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Consideratio motus singularis, qui in filo perfecte flexili locum habere potest

Leonhard Euler

CONSIDERATIO MOTVS PLANE SINGVLARIS, QVI IN FILO PERFECTE FLEXILI LOCVM HABERE POTEST.

Auctore
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§. 1.

Quanquam theoria non solum aequilibrii sed etiam motus pro omnibus filis tam perfecte flexibilibus quam etiam elasticis ita perfecte sit explorata, vt nihil amplius desiderari posse videatur: tamen formulae pro motu determinando traditae etiamnunc omni vsu caruerunt; cum pro nullo adhuc casu motus huiusmodi filorum definiri potuerit exceptis solis illis casibus, quibus talia fila motum reciprocum seu oscillatorium eumque adeo infinite paruum recipere valent. Huius autem defectus causa neutiquam theoriae mechanicae est tribuenda sed vnica imperfectioni analyseos adscribi debet: ita vt ante vix quicquam in hoc genere sperari possit, quam scientia analyseos insignia incrementa acceperit.

§. 2. Quin etiam casus simplicissimus, quo motus fili perfecte flexilis a nullis plane viribus sollicitati in eodem plano concitari potest, nullis adhuc artificiis a me quidem adhibitis

bitis expediri potuit. Quod quidem eo minus eſt mirandum, cum ſi loco fili conſiderentur plures virgae ita inuicem iunctae, vt circa iuncturas liberrime commoueri queant, motus nullo adhuc modo perfecte aſſignari potuerit, ſtatim ac plures duabus virgis hoc modo fuerint coniunctae.

Tab. IV. Fig. 1. §. 3. Quo igitur ſummas has difficultates penitius perſpiciamus, conſideremus filum quodcunque flexile E Y F quod a viribus quibuscunque ſollicitatum in ipſo plano tabulae vtcunque promoueatur, et ſumta in hoc plano recta fixa O A, pro axe habenda, elapſo tempore t teneat filum ſitum in figura exhibitum E Y F, a cuius puncto quocunque indefinito Y ad axem ducatur normalis Y X, vocenturque coordinatae $OX = x$ et $XY = y$, ipſa autem portio fili $EY = s$, vt ſit $\partial x^2 + \partial y^2 = \partial s^2$. Tum vero hoc tempore fili elementum $Yy = \partial s$ ſollicitetur a duabus viribus $YP = P \partial s$ et $YQ = Q \partial s$, quarum directiones ſint coordinatis parallelae. Quibus poſitis manifeſtum eſt, ambas coordinatas x et y ſpectari debere tanquam functiones duarum variabilium, arcus ſcilicet $EY = s$ ac temporis t. Vnde ſumto tempore t conſtante, vt fili figura quam ipſo tempore tenet exploretur, erit per ea quae de functionibus duarum variabilium iam ſatis ſunt explicata, $\partial x = \partial s (\frac{\partial x}{\partial s})$ et $\partial y = \partial s (\frac{\partial y}{\partial s})$, hincque ergo $(\frac{\partial x}{\partial s})^2 + (\frac{\partial y}{\partial s})^2 = 1$. At vero ſumto ſolo tempore t variabili, manente arcu $EY = s$ inuariato, coordinatae x et y pro eodem fili puncto Y ita variabunt, vt ſit $\partial x = \partial t (\frac{\partial x}{\partial t})$ et $\partial y = \partial t (\frac{\partial y}{\partial t})$, vbi notetur formulam $(\frac{\partial x}{\partial t})$ exprimere celeritatem puncti y ſecundum directionem YP, et $(\frac{\partial y}{\partial t})$ celeritatem ſecundum directionem YQ, vnde porro acceleratio motus pro puncto Y ſecundum directionem YP erit $= (\frac{\partial \partial x}{\partial t^2})$ et ſecundum directionem $YQ = (\frac{\partial \partial y}{\partial t^2})$. Praeterea

terea vero hic erit monendum, etiam ipsas vires sollicitantes P et Q vtcunque a tempore t pendere posse.

