1788

De summo usu calculi imaginariorum in analysi

Leonhard Euler

DE SVMMO VSV CALCVLI IMAGINARIORVM IN ANALYSI.

Auctore
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Quanta incrementa Calculo Imaginariorum per vniuersam Analysin accepta sint referenda, nunc quidem amplius nemo dubitabit. Nuper equidem conatus sum integrationem formularum rationalium a Calculo Imaginariorum penitus liberare; veruntamen hoc negotium in casibus, vbi denominator plures habet factores inter se aequales, minus feliciter successit. Quin etiam non ita pridem in tales formulas integrales incidi, quae quomodo sine subsidio Imaginariorum tractari queant, nullo adhuc modo perspicio. Cum enim (*) ostendissem, huius formulae integralis: $\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2 \operatorname{cos.} \theta + x^{-n}}$, valorem a termino $x = 0$ vsque ad $x = 1$ extensum esse $\frac{\pi \operatorname{sin.} \frac{\theta p}{n}}{n \operatorname{sin.} \theta \operatorname{sin.} \frac{\pi p}{n}}$, denotante π peripheriam circuli cuius diameter $= 1$, inde facile deducitur haec conclusio maxime memorabilis: quod huius formulae integralis

(*) Vid. Dissertationem praecedentem pag. 20.

gralis: $\int \frac{\partial x}{x\, l\, x} \cdot \frac{x^p - x^{-p}}{x^n + 2\,\text{cof.}\,\theta + x^{-n}}$ valor, pariter a termino $x = 0$ vsque ad $x = 1$ extenfus, aequetur ifti integrali:

$$\frac{\pi}{n\,\text{fin.}\,\theta} \int \frac{\partial p\,\text{fin.}\,\frac{\theta p}{n}}{\text{fin.}\,\frac{\pi p}{n}},$$

vbi fcilicet quantitas p tanquam variabilis fpectatur, et integrale ita capitur, vt euanefcat pofito $p = 0$. Quodfi ergo nunc faciamus $\frac{p}{n} = \Phi$, integrari oportet huiusmodi formulam differentialem: $\frac{\partial \Phi\,\text{fin.}\,m\,\Phi}{\text{fin.}\,n\,\Phi}$. Quemadmodum igitur ifta integratio auxilio Imaginariorum tractari debeat, hic fum oftenfurus.

De integratione formulae

$$\int \frac{\partial \Phi\,\text{fin.}\,m\,\Phi}{\text{fin.}\,n\,\Phi}.$$

§. 1. Ante omnia hanc formulam ad quantitates algebraicas ordinarias reuocari conuenit, id quod commodius quam per Imaginaria praeftari nequit. Hunc in finem ftatuamus breuitatis gratia $t = \text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi$ et $u = \text{cof.}\,\Phi - \sqrt{-1}\,\text{fin.}\,\Phi$, ita vt fit $t\,u = 1$; tum vero erit

$$\partial t = -\partial \Phi\,(\text{fin.}\,\Phi - \sqrt{-1}\,\text{cof.}\,\Phi) \text{ ideoque}$$
$$\partial t \sqrt{-1} = -\partial \Phi\,(\text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi) = -t\,\partial \Phi,$$

vnde ergo fiet

$$\partial \Phi = -\frac{\partial t \sqrt{-1}}{t} = \frac{\partial t}{t \sqrt{-1}}.$$

§. 2. His autem formulis conftitutis, ex elementis Calculi Imaginariorum conftat effe

$$t^\lambda = \text{cof.}\,\lambda\,\Phi + \sqrt{-1}\,\text{fin.}\,\lambda\,\Phi \text{ et}$$
$$u^\lambda = \text{cof.}\,\lambda\,\Phi - \sqrt{-1}\,\text{fin.}\,\lambda\,\Phi;$$

vnde

vnde ergo colligitur $t^\lambda - u^\lambda = 2\sqrt{-1}\,\text{fin.}\,\lambda\Phi$, ideoque

$$\text{fin.}\,\lambda\Phi = \frac{t^\lambda - u^\lambda}{2\sqrt{-1}}.$$

Hinc ergo fi loco λ fcribamus numeros m et n, erit

$$\frac{\text{fin.}\,m\Phi}{\text{fin.}\,n\Phi} = \frac{t^m - u^m}{t^n - u^n},$$

quocirca, fi integrale quaefitum littera S defignemus, vt fit $S = \int \frac{\partial\Phi\,\text{fin.}\,m\Phi}{\text{fin.}\,n\Phi}$, facta fubftitutione nunc habebimus

$$\partial S = \frac{\partial t}{t\sqrt{-1}} \cdot \frac{t^m - u^m}{t^n - u^n}.$$

Quia autem eft $u = \frac{1}{t} = t^{-1}$, formula propofita ad fpeciem confuetam, folam variabilem t inuoluentem, eft reducta, cum fit

$$\partial S\sqrt{-1} = \frac{\partial t}{t}\,\frac{t^m - t^{-m}}{t^n - t^{-n}},$$

cuius formulae adeo integralis iam paffim euoluta reperitur. Hic autem probe meminiffe oportet, ipfam quantitatem t non effe realem, cum fit $t = \text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi$.

§. 3. Manifeftum hic eft ambos numeros m et n femper tanquam integros fpectari poffe, cum iis ratio indicetur, quam ambo anguli $m\Phi$ et $n\Phi$ inter fe tenent. Hic igitur ante omnia difpiciendum erit, vtrum exponens m maior minorue fit exponente n; quandoquidem notum eft, fi fuerit $m > n$, fractionem noftram effe fpuriam, atque partes integras ante ex ea elici debere quam integratio fufcipiatur. Hos ergo cafus hic primum euolui conueniet. Sit igitur primo $m = n + \lambda$, ita tamen vt fit $\lambda < n$, ac facile patebit, fractionem $\frac{t^{n+\lambda} - t^{-(n+\lambda)}}{t^n - t^{-n}}$ continere partem integram $t^\lambda + t^{-\lambda}$, qua ab ifta fractione fub-

 lata

lata remanet $\frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}}$, quae fractio non amplius est spuria. Ex parte integra autem ducta in $\frac{\partial t}{t}$ oritur integrale $\frac{t^\lambda - t^{-\lambda}}{\lambda}$. At vero est $t^\lambda - t^{-\lambda} = t^\lambda - u^\lambda = 2\sqrt{-1}\,\text{fin.}\,\lambda\Phi$, quod per $\sqrt{-1}$ diuisum dat partem integralis hinc oriundam $= \frac{2\,\text{fin.}\,\lambda\Phi}{\lambda}$.

§. 4. Sin autem fuerit $m > 2n$, siue $m = 2n + \lambda$, tum fractio nostra $\frac{t^{2n+\lambda} - t^{-2n-\lambda}}{t^n - t^{-n}}$ hanc continebit partem integram: $t^{n+\lambda} + t^{-n-\lambda}$, qua ablata remanet adhuc ista fractio: $\frac{t^\lambda - t^{-\lambda}}{t^n - t^{-n}}$, quae iam est genuina ob $\lambda < n$. At vero ex parte integra ducta in $\frac{\partial t}{t}$, oritur integrando

$$\frac{t^{n+\lambda} - t^{-n-\lambda}}{n+\lambda} = \frac{t^{n+\lambda} - u^{n+\lambda}}{n+\lambda},$$

cuius valor est: $\frac{2\sqrt{-1}\,\text{fin.}\,(n+\lambda)\Phi}{n+\lambda}$, qui per $\sqrt{-1}$ diuisus praebet partem integralis hinc natam $= \frac{2\,\text{fin.}\,(n+\lambda)\Phi}{n+\lambda}$.

