligatur aggregatum earum partium finitum habere valorem.

 Euo-

Euolutio prima geometrica.

§. 2. Primo igitur hanc formulam per quadraturas exhibeamus, confiderando lineam curuam, cuius abfciffae x refpondeat applicata $y = \frac{1}{1-x} + \frac{1}{lx}$, tum vero eius area $\int y \partial x$ abfciffae x infiftens ipfum valorem quaefitum repraefentabit, quamobrem formam huius curuae accuratius perpendamus. Ac primo quidem euidens eft, hanc curuam neutiquam in regionem abfciffarum negatiuarum porrigi, fed a termino $x = 0$ incipere. Pofito autem $x = 0$ manifefto fit $y = 1$, ob $lx = \infty$; at exiftente x infinite paruo fiet $y = 1 + x + \frac{1}{lx}$, vbi facile perfpicitur poftremum membrum $\frac{1}{lx}$ effe negatiuum et quafi infinities maius quam x, ita vt fiat $y = 1 - ix$, exiftente i numero maximo; vnde patet, fi curuam ad axem A O referamus in eoque abfciffas x a puncto A capiamus, in ipfo puncto A applicatam fore A C $= 1$, et curuam in C hanc applicatam A C tangere, propterea quod decrementum applicatae infinities fuperat incrementum abfciffae. Curua igitur originem ducet ab ipfo puncto C, hincque continuo propius ad axem inflectetur, quem tandem in diftantia infinita attinget. Pofito enim $x = \infty$ fit $y = -\frac{1}{\infty} + l\frac{1}{\infty}$; vbi notetur prius membrum $\frac{1}{\infty}$ prae altero euanefcere, ita vt ifte valor fit pofitiuus, vnde patet, hanc curuum a puncto C ad axem continuo propius effe acceffuram.

Tab. I. Fig. 1.

§. 3. Confideremus nunc abfciffam A B $= 1$, vbi fumto $x = 1$ fit $y = \frac{1}{0} + \frac{1}{0}$, vnde nihil plane concludere liceret, hanc ob cauffam ftatuamus $x = 1 - \omega$, vt fiat $y = \frac{1}{\omega} + \frac{1}{l(1-\omega)}$. Iam $l(1-\omega)$ in feriem euoluendo fiet

$$y = \frac{1}{\omega} - \frac{1}{\omega + \frac{1}{2}\omega^2 + \frac{1}{3}\omega^3 + \text{etc.}} = \frac{\frac{1}{2} + \frac{1}{3}\omega}{1 + \frac{1}{2}\omega + \frac{1}{3}\omega^2}.$$

Fiat

numeratore loco $l\,x$ ſcribamus, prodibitque

$$y = \frac{-\frac{1}{2}(1-x)^2 - \frac{1}{3}(1-x)^3 - \frac{1}{4}(1-x)^4 - \text{etc.}}{(1-x)\,l\,x}$$

atque hinc

$$y = \frac{-\frac{1}{2}(1-x) - \frac{1}{3}(1-x)^2 - \frac{1}{4}(1-x)^3 - \text{etc.}}{l\,x},$$

hinc igitur per partes integrando valor quaeſitus erit

$$\int y\,\partial x = -\tfrac{1}{2}\int\frac{(1-x)\,\partial x}{l\,x} - \tfrac{1}{3}\int\frac{(1-x)^2\,\partial x}{l\,x} - \tfrac{1}{4}\int\frac{(1-x)^3\,\partial x}{l\,x} - \text{etc.}$$

quae formulae ſingulae facile ad formulam illam generalem reducuntur, qua oſtendi eſſe

$$\int\frac{x^m - x^n}{l\,x}\,\partial x = l\,\frac{m+1}{n+1}.\ (*)$$

Hinc enim ſtatim erit $\int\frac{(1-x)\,\partial x}{l\,x} = l\,\frac{1}{2}$, et quia eſt

$(1-x)^2 = 1 - x - (x - x\,x)$, erit

$$\int\frac{(1-x)^2\,\partial x}{l\,x} = l\,\tfrac{1}{2} - l\,\tfrac{2}{3} = l\,\frac{1.3}{2^2}.$$