§. 4. His expositis secundum praecepta pro motu huius fili tradita ex viribus sollicitantibus deriuentur isti valores:

$$P' = P - \frac{1}{2g}\left(\frac{\partial\partial x}{\partial t^2}\right) \text{ et } Q' = Q - \frac{1}{2g}\left(\frac{\partial\partial y}{\partial t^2}\right),$$

vbi g denotat altitudinem lapsus grauium pro vno minuto secundo, siquidem tempus t in minutis secundis exprimere lubuerit. Tum vero hic littera s non solum nobis longitudinem arcus EY sed etiam eius pondus denotare assumitur, quandoquidem filo per totam longitudinem eandem crassitiem tribuimus.

§. 5. Per has autem quantitates deriuatas P′ et Q′ totus fili motus ex hac aequatione satis simplici inuestigari debet

$$\left(\frac{\partial y}{\partial s}\right)\int P'\partial s - \left(\frac{\partial x}{\partial s}\right)\int Q'\partial s = 0.$$

In quibus formulis integralibus sola quantitas s pro variabili est habenda, tempore t manente constante. Hinc igitur si loco P′ et Q′ substituamus eorum valores, aequatio nostra pro motu determinando erit

$$\left(\frac{\partial y}{\partial s}\right)\int P\partial s - \left(\frac{\partial x}{\partial s}\right)\int Q\partial s = \frac{1}{2g}\left(\frac{\partial y}{\partial s}\right)\int\partial s\left(\frac{\partial\partial x}{\partial t^2}\right) - \frac{1}{2g}\left(\frac{\partial x}{\partial s}\right)\int\partial s\left(\frac{\partial\partial y}{\partial t^2}\right).$$

Praeterea vero si tensio fili hoc tempore in puncto Y ponatur $= T$, erit

$$T = -\left(\frac{\partial x}{\partial s}\right)\int P'\partial s - \left(\frac{\partial y}{\partial s}\right)\int Q'\partial s, \text{ siue}$$

$$T = -\left(\frac{\partial x}{\partial s}\right)\int P\partial s - \left(\frac{\partial y}{\partial s}\right)\int Q\partial s + \frac{1}{2g}\left(\frac{\partial x}{\partial s}\right)\int\partial s\left(\frac{\partial\partial x}{\partial t^2}\right)$$
$$+ \frac{1}{2g}\left(\frac{\partial y}{\partial s}\right)\int\partial s\left(\frac{\partial\partial y}{\partial t^2}\right).$$

§. 6. Quod si ergo filum a nullis plane viribus sollicitari ponamus, ita vt motus fili flexilis super plano horizontali vtcunque proiecti determinari debeat, ob vires $P = 0$ et

$Q = 0$, tota motus determinatio pendebit a resolutione huius aequationis satis simplicis:

$$0 = \left(\frac{\partial y}{\partial s}\right)\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) - \left(\frac{\partial x}{\partial s}\right)\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right),$$

quae autem quomodo tractari debeat nullo plane modo perspicitur. Tum vero tensio euadet:

$$T = \frac{1}{2g}\left(\frac{\partial x}{\partial s}\right)\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) + \frac{1}{2g}\left(\frac{\partial y}{\partial s}\right)\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right).$$

Quamobrem Geometrae erunt hortandi, vt omnes vires intendere velint ad resolutionem huius aequationis expediendam.