§. 5. Simili modo si fuerit $m > 3n$, ac ponatur $m = 3n + \lambda$, fractio nostra erit $\frac{t^{3n+\lambda} - t^{-3n-\lambda}}{t^n - t^{-n}}$, quae continebit partem integram $t^{2n+\lambda} + t^{-2n-\lambda}$; hac autem ablata remanebit adhuc fractio $\frac{t^{n+\lambda} - t^{-n-\lambda}}{t^n - t^{-n}}$; quae etiamnunc est spuria et continet partem integram $t^\lambda + t^{-\lambda}$, qua ablata demum remanet fractio genuina $\frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}}$. Ex partibus autem integris oriuntur hae partes integralis: $\frac{2\,\text{fin.}\,(2n+\lambda)\Phi}{2n+\lambda} + \frac{2\,\text{fin.}\,\lambda\Phi}{\lambda}$.

§. 6.

§. 6. Ponamus quoque esse $m > 4n$, ideoque $m = 4n + \lambda$, et fractio nostra erit $\frac{t^{4n+\lambda} - t^{-4n-\lambda}}{t^n - t^{-n}}$, quae statim continet partem integram $t^{3n+\lambda} + t^{-3n-\lambda}$, hac autem ablata remanet adhuc ista fractio: $\frac{t^{2n+\lambda} - t^{-2n-\lambda}}{t^n - t^{-n}}$, quae denuo continet partem integram $t^{n+\lambda} + t^{-n-\lambda}$, qua subtracta tandem remanet ista fractio genuina: $\frac{t^{\lambda} - t^{-\lambda}}{t^n - t^{-n}}$. Iam vero ex partibus integris obtinentur pro integrali S istae partes:
$\frac{2\,\mathrm{sin.}\,(3n+\lambda)\Phi}{3n+\lambda} + \frac{2\,\mathrm{sin.}\,(n+\lambda)\Phi}{n+\lambda}$.

§. 7. Sit porro etiam $m > 5n$, siue $m = 5n + \lambda$, ac nostra fractio $\frac{t^{5n+\lambda} - t^{-5n-\lambda}}{t^n - t^{-n}}$ primo continebit partem integram $t^{4n+\lambda} + t^{-4n-\lambda}$, qua ablata remanet adhuc ista fractio: $\frac{t^{3n+\lambda} - t^{-3n-\lambda}}{t^n - t^{-n}}$, quae per antecedentia continet adhuc duas partes integras, scilicet $t^{2n+\lambda} + t^{-2n-\lambda}$ et $t^{\lambda} + t^{-\lambda}$, quibus ablatis remanet tandem ista fractio genuina: $\frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}}$.

§. 8. Ex his casibus iam satis perspicitur, quomodo, si exponens n adhuc maior accipiatur, partes integrae in integrale S ingredientes se sint habiturae, quas idcirco hic coniunctim aspectui exponamus.

I. Si $m = n + \lambda$, erit $\int \frac{\partial \Phi\,\mathrm{sin.}\,(n+\lambda)\Phi}{\mathrm{sin.}\,n\Phi} =$
$$\frac{2\,\mathrm{sin.}\,\lambda\Phi}{\lambda} + \int \frac{\partial t}{t\sqrt{-1}} \cdot \frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}};$$

II. Si $m = 2n + \lambda$, erit $\int \frac{\partial \Phi \, \text{fin.} \, (2n + \lambda) \Phi}{\text{fin.} \, n \Phi} =$

$$\frac{2 \, \text{fin.} \, (n + \lambda) \Phi}{n + \lambda} + \int \frac{\partial t}{t \sqrt{-1}} \cdot \frac{t^{\lambda} - t^{-\lambda}}{t^{n} - t^{-n}};$$

III. Si $m = 3n + \lambda$, erit $\int \frac{\partial \Phi \, \text{fin.} \, (3n + \lambda) \Phi}{\text{fin.} \, n \Phi} =$

$$\frac{2 \, \text{fin.} \, (2n + \lambda) \Phi}{2n + \lambda} + \frac{2 \, \text{fin.} \, \lambda \Phi}{\lambda} + \int \frac{\partial t}{t \sqrt{-1}} \cdot \frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^{n} - t^{-n}};$$

IV. Si $m = 4n + \lambda$, erit $\int \frac{\partial \Phi \, \text{fin.} \, (4n + \lambda) \Phi}{\text{fin.} \, n \Phi} =$

$$\frac{2 \, \text{fin.} \, (3n + \lambda) \Phi}{3n + \lambda} + \frac{2 \, \text{fin.} \, (n + \lambda) \Phi}{n + \lambda} + \int \frac{\partial t}{t \sqrt{-1}} \cdot \frac{t^{\lambda} - t^{-\lambda}}{t^{n} - t^{-n}};$$

V. Si $m = 5n + \lambda$, erit $\int \frac{\partial \Phi \, \text{fin.} \, (5n + \lambda) \Phi}{\text{fin.} \, n \Phi} =$

$$\frac{2 \, \text{fin.} \, (4n + \lambda) \Phi}{4n + \lambda} + \frac{2 \, \text{fin.} \, (2n + \lambda) \Phi}{2n + \lambda} + \frac{2 \, \text{fin.} \, \lambda \Phi}{\lambda} + \int \frac{\partial t}{t \sqrt{-1}} \cdot \frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^{n} - t^{-n}};$$

VI. Si $m = 6n + \lambda$, erit $\int \frac{\partial \Phi \, \text{fin.} \, (6n + \lambda) \Phi}{\text{fin.} \, n \Phi} =$

$$\frac{2 \, \text{fin.} \, (5n + \lambda) \Phi}{5n + \lambda} + \frac{2 \, \text{fin.} \, (3n + \lambda) \Phi}{3n + \lambda} + \frac{2 \, \text{fin.} \, (n + \lambda) \Phi}{n + \lambda} + \int \frac{\partial t}{t \sqrt{-1}} \cdot \frac{t^{\lambda} - t^{-\lambda}}{t^{n} - t^{-n}};$$

etc. etc.

§. 9. His igitur casibus, quibus $m > n$, felicissimo cum successu expeditis, totum negotium reducitur ad integrationem formulae $\frac{\partial \Phi \, \text{fin.} \, m \Phi}{\text{fin.} \, n \Phi}$, pro casibus quibus est $m < n$; quandoquidem ex modo allatis manifestum est, quomodo illi casus ad hos facillime reducuntur. Tum igitur ope nostrae substitutionis $t = \text{cos.} \, \Phi + \sqrt{-1} \, \text{fin.} \, \Phi$ peruenitur ad hanc formulam:

$$S \sqrt{-1} = \int \frac{\partial t}{t} \cdot \frac{t^{m} - t^{-m}}{t^{n} - t^{-n}},$$

cuius ergo integrationem data opera instituamus.

Inue-

Inuestigatio integralis

$$\int \frac{\partial t}{t} \cdot \frac{t^m - t^{-m}}{t^n - t^{-n}}$$

existente $m \lessgtr n$.

§. 10. Hic ante omnia cuncti factores trinomiales nostri denominatoris $t^n - t^{-n}$ indagari debebunt, quorum singulorum forma ita exhiberi potest: $t^1 - 2\,\text{cos.}\,\omega + t^{-1}$, vbi angulum ω ita definiri oportet, vt posito $t^1 - 2\,\text{cos.}\,\omega + t^{-1} = 0$ simul ipse denominator euanescat; tum autem exinde colligitur $t = \text{cos.}\,\omega + \sqrt{-1}\,\text{sin.}\,\omega$, vnde statim patet fore

$$t^n = \text{cos.}\,n\,\omega + \sqrt{-1}\,\text{sin.}\,n\,\omega \text{ et}$$
$$t^{-n} = \text{cos.}\,n\,\omega - \sqrt{-1}\,\text{sin.}\,n\,\omega,$$

quamobrem noster denominator reducetur ad hanc formam: $2\sqrt{-1}\,\text{sin.}\,n\,\omega$, qui ergo valor nihilo debet aequari.

