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De viribus centripetis, ad curvas non in eodem plano sitas describendas, requisitis

Leonhard Euler

DE VIRIBVS CENTRIPETIS, AD CVRVAS NON IN EODEM PLANO SITAS DESCRIBENDAS, REQVISITIS.

Auctore
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§. 1.

Notissimum est problema & a summo *Nevtono* egregie solutum, quo vis centripeta ad datum punctum fixum tendens quaeritur, a qua si corpus sollicitetur, per datam lineam curuam promoueatur. Euidens autem est in enunciatione huius problematis tacite assumi, tam totam curuam a corpore describendam in eodem plano sitam esse debere, quam ipsum centrum virium in eodem plano esse statuendum, quando quidem certum est, omne corpus ab vnica vi centripeta sollicitatum semper in eodem plano moveri, quod per ipsum centrum virium transeat. Hinc ergo patet, si curva a corpore describenda non fuerit in eodem plano sita, tum omnino fieri non posse vt talis motus ab vnica vi centripeta producatur, sed ad minimum duo centra virium diuersa constitui debere. Ostendam igitur hic, talia duo centra virium semper pro arbitrio accipi posse, quomodocunque curua descripta extra planum diuagetur, ac perpetuo illas binas vires centripetas ita determinari posse, vt corpus ab iis sollicitatum per curuam propositam promoueri queat.

§. 2.

Tab. III. Fig. 1. §. 2. Sit igitur propoſita curua quaecunque A Z, cuius punctum quoduis Z vtcunque extra planum tabulae ſit poſitum, cuius ergo locus more ſolito per ternas coordinatas $CX = x$, $XY = y$ et $YZ = x$ definiatur, inter quas igitur ob curuam datam duplex relatio detur neceſſe eſt, vnde pro quauis abſciſſa x tam valor ipſius y quam ipſius z aſſignari poſſit. Quin etiam, introducendo ipſius curuae elementum, quod ſit $= \partial s$, poni poterit $\partial x = p \partial s$, $\partial y = q \partial s$ et $\partial z = r \partial s$, ita vt ſit $pp + qq + rr = 1$, et quouis caſu valores harum litterarum vt cogniti ſpectari poterunt.

§. 3. Conſtituatur igitur alterum centrum virium in ipſo puncto C ac ponatur ab eo diſtantia puncti Z ſcilicet $CZ = v$, ita vt ſit $vv = xx + yy + zz$; tum vero denotet V ipſam vim centripetam qua corpus in Z verſus iſtud centrum C vrgeri debet, ex qua ergo ſecundam directiones coordinatarum reſultabunt ternae vires, quae erunt

ſecundum $XC = \frac{Vx}{v}$,

ſecundum $YX = \frac{Vy}{v}$ et

ſecundum $ZY = \frac{Vz}{v}$.

§. 4. Alterum autem centrum virium C′ vbicunque in plano quidem tabulae accipiatur, pro quo ſi punctum Z pariter per ternas coordinatas prioribus parallelas referatur, vocentur eae $C'X' = x'$, $X'Y = y'$ et $YZ = z$ vt ante, quae ergo a praecedentibus tantum quantitate conſtante diſcrepabunt, ita vt ſit $x' = x + a$, et $y' = y + b$. Quod ſi vero centrum C′ extra planum tabulae accipiatur, tum praeterea erit $z' = z + c$; vnde differentialia harum coordinatarum a praecedentibus non diſcrepabunt, ideoque litterae p, q, r perinde ad ambo centra virium C et C′ referentur. Statuatur igitur pro hoc centro C′ di-

distantia $C'Z = v'$ et ipsa vis centripeta huc tendens $= V'$, ita vt nunc fit $v'v' = x'x' + y'y' + z'z'$; at vires pro ternis nostris directionibus hinc oriundae erunt

secundum $X'C' = \frac{V'x'}{v'}$,

secundum $YX' = \frac{V'y'}{v'}$,

secundum $ZY = \frac{V'z'}{v'}$.

§. 5. Introducamus nunc elementum temporis ∂t, quo pro constanti assumto, principia motus sequentes tres aequationes suppeditabunt:

$$\text{I. } \frac{\partial\partial x}{\alpha\,\partial t^2} = -\frac{Vx}{v} - \frac{V'x'}{v'};$$

$$\text{II. } \frac{\partial\partial y}{\alpha\,\partial t^2} = -\frac{Vy}{v} - \frac{V'y'}{v'};$$

$$\text{III. } \frac{\partial\partial z}{\alpha\,\partial t^2} = -\frac{Vz}{v} - \frac{V'z'}{v'};$$

vbi α denotat certam quantitatem constantem, ex ratione qua tam tempus t exprimitur quam ipsae vires V et V' ad mensuras cognitas reuocantur, petendam. Cum igitur hic tres habeantur aequationes, facile intelligitur, ex iis binas vires incognitas V et V' semper determinari posse.