§. 7. Equidem meos conatus etiam irritos hic communicare non dubito dum forte aliis occasionem praebere poterunt feliciori successu hunc laborem exsequendi. Primo igitur mihi erat propositum, hanc aequationem:

$$\left(\frac{\partial y}{\partial s}\right)\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) = \left(\frac{\partial x}{\partial s}\right)\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right),$$

a formulis integralibus liberare, quem in finem loco functionum x et y alias u et v in calculum introduxi, ponendo

$$\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) = \left(\frac{\partial \partial u}{\partial t^2}\right) \text{ et } \int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right) = \left(\frac{\partial \partial v}{\partial t^2}\right),$$

hinc autem differentiando sola variabili adhibita s, prodibit

$$\frac{\partial \partial x}{\partial t^2} = \frac{\partial^3 u}{\partial s \partial t^2} \text{ et } \frac{\partial \partial y}{\partial t^2} = \frac{\partial^3 v}{\partial s \partial t^2}.$$

Hinc autem porro colligemus, dum nunc solam t vt variabilem spectamus, cum sit $\partial t \left(\frac{\partial \partial x}{\partial t^2}\right) = \partial t \left(\frac{\partial^3 u}{\partial s \partial t^2}\right)$, erit integrando $\left(\frac{\partial x}{\partial t}\right) = \left(\frac{\partial \partial u}{\partial s \partial t}\right) + E$, quae constans E etiam arcum s vtcunque in se complecti potest, eodemque modo erit $\left(\frac{\partial y}{\partial t}\right) = \left(\frac{\partial \partial v}{\partial s \partial t}\right) + F$. Hae aequationes porro ducantur in ∂t ac denuo integrentur manente s constante, prodibit

$$x = \left(\frac{\partial u}{\partial s}\right) + E t + G \text{ et } y = \left(\frac{\partial v}{\partial s}\right) + F t + H,$$

vbi E, F, G, H possunt esse functiones ipsius s tantum.

§. 8.

§. 8. Hos valores denuo differentiemus sumta sola s pro variabili ac positis breuitatis gratia $\partial E = E' \partial s$, $\partial F = F' \partial s$, $\partial G = G' \partial s$ et $\partial H = H' \partial s$, obtinebimus

$$\left(\frac{\partial x}{\partial s}\right) = \left(\frac{\partial \partial u}{\partial s^2}\right) + E' t + G' \text{ et } \left(\frac{\partial y}{\partial s}\right) = \frac{\partial \partial v}{\partial s^2} + F' t + H'.$$

Quare cum esse oporteat $\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1$, omissis functionibus adiectis E, F, G, H, requiritur vt fiat $\left(\frac{\partial \partial u}{\partial s^2}\right)^2 + \left(\frac{\partial \partial v}{\partial s^2}\right)^2 = 1$. Tum vero ipsa aequatio pro motu induet hanc formam:

$$\left(\frac{\partial \partial v}{\partial s^2}\right) \cdot \left(\frac{\partial \partial u}{\partial t^2}\right) = \left(\frac{\partial \partial u}{\partial s^2}\right) \cdot \left(\frac{\partial \partial v}{\partial t^2}\right),$$

vbi quidem breuitati consulentes functiones illas arbitrarias ipsius s praetermisimus. Simili modo pro tensione habebimus:

$$T = \frac{1}{2g}\left(\frac{\partial \partial u}{\partial s^2}\right) \cdot \left(\frac{\partial \partial u}{\partial t^2}\right) + \frac{1}{2g}\left(\frac{\partial \partial v}{\partial s^2}\right) \cdot \left(\frac{\partial \partial v}{\partial t^2}\right).$$

§. 9. Totum ergo negotium iam huc est reductum, quemadmodum ambas functiones ipsarum s et t, quas posuimus u et v, comparatas esse oporteat, vt fiat

$$\left(\frac{\partial \partial v}{\partial s^2}\right) \cdot \left(\frac{\partial \partial u}{\partial t^2}\right) = \left(\frac{\partial \partial u}{\partial s^2}\right) \cdot \left(\frac{\partial \partial v}{\partial t^2}\right),$$

siue vt haec proportio non parum elegans locum habeat:

$$\frac{\partial \partial u}{\partial s^2} : \frac{\partial \partial u}{\partial t^2} = \frac{\partial \partial v}{\partial s^2} : \frac{\partial \partial v}{\partial t^2},$$

cui quidem conditioni haud difficulter infinitis modis satisfieri potest. At vero altera conditio adimplenda nunc maximae difficultati videtur obnoxia, vt scilicet euadat $\left(\frac{\partial \partial u}{\partial s^2}\right)^2 + \left(\frac{\partial \partial v}{\partial s^2}\right)^2 = 1$. Hinc igitur manifesto perspicitur, hunc casum, qui sine dubio in hoc genere tanquam simplicissimus est spectandus, tantis difficultatibus ac tenebris etiamnunc esse inuolutum, vt nulla plane via pateat ad scopum optatum perueniendi.