§. 11. Cum igitur debeat esse $\text{sin.}\,n\,\omega = 0$, omnes valores, quos pro $n\,\omega$ accipere licet, erunt $0\,\pi$, π, $2\,\pi$, $3\,\pi$, $4\,\pi$, etc. vnde ipsius anguli ω valores erunt $\frac{0\,\pi}{n}$, $\frac{1\,\pi}{n}$, $\frac{2\,\pi}{n}$, $\frac{3\,\pi}{n}$, etc. et in genere $\frac{i\,\pi}{n}$, denotante i numerum integrum quemcunque. Hinc igitur pro omnibus factoribus nostri denominatoris videntur capi debere n horum valorum; verum manifestum est, quotcunque tales formulae $t^1 - 2\,\text{cos.}\,\omega + t^{-1}$ in se inuicem multiplicentur, vltimum terminum nunquam prodire posse $-t^{-n}$. At vero hic meminisse oportet, quae circa huiusmodi integrationes in genere sunt praecepta: scilicet talem factorem trinomialem $t\,t - 2\,t\,\text{cos.}\,\omega + 1$, casu quo $\omega = 0$, non factorem quadratum $(t-1)^2$, sed tantum simplicem $t - 1$ innui, quod idem quoque euenit si $\omega = \pi$, tum enim quoque non factor quadratus $(t+1)^2$, sed tantum simplex $t + 1$ est sumendus, quare cum hi ipsi casus inter valores ipsius ω occurrant, necesse est vt numerus ho-

horum factorum vnitate augeatur. Hic autem commode vsu venit, vt isti casus ex valoribus $\omega = 0$ et $\omega = \pi$ oriundi e medio tollantur.

§. 12. Cum igitur fractionem nostram $\frac{t^m - t^{-m}}{t^n - t^{-n}}$ in meras fractiones simplices resolui oporteat, quarum denominatores sint $t^1 - 2\,\text{cos}.\,\omega + t^{-1}$, pro vnaquaque harum fractionum statuamus

$$\frac{t^m - t^{-m}}{t^n - t^{-n}} = \frac{\Delta}{t - 2\,\text{cos}.\,\omega + t^{-1}} + R$$

vbi R complectatur omnes reliquas fractiones, et nunc vtrinque multiplicemus per $t - 2\,\text{cos}.\,\omega + t^{-1}$, vt prodeat

$$\frac{(t^m - t^{-m})(t - 2\,\text{cos}.\,\omega + t^{-1})}{t^n - t^{-n}} = \Delta + R(t - 2\,\text{cos}.\,\omega + t^{-1});$$

vnde si iam ponamus $t - 2\,\text{cos}.\,\omega + t^{-1} = 0$, quod fit sumendo $t = \text{cos}.\,\omega + \sqrt{-1}\,\text{fin}.\,\omega$, hinc colligitur numerator nostrae fractionis

$$\Delta = (t^m - t^{-m})\frac{t - 2\,\text{cos}.\,\omega + t^{-1}}{t^n - t^{-n}}.$$

Tum autem manifestum est in hac fractione, ad quam sumus deducti, hoc casu tam numeratorem quam denominatorem in nihilum abire, vnde iuxta regulam notissimam eorum loco sua scribamus differentialia, atque ista fractio induet hanc formam: $\frac{t^1 - t^{-1}}{n(t^n + t^{-n})}$, vbi manifesto erit $t - t^{-1} = 2\sqrt{-1}\,\text{fin}.\,\omega$, at $t^n + t^{-n} = 2\,\text{cos}.\,n\omega$, ita vt nunc valor huius fractionis futurus sit $\frac{\sqrt{-1}\,\text{fin}.\,\omega}{n\,\text{cos}.\,n\omega}$, qui ductus in $t^m - t^{-m} = 2\sqrt{-1}\,\text{fin}.\,m\omega$ dabit numeratorem nostrum quaesitum $\Delta = -\frac{2\,\text{fin}.\,\omega\,\text{fin}.\,m\omega}{n\,\text{cos}.\,n\omega}$.

Quia

Quia autem est sin. $n\omega = 0$, semper erit vel cos. $n\omega = 1$, vel cos. $n\omega = -1$, provti, statuendo in genere $\omega = \frac{i\pi}{n}$, numerus i fuerit vel par, vel impar.

§. 13. Inuenta igitur hac fractione: $-\frac{2\,\text{sin.}\,\omega\,\text{sin.}\,m\omega}{n\,\text{cos.}\,n\omega}$, ea in $\frac{\partial t}{t}$ multiplicetur et integretur, sicque ad istam pertingimus formulam integralem:

$$-\frac{2\,\text{sin.}\,\omega\,\text{sin.}\,m\omega}{n\,\text{cos.}\,n\omega}\cdot\frac{\partial t}{t}\cdot\frac{1}{t - 2\,\text{cos.}\,\omega + t^{-1}},$$

cuius quidem integratio nulla amplius laborat difficultate: perduceret enim ad arcum circuli cuius tangens $= \frac{t\,\text{sin.}\,\omega}{1 - t\,\text{cos.}\,\omega}$; verum quia ipsa quantitas t iam est imaginaria, hinc parum lucraremur, quoniam necesse foret istum arcum imaginarium ad quantitates reales reducere, siquidem constat, arcus imaginarios ad logarithmos reales reduci.

§. 14. Vt igitur hunc laborem euitemus, loco nostrae variabilis t ipsum angulum Φ rursus in calculum reuocemus, et quia iam vidimus esse $\frac{\partial t}{t} = \partial\Phi\sqrt{-1}$, tum vero $t + u = 2\,\text{cos.}\,\Phi$, hisce valoribus substitutis formula integranda erit $-\frac{\text{sin.}\,\omega\,\text{sin.}\,m\omega}{n\,\text{cos.}\,n\omega}\cdot\frac{\partial\Phi\sqrt{-1}}{\text{cos.}\,\Phi - \text{cos.}\,\omega}$, quae formula per $\sqrt{-1}$ diuisa praebet partem ipsius integralis quaesiti S, ita vt sit

$$S = -\frac{\text{sin.}\,\omega\,\text{sin.}\,m\omega}{n\,\text{cos.}\,n\omega}\int\frac{\partial\Phi}{\text{cos.}\,\Phi - \text{cos.}\,\omega},$$

siquidem angulo ω successiue omnes suos valores tribuamus; vbi per se manifestum est, in hac integratione angulum ω esse constantem solumque Φ variabilem.

§. 15. Ex coëfficiente huius formulae statim patet, quod iam supra innuimus, ex valoribus ipsius ω primo et extremo, scilicet $\omega = 0$ et $\omega = \pi$, partes integralis sponte e me-

 dio

dio tolli, ita vt nunc sufficiat loco ω successiue substitui hos valores: $\frac{\pi}{n}$, $\frac{2\pi}{n}$, $\frac{3\pi}{n}$, ... $\frac{(n-1)\pi}{n}$. Vbi recordandum, dum statuitur $\omega = \frac{i\pi}{n}$, quoties i fuerit numerus par, fore $\cos. n\omega = +1$; sin autem sit i numerus impar, tum fore $\cos. n\omega = -1$. Quibus obseruatis totum negotium reductum est ad integrationem huius formulae satis memorabilis: $\int \frac{\partial \Phi}{\cos. \Phi - \cos. \omega}$.

