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De superficie coni scaleni, ubi imprimis intentes difficultates, quae in hac investigatione occurrunt, perpenduntur

Leonhard Euler

DE SVPERFICIE CONI SCALENI, VBI IMPRIMIS INGENTES DIFFICVLTATES, QVAE IN HAC INVESTIGATIONE OCCVRRVNT, PERPENDVNTVR.

Auctore.
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§. 1.

Sit circulus E G F H basis coni scaleni propositi, cuius vertex in sublimi situs sit A, vnde ad planum basis demittatur perpendiculum A B, et ex B per centrum basis C agatur recta F B C E. Vocetur altitudo $AB = a$, deinde vero sit $BC = b$, quae linea exhibet coni obliquitatem; si enim esset $b = 0$, conus foret rectus. Denique vero vocetur radius basis $CE = CF = c$, ac manifestum est his tribus quantitatibus a, b, c naturam coni penitus determinari. Hinc si ad verticem ductae intelligantur rectae E A et F A, ob $BE = c + b$ et $BF = c - b$, erit $AE = \sqrt{aa + (c+b)^2}$, quod est latus coni maximum; latus vero minimum erit $AF = \sqrt{aa + (c-b)^2}$. Praeterea si in basi ducatur diameter G H ad E F normalis, rectae A G et A H erunt latera media coni inter se aequalia; ad quorum quantitatem inueniendam, quoni- Tab. I. Fig. 5.

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am eſt $AC = \surd(aa + bb)$ et triangula ACG et ACH ad C rectangula, erit $AG = AH = \surd(aa + bb + cc)$.

§. 2. Quoniam igitur nobis propoſitum eſt ſuperficiem huius coni ſcaleni indagare, quemadmodum ea ſcilicet per terna elementa a, b et c definiatur, haec inueſtigatio facillime ſequenti modo inſtituetur. Ducto coni latere maximo AE, in baſi coni, ex centro C, capiatur angulus indefinitus $ECS = \phi$, qui ſuo differentiali $SCs = \partial\phi$ augeatur, ac vocetur portio ſuperficiei conicae inter rectas AE et AS atque arcum ES incluſa $= S$, ita vt poſito $\phi = 180°$ punctum S in F perueniat, et iſta quantitas S nobis ſit indicatura ſemiſſem ſuperficiei conicae, eiusque ergo duplum totam ſuperficiem coni quaeſitam. Quodſi iam ex A ducamus rectam proximam As, area trianguli elementaris SAs dabit valorem differentialis ∂S, ita vt totum negotium huc redeat, vt area iſtius trianguli SAs exploretur, quod ob arculum $Ss = c\partial\phi$, ideoque infinite paruum, tanquam triangulum rectilineum ſpectari poteſt.

§. 3. Hunc in finem ducatur ad S tangens circuli SP, ſiue, quod eodem redit, producatur elementum Ss, ita vt recta SP ſit baſis Ss producta; vnde ſi ex A ad eam ducatur perpendicularis AP, erit area trianguli ASs, ſiue

$$\partial S = \frac{Ss \cdot AP}{2} = \tfrac{1}{2} AP \cdot c\,\partial\phi.$$

Conſtat autem hoc perpendiculum AP duci, ſi ex puncto B ad rectam SP demittatur perpendiculum BP, quandoquidem tum etiam recta AP ei erit normalis. Iam ex C ad rectam BP normaliter agatur recta CQ, et quia BP parallela eſt radio CS, erit angulus $CBQ = \phi$, vnde ob $BC = b$ erit $CQ = b\,\text{ſin.}\,\phi$ et $BQ = b\,\text{coſ.}\,\phi$. Quare cum ſit $PQ = CS = c$, erit $BP =$

$c + b$

$c + b \cos\Phi$ et interuallum $SP = CQ = b \sin\Phi$, ideoque ex triangulo APB, quia AB ad BP est perpendicularis, reperietur hypothenusa $AP = \sqrt{aa + (c + b \cos\Phi)^2}$, consequenter hinc elicimus elementum superficiei quaesitum

$$\partial S = \tfrac{1}{2} c \partial \Phi \sqrt{aa + (c + b \cos\Phi^2)}$$

Sicque tota inuestigatio huc est perducta, vt ista formula differentialis integretur.

§. 4. Consideremus primo casum coni recti, qui prodit facta obliquitate $b = 0$. Hoc ergo casu habebimus $\partial S = \tfrac{1}{2} c \partial \Phi \sqrt{(aa + cc)}$, vnde integrando fit $S = \tfrac{1}{2} c \Phi \sqrt{(aa + cc)}$. Fiat nunc $\Phi = 180^\circ$, siue $\Phi = \pi$, et semissis superficiei conicae erit $= \tfrac{1}{2} \pi c \sqrt{(aa + cc)}$, ideoque tota coni superficies $= \pi c \sqrt{(aa + cc)}$; vbi notetur, formulam $\sqrt{(aa + cc)}$ exprimere latus huius coni recti; tum vero, totam basis peripheriam esse $= 2 \pi c$. Constat autem superficiem coni recti inueniri, si latus coni ducatur in dimidiam basis circumferentiam.

§. 5. Hinc autem facile intelligitur pro conis scalenis hanc inuestigationem multo magis fieri arduam; propterea quod ea pendet ab integratione huius formulae:

$$\partial S = \tfrac{1}{2} c \partial \Phi \sqrt{aa + (c + b \cos\Phi)^2},$$

quae euolata praebet

$$\partial S = \tfrac{1}{2} c \partial \Phi \sqrt{aa + cc + 2bc \cos\Phi + bb \cos\Phi^2},$$

quae ob $\cos\Phi^2 = \tfrac{1}{2} + \tfrac{1}{2} \cos 2\Phi$ etiam transmutari potest in hanc formam:

$$\partial S = \tfrac{1}{2} c \partial \Phi \sqrt{(aa + \tfrac{1}{2} bb + cc + 2bc \cos\Phi + \tfrac{1}{2} bb \cos 2\Phi)}.$$

Huius autem formulae integratio absoluta nullo modo sperari potest, siquidem certum est, eam neque per logarithmos, neque

que per arcus circulares expediri posse; quamobrem nobis tantum in approximationibus erit acquiescendum.