§. 6. Elidamus hinc primo vim V', id quod duplici modo fieri poterit; primo scilicet ex aequationum §. praecedentis prima et secunda fiet

$$\frac{y'\partial\partial x - x'\partial\partial y}{\alpha\,\partial t^2} = -\frac{V(y'x - x'y)}{v}$$

deinde secunda et tertia simili modo praebent

$$\frac{z'\partial\partial y - y'\partial\partial z}{\alpha\,\partial t^2} = -\frac{V(z'y - y'z)}{v}$$

quarum haec per illam diuisa perducit ad hanc aequationem:

$$\frac{z'\partial\partial y - y'\partial\partial z}{y'\partial\partial x - x'\partial\partial y} = \frac{z'y - y'z}{y'x - x'y}$$

vnde etiam ipsum temporis elementum ∂t est expulsum; interim

rim tamen conditio, qua ∂t constans est assumtum, etiamnunc inhaeret, vnde ratio differentialium secundi gradus, quae per se forent indefinita, peti debet.

§. 7. Quod si iam postrema ista aequatio euoluatur, peruenietur ad sequentem aequationem

$$y'\partial\partial x(z'y-y'z)+y'\partial\partial y(x'z-z'x)+y'\partial\partial z(y'x-x'y)=0,$$

quae commode per y' diuidi se patitur, et aequatio induet hanc formam:

$$\partial\partial x(z'y-y'z)+\partial\partial y(x'z-z'x)+\partial\partial z(y'x-x'y)=0.$$

Hic iam introducamus valores supra notatos, scilicet $x'=x+a$, $y'=y+b$, $z'=z+c$, et prodibit haec aequatio:

$$\partial\partial x\,(cy-bz)+\partial\partial y\,(az-cx)+\partial\partial z\,(bx-ay)=0,$$

quae secundum litteras a, b, c disposita euadet

$$a(z\partial\partial y-y\partial\partial z)+b(x\partial\partial z-z\partial\partial x)+c(y\partial\partial x-x\partial\partial y)=0.$$

§. 8. Hic igitur singulae partes manifesto sponte sunt integrabiles, vnde integratio dabit

$$a(z\partial y-y\partial z)+b(x\partial z-z\partial x)+c(y\partial x-x\partial y)=\text{const.}$$

quam constantem vtique formam differentialem habere oportet; quare cum elementum ∂t sumtum sit constans, aequatio ita repraesentari debet:

$$a(z\partial y-y\partial z)+b(x\partial z-z\partial x)+c(y\partial x-x\partial y)=C\partial t.$$

Ac si iam introducamus positiones $\partial x=p\partial s$, $\partial y=q\partial s$ et $\partial z=r\partial s$, haec aequatio accipiet hanc formam:

$$a\partial s\,(qz-ry)+b\partial s\,(rx-pz)+c\partial s\,(py-qx)=C\partial t,$$

atque hinc, quia elementum ∂t suppositum est constans, differentiando colligetur differentiale secundum $\partial\partial s$, cum fiat

$a\partial\partial s$

$$\left.\begin{array}{l} a\partial\partial s\,(qz-ry)+b\partial\partial s\,(rx-pz)+c\partial\partial s\,(py-qx)\\ +a\partial s\,(z\partial q-y\partial r)+b\partial s\,(x\partial r-z\partial p)+c\partial s\,(y\partial p-x\partial q)\end{array}\right\}=0.$$

Sicque erit

$$\frac{\partial\partial s}{\partial s}=\frac{a(y\partial r-z\partial q)+b(z\partial p-x\partial r)+c(x\partial q-y\partial p)}{a(qz-ry)+b(rx-pz)+c(py-qx)}.$$

§. 9. Cum igitur formula $\frac{\partial s}{\partial t}$ exprimat celeritatem corporis in puncto Z, si haec celeritas vocetur $=u$, vt sit $\partial t=\frac{\partial s}{u}$, iam adepti sumus formulam pro corporis celeritate u; erit enim

$$u=\frac{C}{a(qz-ry)+b(rx-pz)+c(py-qx)}.$$

Introducendo autem istam celeritatem, cum in nostris formulis principalibus sit $\frac{\partial\partial x}{\partial t}=\partial.\frac{\partial x}{\partial t}$, ob $\partial x=p\,\partial s$ et $\partial t=\frac{\partial s}{u}$, erit $\frac{\partial\partial x}{\partial t}=\partial.pu=p\,\partial u+u\,\partial p$; similique modo erit

$$\frac{\partial\partial y}{\partial t}=\partial.qu=q\,\partial u+u\,\partial q \text{ et}$$

$$\frac{\partial\partial z}{\partial t}=\partial.ru=r\,\partial u+u\,\partial r.$$