§. 16. Facile quidem foret istam formulam ad quantitates reales consuetas reuocare; interim tamen sequenti modo haec integratio facilius et elegantius absolui potest. Ponamus enim breuitatis gratia $\frac{\partial \Phi}{\cos. \Phi - \cos. \omega} = \partial s$, et secundum calculum angulorum iam satis vulgatum nouimus esse

$$\cos. \Phi - \cos. \omega = 2 \sin. \tfrac{\omega + \Phi}{2} . \sin. \tfrac{\omega - \Phi}{2},$$

sicque habebimus:

$$\partial s = \frac{\partial \Phi}{2 \sin. \frac{\omega + \Phi}{2} \sin. \frac{\omega - \Phi}{2}}, \text{ siue}$$

$$\frac{2 \partial s}{\partial \Phi} = \frac{1}{\sin. \frac{\omega + \Phi}{2} . \sin. \frac{\omega - \Phi}{2}},$$

quae fractio, quia denominator duobus constat factoribus, commode resolui potest in duas fractiones huiusmodi:

$$\frac{\alpha \cos. \frac{\omega + \Phi}{2}}{\sin. \frac{\omega + \Phi}{2}} + \frac{\beta \cos. \frac{\omega - \Phi}{2}}{\sin. \frac{\omega - \Phi}{2}},$$

vbi statim patet sumi debere $\beta = \alpha$, tum enim summa harum fractionum prodit $\frac{\alpha \sin. \omega}{\sin. \frac{\omega + \Phi}{2} \sin. \frac{\omega - \Phi}{2}}$, vnde $\alpha = \beta = \frac{1}{\sin. \omega}$. Hinc autem erit

$$\partial s = \frac{1}{2 \sin. \omega} . \left(\frac{\partial \Phi \cos. \frac{\omega + \Phi}{2}}{\sin. \frac{\omega + \Phi}{2}} + \frac{\partial \Phi \cos. \frac{\omega - \Phi}{2}}{\sin. \frac{\omega - \Phi}{2}} \right),$$

in

in quibus formulis numerator manifesto est differentiale denominatoris, vnde concludimus fore

$$s = \frac{1}{\sin.\,\omega}\, l\, \frac{\sin.\frac{\omega+\Phi}{2}}{\sin.\frac{\omega-\Phi}{2}}.$$

§. 17. Inuento iam hoc integrali, in quo cardo totius inuestigationis versabatur, quilibet factor denominatoris in valorem integralem quaesitum S ductus suppeditat istam partem:

$$-\frac{\sin.\,m\,\omega}{n\,\cos.\,n\,\omega}\cdot l\, \frac{\sin.\frac{\omega+\Phi}{2}}{\sin.\frac{\omega-\Phi}{2}},$$

vbi tantum opus est vt loco anguli ω successiue omnes eius valores debiti substituantur, tum enim aggregatum omnium harum formularum praebebit verum valorem integralis $S = \int \frac{\partial\Phi\,\sin.\,m\,\Phi}{\sin.\,n\,\Phi}$.

§. 18. Quo autem totum integrale succinctius repraesentare valeamus, ponamus breuitatis gratia $\frac{\pi}{n} = 2\,\alpha$, ita vt valores ipsius ω futuri sint $2\,\alpha$, $4\,\alpha$, $6\,\alpha$, ... $2\,(n-1)\,\alpha$; tum vero sit $\Phi = 2\,\psi$, atque formulae integralis $\int \frac{2\,\partial\psi\,\sin.\,2\,m\,\psi}{n\,\sin.\,2\,n\,\psi}$ valor completus erit

$$\begin{aligned} S = &\frac{\sin.\,2\,m\,\alpha}{n}\, l\frac{\sin.\,\alpha+\psi}{\sin.\,\alpha-\psi} - \frac{\sin.\,4\,m\,\alpha}{n}\, l\frac{\sin.\,2\,\alpha+\psi}{\sin.\,2\,\alpha-\psi} \\ &+\frac{\sin.\,6\,m\,\alpha}{n}\, l\frac{\sin.\,3\,\alpha+\psi}{\sin.\,3\,\alpha-\psi} - \frac{\sin.\,8\,m\,\alpha}{n}\, l\frac{\sin.\,4\,\alpha+\psi}{\sin.\,4\,\alpha-\psi} \\ &+\frac{\sin.\,10\,m\,\alpha}{n}\, l\frac{\sin.\,5\,\alpha+\psi}{\sin.\,5\,\alpha-\psi} - \frac{\sin.\,12\,m\,\alpha}{n}\, l\frac{\sin.\,6\,\alpha+\psi}{\sin.\,6\,\alpha-\psi}, \\ &\text{etc.} \end{aligned}$$

donec horum membrorum numerus fit $n-1$. Haec autem formula tantum valet quando $m < n$: si enim fuerit $m > n$, iam ante ostendimus, cuiusmodi termini insuper debeant adiungi.

§. 19. Hic obseruandum est haec integralia ita esse sumta, vt euanescant posito $\Phi = 0$, quoniam hoc casu omnes

 loga-

logarithmi ad vnitatem referuntur. Deinde etiam euidens eſt, ſi angulus ψ augeatur vsque ad α, tum integrale iam in infinitum excreſcere; vnde patet hunc angulum non vltra iſtum terminum augeri conuenire. Verum etiam caſus initio memoratus, qui ad hanc formulam integralem ducit, non poſtulat vt iſte angulus vltra hunc terminum augeatur, quamobrem operae pretium erit integrationem inuentam ad hunc ipſum caſum accommodare.

Problema.

Valorem iſtius formulae integralis:

$$\int \frac{\partial x}{x\, l\, x} \cdot \frac{x^{p} - x^{-p}}{x^{n} + 2 \operatorname{cos.} \theta + x^{-n}},$$

a termino $x = 0$ *ad* $x = 1$ *extenſum, per expreſſionem finitam aſſignare.*

Solutio.

§. 20. Quoniam iſtum valorem quaeſitum reduxi ad hanc formulam integralem: $\frac{\pi}{n \operatorname{ſin.} \theta} \int \frac{\partial p \operatorname{ſin.} \frac{\theta p}{n}}{\operatorname{ſin.} \frac{\pi p}{n}}$, primum tenendum eſt, eum finite exprimi non poſſe, niſi angulus θ ad π habeat rationem rationalem. Ponamus ergo hanc rationem eſſe $\theta : \pi = \mu : \nu$, ita vt μ et ν ſint numeri integri, quamobrem pro formula ante tractata ſtatuamus $m = \mu$ et $n = \nu$, vnde fiet angulus $\nu \Phi = \frac{\pi p}{n}$. Ponamus hic breuitatis gratia $\frac{p}{n} = r$, vt habeamus $\Phi = \frac{\pi r}{\nu}$, et valor, quem quaerimus, ob $p = n r$, erit $\frac{\pi}{\operatorname{ſin.} \theta} \int \frac{\partial r \operatorname{ſin.} \theta r}{\operatorname{ſin.} \pi r}$, quare cum hinc fiat $\Phi = \frac{\pi r}{\nu}$, formula ſupra tractata $\int \frac{\partial \Phi \operatorname{ſin} m \Phi}{\operatorname{ſin.} n \Phi}$ abibit in hanc: $S = \frac{\pi}{\nu} \int \frac{\partial r \operatorname{ſin.} \frac{\mu \pi r}{\nu}}{\operatorname{ſin.} \pi r}$, ſicque valor, quem hic quaerimus, erit $\frac{\nu . S}{\operatorname{ſin.} \theta}$, ita vt tantum opus ſit valorem ipſius S pro hoc caſu euoluere.

§. 21.