Simili modo facile patebit fore

$$\int\frac{(1-x)^3\,\partial x}{l\,x} = l\,\frac{1.3^3}{2^3.4};$$

$$\int\frac{(1-x)^4\,\partial x}{l\,x} = l\,\frac{1.3^6.5}{2^4.4^4};$$

$$\int\frac{(1-x)^5\,\partial x}{l\,x} = l\,\frac{1.3^{10}.5^5}{2^5.4^{10}.6};$$

$$\int\frac{(1-x)^6\,\partial x}{l\,x} = l\,\frac{1.3^{15}.5^{15}.7}{2^6.4^{20}.6^6};\ \text{etc.}$$

§. 6. Ex his igitur valor noſtrae formulae $\int y\,\partial x$ per ſeriem logarithmicam prorſus ſingularem ſequenti modo exprimetur:

$$\int y\,\partial x = \tfrac{1}{2}\,l\,2 + \tfrac{1}{3}\,l\,\frac{2^2}{1.3} + \tfrac{1}{4}\,l\,\frac{2^3.4}{1.3^3} + \tfrac{1}{5}\,l\,\frac{2^4.4^4}{1.3^6.5}$$
$$+ \tfrac{1}{6}\,l\,\frac{2^5.4^{10}.6}{1.3^{10}.5^5} + \tfrac{1}{7}\,l\,\frac{2^6.4^{20}.6^6}{1.3^{15}.5^{15}.7} + \text{etc.}$$

Vbi

(*) Hoc integrale duplici modo ab Ill. huius diſſertationis Auctore fuit inventum in Tomo XIX. Nouorum Commentariorum pag. 70 et 79. F.

Vbi probe notandum eſt omnes logarithmos capi debere hyperbolicos; facile autem intelligitur, terminos huius ſerici continuo prodire minores, neque tamen hanc ſeriem tantopere conuergere, vt ex ea valor quaeſitus commode computari poſſit.

Euolutio tertia.

§. 7. Vtamur eadem reſolutione logarithmi x in ſeriem infinitam, ac ponamus breuitatis gratia $1 - x = t$, vt ſit

$$l\,x = -t - \tfrac{1}{2}tt - \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4 - \text{etc. eritque}$$

$$\frac{1}{l\,x} = \frac{-1}{t(1 + \frac{1}{2}t + \frac{1}{3}tt + \frac{1}{4}t^3 + \frac{1}{5}t^4 + \text{etc.})}.$$

Iam fractionem $\dfrac{1}{1 + \frac{1}{2}t + \frac{1}{3}tt + \text{etc.}}$ conuertamus more ſolito in ſeriem recurrentem, quae ſit

$$1 + \alpha t + \beta tt + \gamma t^3 + \delta t^4 + \varepsilon t^5 + \zeta t^6 + \text{etc.}$$

vbi coëfficientes α, β, γ, δ, etc. ita erunt comparati, vt ſit

$$\alpha + \tfrac{1}{2} = 0, \quad \text{hincque} \quad \alpha = -\tfrac{1}{2},$$
$$\beta + \tfrac{1}{2}\alpha + \tfrac{1}{3} = 0, \qquad \beta = -\tfrac{1}{12},$$
$$\gamma + \tfrac{1}{2}\beta + \tfrac{1}{3}\alpha + \tfrac{1}{4} = 0, \qquad \gamma = -\tfrac{1}{24},$$
$$\delta + \tfrac{1}{2}\gamma + \tfrac{1}{3}\beta + \tfrac{1}{4}\alpha + \tfrac{1}{5} = 0, \qquad \delta = -\tfrac{19}{720},$$
$$\text{etc.} \qquad \text{etc.}$$

vnde hanc ſeriem tanquam cognitam ſpectare licet.