§. 10. His inuentis ambas vires quaesitas V et V′ definire licebit; cum enim inuenerimus:

$$\frac{y'\partial\partial x-x'\partial\partial y}{\alpha\,\partial t^2}=-\frac{V}{v}(y'x-x'y)=-\frac{V'}{v}(bx-ay),$$

multiplicando per $\alpha\,\partial t$ et loco $\frac{\partial\partial x}{\partial t}$ et $\frac{\partial\partial y}{\partial t}$ valores ante inuentos substituendo, erit

$$y'(p\,\partial u+u\,\partial p)-x'(q\,\partial u+u\,\partial q)=-\frac{\alpha V\partial t}{v}(bx-ay),$$

ergo ob $x'=x+a$ et $y'=y+b$, erit

$$V=\frac{(x+a)(q\,\partial u+u\,\partial q)-(y+b)(p\,\partial u+u\,\partial p)}{bx-ay}\cdot\frac{v}{\alpha\,\partial t}.$$

§. 11. Simili modo cum ex aequationibus principalibus elici queat haec aequatio y'

 $y\partial\partial x$

$$\frac{y\partial\partial x - x\partial\partial y}{\alpha\,\partial t^2} = -\frac{V'}{v'}(x'y - y'x) = -\frac{V'}{v'}(ay - bx),$$

factis iisdem substitutionibus perueniemus ad sequentem formam:

$$V' = \frac{v'}{\alpha\,\partial t} \cdot \frac{x(q\partial u + u\partial q) - y(p\partial u + u\partial p)}{ay - bx}.$$

Sicque pro quouis casu oblato, non solum ambas vires centripetas V et V′, sed etiam celeritatem corporis in singulis punctis curuae propositae assignare licet.

§. 12. Quoniam ternas coordinatas x, y et z inter se permutare licet, etiam pro vtraque vi centripeta V et V′ ternae expressiones exhiberi poterunt, quae erunt pro vi V ad centrum C tendente

$$\text{I. } V = \frac{v}{\alpha\,\partial t}\left(\frac{(x+a)(q\partial u + u\partial q) - (y+b)(p\partial u + u\partial p)}{bx - ay}\right),$$

$$\text{II. } V = \frac{v}{\alpha\,\partial t}\left(\frac{(y+b)(r\partial u + u\partial r) - (z+c)(q\partial u + u\partial q)}{cy - bz}\right),$$

$$\text{III. } V = \frac{v}{\alpha\,\partial t}\left(\frac{(z+c)(p\partial u + u\partial p) - (x+a)(r\partial u + u\partial r)}{az - cx}\right),$$

Simili modo pro altera vi centripeta V′ ad centrum C′ tendente

$$\text{I. } V' = \frac{v'}{\alpha\,\partial t} \cdot \frac{x(q\partial u + u\partial q) - y(p\partial u + u\partial p)}{ay - bx},$$

$$\text{II. } V' = \frac{v'}{\alpha\,\partial t} \cdot \frac{y(r\partial u + u\partial r) - z(q\partial u + u\partial q)}{bz - cy},$$

$$\text{III. } V' = \frac{v'}{\alpha\,\partial t} \cdot \frac{z(p\partial u + u\partial p) - x(r\partial u + u\partial r)}{cx - az},$$

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eiusdem quaestionis multo succinctior.

Tab. III. Fig. 2. §. 13. Quia ternas directiones fixas, quibus ternas coordinatas x, y, z parallelas statuimus, pro lubitu accipere licet, solutio multo fiet simplicior, si axem, in quo abscissas x, capimus, per ipsa bina centra virium C et C′ ducamus. Sint igitur C et C′ bina centra virium, et ne alteri prae altero vllam

lam praerogatiuam tribuamus, abſciſſas OX a puncto medio O computemus, quem in finem ponamus $OC = OC' = k$, vt pro centro C ſit abſciſſa $CX = x + k$, pro altero vero centro $C'X = x - k$, quia in partem contrariam vergit, binae autem reliquae coordinatae XY et YZ aeque referuntur. Hinc ductis diſtantiis CZ et C'Z, erit

$$CZ = v = \sqrt{(x+k)^2 + yy + zz} \text{ et}$$
$$C'Z = v' = \sqrt{(x-k)^2 + yy + zz},$$

ita vt ſit $vv + v'v' = 4kx$.

§. 14. Quod ſi nunc vt ante vis ad centrum C tendens vocetur $= V$, altera vero vis ad centrum C' tendens $= V'$, ſumto elemento temporis ∂t conſtante tres aequationes motum determinantes erunt:

$$\text{I. } \frac{\partial\partial x}{\alpha\,\partial t^2} = -\frac{V(x+k)}{v} - \frac{V'(x-k)}{v'};$$
$$\text{II. } \frac{\partial\partial y}{\alpha\,\partial t^2} = -\frac{Vy}{v} - \frac{V'y}{v'};$$
$$\text{III. } \frac{\partial\partial z}{\alpha\,\partial t^2} = -\frac{Vz}{v} - \frac{V'z}{v'};$$

vbi ſtatim commode vſu venit, vt ſecunda per tertiam diuiſa praebeat hanc ſimplicem aequalitatem $\frac{\partial\partial y}{\partial\partial z} = \frac{y}{z}$, ideoque $z\partial\partial y - y\partial\partial z = 0$, hinc igitur integrando fiet $z\partial y - y\partial z = C\partial t = \frac{C\partial s}{u}$, poſito ſcilicet elemento curuae deſcriptae $= \partial s$ et celeritate $= u$.