§. 21. Confideremus nunc primum valorem ipfius ω, qui erat $\omega = \frac{\pi}{n} = \frac{\pi}{\nu}$, qui pro S produxit partem integralem

$$-\frac{\text{fin.}\, m\omega}{n\, \text{cof.}\, n\omega}\, l\, \frac{\text{fin.}\frac{\omega+\Phi}{2}}{\text{fin.}\frac{\omega-\Phi}{2}},$$

erit hic $m\omega = \frac{\mu\pi}{\nu} = \theta$ et $\text{cof.}\, n\omega = -1$; tum vero

$$\omega + \Phi = \frac{\pi}{\nu}(1 + r) \text{ et } \omega - \Phi = \frac{\pi}{\nu}(1 - r);$$

Primum igitur hic fumi debet angulus $\frac{\pi}{2\nu}$, quem breuitatis gratia ponamus $= \varrho$, vt fit $\varrho = \frac{\pi}{2\nu}$, et prima pars noftrae formulae S erit $\frac{\text{fin.}\,\theta}{\nu}\, l\, \frac{\text{fin.}\,\varrho(1+r)}{\text{fin.}\,\varrho(1-r)}$; fequentes autem partes erunt

$$-\frac{\text{fin.}\,2\theta}{\nu}\, l\, \frac{\text{fin.}\,\varrho(2+r)}{\text{fin.}\,\varrho(2-r)};$$

$$+\frac{\text{fin.}\,3\theta}{\nu}\, l\, \frac{\text{fin.}\,\varrho(3+r)}{\text{fin.}\,\varrho(3-r)};$$

etc.

quae partes ductae in $\frac{\nu}{\text{fin.}\,\theta}$ praebent ipfum valorem quem noftrum problema poftulat, qui ergo erit

$$\frac{\text{fin.}\,\theta}{\text{fin.}\,\theta}\, l\, \frac{\text{fin.}\,\varrho(1+\varrho)}{\text{fin.}\,\varrho(1-\varrho)} - \frac{\text{fin.}\,2\theta}{\text{fin.}\,\theta}\, l\, \frac{\text{fin.}\,\varrho(2+r)}{\text{fin.}\,\varrho(2-r)}$$
$$+\frac{\text{fin.}\,3\theta}{\text{fin.}\,\theta}\, l\, \frac{\text{fin.}\,\varrho(3+\varrho)}{\text{fin.}\,\varrho(3-\varrho)} - \frac{\text{fin.}\,4\theta}{\text{fin.}\,\theta}\, l\, \frac{\text{fin.}\,\varrho(4+r)}{\text{f.n.}\,\varrho(4-r)} \text{ etc.}$$

quae membra eo vsque continuari debent, donec eorum numerus fiat $\nu - 1$, vbi pro noftro problemate tantum notetur effe $r = \frac{p}{n}$ et $\varrho = \frac{\pi}{2\nu}$, exiftente $\theta : \pi = \mu : \nu$, fiue $\theta = \frac{\mu\pi}{\nu}$, ita vt μ fit numerus integer. Cum igitur in formula propofita exponens p neceffario minor fit quam n, erit r vnitate minor, ideoque omnes iftae formulae finitae.

§. 22. Forma igitur generalis omnium partium, ex quibus hoc integrale conftat, eft $\pm \frac{\text{fin.}\, i\theta}{\text{fin.}\,\theta}\, l\, \frac{\text{fin.}\,\varrho(i+r)}{\text{fin.}\,\varrho(i-r)}$, vbi fignum fuperius + valet, quoties i fuerit numerus impar, inferius vero

—, ſi par. Pro vltima igitur harum partium erit $i = \nu - 1$. Vbi probe notetur, ſi ſumeremus $i = \nu$, partem hinc reſultantem ſponte eſſe euanituram, propterea quod $i\rho = \nu\rho = \frac{\pi}{2}$, ideoque ambo ſinus poſt logarithmum inter ſe aequales, ita vt perinde ſit, ſiue membrorum numerus ſtatuatur $= \nu - 1$, ſiue $= \nu$.

§. 23. Conſideremus nunc vltimum membrum noſtri valoris integralis, ſumendo $i = \nu - 1$, vnde fiet $\text{ſin.}\,(\nu - 1)\,\theta = \text{ſin.}\,(\mu\pi - \theta)$, qui erit $= \text{ſin.}\,\theta$, ſi μ fuerit numerus impar, ſin autem μ fuerit numerus par, is erit $= -\text{ſin.}\,\theta$. Tum vero erit $i\rho = (\nu - 1)\rho = \frac{\pi}{2} - \frac{\pi}{2\nu}$, ideoque

$$\text{ſin.}\,(\nu - 1)\rho = \text{ſin.}\,(\tfrac{\pi}{2} - \rho(1 - r)) = \text{coſ.}\,\rho(1 - r).$$

Simili modo pro denominatore erit

$$\text{ſin.}\,\rho(i - r) = \text{ſin.}\,(\tfrac{\pi}{2} - \rho(1 + r)) = \text{coſ.}\,\rho(1 + r);$$

ita vt in vltimo membro coſinus eorundem angulorum occurrant, quorum ſinus occurrunt in primo membro, quae permutatio etiam reperietur in membro penultimo et ſecundo, tum vero etiam in antepenultimo et tertio, vnde bina huiusmodi membra in vnum coniungi poterunt.

Caſus I. ν par. μ par. §. 24. Hic autem quatuor caſus examinari conuenit, prouti ambo numeri μ et ν fuerint numeri vel pares vel impares. Sint igitur primo ambo pares, vnde coëfficiens vltimi membri erit $+\frac{\text{ſin.}\,\mu\pi - \theta}{\text{ſin.}\,\theta} = -\frac{\text{ſin.}\,\theta}{\text{ſin.}\,\theta}$, ideoque totum membrum vltimum $= -\frac{\text{ſin.}\,\theta}{\text{ſin.}\,\theta}\, l\, \frac{\text{coſ.}\,\rho(1 - r)}{\text{coſ.}\,\rho(1 + r)}$, quamobrem primum membrum cum vltimo coniunctum dabit

$$\frac{\text{ſin.}\,\theta}{\text{ſin.}\,\theta}\, l\, \frac{\text{ſin.}\,\rho(1 + r)}{\text{ſin.}\,\rho(1 - r)} \cdot \frac{\text{coſ.}\,\rho(1 + r)}{\text{coſ.}\,\rho(1 - r)} = \frac{\text{ſin.}\,\theta}{\text{ſin.}\,\theta}\, l\, \frac{\text{ſin.}\,2\rho(1 + r)}{\text{ſin.}\,2\rho(1 - r)}.$$

Simili modo ſecundum membrum et penultimum coaleſcent in $-\frac{\text{ſin.}\,2\theta}{\text{ſin.}\,\theta}\, l\, \frac{\text{ſin.}\,2\rho(2 + r)}{\text{ſin.}\,2\rho(2 - r)}$; tum vero etiam membrum tertium cum

ante-

antepenultimo dabit $+\frac{\sin. 3\theta}{\sin.\theta}\, l\frac{\sin. 2\rho(3+r)}{\sin. 2\rho(3-r)}$. Sicque de ceteris, ita vt hoc modo numerus membrorum ad ſemiſſem reducatur.

§. 25. Maneat nunc ν numerus par, ſit vero μ numerus impar, eritque coëfficiens vltimi membri $\frac{\sin.\theta}{\sin.\theta}$, quod ergo cum primo coniunctum dabit Caſus II. ν par. μ impar.

$$+\frac{\sin.\theta}{\sin.\theta}\, l\frac{\sin.\rho(1+r)}{\sin.\rho(1-r)}\cdot\frac{\cos.\rho(1-r)}{\cos.\rho(1+r)}=\frac{\sin.\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\rho(1+r)}{\text{tang.}\,\rho(1-r)}.$$

Eodem modo membrum ſecundum cum penultimo contrahetur in hanc formam: $-\frac{\sin. 2\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\rho(2+r)}{\text{tang.}\,\rho(2-r)}$; at tertium membrum cum antepenultimo coniunctum dabit $\frac{\sin. 3\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\rho(3+r)}{\text{tang.}\,\rho(3-r)}$.