§. 8. Hoc igitur valore ſubſtituto erit

$$\frac{1}{l\,x} = -\frac{1}{t} - \alpha - \beta t - \gamma tt - \delta t^3 - \varepsilon t^4 + \text{etc.},$$

quare cum ſit $\frac{1}{1-x} = \frac{1}{t}$, erit

$$y = -\alpha - \beta t - \gamma tt - \delta t^3 - \varepsilon t^4 - \text{etc. ſiue}$$
$$y = -\alpha - \beta(1-x) - \gamma(1-x)^2 - \delta(1-x)^3 - \text{etc.}$$

Cum

Cum nunc in genere fit

$$\int \partial x\,(1-x)^n = C - \frac{(1-x)^{n+1}}{n+1} = \frac{1}{n+1} - \frac{(1-x)^n}{n+1},$$

pofito $x = 1$, quemadmodum affumfimus, erit

$$\int \partial x\,(1-x)^n = \frac{1}{n+1}.$$

Hinc igitur fingulis integralibus collectis reperietur

$$\int y\,\partial x = -\frac{\alpha}{2} - \frac{\beta}{3} - \frac{\gamma}{4} - \frac{\delta}{5} - \frac{\varepsilon}{6} - \text{etc.},$$

vnde per valores ante euolutos fiet

$$\int y\,\partial x = \frac{1}{4} + \frac{1}{36} + \frac{1}{96} + \frac{19}{3600} + \text{etc.}$$

quae feries vtique parum eft conuergens.

Euolutio quarta.

§. 9. Cum habeamus $y = \frac{lx + 1 - x}{(1-x)\,lx}$, quemadmodum ante partem lx in feriem infinitam refoluimus, ita nunc viciffim ipfam quantitatem in feriem per logarithmos ipfius x procedentem euoluamus. Quia enim eft $x = e^{lx}$, erit

$$x = 1 + lx + \tfrac{1}{2}(lx)^2 + \tfrac{1}{6}(lx)^3 + \tfrac{1}{24}(lx)^4 + \text{etc.}$$

vbi loco lx breuitatis ergo fcribamus u, atque hanc feriem tantum in numeratorem introducamus, vt fiat

$$y = \frac{-\tfrac{1}{2}uu - \tfrac{1}{6}u^3 - \tfrac{1}{24}u^4 - \tfrac{1}{120}u^5 - \text{etc.}}{u(1-x)}, \text{ fiue}$$

$$y = \frac{-\tfrac{1}{2}u - \tfrac{1}{6}uu - \tfrac{1}{24}u^3 - \tfrac{1}{120}u^4 - \text{etc.}}{1-x},$$

ideoque

$$\int y\,\partial x = -\tfrac{1}{2}\int\frac{\partial x\,lx}{1-x} - \tfrac{1}{6}\int\frac{\partial x\,(lx)^2}{1-x} - \tfrac{1}{24}\int\frac{\partial x\,(lx)^3}{1-x} - \tfrac{1}{120}\int\frac{\partial x\,(lx)^4}{1-x} - \text{etc.}$$

§. 10.

§. 10. Cum nunc in genere, ſumto ſcilicet integrali ab $x = 0$ ad $x = 1$, ſit $\int \partial x (l x)^n = \pm 1 . 2 . 3 . 4 . 5 . \ldots . n$, vbi ſignum $+$ valet quando n eſt numerus par, contra vero ſignum $-$, erit porro

$$\int x^{n-1} \partial x (l x)^\lambda = \pm \frac{1 . 2 . 3 . 4 . 5 . \ldots . \lambda}{n^{\lambda+1}},$$

vbi ſignum $+$ valet ſi λ fuerit numerus par, inferius vero ſi impar. Hinc igitur ſingulas noſtras formulas per ſeries integremus, dum loco $\frac{1}{1-x}$ ſeriem ſcribimus

$$1 + x + x x + x^3 + x^4 + x^5 + \text{etc.}$$

atque hinc primo nanciſcemur

$$\int \frac{\partial x \, l x}{1-x} = -1 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.}\right)$$

$$\int \frac{\partial x (l x)^2}{1-x} = 1 . 2 \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \text{etc.}\right)$$

$$\int \frac{\partial x (l x)^3}{1-x} = -1 . 2 . 3 \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.}\right)$$

$$\int \frac{\partial x (l x)^4}{1-x} = 1 . 2 . 3 . 4 \left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \frac{1}{6^5} + \text{etc.}\right)$$

etc. etc.