§. 15. Ponamus nunc vt ante $\partial x = p\partial s$, $\partial y = q\partial s$ et $\partial z = r\partial s$, ita vt ſit $pp + qq + rr = 1$, eritque $C\partial t = \partial s(qz - ry)$, ideoque $u = \frac{C}{qz - ry}$. Porro vero ob ſumtum ∂t conſtans fiet $\partial\partial s = \frac{\partial s(y\partial r - z\partial q)}{qz - ry}$, vel etiam per celeritatem habebimus $\partial\partial s = \frac{\partial u\,\partial s}{u}$, vnde erit

$$\partial\partial x = \frac{\partial s(p\,\partial u + u\,\partial p)}{u},$$

$$\partial\partial y = \frac{\partial s(q\,\partial u + u\,\partial q)}{u} \text{ et}$$

$$\partial\partial z = \frac{\partial s(r\,\partial u + u\,\partial r)}{u},$$

vnde porro colligitur: $\frac{\partial\partial x}{\partial t^2} = \frac{u(r\,\partial u + u\,\partial r)}{\partial s}$, $\frac{\partial\partial y}{\partial t^2} = \frac{u(q\,\partial u + u\,\partial q)}{\partial s}$ et $\frac{\partial\partial z}{\partial t^2} = \frac{u(r\,\partial u + u\,\partial r)}{\partial s}$, quae ternae expressiones breuitatis gratia per litteras P, Q et R indicentur, vbi notasse iuuabit fore $\frac{Q}{R} = \frac{y}{z}$, ob $u = \frac{c}{qz - ry}$.

§. 16. Nunc pro ipsis viribus centripetis inueniendis combinemus primam aequationem cum secunda, ac primo I. y — II. $(x - k)$ praebet

$$\frac{P.y - Q(x-k)}{\alpha} = -\frac{2Vky}{v}, \text{ ideoque}$$

$$\text{vis } V = \frac{vQ(x-k) - vPy}{2\alpha ky}.$$

Simili modo combinatio I. y — II. $(x + k)$ praebet

$$\frac{yP - Q(x+k)}{\alpha} = \frac{2V'ky}{v'} \text{ ideoque vis}$$

$$V' = \frac{v'Py - v'Q(x+k)}{2\alpha ky},$$

quas ambas formulas etiam ita repraesentare licet

$$\frac{2\alpha V}{v} = \frac{Qx - Py}{ky} - \frac{Q}{y} \text{ et } \frac{2\alpha V'}{v'} = \frac{Py - Qx}{ky} - \frac{Q}{y}.$$

Ad illustrationem huius solutionis quaedam exempla subiungamus.

Problema I.

§. 17. *Si corpus vtcunque in superficie cylindri moueri debeat, in cuius axe ambo centra virium C et C′ accipiantur inuenire tam celeritatem corporis* u *quam ambas vires centripetas V et V′.*

Solutio.

Solutio.

Sit radius cylindri $= a$ et distantia centrorum $CC = 2k$, eritque $yy + zz = aa$; ponatur ergo

$$y = a \cos. \Phi \text{ et } z = a \sin. \Phi,$$

tum vero, quia motum quemcunque in superficie cylindri statuimus, abscissa x tanquam certa functio anguli Φ spectari poterit, vnde fiat $\partial x = \Pi \partial \Phi$, inde autem colligentur distantiae corporis a centris virium

$$v = \sqrt{(x + k)^2 + aa} \text{ et } v' = \sqrt{(x - k)^2 + aa};$$

deinde vero ob $\partial y = - a \partial \Phi \sin. \Phi$ et $\partial z = a \partial \Phi \cos. \Phi$, erit $\partial s = \partial \Phi \sqrt{(\Pi \Pi + aa)}$, ex quo porro fiet $p = \frac{\pi}{\sqrt{(\Pi \Pi + aa)}}$, $q = \frac{a \sin. \Phi}{\sqrt{(\Pi \Pi + aa)}}$ et $r = \frac{a \cos. \Phi}{\sqrt{(\Pi \Pi + aa)}}$.