§. 6. Ponamus breuitatis gratia $aa + \frac{1}{2}bb + cc = ff$, vt habeamus:

$$\partial S = \tfrac{1}{2} c \partial \Phi \sqrt{ff + 2bc \operatorname{cos.} \Phi + \tfrac{1}{2} bb \operatorname{cos.} 2\Phi},$$

vbi primo obseruandum occurrit, si quantitas ff fuerit valde magna prae binis reliquis terminis, tum approximationem nullam moram facessere; si enim ponamus

$$2bc \operatorname{cos.} \Phi + \tfrac{1}{2} bb \operatorname{cos.} 2\Phi = v,$$

vt fit $\partial S = \frac{1}{2} c \partial \Phi \sqrt{(ff + v)}$, facta euolutione erit

$$\sqrt{(ff + v)} = f + \tfrac{1}{2} \tfrac{v}{f} - \tfrac{1.1}{2.4} \tfrac{vv}{f^3} + \tfrac{1.1.3}{2.4.6} \tfrac{v^3}{f^5} - \tfrac{1.1.3.5}{2.4.6.8} \tfrac{v^4}{f^7} \text{ etc.}$$

quae series eo magis conuergit, quo minor erit quantitas v prae ff; vnde sufficiet huius seriei vel tantum binos terminos priores accipere, vel insuper tertium, vel adeo etiam quartum pluresue admittere, vnde aliquot casus euoluamus.

Casus I.

quo approximatio in secundo termino subsistit.

§. 7. Hoc igitur casu habebimus

$$\partial S = \tfrac{1}{2} c \partial \Phi (f + \tfrac{v}{2f}),$$

vbi primus terminus integratus dat $\frac{1}{2} fc\Phi$, secundus vero terminus, ob

$$v = 2bc \operatorname{cos.} \Phi + \tfrac{1}{2} bb \operatorname{cos.} 2\Phi,$$

integratus praebet

$$\tfrac{c}{4f} \int \partial \Phi (2bc \operatorname{cos.} \Phi + \tfrac{1}{2} bb \operatorname{cos.} 2\Phi)$$
$$= \tfrac{c}{4f} (2bc \operatorname{sin.} \Phi + \tfrac{1}{4} bb \operatorname{sin.} 2\Phi),$$

ita

ita vt iam ſit

$$S = \tfrac{1}{2} c f \Phi + \frac{b c c \sin. \Phi}{2 f} + \frac{b b c \sin. 2\Phi}{16 f}.$$

Fiat nunc $\Phi = \pi$, ac formula duplicata dabit totam coni ſuperficiem $= \pi c f$, quae reſtituto pro f valore erit

$$S = \pi c \sqrt{(a a + \tfrac{1}{2} b b + c c)},$$

quae ergo ſufficere poteſt, quoties quantitates $2 b c$ et $\frac{1}{2} b b$ fuerint quam minimi reſpectu quantitatis $a a + \frac{1}{2} b b + c c$. Haec conditio imprimis locum habet, quando altitudo coni fuerit permagna prae obliquitate b atque etiam radio baſis c. Ante autem vidimus, ſi obliquitas coni prorſus euaneſceret, ſuperficiem coni recti eſſe $= \pi c \sqrt{(a a + c c)}$, nunc igitur ſuperficies tantillo eſt maior in ratione

$$\sqrt{(a a + c c)} : \sqrt{(a a + \tfrac{1}{2} b b + c c)}.$$

Caſus II.

quo approximatio in tertio termino ſubſiſtit.

§. 8. Quoniam hic tantum ſuperficiem coni quaerimus, ſtatim ponere poſſumus $S = c \partial \Phi \sqrt{(f f + v)}$; tum enim integratione peracta tantum opus eſt facere $f = \pi$. Praeſenti igitur caſu erit

$$\partial S = c \partial \Phi \left(f + \frac{v}{2 f} - \frac{v v}{8 f^3}\right);$$

modo autem vidimus binos terminos priores dare $\pi c f$, ita vt ſit $S = \pi c f - \frac{c}{8 f^3} \int v v \partial \Phi$. Eſt vero

$$v v = 4 b b c c \operatorname{cof.} \Phi^2 + 2 b^3 c \operatorname{cof.} \Phi \operatorname{cof.} 2\Phi + \tfrac{1}{4} b^4 \operatorname{cof.} 2\Phi^2,$$

quae formula ob

$\operatorname{cof.} \Phi^2 = \frac{1}{2} + \frac{1}{2} \operatorname{cof.} 2\Phi$;

$\operatorname{cof.} \Phi \operatorname{cof.} 2\Phi = \frac{1}{2} \operatorname{cof.} \Phi + \frac{1}{2} \operatorname{cof.} 3\Phi$ et

$\operatorname{cof.} 2\Phi^2 = \frac{1}{2} + \frac{1}{2} \operatorname{cof.} 4\Phi$,

 trans-

transformatur in hanc:

$$vv = 2bbcc + \tfrac{1}{8}b^4 + b^3 c \operatorname{cof.}\Phi + 2bbcc \operatorname{cof.} 2\Phi$$
$$+ b^3 c \operatorname{cof.} 3\Phi + \tfrac{1}{8} b^4 \operatorname{cof.} 4\Phi,$$

quae ergo formula conſtat quatuor membris, quorum primum tantum in integratione eſt conſiderandum, propterea quod ſequentes termini integrati darent ſin. Φ; ſin. 2Φ; ſin. 3Φ et ſin. 4Φ, qui poſito $\Phi = \pi$ omnes in nihilum abeunt, ita vt pro hoc caſu ſit $\int vv\partial\Phi = \pi(2bbcc + \frac{1}{8}b^4)$, quamobrem tota coni ſuperficies erit $S = \pi cf - \frac{\pi bbc^3}{4f^3} - \frac{\pi b^4 c}{64 f^3}$, quae formula iam multo propius ad veritatem accedit, quam ea quae caſu primo eſt inuenta.

Caſus III.

quo approximatio in quarto termino ſiſtitur.

§. 9. Hic igitur ad expreſſionem modo inuentam inſuper adiici debet valor, qui ex hac formula integrali reſultat: $\frac{1.1.3}{2.4.6}\frac{c}{f^5}\int v^3\partial\Phi$, poſtquam ſcilicet poſitum fuerit $\Phi = \pi$. Modo autem vidimus eſſe

$$vv = 2bbcc + \tfrac{1}{8}b^4 + b^3 c \operatorname{cof.}\Phi + 2bbcc \operatorname{cof.} 2\Phi$$
$$+ b^3 c \operatorname{cof.} 3\Phi + \tfrac{1}{8} b^4 \operatorname{cof.} 4\Phi,$$

quae forma per $v = 2bc \operatorname{cof.}\Phi + \frac{1}{2}bb \operatorname{cof.} 2\Phi$ multiplicata, retentis tantum terminis conſtantibus, qui facta reductione ſupererunt, dabit

$$v^3 = b^4 cc + \tfrac{1}{2} b^4 cc = \tfrac{3}{2} b^4 cc,$$

vnde fit $\int v^3\partial\Phi = \frac{3}{2}\pi b^4 cc$, ita vt pars adiicienda ſit $\frac{3\pi b^4 c^3}{32 f^5}$, conſequenter adiecta etiam hac parte habebimus accuratius

$$S = \pi cf - \frac{\pi bbc^3}{4f^3} - \frac{\pi b^4 c}{64 f^3} + \frac{3\pi b^4 c^3}{32 f^5}.$$

§. 10.