His igitur ſeriebus ſubſtitutis reperiemus

$$\int y \partial x = \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.}\right)$$
$$- \frac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.}\right)$$
$$+ \frac{1}{4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.}\right)$$
$$- \frac{1}{5}\left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.}\right)$$

cuius expreſſionis iam nuper oſtendi valorem eſſe numerum illum memorabilem 0, 5772156649015325. (*)

Euo-

(*) V. Diſſertatio: *De numero memorabili in ſummatione progreſſionis harmonicae naturalis occurrente.* Acta Acad. pro Anno 1781. Pars poſterior, pag. 49. ſeqq. F.

Euolutio quinta.

§. 11. Vtamur hic eadem resolutione in seriem ipsius numeri x, sed eam alio modo adhibeamus. Scilicet cum posito $l\,x = u$ sit

$$x = 1 + u + \tfrac{1}{2}\,u\,u + \tfrac{1}{6}\,u^3 + \tfrac{1}{24}\,u^4 + \tfrac{1}{120}\,u^5 + \text{etc.}$$

erit formulae nostrae ipsa pars prior

$$\frac{1}{1 - x} = \frac{-1}{u + \tfrac{1}{2}\,u\,u + \tfrac{1}{6}\,u^3 + \tfrac{1}{24}\,u^4 + \tfrac{1}{120}\,u^5 \text{ etc.}} =$$

$$-\frac{1}{u}\cdot\frac{1}{1 + \tfrac{1}{2}\,u + u\,u + \tfrac{1}{24}\,u^3 + \text{etc.}}.$$

Hanc fractionem:

$$\frac{1}{1 + \tfrac{1}{2}\,u + \tfrac{1}{6}\,u\,u + \tfrac{1}{24}\,u^3 + \text{etc.}}$$

more solito in seriem recurrentem conuertamus, quae sit

$$1 - A\,u + B\,u\,u - C\,u^3 + D\,u^4 - E\,u^5 + \text{etc.}$$

eritque facta comparatione:

$A = \frac{1}{2}$ ergo $A = \frac{1}{2}$.

$B = \frac{1}{2}A - \frac{1}{6}$ $B = \frac{1}{12}$

$C = \frac{1}{2}B - \frac{1}{6}A + \frac{1}{24}$ $C = 0$

$D = \frac{1}{2}C - \frac{1}{6}B + \frac{1}{24}A - \frac{1}{120}$ $D = -\frac{1}{720}$

$E = \frac{1}{2}D - \frac{1}{6}C + \frac{1}{24}B - \frac{1}{120}A + \frac{1}{720}$ $E = 0.$

§. 12. Hac igitur serie introducta et loco u restituto valore $l\,x$, formula nostra $\frac{1}{1-x} + \frac{1}{l\,x} = y$ sequentem induet formam:

$$y = -\tfrac{1}{u} + A - B\,u + C\,u\,u - D\,u^3 + \text{etc.} + \tfrac{1}{u}, \text{ siue}$$

$$y = A - B\,l\,x + C\,(l\,x)^2 - D\,(l\,x)^3 + E\,(l\,x)^4 - \text{etc.}$$

vnde cum in genere sit

$\int\partial x$

$$\int \partial x\,(l\,x)^n = \pm 1.\ 2.\ 3.\ 4.\ \ldots\ n,$$

poſtquam ſcilicet abſoluta integratione poſitum fuerit $x = 1$, vbi ſignum ſuperius valet, quando n eſt numerus par, inferius vero ſi n impar: hoc obſeruato nanciſcimur valorem quaeſitum $\int y\,\partial x$ ſequenti modo expreſſum:

$$\int y\,\partial x = A + 1B + 1.2\,C + 1.2.3\,D + 1.2.3.4\,E + 1\ldots 5\,F + \text{etc.}$$

quae ſeries vtique parum conuergit, ob coëfficientes litterarum A, B, C, D; verum perpendendum eſt, ipſos valores harum litterarum continuo magis decreſcere, quandoquidem certum eſt, feriei valorem eſſe debere 0,5772156649015325, quare operae pretium erit harum litterarum ſeriem accuratius euoluere.