His positis celeritas corporis in Z reperietur

$$u = \frac{c \sqrt{(\Pi \Pi + aa)}}{aa}.$$

Deinde pro ipsis viribus inueniendis quaerantur ante omnia valores litterarum P, Q, R; reperieturque

$$P = \frac{cc \partial \Pi}{a^4 \partial \Phi} = \frac{cc}{a^4} \Pi,$$

posito $\partial \Pi = \Pi' \partial \Phi$. Porro erit $Q = - \frac{cc \cos. \Phi}{a^3}$, ac denique $R = - \frac{cc \sin. \Phi}{a^3}$, vnde vtique habetur $\frac{Q}{R} = \frac{y}{z} = \frac{\cos. \Phi}{\sin. \Phi}$.

His valoribus inuentis pro ipsis viribus quaeramus primo formulam:

$$\frac{Qx - Py}{y} = - \frac{cc}{a^4} (x + \Pi),$$

ac tum ob $\frac{Q}{y} = - \frac{cc}{a^4}$ habebimus:

$$\frac{2aV}{v} = \frac{cc}{a^4} - \frac{cc}{a^4} \frac{(x + \Pi)}{k} = \frac{cc}{a^4} \left(1 - \frac{x - \Pi}{k}\right),$$

$$\frac{2aV'}{v'} = \frac{c}{a^4} + \frac{cc}{a^4} \frac{(x + \Pi)}{k} = \frac{cc}{a^4} \left(1 + \frac{x + \Pi}{k}\right),$$

vnde

vnde ipſae vires erunt

$$V = \frac{ccv}{2\alpha a^4}\left(1 - \frac{x - \Pi}{k}\right) \text{ et } V' = \frac{ccv'}{2\alpha a^4}\left(1 + \frac{x + \Pi}{k}\right).$$

Corollarium 1.

§. 18. Quod ſi velimus vt corpus in ſuperficie cylindri deſcribat helicem Archimedis, ſeu eam lineam, quae intra ſuos terminos eſt breuiſſima, poni debet $x = na\Phi$, ideoque $\Pi = na$ et $\Pi' = 0$, vnde ſequitur celeritas corporis $u = \frac{c\sqrt{(1+nn)}}{a}$, quae ergo perpetuo manebit conſtans, vnde ſi vocetur $u = c$, erit $C = \frac{ac}{\sqrt{(1+nn)}}$, tum vero ambae vires centripetae ita erunt comparatae

$$V = \frac{ccv}{2\alpha a^4}\left(1 - \frac{na(1+\Phi)}{k}\right) \text{ et } V' = \frac{ccv'}{2\alpha a^4}\left(1 + \frac{na(1+\Phi)}{k}\right),$$

quare ſi loco C ſcribamus valorem modo aſſignatum, erit

$$V = \frac{ccv}{2\alpha aa(1+nn)}\left(1 - \frac{na(1+\Phi)}{k}\right) \text{ et}$$

$$V' = \frac{ccv'}{2\alpha aa(1+nn)}\left(1 + \frac{na(1+\Phi)}{k}\right).$$

Corollarium 2.

§. 19. Quod ſi factorem conſtantem $\frac{cc}{2\alpha aa(1+nn)}$ deſignemus litera, A habemus

$$V = Av\left(1 - \frac{na(1+\Phi)}{k}\right) \text{ et } V' = Av'\left(1 + \frac{na(1+\Phi)}{k}\right),$$

vnde patet vtramque vim centripetam conſtare duabus partibus, quarum prior ſimpliciter proportionalis eſt diſtantiae corporis a centro, poſterior vero eidem diſtantiae per angulum Φ multiplicatae, quandoquidem erit

$$V = A\left(1 - \frac{na}{k}\right)v - \frac{Ana\Phi v}{k} \text{ et}$$

$$V' = A\left(1 + \frac{na}{k}\right)v' + \frac{Ana\Phi v'}{k},$$

vbi poſterior vis V′ maior eſt priore, propterea quod corpus a cena centro C recedere, contra vero ad centrum C′ accedere assumsimus.

Problema II.

§. 20. *Si corpus vtcunque in superficie globi moueri debeat, et ambo centra virium in ipsis polis huius globi statuantur, inuenire tam celeritatem corporis u, quam ambas vires centripetas V et V′.*

Solutio.

Sint igitur C et C′ poli nostrae Sphaerae, cuius radius ponatur $= a$, eritque O eius centrum et $k = a$. Cum igitur esse debeat $xx + yy + zz = aa$, statuamus $x = a \sin. \eta$, tum vero $y = a \cos. \eta \cos. \Phi$, et $z = a \cos. \eta \sin. \Phi$, vbi in genere ambo anguli η et Φ relationem quamcunque inter se tenere possunt, ex quo statuamus $\partial \Phi = \Pi \partial \eta$ et $\partial \Pi = \Pi' \partial \eta$. Primo igitur distantiae corporis ab vtroque polo erunt

$$v = a \sqrt{(2 + 2 \sin. \eta)} = 2 a \cos. (45^\circ - \tfrac{1}{2} \eta) \text{ et}$$

$$v' = a \sqrt{(2 - 2 \sin. \eta)} = 2 a \sin. (45^\circ - \tfrac{1}{2} \eta).$$

Cum nunc sit $\partial s = \sqrt{(\partial x^2 + \partial y^2 + \partial z^2)}$, ob