§. 10. Contemplemur, hic casum, quo obliquitas b ipsi radio baseos est aequalis, siue vbi perpendiculum AB in ipsum punctum F incidit. Facto igitur $b = c$, superficies huius coni, dum approximatio vsque ad quartum terminum producitur, erit

$$S = \pi c f - \frac{17 \pi c^5}{64 f^3} + \frac{3 \pi c^7}{32 f^5}, \text{ siue}$$

$$S = \pi c f \left(1 - \frac{17 c^4}{64 f^4} + \frac{3 c^6}{32 f^6}\right),$$

vbi notetur esse $ff = aa + \frac{3}{2} cc$. Haec expressio eo propius ad veritatem accedit, quo maior fuerit quantitas f prae radio basis c. Ita si altitudo coni diametro baseos aequetur, ita vt sit $ff = \frac{11}{2} cc$, tum superficies huius coni erit

$$S = \pi c \sqrt{\tfrac{11}{2}} \cdot \left(1 - \frac{17}{16 \cdot 121} + \frac{3}{4 \cdot 1331}\right),$$

quae partes in vnam contractae praebent superficiem coni

$$S = \frac{21121}{21296} \pi c \sqrt{\tfrac{11}{2}}.$$

§. 11. Hanc autem approximandi methodum non ad plures terminos prosequimur, quoniam calculus nimis fieret molestus, neque vlla lex progressionis perspici posset. Plerumque autem approximatio postrema sufficere posse videtur, dummodo quantitas f notabiliter superet ambas quantitates b et c. Tentemus autem aliam methodum, quae quidem pariter postulat vt altitudo coni a plurimum superet bina reliqua elementa b et c, quae autem quandam legem progressionis pollicetur, ita vt approximationem pro lubitu continuo vlterius persequi liceat.

Alia methodus
approximandi quando a multum superat b et c.

§. 12. Hic scilicet formulam $(c + b \operatorname{cos.}\Phi)^2$ non euoluemus, sed cum sit per seriem

 $\sqrt{aa}$

$$\sqrt{[a\,a + (c + f\,\text{cof.}\,\Phi)^2]} = a + \tfrac{1}{2}\,\frac{(c + b\,\text{cof.}\,\Phi)^2}{a} - \frac{1.1}{2.4}\,\frac{(c + b\,\text{cof.}\,\Phi)^4}{a^3}$$
$$+ \frac{1.1.3}{2.4.6}\,\frac{(c + b\,\text{cof.}\,\Phi)^6}{a^5} - \frac{1.1.3.5}{2.4.6.8}\,\frac{(c + b\,\text{cof.}\,\Phi)^8}{a^7} + \text{etc.}$$

fingulas poteftates pares ipfius $c + b$ cof. Φ ita euoluamus, vt ftatim omnes poteftates ipfius cof. Φ ad cofinus fimplices reuocemus; tum enim omnia membra per quempiam cofinum affecta tuto reiicere poterimus, propterea quod in integratione praebent finus angulorum multiplorum ipfius Φ, qui pofito $\Phi = \pi$ omnes in nihilum effent abituri.

§. 13. Quo igitur hoc negotium facilius expediri queat, ante omnia obferuaffe iuuabit, omnes poteftates impares ipfius cof. Φ nullam fuppeditare quantitatem abfolutam, ita vt has poteftates penitus omittere liceat; ex poteftatibus autem paribus fequentes nafcuntur quantitates abfolutae:

$$\text{cof.}\,\Phi^2 = \tfrac{1}{2};$$
$$\text{cof.}\,\Phi^4 = \tfrac{1.3}{2.4};$$
$$\text{cof.}\,\Phi^6 = \tfrac{1.3.5}{2.4.6};$$
$$\text{cof.}\,\Phi^8 = \tfrac{1.3.5.7}{2.4.6.8};\ \text{etc.}$$

Iuxta hanc igitur regulam poteftates pares binomii $c + b$ cof. Φ euoluamus eritque:

$$(c + b\,\text{cof.}\,\Phi)^2 = c\,c + \tfrac{1}{2}\,b\,b,$$
$$(c + b\,\text{cof.}\,\Phi)^4 = c^4 + 3\,b\,b\,c\,c + \tfrac{1.3}{2.4}\,b^4,$$
$$(c + b\,\text{cof.}\,\Phi)^6 = c^6 + \tfrac{15}{2}\,b\,b\,c^4 + \tfrac{1.3.15}{2.4}\,b^4\,c\,c + \tfrac{1.3.5}{2.4.6}\,b^6.$$

Quin etiam res in genere hoc modo expedietur:

$$(c + b\,\text{cof.}\,\Phi)^{2n} = c^{2n} + \frac{2n}{1}\,\frac{(2n-1)}{2}\cdot\tfrac{1}{2}\,b^2\,c^{2n-2}$$
$$+ \frac{2n}{1}\cdot\frac{2n-1}{2}\cdot\frac{2n-2}{3}\cdot\frac{2n-3}{3}\cdot\frac{1.3}{2.4}\,b^4\,c^{2n-4}$$
$$+ \frac{2n}{1}\cdot\frac{2n+1}{2}\cdot\frac{2n-2}{3}\cdot\frac{2n-3}{4}\cdot\frac{2n-4}{5}\cdot\frac{1.3.5}{2.4.6}\,b^6\,c^{2n-6} + \text{etc.}$$

§. 14. Introducamus nunc iſtos valores in ſeriem pro $\sqrt{[aa + (c + b \operatorname{coſ.} \Phi)^2]}$ exhibitam, et ſtatim per πc multiplicemus, atque integra coni ſuperficies ſequenti modo exprimetur:

$$S = \pi c a + \frac{1 \pi c}{2 a}(cc + \tfrac{1}{2} bb) - \frac{1.1.\pi c}{2.4 a^3}(c^4 + 3 bbcc + \frac{1.3}{2.4} b^4)$$
$$+ \frac{1.1.3 \pi c}{2.4.6 a^5}(c^6 + \tfrac{15}{2} bbc^4 + \frac{1.3.15}{2.4} . b^4 cc + \frac{1.3.5}{2.4.6} b^6)$$
$$- \frac{1.1.3.5 \pi c}{2.4.6.8 a^7}(c^8 + \frac{8.7.1}{1.2.2} bbc^6 + \frac{8.7.6.5}{1.2.3.4} . \frac{1.3}{2.4} b^4 c^4$$
$$+ \frac{8.7.6.5.4.3}{1.2.3.4.5.6} . \frac{1.3.5}{2.4.6} b^6 cc + \frac{1.3.5.7}{2.4.6.8} b^8).$$