§. 10. Talis reductio etiam in genere fieri potest in aequatione latissime patente:

$2g(\frac{\partial y}{\partial s})\int P\partial s - 2g(\frac{\partial x}{\partial s})\int Q\partial s = (\frac{\partial y}{\partial s})\int\partial s(\frac{\partial\partial x}{\partial t^2}) - (\frac{\partial x}{\partial s})\int\partial s(\frac{\partial\partial y}{\partial t^2})$, atque adeo facilius ita instituetur. Ponatur statim $x = (\frac{\partial u}{\partial s})$ et $y = (\frac{\partial v}{\partial s})$. Hinc igitur erit $(\frac{\partial x}{\partial s}) = (\frac{\partial\partial u}{\partial s^2})$ et $(\frac{\partial y}{\partial s}) = (\frac{\partial\partial v}{\partial s^2})$, ita vt nunc esse debeat $(\frac{\partial\partial u}{\partial s^2})^2 + (\frac{\partial\partial v}{\partial s^2}) = 1$. Porro vero erit $(\frac{\partial x}{\partial t}) = (\frac{\partial\partial u}{\partial s\,\partial t})$ et $(\frac{\partial y}{\partial t}) = (\frac{\partial\partial v}{\partial s\,\partial t})$, quae formulae exprimunt celeritates puncti Y secundum directiones YP et YQ. Tum vero habebimus insuper $(\frac{\partial\partial x}{\partial t^2}) = (\frac{\partial^3 u}{\partial s\,\partial t^2})$ et $(\frac{\partial\partial y}{\partial t^2}) = (\frac{\partial^3 v}{\partial s\,\partial t^2})$, atque nunc integratio succedit: erit enim

$$\int\partial s(\frac{\partial\partial x}{\partial t^2}) = (\frac{\partial\partial u}{\partial t^2}) + \Gamma : t \text{ et}$$

$$\int\partial s(\frac{\partial\partial y}{\partial t^2}) = (\frac{\partial\partial v}{\partial t^2}) + \Delta : t,$$

vbi functiones quascunque temporis loco constantium sunt adiectae, propterea quod in istis integrationibus tempus t vt constans est spectatum. Quamobrem si vires P et Q etiam x vel y inuoluunt, hoc modo tota aequatio inter binas functiones u et v subsistet.

§. 11. Nihilo vero minus nullum adhuc fructum mihi quidem hinc percipere licuit, ac praecipua huius difficultatis caussa in hoc sita esse videtur: quod innumeras figuras diversas quas filum successiue induit, vix vllo modo ita per calculum exprimere licet, vt ad quoduis tempus definiri queat quales functiones ipsarum s et t binae coordinatae x et y sint futurae. Hanc ob rem istud argumentum ordine inuerso tractare institui, dum scilicet ad quoduis tempus figuram fili tanquam datam spectabo atque in vires P et Q inquiram, quae filo talem motum imprimere valeant.

Status quaestionis.