$$\partial x = a \partial \eta \cos. \eta$$

$$\partial y = - a \partial \eta (\sin. \eta \cos. \Phi + \Pi \cos. \eta \sin. \Phi) \text{ et}$$

$$\partial z = a \partial \eta (\Pi \cos. \eta \cos. \Phi - \sin. \eta \sin. \Phi),$$

habebitur

$$\partial s = a \partial \eta \sqrt{(1 + \Pi \Pi \cos. \eta^2)}.$$

Hinc iam porro deducimus

$$p = \frac{\cos. \eta}{\sqrt{(1 + \Pi \Pi \cos. \eta^2)}},$$

$$q = - \frac{(\sin. \eta \cos. \Phi + \Pi \cos. \eta \sin. \Phi)}{(1 + \Pi \Pi \cos. \eta^2)} \text{ et}$$

$$r = \frac{\Pi \cos. \eta \cos. \Phi - \sin. \eta \sin. \Phi}{\sqrt{(1 + \Pi \Pi \cos. \eta^2)}},$$

quamobrem hinc celeritas corporis ita definietur, vt fit

$$u = \frac{c\sqrt{(1 + \Pi\Pi \cos.\eta^2)}}{a\,\Pi \cos.\eta^2},$$

vnde patet, dum corpus ad alterutrum polum pertingit, quod fit quando angulus η fit rectus, celeritatem corporis ibi fieri infinitam, nisi forte quantitas Π euadat infinita, ita vt $\Pi \cos.\eta^2$ euadat quantitas finita puta A, interim tamen etiam hoc casu, quoniam fit $u = \frac{C\sqrt{(1 + \frac{AA}{\cos.\eta^2})}}{a A}$, manifesto celeritas fit infinita.

Nunc vt etiam valores litterarum P, Q, R eruamus, pro P habemus primo $p u = \frac{c}{a\,\Pi \cos.\eta}$, hincque

$$\partial.\, p u = \frac{c\,\partial\eta}{a} \frac{(\Pi \sin.\eta - \Pi' \cos.\eta)}{\Pi\Pi \cos.\eta^2},$$ vnde ob

$$\frac{u}{\partial s} = \frac{c}{a a \Pi\, \partial\eta \cos.\eta^2}$$ reperietur

$$P = \frac{c c}{a^3} \frac{(\Pi \sin.\eta - \Pi' \cos.\eta)}{\Pi^3 \cos.\eta^4}.$$

Deinde ob

$$q u = -\frac{c(\sin.\eta \cos.\Phi + \Pi \cos.\eta \sin.\Phi)}{a\,\Pi \cos.\eta^2}$$ erit

$$\partial.\, q u = -\frac{c\,\partial\eta \cos.\Phi}{a} \frac{(\Pi(1 + \Pi\Pi)\cos.\eta^2 + 2\Pi \sin.\eta^2 - \Pi' \sin.\eta \cos.\Phi)}{\Pi\Pi \cos.\eta^3},$$

quae expressio ducta in $\frac{u}{\partial s}$ dabit

$$Q = -\frac{c c \cos.\Phi}{a^3} \frac{(\Pi(1 + \Pi\Pi)\cos.\eta^2 + 2\Pi \sin.\eta^2 - \Pi' \sin.\eta \cos.\eta)}{\Pi^3 \cos.\eta^5}.$$

Denique cum fit

$$r u = -\frac{c}{u} \frac{(\sin.\eta \sin.\Phi - \Pi \cos.\eta \cos.\Phi)}{\Pi \cos.\eta^2}$$ erit

$$\partial.\, r u = -\frac{c\,\partial\eta \sin.\Phi}{a} \frac{(\Pi(1 + \Pi\Pi)\cos.\eta^2 + 2\Pi \sin.\eta^2 - \Pi' \sin.\eta \cos.\eta)}{\Pi\Pi \cos.\eta^3},$$

quae formula ducta in $\frac{u}{\partial s}$ dabit

$$R = -\frac{c c \sin.\Phi}{a^3} \frac{(\Pi(1 + \Pi\Pi)\cos.\eta^2 + 2\Pi \sin.\eta^2 - \Pi' \sin.\eta \cos.\eta}{\Pi^3 \cos.\eta^5},$$

vnde manifestum est esse $Q : R = y : z$ hoc est $= \cos.\Phi : \sin.\Phi$.