§. 26. Sit nunc ν numerus impar, at μ numerus par, et ob priorem conditionem coëfficiens vltimi termini erit $-\frac{\sin.\mu\pi-\theta}{\sin.\theta}$, qui ob μ numerum parem fiet $+\frac{\sin.\theta}{\sin.\theta}$, ideoque vti in caſu ſecundo, vnde etiam primum membrum cum vltimo iunctum dabit $\frac{\sin.\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\rho(1+r)}{\text{tang.}\,\rho(1-r)}$; ſecundum vero cum penultimo iunctum $-\frac{\sin. 2\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\rho(2+r)}{\text{tang.}\,\rho(2-r)}$; tum vero etiam tertium cum antepenultimo iunctum dat $\frac{\sin. 3\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\rho(3+r)}{\text{tang.}\,\rho(3-r)}$. Caſus III. ν impar. μ par.

§. 27. Sint denique ambo numeri μ et ν impares, atque euidens eſt hunc caſum ad primum eſſe rediturum, ideoque primum et vltimum membrum contrahi in $\frac{\sin.\theta}{\sin.\theta}\, l\frac{\sin. 2\rho(1+r)}{\sin. 2\rho(1-r)}$, ſecundum et penultimum in $\frac{\sin. 2\theta}{\sin.\theta}\, l\frac{\sin. 2\rho(2+r)}{\sin. 2\rho(2-r)}$, tertium et antepenultimum in $\frac{\sin. 3\theta}{\sin.\theta}\, l\frac{\sin. 2\rho(3+r)}{\sin. 2\rho(3-r)}$. Vnde patet hos quatuor caſus ad duos reduci poſſe, prouti ambo numeri μ et ν fuerint vel eiusdem indolis, ſcilicet ambo vel pares vel impares, vel diuerſae indolis: alter par, alter impar. Priore caſu eadem contractio locum habebit, quam caſu primo dedimus, poſteriore vero quam pro ſecundo dedimus. Caſus IV. ν impar. μ impar.

§. 28.

§. 28. Ex his intelligitur, si numerus ν fuerit impar ideoque numerus membrorum primum inuentorum $\nu - 1$ par, tum omnia illa membra contrahi in numerum duplo minorem, scilicet $\frac{\nu - 1}{2}$. At vero si ν fuerit numerus par, ob $\nu - 1$ imparem, facta illa contractione remanebit vnum membrum medium respondens valori $i = \frac{\nu}{2}$, pro quo iste reperietur logarithmus:

$$l \frac{\text{sin.}\, \varrho (\frac{\nu}{2} + r)}{\text{sin.}\, \varrho (\frac{\nu}{2} - r)} = l \frac{\text{sin.}\, (\frac{\pi}{4} + \varrho r)}{\text{sin.}\, (\frac{\pi}{4} - \varrho r)}.$$

Quia igitur est $\text{sin.}\, (\frac{\pi}{4} - \varrho r) = \text{cos.}\, (\frac{\pi}{4} + \varrho r)$, euidens est hoc casu haberi $l\, \text{tang.}\, (\frac{\pi}{4} + \varrho r)$, coëfficiens autem erit $\pm \frac{\text{sin.}\, \frac{\nu}{2} \theta}{\text{sin.}\, \theta}$, vbi signum superius valebit si $\frac{\nu}{2}$ fuerit impar, inferius vero si par. Est vero $\text{sin.}\, \frac{\nu}{2} \theta = \text{sin.}\, \frac{\mu \pi}{2}$; vnde patet, si fuerit μ numerus par, hoc membrum penitus e medio tolli; sin autem μ fuerit numerus par, tum $\text{sin.}\, \frac{\mu \pi}{2}$ erit vel $+1$ vel -1. Ista ambiguitas autem iam ante est sublata. His notatis sequentia exempla simpliciora percurramus; vbi notasse iuuabit, numerum μ semper minorem esse debere quam ν, neque tamen sumi posse $\mu = 0$.

§. 29. Quo autem euolutionem casuum specialium faciliorem reddamus, denotet Σ formulam illam integralem, cuius valorem hactenus per partes euoluimus, ita vt sit

$$\Sigma = \frac{\pi}{\text{sin.}\, \theta} \int \frac{\partial r\, \text{sin.}\, \theta r}{\text{sin.}\, \pi r};$$

tum igitur duos casus distingui conueniet, prouti ambo numeri μ et ν fuerint eiusdem vel diuersae indolis.

I. Sint μ et ν eiusdem indolis, eritque

$$\Sigma = \frac{\text{sin.}\, \theta}{\text{sin.}\, \theta} l \frac{\text{sin.}\, 2\varrho (1 + r)}{\text{sin.}\, 2\varrho (1 - r)} - \frac{\text{sin.}\, 2\theta}{\text{sin.}\, \theta} l \frac{\text{sin.}\, 2\varrho (2 + r)}{\text{sin.}\, 2\varrho (2 - r)} + \frac{\text{sin.}\, 3\theta}{\text{sin.}\, \theta} l \frac{\text{sin.}\, 2\varrho (3 + r)}{\text{sin.}\, 2\varrho (3 - r)}$$
$$- \frac{\text{sin.}\, 4\theta}{\text{sin.}\, \theta} l \frac{\text{sin.}\, 2\varrho (4 + r)}{\text{sin.}\, 2\varrho (4 - r)} \text{ etc.}$$

quas

quas formulas non vltra multitudinem $\frac{\nu-1}{2}$ continuari necesse est; neque enim hic terminus medius locum habet: si enim fuerit ν numerus par, erit etiam μ par, ideoque termini medii coëfficiens euanescit.

II. Sint numeri μ et ν diuersae indolis, vidimusque fore

$$\Sigma = \frac{\sin.\theta}{\sin.\theta} l \frac{\tang.\varrho(1+r)}{\tang.\varrho(1-r)} - \frac{\sin.2\theta}{\sin.\theta} l \frac{\tang.\varrho(2+r)}{\tang.\varrho(2-r)} + \frac{\sin.3\theta}{\sin.\theta} l \frac{\tang.\varrho(3+r)}{\tang.\varrho(3-r)} - \text{etc.}$$

quos terminos non vltra multitudinem $\frac{\nu-1}{2}$ continuari oportet. Hic autem, quoties ν numerus par, ideoque μ impar, occurret terminus medius, qui nunc vltimum locum occupabit, eritque $\pm \frac{1}{\sin.\theta} l \tang.(\frac{\pi}{4} + \varrho r)$, vbi signorum ambiguitas sequitur alternationem signorum. Ceterum hic vbique recordandum est esse $\varrho = \frac{\pi}{2\nu}$ et $\theta = \frac{\mu\pi}{\nu}$.

Exemplum I, quo $\nu = 2$.

§. 30. Hic igitur erit $\varrho = \frac{\pi}{4} = 45°$; at numerus μ necessario est $= 1$. Quia igitur $\frac{\nu-1}{2} = \frac{1}{2}$, hic solus terminus, quem medium vocamus, occurrit, ita vt nunc habeamus

$$\Sigma = l \tang.\tfrac{\pi}{4}(1+r) = l \tang. 45°(1+r),$$

qui valor sponte ex forma generali deducitur, cum sit

$$\Sigma = \pi \int \frac{\partial r \sin.\frac{\pi r}{2}}{\sin.\pi r};$$

est vero $\sin.\pi r = 2 \sin.\frac{\pi r}{2} \cos.\frac{\pi r}{2}$, vnde fit

$$\Sigma = \frac{\pi}{2} \int \frac{\partial r}{\cos.\frac{\pi r}{2}}.$$

Quod si iam ponamus $\frac{\pi r}{2} = \Phi$, ob $\frac{\pi \partial r}{2} = \partial \Phi$, erit

 $\Sigma =$

$$\Sigma = \int \frac{\partial \Phi}{cos. \Phi} = l \text{ tang.} (45^\circ + \tfrac{1}{2}\Phi).$$

Reſtituto ergo pro Φ valore aſſumto erit $\Sigma = l \text{ tang.} 45^\circ (1 + r)$, vti inuenimus.