Tab. IV. Fig. 2. §. 12. Sumamus igitur initio, vbi erat $t = 0$, filum super plano horizontali in directum fuisse extensum, ita vt situm

tum tenuerit E F, eiusque longitudinem E F ſtatuamus $= a$. Hinc vero elapſo tempore $= t$ acceperit figuram E Y F, quae ſit arcus circularis rectam E F pro axe aſſumtam tangens in ipſo puncto E, ita vt fili terminus E perpetuo maneat immotus. Radius autem huius circuli ſit E O $= r$, functio quaecunque data temporis t, vnde neceſſe eſt vt poſito $t = 0$ iſta functio r euadat infinita. Sit nunc E Y portio quaecunque indefinita fili $= s$, ductoque radio O Y erit angulus E O Y $= \frac{s}{r}$, cuius ſinus erit $\frac{EX}{EO} = \frac{x}{r}$, coſinus vero $1 - \frac{y}{r}$, vnde coordinatae E X $= x$ et X Y $= y$ ita per binas variabiles s et t exprimentur, vt ſit $x = r\,\text{ſin.}\,\frac{s}{r}$ et $y = r(1 - \text{coſ.}\,\frac{s}{r})$. Quibus poſitis quaeſtio ſoluenda huc redit: vt inueſtigentur vires P et Q, quae filo talem motum qualem hic deſcripſimus inducere valeant. Quae quidem quaeſtio maxime adhuc erit indeterminata, propterea quod pro motu determinando vnicam tantum habemus aequationem: Tab. II. Fig. 2.

$$2g\left(\frac{\partial y}{\partial s}\right)\int P\,\partial s - 2g\left(\frac{\partial x}{\partial s}\right)\int Q\,\partial s = \left(\frac{\partial y}{\partial s}\right)\int \partial s\left(\frac{\partial\partial x}{\partial t^2}\right) - \left(\frac{\partial x}{\partial s}\right)\int \partial s\left(\frac{\partial\partial y}{\partial t^2}\right),$$

vnde alterutra quantitatum P et Q arbitrio noſtro relinquetur.

Euolutio formularum
in hanc aequationem ingredientium.

§. 13. Cum littera r ſit functio temporis t tantum, ſumta ſola s variabili impetrabimus has formulas $\left(\frac{\partial x}{\partial s}\right) = \text{coſ.}\,\frac{s}{r}$ et $\left(\frac{\partial y}{\partial s}\right) = \text{ſin.}\,\frac{s}{r}$, vnde ſponte fit $\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1$, vti rei natura poſtulat. Sumto autem ſolo tempore t variabili ponamus breuitatis gratia $\partial r = r'\,\partial t$, ac differentiando reperiemus

$$\left(\frac{\partial x}{\partial t}\right) = r'\,\text{ſin.}\,\frac{s}{r} - \frac{r's}{r}\,\text{coſ.}\,\frac{s}{r} \text{ et}$$

$$\left(\frac{\partial y}{\partial t}\right) = r'\left(1 - \text{coſ.}\,\frac{s}{r}\right) - \frac{sr'}{r}\,\text{ſin.}\,\frac{s}{r}.$$

§. 14. Hae formulae cum ambas celeritates puncti Y exprimant, hinc istas celeritates pro statu fili initiali, vbi erat $t = 0$ filumque in directum extensum, cognoscere licebit, id quod patebit si statuamus $r = \infty$. Tum igitur erit fin. $\frac{s}{r} = \frac{s}{r}$ et cof. $\frac{s}{r} = 1 - \frac{ss}{2rr}$, ex quo pro hoc casu erit

$$\left(\frac{\partial x}{\partial t}\right) = \frac{r' s^3}{2 r^3} \text{ et } \left(\frac{\partial y}{\partial t}\right) = - \frac{r' ss}{2rr}.$$

Videndum igitur est, num istae formulae casu $r = \infty$ seu $t = 0$ valores finitos recipere queant nec ne, id quod ab indole functionis r pendet. Veluti si sit $r = \frac{1}{t^n}$ ita vt exponens n sit positiuus, quoniam posito $t = 0$ fieri debet $r = \infty$, eritque $r' = - \frac{n}{t^{n+1}}$, hoc casu habebitur

$$\left(\frac{\partial x}{\partial t}\right) = - \tfrac{1}{2} n s^3 t^{2n-1} \text{ et } \left(\frac{\partial y}{\partial t}\right) = + \tfrac{1}{2} n s s t^{n-1}.$$