Cum

Cum nunc ob $k = a$ ſit

$$\frac{2\alpha V}{v'} = \frac{Qx}{ay} - \frac{P}{a} - \frac{Q}{y} \text{ et } \frac{2\alpha V'}{v'} = \frac{P}{a} - \frac{Qx}{ay} - \frac{Q}{y}, \text{ erit}$$

$$\frac{Q}{y} = -\frac{cc}{a^4} \frac{(\Pi(1+\Pi\Pi)\,\textit{coſ}.\,\eta^2 + 2\Pi\,\textit{ſin}.\,\eta^2 - \Pi'\,\textit{ſin}.\,\eta\,\textit{coſ}.\,\eta)}{\Pi^3\,\textit{coſ}.\,\eta^6} \text{ et}$$

$$\frac{Qx}{qy} - \frac{P}{a} = -\frac{cc}{a^4} \frac{(\Pi(1+\Pi\Pi)\,\textit{ſin}.\,\eta\,\textit{coſ}.\,\eta^2 + 2\Pi\,\textit{ſin}.\,\eta^3 - \Pi'\,\textit{ſin}.\,\eta^2\,\textit{coſ}.\,\eta + \Pi\,\textit{ſin}.\,\eta\,\textit{coſ}.\,\eta^2 - \Pi'\,\textit{coſ}.\,\eta^3)}{\Pi^3\,\textit{coſ}.\,\eta^6},$$

quae expreſſiones etiam ita repraeſentari poſſunt

$$\frac{Q}{y} = -\frac{cc}{a^4\,\Pi^4\,\textit{coſ}.\,\eta^6} \left(\Pi\,(1+\Pi\Pi\,\text{coſ}.\,\eta^2) - \text{ſin}.\,\eta\,\frac{\partial.\,\Pi\,\textit{coſ}.\,\eta}{\partial\eta}\right) \text{ et}$$

$$\frac{P}{a} = \frac{-cc}{a^4\,\Pi^3\,\textit{coſ}.\,\eta^4}\,\frac{\partial.\,\Pi\,\textit{coſ}.\,\eta}{\partial\eta} \text{ et hinc}$$

$$\frac{Qx}{ay} - \frac{P}{a} = \frac{-cc}{a^4\,\Pi^3\,\textit{coſ}.\,\eta^6} \left(\Pi\,\text{ſin}.\,\eta\,(1+\Pi\Pi\,\text{coſ}.\,\eta^2) - \frac{\partial.\,\Pi\,\textit{coſ}.\,\eta}{\partial\eta}\right).$$

His autem valoribus ſubſtitutis fiet

$$\frac{2\alpha V}{v} = \frac{cc\,(1-\textit{ſin}.\,\eta)}{a^4\,\Pi^3\,\textit{coſ}.\,\eta^6} \left(\Pi\,(1+\Pi\Pi\,\text{coſ}.\,\eta^2) + \frac{\partial.\,\Pi\,\textit{coſ}.\,\eta}{\partial\eta}\right) \text{ et}$$

$$\frac{2\alpha V'}{v'} = \frac{cc\,(1+\textit{ſin}.\,\eta)}{a^4\,\Pi^3\,\textit{coſ}.\,\eta^6} \left(\Pi\,(1+\Pi\Pi\,\text{coſ}.\,\eta^2) - \frac{\partial.\,\Pi\,\textit{coſ}.\,\eta}{\partial\eta}\right),$$

quibus formulis ergo ambae vires centripetae V et V' ſatis commode exprimuntur, vbi meminiſſe iuuabit eſſe

$$v = a\sqrt{2\,(1+\text{ſin}.\,\eta)} \text{ et } v' = a\sqrt{2\,(1-\text{ſin}.\,\eta)}.$$

EVOLVTIO CASVS
quo corpus in ſuperficie Sphaerica Loxodromiam deſcribit.

§. 21. Angulus ergo, ſub quo via corporis ſingulos meridianos interſecat, debet eſſe conſtans, cuius tangens ſi ponatur $= n$, reperietur $\frac{\partial\Phi\,\textit{coſ}.\,\eta}{\partial\eta} = n$, vnde ergo fit $\partial\Phi = \frac{n\,\partial\eta}{\textit{coſ}.\,\eta}$, ſicque habebitur $\Pi = \frac{n}{\textit{coſ}.\,\eta}$, ac propterea formula $\Pi\,\text{coſ}.\,\eta = n$ erit conſtans eiusque differentiale euaneſcet. Hic ergo erit vt ante $v = a\sqrt{2\,(1+\text{ſin}.\,\eta)}$ et $v' = a\sqrt{2\,(1-\text{ſin}.\,\eta)}$; tum vero prodibit celeritas $u = \frac{c\sqrt{(1+nn)}}{a\,n\,\textit{coſ}.\,\eta}$, quae ergo in ipſis polis vbi $\eta = 90°$ fiet infinita. Deinde autem ambae vires centripetae ita erunt comparata vt sit

$$\frac{2\alpha V}{v} = \frac{CC(1-\sin.\eta)(1+nn)}{nn\,a^4\cos.\eta^4} \text{ et } \frac{2\alpha V'}{v'} = \frac{CC(1+\sin.\eta)(1+nn)}{nn\,a^4\cos.\eta^4},$$

vnde loco v et v' valoribus substitutis colligetur

$$V = \frac{(1+nn)CC(1-\sin.\eta)\sqrt{2(1+\sin.\eta)}}{2\alpha nn\,a^3\cos.\eta^4} \text{ et}$$

$$V' = \frac{(1+nn)CC(1+\sin.\eta)\sqrt{2(1-\sin.\eta)}}{2\alpha nn\,a^3\cos.\eta^4}.$$