TRANSFORMATIO FRACTIONIS

$$\frac{1}{1 + \frac{1}{2}u + uu + \frac{1}{24}u^3 + u^4 + \text{etc.}}$$

IN SERIEM

$$1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + \text{etc.}$$

§. 13. Deſignet littera s ſummam huius ſeriei, eritque $s = \frac{u}{e^u - 1}$, vnde fit $e^u = \frac{u+s}{s}$, hincque $u = l\,(u+s) - l\,s$, ergo differentiando erit

$$\partial u = \frac{\partial u + \partial s}{u+s} - \frac{\partial s}{s} = \frac{s\,\partial u - u\,\partial s}{s\,(u+s)},$$

ſiue ſtatim ponatur $s = p\,u$, vt fit $u = l\,\frac{1+p}{p}$, vnde fit $\partial u = -\frac{\partial p}{p\,(p+1)}$, quae expreſſio quo ſeriem praebeat concinniorem, ſtatuamus $p = q - \frac{1}{2}$, vt iam fit $s = (q - \frac{1}{2})\,u$, tum vero erit $\partial u = -\frac{\partial q}{qq - \frac{1}{4}}$, vnde colligitur haec aequatio:

$$qq - \tfrac{1}{4} + \tfrac{\partial q}{\partial u} = 0.$$

 §. 14.

§. 14. Ex hac igitur aequatione inueſtigari debet ſeries valorem ipſius q exhibens, vbi ante omnia principium huius ſeriei inde conſtitui oportet, quod poſito $u = 0$ fieri debeat $s = 1$ et $q = \frac{1}{u} + \frac{1}{2}$; vnde patet ſeriei pro q fingendae primum terminum eſſe debere $\frac{1}{u}$; tum vero facile perſpicitur in hac ſerie poteſtates ipſius u tantum impares aſſumi debere. Quamobrem fingatur iſta ſeries:

$$q = \frac{1}{u} + a u + b u^3 + c u^5 + d u^7 + e u^9 + \text{etc. eritque}$$

$$\begin{aligned} qq = \frac{1}{uu} &+ 2a + 2bu^2 + 2cu^4 + 2du^6 + 2eu^8 + 2fu^{10} + \text{etc.}\\ &+ aa + 2ab + 2ac + 2ad + 2ae + \text{etc.}\\ &\qquad + bb + 2bc + 2bd + \text{etc.}\\ &\qquad\qquad + cc \end{aligned}$$

$$\frac{\partial q}{\partial u} = -\frac{1}{uu} + a + 3b + 5c + 7d + 9e + 11f + \text{etc.}$$

harum ergo ſerierum ſumma debet eſſe $\frac{1}{4}$, vnde deducuntur ſequentes determinationes:

$3a = \frac{1}{4}$	ergo $a = \frac{1}{12}$
$5b + aa = 0$	$b = -\frac{aa}{5}$
$7c + 2ab = 0$	$c = -\frac{2ab}{7}$
$9d + 2ac + bb = 0$	$d = -\frac{2ac - bb}{9}$
$11e + 2ad + 2bc = 0$	$e = -\frac{2ad - 2bc}{11}$
$13f + 2ae + 2bd + cc = 0$	$f = -\frac{2ae - 2bd - cc}{13}$
etc.	etc.

ex quibus formulis valores numerici litterarum a, b, c, d, computari poterunt.

§. 15. His autem litteris a, b, c, d, etc. definitis ipſa ſeries pro s quaeſita erit

$s = 1$

$$s = 1 - \tfrac{1}{2}u + auu + bu^4 + cu^6 + du^8 + eu^{10} + fu^{12} + gu^{14} + \text{etc.}$$

quare cum fupra pofuerimus

$$s = 1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + Fu^6 - Gu^7 + Hu^8 - \text{etc.}$$

valores harum litterarum maiuscularum per minusculas fequenti modo definientur:

$$A = \tfrac{1}{2},\ B = a,\ C = 0,\ D = b,\ E = 0,\ F = c,\ G = 0,\ H = d,\ \text{etc.}$$

ficque poteftatum imparium coëfficientes per fe euanefcunt. Evidens autem eft ope formularum hic inuentarum valores litterarum a, b, c, d, etc. multo facilius et promptius affignari poffe quam per relationes fupra allatas, fcilicèt erit $A = \frac{1}{2}$, $B = a = \frac{1}{12}$, $C = 0$, $D = -\frac{1}{720}$, $E = 0$, $F = \frac{1}{30240}$, etc.