§. 15. Quodſi ex ſingulis membris terminos tantum primos excerpamus, ii conſtituent hanc ſeriem:

$$\pi c (a + \frac{1cc}{2a} - \frac{1.1}{2.4} . \frac{c^4}{a^3} + \frac{1.1.3}{2.4.6} . \frac{c^6}{a^5} - \frac{1.1.3.5}{2.4.6.8} . \frac{c^8}{a^7} + \text{etc.})$$

quae ſeries manifeſto conuenit cum ea quam formula $\sqrt{(aa + cc)}$ producit, quamobrem loco omnium terminorum primorum ſcribere licebit $\pi c \sqrt{(aa + cc)}$. Simili modo ſecundos terminos ſingulorum membrorum excerpamus, qui dabunt hanc ſeriem:

$$\frac{\pi bbc}{2}(\frac{1}{2a} - \frac{1.1.6cc}{2.4a^3} + \frac{1.1.3.15c^4}{2.4.6a^5} - \frac{1.1.3.5.28c^6}{2.4.6.8a^7} \text{ etc.}), \text{ ſiue}$$
$$\frac{\pi bbc}{2a}(\tfrac{1}{2} - \frac{1.1}{2.4} . \frac{4.3}{1.2} . \frac{cc}{aa} + \frac{1.1.3}{2.4.6} . \frac{6.5}{1.2} . \frac{c^4}{a^4} - \frac{1.1.3.5}{2.4.6.8} . \frac{8.7}{1.2} . \frac{c^6}{a^6} \text{ etc.})$$

quae etiam hoc modo repraeſentari poteſt:

$$\frac{\pi bbc}{4a}(1 - \tfrac{3}{2} . \frac{cc}{aa} + \frac{3.5}{2.4} . \frac{c^4}{a^4} - \frac{3.5.7}{2.4.6} . \frac{c^6}{a^6} + \frac{3.5.7.9}{2.4.6.8} . \frac{c^8}{a^8} \text{ etc.})$$

cuius ſeriei valor manifeſto eſt $(1 + \frac{cc}{aa})^{-\frac{3}{2}}$, ita vt ſumma omnium terminorum ſecundorum ſit $= \dfrac{\pi aabbc}{4(aa + cc)^{\frac{3}{2}}}$.

§. 16. Colligamus eodem modo omnia tertia membra ſingulorum terminorum, qui omnes affecti ſunt poteſtate b^4 et conſtituunt hanc ſeriem:

$$-\frac{1.1}{2.4}\pi c.\frac{1.5}{2.4}.\frac{b^4}{a^3}+\frac{1.1.3}{2.4.6}\pi c.\frac{1.3}{2.4}.\frac{6.5.4.3}{1.2.3.4}.\frac{b^4cc}{a^5}$$
$$-\frac{1.1.3.5}{2.4.6.8}\pi c.\frac{1.3}{2.4}.\frac{8.7.6.5}{1.2.3.4}.\frac{b^4c^4}{a^7}+\frac{1.1.3.5.7}{2.4.6.8.10}\pi c.\frac{1.3}{2.4}.$$
$$\frac{10.9.8.7}{1.2.3.4}.\frac{b^4c^6}{a^9}.\ \text{etc.}$$

qui termini reducuntur ad ſequentem expreſſionem:

$$-\frac{1.3}{2.4}.\frac{\pi b^4 c}{24a^3}\left(3.1.-\frac{3.5.3}{2}.\frac{cc}{aa}+\frac{3.5.7.5}{2.4}.\frac{c^4}{a^4}-\frac{3.5.7.9.7}{2.4.6}.\frac{c^6}{a^6}+\text{etc.}\right)$$

§. 17. Iſta ſeries ſequenti modo in clariorem ordinem redigi poterit:

$$-\frac{1.3}{2.4}.\frac{\pi b^4 c}{8a^3}\left(1-\frac{5.3}{2}.\frac{cc}{aa}+\frac{5.7.5}{2.4}.\frac{c^4}{a^4}-\frac{5.7.9.7}{2.4.6}.\frac{c^6}{a^6}+\text{etc.}\right).$$

Ponamus hic breuitatis gratia $\frac{cc}{aa}=xx$, atque factorem communem $-\frac{1.3}{2.4}.\frac{\pi b^4 c}{8a^3}$ multiplicari oportebit per hanc ſeriem:

$$s=1-\frac{5.3}{2}xx+\frac{5.7.5}{2.4}x^4-\frac{5.7.9.7}{2.4.6}x^6+\frac{5.7.9.11.9}{2.4.6.8}x^8-\text{etc.}$$

Haec ſeries iam ſatis eſt regularis, et niſi poſtremi factores numerici adeſſent, eius ſummatio in promptu foret. Ad hos igitur factores tollendos vtamur integratione, ac reperiemus

$$\int s\,\partial x=x-\tfrac{5}{2}x^3+\frac{5.7}{2.4}x^5-\frac{5.7.9}{2.4.6}x^7+\frac{5.7.9.11}{2.4.6.8}x^9-\text{etc.}$$

Nouimus autem eſſe

$$(1+xx)^{-\frac{5}{2}}=1-\tfrac{5}{2}xx+\frac{5.7}{2.4}x^4-\frac{5.7.9}{2.4.6}x^6+\text{etc.},$$

vnde patet fore $\int s\,\partial x=x(1+xx)^{-\frac{5}{2}}$, hincque differentiando colligitur

$$s=(1+xx)^{-\frac{5}{2}}-5xx(1+xx)^{-\frac{7}{2}},\ \text{ſiue}$$
$$s=\frac{1-4xx}{(1+xx)^{\frac{7}{2}}}.$$

§. 18.