Hinc ergo intelligitur si n sit 1 fore

$$\left(\frac{\partial x}{\partial t}\right) = - \tfrac{1}{2} s^3 t = 0 \text{ et } \left(\frac{\partial y}{\partial t}\right) = \tfrac{1}{2} n s s.$$

Quo igitur casu sola celeritas $\left(\frac{\partial x}{\partial t}\right)$ euanescit. At si fuerit $n = \frac{1}{2}$, fiet $\left(\frac{\partial x}{\partial t}\right) = - \frac{1}{4} s^3$. Altera vero $\left(\frac{\partial y}{\partial t}\right) = \frac{ss}{4\sqrt{t}} = \infty$. Hinc igitur patet, pro indole functionis r euenire posse vt celeritates initiales modo fiant $= 0$, modo determinatum obtineant valorem, modo etiam in infinitum excrescant, solo termino E ipso excepto vbi $s = 0$, ille enim certe quieuisse necesse est.

§. 15. Progrediamur nunc etiam ad differentialia secunda sumendo solum t variabile, quem in finem statuamus $\partial r' = r'' \partial t$, et subducto calculo reperiemus:

$$\left(\frac{\partial \partial x}{\partial t^2}\right) = r'' \text{ fin. } \frac{s}{r} - \frac{r'' s}{r} \text{ cof. } \frac{s}{r} - \frac{r' r' ss}{r^3} \text{ fin. } \frac{s}{r} \text{ et}$$

$$\left(\frac{\partial \partial y}{\partial t^2}\right) = r'' \left(1 - \text{cof. } \frac{s}{r}\right) - \frac{r''}{r} s \text{ fin. } \frac{s}{r} + \frac{r' r' ss}{r^3} \text{ cof. } \frac{s}{r}.$$

§. 16.

§. 16. Nunc igitur has formulas ducamus in ∂s easque ita integremus vt sola quantitas s pro variabili habeatur, ac reperiemus:

$$\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) = r'' \int \partial s \text{ fin. } \frac{s}{r} - \frac{r''}{r} \int s \, \partial s \text{ cof. } \frac{s}{r}$$
$$- \frac{r' r'}{r s} \int s s \, \partial s \text{ fin. } \frac{s}{r} + \Gamma : t,$$

eodemque modo

$$\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right) = r'' s - r'' \int \partial s \text{ cof. } \frac{s}{r} - \frac{r''}{r} \int s \, \partial s \text{ fin. } \frac{s}{r}$$
$$+ \frac{r' r'}{r^3} \int s s \, \partial s \text{ cof. } \frac{s}{r} + \Delta : t,$$

vbi loco constantium adiecimus functiones quascunque ipsius t, propterea quod tempus spectatum est vt constans.

§. 17. Superest igitur tantum vt formulas integrales euoluamus, hoc modo:

$$\int \partial s \text{ fin. } \frac{s}{r} = - r \text{ cof. } \frac{s}{r}; \int \partial s \text{ cof. } \frac{s}{r} = r \text{ fin. } \frac{s}{r};$$
$$\int s \, \partial s \text{ cof. } \frac{s}{r} = r s \text{ fin. } \frac{s}{r} + r r \text{ cof. } \frac{s}{r};$$
$$\int s \, \partial s \text{ fin. } \frac{s}{r} = - r s \text{ cof. } \frac{s}{r} + r r \text{ fin. } \frac{s}{r};$$
$$\int s s \partial s \text{ fin. } \frac{s}{r} = - r s s \text{ cof. } \frac{s}{r} + 2 r r s \text{ fin. } \frac{s}{r} + 2 r^3 \text{ cof. } \frac{s}{r} \text{ et}$$
$$\int s s \partial s \text{ cof. } \frac{s}{r} = r s s \text{ fin. } \frac{s}{r} + 2 r r s \text{ cof. } \frac{s}{r} - 2 r^3 \text{ fin. } \frac{s}{r}:$$