Exemplum II, quo $\nu = 3$.

§. 31. Hic ergo erit $\varrho = \frac{\pi}{6} = 30^\circ$, et quia $\frac{\nu - 1}{2} = 1$, integrale noſtrum vnico conſtabit termino. Nunc autem numerus μ duos valores habere poteſt: 1 et 2. Sit primo $\mu = 1$ hincque $\theta = \frac{\pi}{3} = 60$, et quia ambo numeri ſunt impares, ex caſu primo colligimus $\Sigma = l \frac{\text{ſin.} 60^\circ (1 + r)}{\text{ſin.} 60^\circ (1 - r)}$. At ſi fuerit $\mu = 2$ ideoque $\theta = 120^\circ$, quia numeri μ et ν habent diſparia ſigna, ex caſu ſecundo habebimus $\Sigma = l \frac{\text{tang.} 30^\circ (1 + r)}{\text{tang.} 30^\circ (1 - r)}$.

Exemplum III, quo $\nu = 4$.

§. 32. Hic ergo erit $\varrho = \frac{\pi}{8} = 22\tfrac{1}{2}^\circ$, et quia $\frac{\nu - 1}{2} = 1\tfrac{1}{2}$, integrale vnico tantum membro integro conſtabit, niſi forte terminus medius accedat, quemadmodum ſingulis caſibus pro μ aſſumtis videbimus.

1°. Sit igitur $\mu = 1$, erit $\theta = 45^\circ$ et $2\theta = 90^\circ$. Hinc ergo ob numeros μ et ν diſpares, ex caſu ſecundo habebimus

$$\Sigma = l \frac{\text{tang.} 22\tfrac{1}{2}^\circ (1 + r)}{\text{tang.} 22\tfrac{1}{2}^\circ (1 - r)} - \sqrt{2}. \, l \text{ tang.} 22\tfrac{1}{2}^\circ (2 + r).$$

2°. Sit $\mu = 2$, eritque $\theta = 90^\circ$ et $2\theta = 180^\circ$. Hinc ex caſu primo nanciſcimur $\Sigma = l \frac{\text{ſin.} 45^\circ (1 + r)}{\text{ſin.} 45^\circ (1 - r)}$. Cum autem ſit

$$\text{ſin.} 45^\circ (1 - r) = \text{coſ.} 45^\circ (1 + r),$$

euidens eſt fore $\Sigma = l \text{ tang.} 45^\circ (1 + r)$, qui caſus vtique conuenit cum ratione $\mu : \nu = 1 : 2$.

3°.

3°. At si $\mu = 3$, ideoque $\theta = 135°$ et $2\theta = 270°$, cuius anguli sinus est -1, ob signa disparia habebimus ex casu secundo:

$$\Sigma = l \frac{\text{tang. } 22\tfrac{1}{2}°(1+r)}{\text{tang. } 22\tfrac{1}{2}°(1-r)} + \sqrt{2} . l \text{ tang. } 22\tfrac{1}{2}°(2+r).$$

Exemplum IV, quo $\nu = 5$.

§. 33. Hic ergo erit $\varrho = 18°$, et quia $\frac{\nu-1}{2} = 2$, integralia ex duobus membris integris constabunt, quia terminus medius, quem quasi dimidium spectamus, hic non occurrit.

1°. Sit $\mu = 1$, eritque $\theta = 36°$ et $2\theta = 72°$; hinc ob ambo signa eadem casus primus nobis dat

$$\Sigma = l \frac{\textit{sin.}\, 36°(1+r)}{\textit{sin.}\, 36°(1-r)} - \frac{\textit{sin.}\, 72°}{\textit{sin.}\, 36°} l \frac{\textit{sin.}\, 36°(2+r)}{\textit{sin.}\, 36°(2-r)}.$$

2°. Sit $\mu = 2$, eritque $\theta = 72°$, ideoque sin. $2\theta =$ sin. $36°$; vnde ob signa disparia casus secundus dat

$$\Sigma = l \frac{\textit{tang.}\, 18°(1+r)}{\textit{tang.}\, 18°(1-r)} - \frac{\textit{sin.}\, 36°}{\textit{sin.}\, 72°} l \frac{\textit{tang.}\, 18°(2+r)}{\textit{tang.}\, 18°(2-r)}.$$

3°. Sit $\mu = 3$, ideoque $\theta = 108°$, siue sin. $\theta =$ sin. $72°$ et sin. $2\theta = -$ sin. $36°$; vnde ob signa paria casus primus dat

$$\Sigma = l \frac{\textit{sin.}\, 36°(1+r)}{\textit{sin.}\, 36°(1-r)} + \frac{\textit{sin.}\, 36°}{\textit{sin.}\, 72°} l \frac{\textit{sin.}\, 36°(2+r)}{\textit{sin.}\, 36°(2-r)}.$$

4°. Sit denique $\mu = 4$ et $\theta = 144°$, hincque sin. $\theta =$ sin. $36°$ et sin. $2\theta = -$ sin. $72°$; vnde ob signa disparia casus II. praebet

$$\Sigma = l \frac{\textit{tang.}\, 18°(1+r)}{\textit{tang.}\, 18°(1-r)} + \frac{\textit{sin.}\, 72°}{\textit{sin.}\, 36°} l \frac{\textit{tang.}\, 18°(2+r)}{\textit{tang.}\, 18°(2-r)}.$$

Exemplum V, quo $\nu = 6$.

§. 34. Hic igitur est $\varrho = \frac{\pi}{12} = 15°$, et quia $\frac{\nu-1}{2} = \frac{5}{2}$, integralia duobus membris integris constabunt, quibus accedere potest terminus medius, siue membrum dimidium, quando scilicet μ est numerus impar.

 1°.

1°. Sit $\mu = 1$, erit $\theta = \frac{1}{6}\pi = 30^\circ$, hinc fin. $\theta = \frac{1}{2}$, fin. $2\theta = \frac{\sqrt{3}}{2}$ et fin. $3\theta = +1$; quare ob figna difparia fecundus cafus nobis fuppeditat

$$\Sigma = l\frac{\text{tang.}\,15^\circ(1+r)}{\text{tang.}\,15^\circ(1-r)} - \sqrt{3}\,.\,l\frac{\text{tang.}\,15^\circ(1+r)}{\text{tang.}\,15^\circ(1-r)} + 2\,l\,\text{tang.}\,15^\circ(3+r).$$

2°. Sit $\mu = 2$, ideoque $\theta = 60^\circ$, vnde fit fin. $\theta = \frac{\sqrt{3}}{2}$, fin. $2\theta = \frac{\sqrt{3}}{2}$ et fin. $3\theta = 0$; vnde ob figna paria ex cafu primo colligimus

$$\Sigma = l\frac{\text{fin.}\,30^\circ(1+r)}{\text{fin.}\,30^\circ(1-r)} - l\frac{\text{fin.}\,30^\circ(2+r)}{\text{fin.}\,30^\circ(2-r)},$$

quae expreffio perfecte aequalis prodiit ei quam fupra inuenimus pro cafu $\nu = 3$ et $\mu = 1$.

3°. Sit $\mu = 3$, ideoque $\theta = 90$, hinc fin. $\theta = 1$, fin. $2\theta = 0$ et fin. $3\theta = -1$; vnde ob figna difparia cafus fecundus nobis praebet

$$\Sigma = l\frac{\text{tang.}\,15^\circ(1+r)}{\text{tang.}\,15^\circ(1-r)} + * - l\,\text{tang.}\,15^\circ(3+r),\ \text{fiue}$$

$$\Sigma = l\frac{\text{tang.}\,15^\circ(1+r)}{\text{tang.}\,15^\circ(1-r)\,\text{tang.}\,15^\circ(3+r)},$$

quae expreffio aequalis effe debet ei, quae in primo exemplo prodiit, quia vtroque cafu eft $\mu : \nu = 1 : 2$.