§. 18. Restituamus nunc loco xx valorem $\frac{cc}{aa}$, fietque $s = \frac{a^5(aa - 4cc)}{(aa + cc)^{\frac{7}{2}}}$, qui valor multiplicatus per factorem communem $-\frac{1.3}{2.4} \cdot \frac{\pi b^4 c}{8 a^3}$, dabit summam omnium terminorum tertiorum, quae ergo erit

$$= -\frac{1.3}{2.4} \cdot \frac{\pi aab^4 c}{8.} \cdot \frac{aa - 4cc}{(aa + cc)^{\frac{7}{2}}};$$

quamobrem si istae summae terminorum primorum, secundorum ac tertiorum coniungantur, pro superficie nostri coni scaleni nanciscemur sequentem expressionem:

$$S = \pi c \surd(aa + cc) + \frac{\pi aabbc}{4(aa + cc)^{\frac{3}{2}}} - \frac{1.3}{2.4} \cdot \frac{\pi aab^4 c(aa - 4cc)}{8(aa + cc)^{\frac{7}{2}}},$$

ita vt tantum supersit insuper terminos quartos, quintos etc. inuestigare, quos autem plerumque negligere licebit. Facile autem intelligitur, si etiam hos terminos summare voluerimus, denominatores futuros esse $(aa + cc)^{\frac{11}{2}}$; $(aa + cc)^{\frac{15}{2}}$ etc. verum numeratores nimis operosum foret explorare.

§. 19. Tentemus igitur summationem terminorum quartorum, qui adhibita simili operatione talem progressionem suppeditant, cuius factor communis est

$$\frac{1.3.5}{2.4.6} \cdot \frac{3.5}{1.2\ldots6} \cdot \frac{\pi b^6 c}{a^5} = \frac{1.3.5}{2^2.4^2.6^2} \cdot \frac{\pi b^6 c}{a^5}$$

in quem duci debebit haec series:

$$3 - \frac{7.5.3}{2} \cdot \frac{c^2}{a^2} + \frac{7.9.7.5}{2.4} \cdot \frac{c^4}{a^4} - \frac{7.9.11.9.7}{2.4.6} \cdot \frac{c^6}{a^6} + \frac{7.9.11.13.11.9}{2.4.6.8} \cdot \frac{c^8}{a^8} \text{ etc.}$$

Fiat igitur iterum $\frac{cc}{aa} = xx$, ac ponatur

$$s = 3 - \frac{7.5.3}{2} xx + \frac{7.9.7.5}{2.4} x^4 - \frac{7.9.11.9.7}{2.4.6} x^6 + \frac{7.9.11.13.11.9}{2.4.6.8} x^8 \text{ etc.}$$

cuius

cuius ergo feriei fummam indagari oportet, id quod fequenti modo fumus expedituri.

§. 20. Primo fcilicet, vt factores poftremi tollantur, per integrationem formetur ifta feries:

$$\int s\,\partial x = 3\,x - \frac{7.5}{2}\,x^3 + \frac{7.9.7}{2.4}\,x^5 - \frac{7.9.11.9}{2.4.6}\,x^7 + \frac{7.9.11.13.11}{2.4.6.8}\,x^9 \text{ etc.}$$

Vt nunc hinc denuo vltimos factores tollamus, multiplicemus per $x\,\partial x$ et integrando reperiemus

$$\int x\,\partial x \int s\,\partial x = x^3 - \tfrac{7}{2}\,x^5 + \frac{7.9}{2.4}\,x^7 - \frac{7.9.11}{2.4.6}\,x^9 + \frac{7.9.11.13}{2.4.6.8}\,x^{11} \text{ etc.}$$

Cum igitur fit

$$(1 + x\,x)^{-\frac{7}{2}} = 1 - \tfrac{7}{2}\,x\,x + \frac{7.9}{2.4}\,x^4 - \frac{7.9.11}{2.4.6}\,x^6 + \frac{7.9.11.13}{2.4.6.8}\,x^8 \text{ etc.}$$

manifeftum eft fore

$$\int x\,\partial x \int s\,\partial x = x^3\,(1 + x\,x)^{-\frac{7}{2}}$$

cuius differentiale per $x\,\partial x$ diuifum dabit

$$\int s\,\partial x = 3\,x\,(1 + x\,x)^{-\frac{7}{2}} - 7\,x^3\,(1 + x\,x)^{-\frac{9}{2}}$$

haecque formula denuo differentiata praebet

$$s = 3\,(1 + x\,x)^{-\frac{7}{2}} - 42\,x\,x\,(1 + x\,x)^{-\frac{9}{2}} + 63\,x^4\,(1 + x\,x)^{-\frac{11}{2}}$$

quae expreffio porro reducitur ad hanc:

$$s = \frac{3 - 36\,x\,x + 24\,x^4}{(1 + x\,x)^{\frac{11}{2}}}.$$

Scribendo igitur $\frac{c\,c}{a\,a}$ loco $x\,x$ erit

$$s = \frac{a^7\,(3\,a^4 - 36\,a\,a\,c\,c + 24\,c^4)}{(a\,a + c\,c)^{\frac{11}{2}}}$$

quae formula, ducta in factorem communem $\frac{1.3.5}{2^2.4^2.6^2} \cdot \frac{\pi\,b^6\,c}{a^5}$, praebet

bet summam omnium terminorum quartorum

$$= \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{\pi\, a\, a\, b^6\, c\, (3\, a^4 - 36\, a\, a\, c\, c + 24\, c^4)}{(a\, a + c\, c)^{\frac{11}{2}}}.$$

§. 21. Euolutio ista postrema nobis hoc eximium commodum praestat, vt etiam legem, qua sequentium terminorum summae progrediuntur, patefaciat. Quemadmodum enim, si summa terminorum tertiorum statuatur $= - \frac{1 \cdot 3}{2^2 \cdot 4^2} \cdot \frac{\pi\, b^4\, c}{a^3} \cdot s$, posito $\frac{c\, c}{a\, a} = x\, x$, pro s peruenimus ad hanc aequationem: $\int s\, \partial x = x\, (1 + x\, x)^{-\frac{5}{2}}$, ita pro terminis quartis, si earum summa ponatur $= \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{\pi\, b^6\, c}{a^5}\, s$, pro s inuenimus hanc aequationem: $\int x\, \partial x \int s\, \partial x = x^3\, (1 + x\, x)^{-\frac{7}{2}}$. Hoc modo facile patet, si summa terminorum quintorum ponatur $= - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} \cdot \frac{\pi\, b^8\, c}{a^7} \cdot s$, tum pro quantitate s inuenienda prodituram esse hanc aequationem:

$$\int x\, \partial x \int x\, \partial x \int s\, \partial x = x^5\, (1 + x\, x)^{-\frac{9}{2}}.$$

Eodemque modo pro terminis sextis, si eorum summa statuatur $= \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} \cdot \frac{\pi\, b^{10}\, c}{a^9}\, s$, tum quantitas s ex hac aequatione definiri debebit:

$$\int x\, \partial x \int x\, \partial x \int x\, \partial x \int s\, \partial x = x^7\, (1 + x\, x)^{-\frac{11}{2}};$$

sicque lex progressionis in infinitum penitus est manifesta.