§. 16. Quoniam hoc modo calculus iftarum litterarum mox ad numeros vehementer magnos deduceret, loco litterarum a, b, c, d, etc. quaeramus alias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. quarum figna iam alternentur et quarum valores ad illos fequenti modo referantur:

$$a = \frac{\mathfrak{A}}{12},\ b = -\frac{\mathfrak{B}}{12^2},\ c = +\frac{\mathfrak{C}}{12^3},\ d = -\frac{\mathfrak{D}}{12^4},\ e = +\frac{\mathfrak{E}}{12^5},\ \text{etc.}$$

ita vt iam fit noftra feries

$$s = 1 - \tfrac{1}{2}u + \frac{\mathfrak{A}uu}{12} - \frac{\mathfrak{B}u^4}{12^2} + \frac{\mathfrak{C}u^6}{12^3} - \frac{\mathfrak{D}u^8}{12^4} + \text{etc.}$$

atque iftae nouae litterae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ fequenti modo determinabuntur:

$$\mathfrak{A} = 1 \qquad \mathfrak{D} = \frac{2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}}{9},$$

$$\mathfrak{B} = \frac{\mathfrak{A}\mathfrak{A}}{5} \qquad \mathfrak{E} = \frac{2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}}{11},$$

$$\mathfrak{C} = \frac{2\mathfrak{A}\mathfrak{B}}{7} \qquad \mathfrak{F} = \frac{2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}}{13},\ \text{etc.}$$

qui valores nunc haud difficulter in numeris euoluentur ac reperientur:

$$\mathfrak{A} = 1,\ \mathfrak{B} = \tfrac{1}{5},\ \mathfrak{C} = \tfrac{2}{35},\ \mathfrak{D} = \tfrac{3}{175},\ \mathfrak{E} = \tfrac{2}{385},\ \mathfrak{F} = \tfrac{1382}{875875},\ \text{etc.}$$

§. 17. Introducamus igitur iſtas nouas litteras $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. in ſeriem §. 12. pro $\int y\,\partial x$ inuentam eritque

$$\int y\,\partial x = \frac{1}{2} + \frac{1.\mathfrak{A}}{12} - \frac{1.2.3\,\mathfrak{B}}{12^2} + \frac{1.2.3.4.5\,\mathfrak{C}}{12^3} - \frac{1.\ldots\ldots 7\,\mathfrak{D}}{12^4} + \text{etc.}$$

hanc autem ſeriem non ſatis conuergere iam ſupra obſeruauimus.

TRANSFORMATIO FRACTIONIS

$$\frac{1}{1 + \frac{1}{2}t + \frac{1}{3}tt + \frac{1}{4}t^3 + \frac{1}{5}t^4 + \text{etc}}$$

IN SERIEM

$$1 + \alpha t + \beta tt + \gamma t^3 + \delta t^4 + \varepsilon t^5 + \zeta t^6 + \text{etc.}$$

§. 17. Ad hanc transformationem perducti ſumus ſupra in §. 7. vbi euolutio litterarum α, β, γ, δ, etc. mox fiebat nimis moleſta. Nunc igitur ſimili modo vtamur quo ante, poſitaque hac ſerie quam quaerimus $= s$, erit $s = \frac{-t}{l(1-t)}$, ideoque $l(1-t) = -\frac{t}{v}$, ergo differentiando $-\frac{\partial t}{1-t} = \frac{\partial v}{vv}$, ſeu $\frac{\partial v}{vv} + \frac{\partial t}{1-t} = 0$, cui hanc formam tribuamus:

$$vv + \frac{\partial v}{\partial t}(1-t) = 0,$$

ex qua aequatione ſeries idonea pro v elici debet.