Cum nunc sit $\cos.\eta^2 = (1+\sin.\eta)(1-\sin.\eta)$, hae formulae reducentur ad sequentes

$$V = \frac{(1+nn)CC\sqrt{2}}{2\alpha nn\,a^3\cos.\eta^2\sqrt{(1+\sin.\eta)}} = \frac{(1+nn)CC}{\alpha nn\,a^3\cos.\eta^2\sqrt{2(1+\sin.\eta)}} \text{ et}$$

$$V' = \frac{(1+nn)CC}{\alpha nn\,a^3\cos.\eta^2\sqrt{2(1-\sin.\eta)}};$$

vnde patet fore $V : V' = v' : v$ ita vt sit $V'v' = Vv$.

§. 22. Quod si fuerit $n = 0$, quo casu corpus movebitur in ipso meridiano, erit primo celeritas $u = \infty$; quia autem C est constans arbitraria, si ponamus $C = nA$, fiet $u = \frac{A}{a\cos.\eta}$; ipsae autem vires erunt

$$V = \frac{AA}{\alpha a^3\cos.\eta^2\sqrt{2(1+\sin.\eta)}} \text{ et } V' = \frac{AA}{\alpha a^3\cos.\eta^2\sqrt{2(1-\sin.\eta)}},$$

vbi notetur, dum corpus per aequatorem transit vbi $\eta = 0$, tum fore $u = \frac{A}{a}$, atque vires $V = \frac{AA}{\alpha a^3\sqrt{2}}$ et $V' = \frac{AA}{\alpha a^3\sqrt{2}}$, sicque hae vires erunt aequales; quo longius autem corpus ab aequatore recedit, tam celeritas quam ambae vires continuo fiunt maiores. Ponamus corpus iam tam prope ad polum accessisse, vt sit $\eta = 90° - \omega$, existente ω quasi infinite paruo; eritque $\cos.\eta^2 = \omega\omega$, at $1 + \sin.\eta = 2$, et $1 - \sin.\eta = \frac{1}{2}\omega\omega$; tum ergo erit $u = \frac{A}{a\omega}$, $V = \frac{AA}{2\alpha a^3\omega\omega}$ et $V' = \frac{AA}{\alpha a^3\omega^3}$.

§. 23. Sumamus nunc angulum Loxodromiae esse rectum, quo corpus vel in aequatore vel in quouis circulo aequatori parallelo movetur, cuius distantia ab aequatore angulo η indi-

indicatur; tum igitur erit $n = \infty$, hincque celeritas $u = \frac{c}{\alpha\, cos\, \eta}$, quae ergo erit conſtans ob angulum η conſtantem; tum vero vires erunt

$$V = \frac{cc}{\alpha a^3\, cos.\, \eta^2 \sqrt{2}(1 + sin.\, \eta)} \quad \text{et} \quad V' = \frac{cc}{\alpha a^3\, cos.\, \eta^2 \sqrt{2}(1 - sin.\, \eta)},$$

quae ergo etiam ambae erunt conſtantes.

§. 24. Videamus vero etiam, quomodo in genere vires futurae ſint comparatae, quando corpus iam proxime ad polum acceſſerit ita vt ſit $\eta = 90° - \omega$; tum autem erit celeritas $u = \frac{c\sqrt{(1 + nn)}}{n\, a\, \omega}$, atque vires

$$V = \frac{(1 + nn)cc}{\alpha\, n\, n\, a^3\, \omega\, \omega} \quad \text{et} \quad V' = \frac{(1 + nn)cc}{\alpha\, n\, n\, a^3\, \omega^3};$$

vnde patet, vim V infinities eſſe minorem quam alteram V′, ita vt corpus a ſola vi V′ vrgeri cenſeri queat, et quia hic ω exprimet diſtantiam corporis a polo C′, vis iſta reciproce erit cubo huius diſtantiae proportionalis, corpus autem iam circa polum C′ ſpiralem logarithmicam deſcribet, quae vtique talem vim centripetam poſtulat.

§. 25. Superfluum foret plures caſus euoluere, quandoquidem inde nullae expreſſiones elegantes et attentione dignae forent exſpectandae; imprimis enim mihi erat propoſitum oſtendere, ſi corpus in quacunque curua non in eodem plano ſita moueri debeat, ſemper duas vires centripetas aſſignari poſſe, quae adeo ad data duo centra virium diriguntur, ita vt corpus ab iſtis viribus ſollicitatum per ipſam curuam propoſitam promoueatur.