§. 22. Quoniam igitur summam terminorum quartorum nobis pariter euoluere licuit, eam insuper ad summam praecedentium addamus, atque superficies nostri coni scaleni nunc accuratius sequenti forma exprimetur:

$$\pi c\sqrt{(aa+cc)}+\frac{\pi aabbc}{2^2(aa+cc)^{\frac{3}{2}}}-\frac{1.3}{2^2.4^2}.\frac{\pi aab^4c(aa-4cc)}{(aa+cc)^{\frac{7}{2}}}$$
$$+\frac{1.3.5}{2^2.4^2.6^2}.\frac{\pi aab^6c(3a^4-36aacc+24c^4)}{(aa+cc)^{\frac{11}{2}}}$$

quam formam semper adhibere licebit, quoties bb fuerit, valde paruum prae $aa+cc$, id quod duplici modo contingere poteſt, vel quando altitudo coni a plurimum ſuperat eius obliquitatem b, vel quando radius baſis c multum excedit obliquitatem b: atque ſi haec vtraque conditio locum habeat, iſta formula eo magis ad veritatem appropinquabit.

§. 23. Sin autem neutra harum conditionum locum inueniat, atque obliquitas b tam ratione altitudinis a quam radii baſeos c notabilem habeat magnitudinem, vel adeo hos terminos ſuperet, tum formula noſtra inuenta nullum plane vſum praeſtare poterit. His igitur caſibus maxima difficultas occurrit ſuperficiem coni definiendi, atque longe alia artificia deſiderantur, quorum beneficio iſta quaeſtio enodari queat.

§. 24. Conſideremus primo caſum, quo altitudo coni a penitus euaneſcit, ita vt pro elemento ſuperficiei habeamus hanc formulam: $\partial S = c\partial\Phi\sqrt{(c+b\,\text{coſ.}\,\Phi)^2}$, quam iam duplicauimus, ita vt integratione peracta tantum ſuperſit ſtatuere $\Phi = 180^\circ = \pi$. Cum igitur ſignum radicale quadrato ſit praefixum, erit vtique $\partial S = c\partial\Phi(c+b\,\text{coſ.}\,\Phi)$, vnde integrando elicitur $S = cc\Phi + bc\,\text{ſin.}\,\Phi$, vnde facto $\Phi = 180$ tota ſuperficies prodit $=\pi cc$, ſicque ipſi areae baſis erit aequalis, id quod per ſe eſt perſpicuum, quoties vertex coni intra baſin cadit; ſin autem extra baſin incidat, manifeſtum eſt ſuperficiem coni multo maiorem fore quam aream baſeos. Si enim talem conum charta ob-

obducere voluerimus, euidens eſt eo maius ſpatium requiri, quo longius vertex coni extra baſin fuerit remotus.

§. 25. Ponamus igitur verticem coni extra baſin in A incidere, ita vt ſit $CA = b$, exiſtente radio $CE = CF = c$, tum vero ex A ducantur rectae AM et AN baſis tangentes ac manifeſtum eſt ex baſis portione MEN, ſi ex ſingulis punctis ad A rectae ductae intelligantur, produci aream ex area circuli et trilineo AMFN compoſitam. Deinde ex altera baſeos parte MFN, ſi pariter ex ſingulis punctis ad A rectae agerentur, area prodibit itidem trilineo AMFN aequalis, ita vt tota coni ſuperficies aequalis ſit areae baſeos vna cum hoc trilineo bis ſumto. Ad hanc igitur aream inueniendam vocemus angulum $ACM = \zeta$, et cum ſit $AC = b$, erit recta tangens $AM = b$ ſin. ζ, ideoque area trianguli $ACM = \frac{1}{2} bc$ ſin. ζ, a quo auferatur area ſectoris $FCM = \frac{1}{2} cc\zeta$, et remanebit area trilinei $AMF = \frac{1}{2} bc$ ſin. $\zeta - \frac{1}{2} cc\zeta$, cuius duplum dabit aream trilinei $AMFN = bc$ ſin. $\zeta - cc\zeta$, quamobrem tota ſuperficies huius coni, cuius altitudo a eſt quaſi infinite parua, erit $= \pi cc + 2bc$ ſin. $\zeta - 2cc\zeta$. Tab. I. Fig. 7.

§. 26. Cum igitur ſuper hac determinatione nullum dubium ſupereſſe poſſit, quaeritur, cur calculus hoc caſu tantopere a veritate abludat? Cauſa autem ſine vllo dubio in formula radicali $\sqrt{(c + b \text{ coſ. } \phi)^2}$ latet, quae cum duplicem ſignificationem inuoluat, alteram poſitiuam, alteram negatiuam, natura noſtrae quaeſtionis manifeſto tantum valorem poſitiuum poſtulat. Quare cum poſuerimus $\partial S = c\,\partial\phi\,(c + b \text{ coſ. } \phi)$, haec poſitio eatenus tantum valet, quatenus quantitas $c + b$ coſ. ϕ eſt poſitiua, at vero, dum angulus ϕ vltra rectum augetur, quia coſ. ϕ fit negatiuus, euadere poterit $c + b$ coſ. $\phi = 0$, quando ſcilicet fit coſ. $\phi = -\frac{c}{b}$. Quare cum ſupra, ducta tan-

 gente

gente A M, fuerit cof. A C M = cof. $\zeta = \frac{c}{b}$, fequitur, fumto $\phi = \pi - \zeta$ formulam $c + b$ fof. ϕ euanefcere; fin autem angulus ϕ vltra hunc terminum augeatur, eius valor euadet negatiuus, atque in locum formulae radicalis fubftitui debebit $-c - b$ cof. ϕ.

§. 27. Ob hunc duplicem vfum formulae radicalis perfpicuum eft, integrationem formulae noftrae differentialis in duas partes diftribui debere, quarum prior petenda erit ex formula $\partial S = c\,\partial\phi\,(c + b\,\text{cof.}\,\phi)$, cuius integrale a $\phi = 0$ tantum vsque ad terminum $\phi = \pi - \zeta$ extendi debet, hinc ergo colligetur

$$S = c\,c\,(\pi - \zeta) + b\,c\,\text{fin.}\,\zeta;$$

alteram vero partem ex formula

$$\partial S = -c\,\partial\phi\,(c + b\,\text{cof.}\,\phi)$$

deduci oportet, cuius integrale a termino $\phi = \pi - \zeta$ vsque ad terminum $\phi = \pi$ extendi debet. Cum igitur integrale hinc oriundum fit $S = C - c\,c\,\phi - b\,a\,\text{fin.}\,\phi$, conftans ita definiatur, vt hoc integrale euanefcat, fumto $\phi = \pi - \zeta$; eritque idcirco $C = c\,c\,(\pi - \zeta) + b\,c\,\text{fin.}\,\zeta$. Fiat igitur nunc $\phi = \pi$, atque altera pars noftri integralis erit $= b\,c\,\text{fin.}\,\zeta - c\,c\,\zeta$, quae cum parte prius inuenta praebet totam huius coni fuperficiem

$$\pi\,c\,c + 2\,b\,c\,\text{fin.}\,\zeta - 2\,c\,c\,\zeta,$$

quia iam valor cum veritate egregie confpirat.