§. 18. Cum igitur poſito $t = 0$ fiat $s = 1$, hoc caſu eſſe debebit $v = \frac{t}{t}$, quamobrem fingamus iſtam ſeriem:

$$v = \frac{1}{t} + a + bt + ctt + dt^3 + et^4 + \text{etc.},$$

qui valor ſequenti modo ſubſtituatur

∂v

$$\frac{\partial v}{\partial t} = -\frac{1}{tt} + * + b + 2ct + 3dtt + 4et^3 + 5ft^4 + 6gt^5 + \text{etc.}$$
$$-\frac{t\,\partial v}{\partial t} = +\frac{1}{t} - * - b - 2c - 3d - 4e - 5f - \text{etc.}$$
$$+vv = +\frac{1}{tt} + \frac{2a}{t} + 2b + 2c + 2d + 2e + 2f + 2g + \text{etc.}$$
$$+aa + 2ab + 2ac + 2ad + 2ae + 2af + \text{etc.}$$
$$+bb + 2bc + 2bd + 2be + \text{etc.}$$
$$+cc + 2cd + \text{etc.}$$

Hinc igitur prodeunt ſequentes determinationes:

$$1 + 2a = 0,$$
$$3b + aa = 0,$$
$$4c + 2ab - b = 0,$$
$$5d + 2ac - 2c + bb = 0,$$
$$6e + 2ad + 2bc - 3d = 0,$$
$$7f + 2ae + 2bd - 4e + cc = 0,$$

quae formulae ob $a = -\frac{1}{2}$ contrahuntur in ſequentes:

$$3b = -\tfrac{1}{4} \qquad \text{ergo } a = -\tfrac{1}{2}$$
$$4c = 2b \qquad b = -\tfrac{1}{12}$$
$$5d = 3c - bb \qquad c = -\tfrac{1}{24}$$
$$6e = 4d - 2bc \qquad d = -\tfrac{19}{720}$$
$$7f = 5e - 2bd - cc \qquad e = -\tfrac{3}{160}$$
$$8g = 6f - 2be - 2c\partial \qquad f = -\tfrac{827}{32.2{,}0.7}.$$

§. 19. Hinc igitur erit ſeries quaeſita

$$s = 1 + at + btt + ct^3 + dt^4 + \text{etc.},$$

quae ſupra poſita fuerat

$$s = 1 + \alpha t + \beta tt + \gamma t^3 + \delta t^4 + \text{etc.}$$

litterae igitur latinae et graecae prorſus conueniunt, eritque ergo

$$\int y\,\partial x = -\frac{a}{2} - \frac{b}{3} - \frac{c}{4} - \frac{d}{5} - \frac{e}{6} - \text{etc.}$$

et

et valoribus modo inuentis ſubſtitutis

$$\int y\,\partial x = \frac{1}{2.2} + \frac{1}{3.12} + \frac{1}{4.24} + \frac{19}{5.720} + \frac{3}{6.160} + \text{etc.}$$

§. 20. Quo autem calculus harum litterarum expeditior reddatur, ponamus $a = -\frac{A}{2}$, $b = -\frac{B}{4}$, $c = -\frac{C}{8}$, $d = -\frac{D}{16}$, $e = -\frac{E}{32}$, etc. vt ſit

$$\int y\,\partial x = \frac{A}{2.2} + \frac{B}{3.4} + \frac{C}{4.8} + \frac{D}{5.16} + \frac{E}{6.32} + \text{etc.}$$

pro his autem litteris habebimus ſequentes determinationes:

$A = 1$,	$E = \frac{8D + 2BC}{6}$,
$B = \frac{1}{3}$,	$F = \frac{10E + 2BD + CC}{7}$,
$C = \frac{4B}{4}$,	$G = \frac{12F + 2BE + 2CD}{8}$,
$D = \frac{6C + BB}{5}$,	etc.

vnde colligitur

$$A = 1,\; B = \frac{1}{3},\; C = \frac{1}{3},\; D = \frac{19}{45},\; E = \frac{32}{50}, \text{ etc.}$$

Haec igitur ad calculos ſuperiores ſubleuandos ſufficere poterunt.

VBE-

Fig. 1.