4°. Sit $\mu = 4$, ideoque $\theta = 120^\circ$, hinc fin. $\theta = \frac{\sqrt{3}}{2}$, fin. $2\theta = -\frac{\sqrt{3}}{2}$, fin. $3\theta = 0$; vnde ob figna paria cafus primus praebet

$$\Sigma = l\frac{\text{fin.}\,30^\circ(1+r)}{\text{fin.}\,30^\circ(1-r)} + l\frac{\text{fin.}\,30^\circ(2+r)}{\text{fin.}\,30^\circ(2-r)},$$

quae conuenire debet cum fuperiore pro cafu quo $\mu : \nu = 2 : 3$.

5°. Sit $\mu = 5$, ideoque $\theta = 150^\circ$, ergo fin. $\theta = \frac{1}{2}$, fin. $2\theta = -\frac{\sqrt{3}}{2}$, fin. $3\theta = 1$; vnde ob figna difparia fecundus cafus nobis dat

$\Sigma =$

$$\Sigma = l\frac{tang.15^\circ(1+r)}{tang.15^\circ(1-r)} + \sqrt{3}.l\frac{tang.15^\circ(2+r)}{tang.15^\circ(2-r)} + 2l\,\text{tang}.\,15^\circ(3+r).$$

Exemplum VI, quo $\nu = \infty$.

§. 35. Quia igitur fractio $\frac{\mu}{\nu}$ vt euanefcens fpectatur, ponamus $\mu = 1$, ficque angulus θr prae πr euanefcet; vnde cum loco finuum angulorum θ et θr ipfos angulos ponere liceat, erit nofter valor $\Sigma = \frac{\pi \int r \partial r}{\text{fin}.\pi r}$. Deinde quia etiam angulus $\varrho = \frac{\pi}{2\nu}$ in nihilum abit, loco omnium finuum, in expreffione pro Σ inuenta occurrentium, ipfos angulos fcribere licebit, quo obferuato valor quantitatis Σ fequenti modo exprimetur:

$$l\frac{1+r}{1-r} - 2l\frac{2+r}{2-r} + 3l\frac{3+r}{3-r} - 4l\frac{4+r}{4-r} + \text{etc.}$$

§. 36. Singuli hi logarithmi commode in feries refolui poffunt. Cum enim forma generalis omnium terminorum fit $il\frac{i+r}{i-1}$, tum vero per notam refolutionem fit

$$l\frac{i+r}{i-1} = \frac{2r}{i} + \frac{2r^3}{3i^3} + \frac{2r^5}{5i^5} + \frac{2r^7}{7i^7} + \text{etc.}$$

erit totum membrum

$$= 2r\left(1 + \frac{rr}{3ii} + \frac{r^4}{5i^4} + \frac{r^6}{7i^6} + \text{etc.}\right)$$

quamobrem fingulis partibus hoc modo euolutis fiet

$$\begin{aligned}\frac{\Sigma}{2r} = &+1 + \frac{rr}{3} + \frac{r^4}{5} + \frac{r^6}{7} + \frac{r^8}{9} + \text{etc.}\\ &-1 - \frac{rr}{3.4} - \frac{r^4}{5.4^2} - \frac{r^6}{7.4^3} - \frac{r^8}{9.4^4} - \text{etc.}\\ &+1 + \frac{rr}{3.9} + \frac{r^4}{5.9^2} + \frac{r^6}{7.9^3} + \frac{r^8}{9.9^4} + \text{etc.}\\ &-1 - \frac{rr}{3.16} - \frac{r^4}{5.16^2} - \frac{r^6}{7.16^3} - \frac{r^8}{9.16^4} - \text{etc.}\\ &\text{etc.}\end{aligned}$$

§. 37. Quod fi iam iftas feries fecundum columnas verticales difponamus, quia prima columna dat

SPECIMEN SINGVLARE ANALYSEOS INFINITORVM INDETERMINATAE.

Auctore

L. EVLERO.

Conuent. exhib. d. 18 *Mart.* 1776.

§. 1.

Iam ante complures annos nouum prorſus Calculi genus adumbraui, cui *Analyſeos Infinitorum indeterminatae* nomen inpoſueram, quoniam ad Analyſin Infinitorum ordinariam eodem modo refertur, quo Analyſis Diophantea ad Algebram communem. Indoles ſcilicet huius Calculi in eo conſiſtit, vt eiusmodi relatio inter binas variabiles inueſtigetur, vnde vna pluresue formulae integrales nanciſcantur valores ſiue algebraicos, ſiue datas quadraturas inuoluentes. Veluti ſi talis definiri debeat relatio inter binas variabiles x et y, vt iſta formula integralis: $\int \surd(\partial x^2 + \partial y^2)$, algebraicum valorem adipiſcatur, vel etiam datas quantitates tranſcendentes inuoluat. Hinc enim euidens eſt curuas algebraicas obtineri, quae ſint vel rectificabiles, vel quarum rectificatio a datis quadraturis pendeat; atque hinc problema illud *Hermannianum* celeberrimum methodo directa ſolutum dedi, quo requirebantur curuae algebraicae non rectificabiles, ſed quarum rectificatio datas quadraturas inuolveret, in quibus tamen nihilominus vel vnus, vel duo, vel adeo quotquis voluerit arcus aſſignari poſſent abſolute rectificabiles,

$$1 - 1 + 1 - 1 + 1 - 1 + \text{etc.} = \tfrac{1}{2},$$

prodibit haec expreſſio:

$$\frac{\Sigma}{2r} = \tfrac{1}{2} + \tfrac{1}{3}rr\left(1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \text{etc.}\right)$$
$$+ \tfrac{1}{5}r^4\left(1 - \tfrac{1}{4^2} - \tfrac{1}{9^2} - \tfrac{1}{16^2} + \tfrac{1}{25^2} - \text{etc.}\right)$$
$$+ \tfrac{1}{7}r^6\left(1 - \tfrac{1}{4^3} + \tfrac{1}{9^3} - \tfrac{1}{16^3} + \tfrac{1}{25^3} - \text{etc.}\right)$$
$$\text{etc.}$$

Quoniam igitur harum ſerierum omnium ſummae ſunt cognitae, hinc per approximationem eo facilius valor litterae Σ definiri poterit, quia littera r ſemper denotat fractionem vnitate maiorem.

§. 38. Quod ſi ergo in ſubſidium vocemus ea quae olim circa ſummas harum poteſtatum erueram, atque iisdem denominationibus vtamur, ponendo:

$$A\pi^2 = 1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.}$$
$$B\pi^4 = 1 + \tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \tfrac{1}{25^2} + \text{etc.}$$
$$C\pi^6 = 1 + \tfrac{1}{4^3} + \tfrac{1}{9^3} + \tfrac{1}{16^3} + \tfrac{1}{25^3} + \text{etc.}$$
$$\text{etc.}$$

quoniam hinc facile ſummae deriuantur, quando terminorum ſigna alternantur, habebitur:

$$\frac{\Sigma}{2r} = \tfrac{1}{2} + \tfrac{1}{3}\left(1 - \tfrac{1}{2}\right)A\pi\pi rr + \tfrac{1}{5}\left(1 - \tfrac{1}{8}\right)B\pi^4 r^4$$
$$+ \tfrac{1}{7}\left(1 - \tfrac{1}{32}\right)C\pi^6 r^6 + \text{etc.}$$

Vbi meminiſſe conuenit eſſe

$$A = \tfrac{1}{6},\ B = \tfrac{1}{90},\ C = \tfrac{1}{945},\ D = \tfrac{1}{9450},\ E = \tfrac{1}{93555},\ \text{etc.}$$

Horum autem valorum ratio iam ſaepius abunde eſt expoſita.

SPECI-