§. 28. Hoc cafu, quo $a = 0$ expedito, facile patet, etiam illis cafibus, quibus altitudo a eft valde parua, refolutionem bipartitam inftitui debere. Verum hic ftatim maxima fe offert difficultas in euolutione formulae radicalis

$$\sqrt{a\,a + (c + b\,\text{cof.}\,\phi)^2}.$$

Cum

Cum enim altitudo a ſit valde exigua, ſeries more ſolito hinc nata prodit ita expreſſa:

$$c+b\,\text{coſ.}\,\Phi+\frac{1}{2}\cdot\frac{aa}{c+b\,\text{coſ.}\,\Phi}-\frac{1.1}{2.4}\frac{a^4}{(c+b\,\text{coſ.}\,\Phi)^3}+\frac{1.1.3}{2.4.6}\frac{a^6}{(c+b\,\text{coſ.}\,\Phi)^5}\text{ etc.}$$

quae ſeries vtique valde conuergit, quando formula $c+b\,\text{coſ.}\Phi$ multum ſuperat altitudinem a. Quoniam autem pariter transeundum eſt per eos caſus, quibus eſt $c+b\,\text{coſ.}\,\Phi=0$, poſt primum terminum ſequentes omnes in infinitum abeunt, ideoque a veritate maxime abhorrent, atque adeo nullum adhuc artificium in Analyſi eſt repertum, quo huic incommodo medela afferri poſſet. His igitur caſibus recurrendum erit ad dimenſionem practicam, qua totam ſuperficiem coni in plures partes partiri et ſingularum areas ſeorſim exquirere ſolemus, id quod commodiſſime fiet, ſi ſuperficies coni in planum explicetur, cui operationi ſequens problema eſt deſtinatum.

Problema.

Si ſuperficies coni ſcaleni in planum explicetur, indolem figurae, quae hinc naſcetur, explorare.

Solutio.

§. 29. Concipiamus cono A E G F H, quem in figura 5 et 6 ſumus contemplati, chartam circumuolui, eamque iterum explicari in planum, veluti fig. 8. indicat, vbi A reſpondeat vertici coni, rectae autem A E et A F exhibeant latus maximum et minimum coni, ita vt area figurae E A F dimidiae ſuperficiei conicae ſit aequalis. Manentibus igitur denominationibus ſupra adhibitis, ſcilicet altitudine coni $AB=a$, obliquitate $BC=b$ et radio baſis $CE=CF=c$, erit in praeſenti figura latus maximum $AE=\sqrt{aa+(b+c)^2}$, latus vero minimum $AF=\sqrt{aa+(b-c)^2}$, longitudo autem curuae Tab. I. Fig. 8.

 ESF

ESF aequabitur semiperipheriae baseos coni, quae est πc. Euidens autem est istam curuam plurimum a natura circuli recedere, cuius ergo indolem et proprietates hic indagari oportet.

§. 30. Cum triangulum elementare ASs (fig. 6.) in ipsa superficie coni sit assumtum, id nunc in nostro plano reperietur, et quoniam rectae SP et AP in plano trianguli erant sitae, eae etiamnunc in nostrum planum incident, eritque recta SP tangens curuae in puncto S, recta vero AP erit perpendiculum ex puncto A in hanc tangentem demissum; portio vero curuae ES aequabitur arcui circulari $ES = c\Phi$, (fig. 6.) posito scilicet angulo $ECS = \Phi$. Quodsi ergo nunc has rectas vocemus $AS = v$, $AP = p$ et $SP = q$, erit ex iis quae supra attulimus $p = aa + (c + b \operatorname{cos}.\Phi)^2$ et $qq = bb \operatorname{sin}.\Phi_2$, siue $q = b \operatorname{sin}.\Phi$, vnde sit

$$vv = pp + qq = aa + bb + cc + 2bc \operatorname{cos}.\Phi.$$

Hinc autem si vocemus aream $EAS = S$, vt ∂S exprimat aream trianguli elementaris ASs, erit vti supra inuenimus

$$\partial S = \tfrac{1}{2} c \partial \Phi \sqrt{aa + (c + b \operatorname{cos}.\Phi)^2} = \tfrac{1}{2} c p \partial \Phi.$$

Quodsi iam vocemus angulum $EAS = \omega$, vt sit angulus $SAs = \partial\omega$, ob $AS = v$ area eiusdem trianguli erit $= \tfrac{1}{2} vv\partial\omega$, quamobrem habebitur haec aequatio: $vv\partial\omega = cp\partial\Phi$, ideoque $\partial\omega = \frac{cp\partial\Phi}{vv}$, siue habebimus

$$\partial\omega = \frac{c\partial\Phi\sqrt{aa + (c + b \operatorname{cos}.\Phi)^2}}{aa + bb + cc + 2bc \operatorname{cos}.\Phi},$$

cuius ergo integrale nobis praebebit ipsum angulum EAS, angulo Φ respondentem; ac si tum fiat $\Phi = 180° = \pi$, prodibit angulus EAF, cuius ergo determinatio maxime est difficilis, cum neque per logarithmos neque per arcus circulares expediri queat.

§. 31. At vero haec figura continet alia ſymptomata, quae ſatis concinne exprimere licet. Primo ſcilicet, ſi angulus, quem tangens S P cum recta A S conſtituit, vocetur A S P $= \theta$, ſtatim habemus

$$\sin.\ \theta = \frac{p}{v} = \frac{\sqrt{aa + (c + b\cos.\Phi)^2}}{\sqrt{(aa + bb + cc + 2bc\cos.\Phi)}} \text{ et}$$

$$\cos.\ \theta = \frac{q}{v} = \frac{b\sin.\Phi}{\sqrt{(aa + bb + cc + 2bc\cos.\Phi)}},$$

vnde patet in ipſo puncto E, vbi $\Phi = 0$, fieri $\cos.\ \theta = 0$, ideoque rectam A E ad curuam in E eſſe normalem, quod idem quoque euenit in puncto F, vbi $\Phi = \pi$, ita vt in ambobus terminis E et F rectae A E et A F curuae normaliter inſiſtant; in punctis autem intermediis rectae A S cum curua angulos obliquos conſtituent, quemadmodum ex quantitate tangentis S P eſt manifeſtum. Vbi imprimis notaſſe iuuabit, ſi punctum S capiatur in ipſo puncto G (fig. 5.), vbi eſt $\Phi = 90°$, tum quantitatem tangentis S P $= q$ fore $= b$, ideoque ipſi obliquitati coni aequalem. In omnibus autem reliquis punctis iſta tangens S P $= q$ minor erit quam obliquitas b.