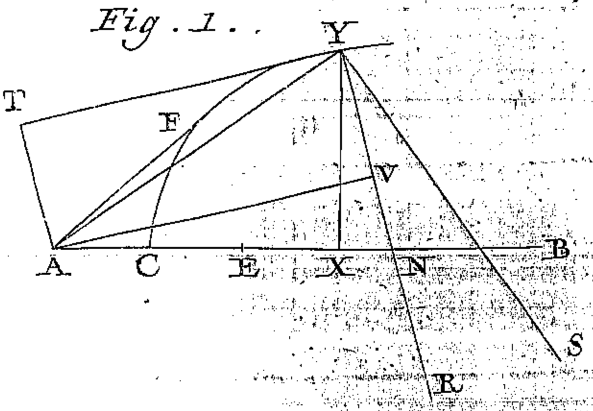

Fig. 2.

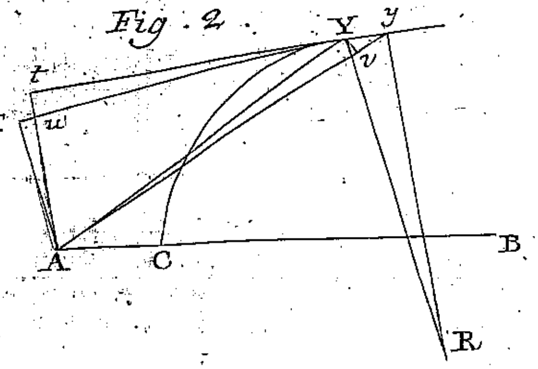

Fig. 3.

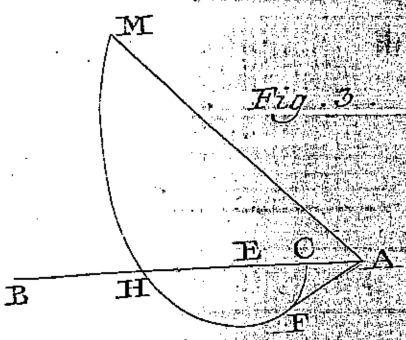

Fig. 4.

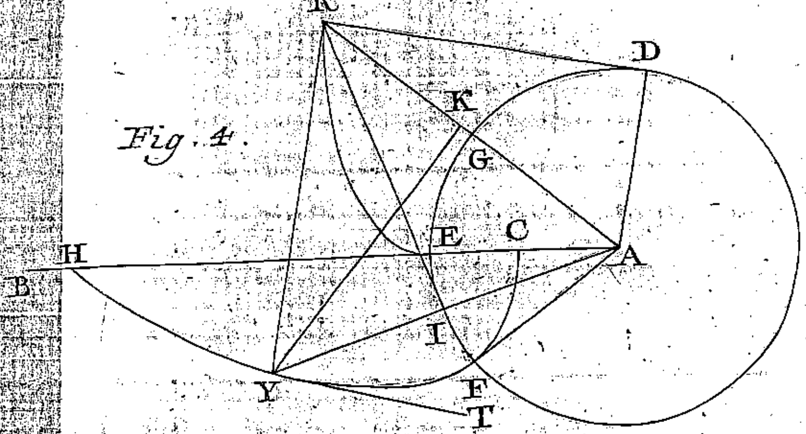

Fig. 5.

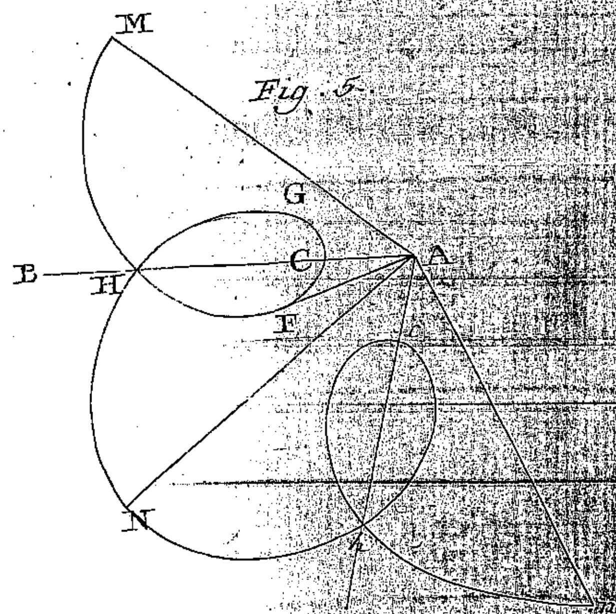

Fig. 6.