§. 32. Praeterea vero etiam ipſam curuaturam noſtrae curuae E S F in ſingulis punctis S ſatis concinne exprimere licet. Si enim radium oſculi in puncto S deſignemus littera r, conſtat, eum ex perpendiculo in tangentem A P $= p$ ita exprimi, vt ſit $r = \frac{v\,\partial v}{\partial p}$. Cum igitur ſit

$$v\,\partial v = -\,b\,c\,\partial\Phi \sin.\Phi \text{ et}$$

$$p\,\partial p = -\,b\,\partial\Phi \sin.\Phi\,(c + b\cos.\Phi), \text{ ideoque}$$

$$\partial p = -\,\frac{b\,\partial\Phi \sin.\Phi\,(c + b\cos.\Phi)}{p},$$

his valoribus ſubſtitutis reperitur radius oſculi $r = \frac{cp}{c + b\cos.\Phi}$, vnde ſequitur in ipſo puncto E, vbi $\Phi = 0$, radium oſculi fore

$$r = \frac{cp}{c + b} = \frac{c\sqrt{aa + (c + b)^2}}{c + b};$$

at

at vero in altero termino F, vbi $\Phi = \pi$, radius oſculi erit

$$r = \frac{cp}{c-b} = \frac{c\sqrt{aa+(c-b)^2}}{c-b}.$$

Vnde patet, ſi fuerit $b > c$, hoc eſt iis caſibus, quibus altitudo A B extra baſin cadit, tum radium oſculi in F fore negatiuum, ideoque curuam in hoc loco conuexitatem verſus A obvertere; contra autem, quamdiu fuerit $b < c$, tum totam curvam vbique verſus A fore concauam.

§. 33. Quodſi porro longitudinem curuae E S ponamus $= s$ ita vt ſit $s = c\Phi$, notum eſt formulam integralem $\int \frac{\partial s}{r}$ exprimere amplitudinem arcus curuae E S, quae ſi deſignetur littera ψ, erit $\partial \psi = \frac{\partial s}{r}$, quamobrem ſubſtitutis valoribus pro ∂s et r inuentis habebimus

$$\partial \psi = \frac{\partial \Phi (c + b \cos \Phi)}{\sqrt{aa + (c + b \cos \Phi)^2}},$$

cuius formulae integratio, etiamſi pariter expediri nequeat, tamen multo ſimplicior eſt cenſenda illa, qua $\partial \omega$ exprimebatur. Inuento autem hoc angulo ψ, ex eo quoque ipſum illum angulum ω definire licebit. Ducta enim ex S ad rectam A E perpendiculari S X, angulus E S X ipſam curuae amplitudinem metitur; quare cum etiam angulus A S P $= \theta$ ſit cognitus, erit angulus A S X $= 180 - \theta - \psi$, qui cum etiam ſit $= 90° - \omega$, reperietur ipſe angulus $\omega = \theta + \psi - 90°$, ſicque integratione formulae illius difficillimae pro $\partial \omega$ inuentae ſuperſedere poterimus.

§. 34. Ex his iam, quae hactenus ſunt allata, ipſa curua E S F haud difficulter in plano deſcribi poterit, quae ſi in plures partes diuidatur, ſingularum partium areae facili negotio practice menſurari poterunt, quae in vnam ſummam collectae dabunt ſuperficiem coni ſcaleni propoſiti. Caeterum hic

ſilentio

ſilentio non eſt praetereundum, quoniam haec figura per explicationem chartae tam facile exhiberi poteſt, hinc eximium exemplum curuae maxime transcendentis obtineri, cuius nihilominus deſcriptio facillime expediri queat.

Additamentum ad §. 21.

Quodſi formulas in §. 21. traditas euoluamus, atque ſimili modo, vt ibi coepimus, ſummas terminorum quintorum, ſextorum et ſequentium actu definiamus, ſeriem haud ineleganten pro ſuperficie coni ſcaleni exhibere poterimus. Quodſi enim breuitatis gratia ponamus $c = a x$ et $\sqrt{(1 + x x)} = u$, tota coni ſcaleni ſuperficies erit $= \pi a a x u . V$, denotante V ſummam ſequentis ſeriei:

$$V = 1 + \frac{1}{2^2} \frac{b b}{1 . a a} . \frac{1}{u^4} - \frac{1 . 3}{2^2 . 4^2} . \frac{b^4}{3 . a^4} . \left(\frac{1 . 3}{u^6} - \frac{3 . 5 x x}{u^8}\right)$$

$$+ \frac{1 . 3 . 5}{2^2 . 4^2 . 6^2} . \frac{b^6}{5 . a^6} \left(\frac{1 . 3 . 5}{u^8} - 2 . \frac{3 . 5 . 7 . x x}{u^{10}} + \frac{5 . 7 . 9 . x^4}{u^{12}}\right)$$

$$+ \frac{1 . 3 . 5 . 7}{2^2 . 4^2 . 6^2 . 8^2} . \frac{b^8}{7 . a^8} \left(\frac{1 . 3 . 5 . 7}{u^{10}} - 3 . \frac{3 . 5 . 7 . 9 . x x}{u^{12}} + \frac{3 . 5 . 7 . 9 . 11 . x^4}{u^{14}} - \frac{7 . 9 . 11 . 13 . x^6}{u^{16}}\right)$$

$$+ \frac{1 . 3 . 5 . 7 . 9}{2^2 . 4^2 . 6^2 . 8^2 . 10^2} \frac{b^{10}}{9 . a^{10}} \left(\frac{1 . 3 . 5 . 7 . 9}{u^{12}} - 4 . \frac{3 . . . . 11 . x x}{u^{14}} + 6 . \frac{5 . . . . 13 . x^4}{u^{16}} \right.$$

$$\text{etc.} \qquad \left. - 4 . \frac{7 . . . . 15 . x^6}{u^{18}} + \frac{9 . . . . 17 . x^8}{u^{20}}\right)$$

$$\text{etc.}$$

Euidens autem eſt hanc ſeriem iis tantum caſibus vſum praeſtare, quibus quantitas $b b$ multo minor eſt quam formula $a a + c c$, quando autem propemodum eſt aequalis, vel adeo maior, tum neceſſario confugiendum erit ad deſcriptionem illam practicam, quam ſupra expoſuimus.