hinc igitur erit

$$\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) = - 2 \text{ cof. } \frac{s}{r} (r r'' + r' r')$$
$$- s \text{ fin. } \frac{s}{r} \left(r'' + \frac{2 r' r'}{r}\right) + \frac{r' r' s s}{r r} \text{ cof. } \frac{s}{r} \text{ et}$$
$$\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right) = - 2 \text{ fin. } \frac{s}{r} (r r'' + r' r')$$
$$+ s \left(r'' + \left(r'' + \frac{2 r' r'}{r}\right) \text{ cof. } \frac{s}{r}\right) + \frac{r' r' s s}{r r} \text{ fin. } \frac{s}{r} + \Delta : t.$$

§. 18. Nunc igitur ad aequationem nostram constituendam prior formula ducatur in $\left(\frac{\partial y}{\partial s}\right) = \text{fin. } \frac{s}{r}$ altera vero in $-(\partial x)$

$-\left(\frac{\partial x}{\partial s}\right) = -\operatorname{cos}.\frac{s}{r}$, et membrum dextrum aequationis noſtrae euadet

$$-r'' s \operatorname{cos}.\frac{s}{r} - \left(r'' + \frac{2 r' r'}{r}\right) s + \operatorname{fin}.\frac{s}{r}\, \Gamma : t - \operatorname{cos}.\frac{s}{r}\, \Delta : t,$$

quoniam igitur membrum ſiniſtrum eſt

$$2 g \operatorname{fin}.\frac{s}{r} \int P\, \partial s - 2 g \operatorname{cos}.\frac{s}{r} \int Q\, \partial s,$$

aequatio, ex qua tota motus natura eſt definienda, erit

$$2 g \operatorname{fin}.\frac{s}{r} \int P\, \partial s - 2 g \operatorname{cos}.\frac{s}{r} \int Q\, \partial s = -r'' s \operatorname{cos}.\frac{s}{r} - \left(r'' + \frac{2 r' r'}{r}\right) s + \operatorname{fin}.\frac{s}{r}\, \Gamma : t - \operatorname{cos}.\frac{s}{r}\, \Delta : t,$$

vnde cum duae adhuc inſint incognitae P et Q, alteram pro lubitu accipere licebit.

§. 19. Conſideremus etiam tenſionem T, quam filum in ſingulis punctis ſuſtinebit, quae cum in genere fuerit

$$T = -\left(\frac{\partial x}{\partial s}\right) \int P\, \partial s - \left(\frac{\partial y}{\partial s}\right) \int Q\, \partial s + \frac{1}{2g}\left(\frac{\partial x}{\partial s}\right) \int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) + \frac{1}{2g}\left(\frac{\partial y}{\partial s}\right) \int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right),$$

ſubſtitutis valoribus modo inuentis fiet

$$T = -\operatorname{cos}.\frac{s}{r} \int P\, \partial s - \operatorname{fin}.\frac{s}{r} \int Q\, \partial s - \frac{1}{g}(r r'' + r' r') + \frac{1}{2g} r'' s \operatorname{fin}.\frac{s}{r} + \frac{1}{2g} \cdot \frac{r' r' s s}{r r} + \frac{1}{2g}\, \Gamma : t \operatorname{cos}.\frac{s}{r} + \frac{1}{2g}\, \Delta : t \operatorname{fin}.\frac{s}{r}.$$

§. 20. Cum igitur ex priore aequatione ſit

$$\int Q\, \partial s = \operatorname{tang}.\frac{s}{r} \int P\, \partial s + \frac{1}{2g} r'' s + \frac{1}{2g}\left(r'' + \frac{2 r' r'}{r}\right) \frac{s}{\operatorname{cos}.\frac{s}{r}} - \frac{1}{2g} \operatorname{tang}.\frac{s}{r}\, \Gamma : t + \frac{1}{2g}\, \Delta : t,$$

ſi hic valor in expreſſione tenſionis ſubſtituatur, prodibit

T =

$$T = -\frac{\int P \partial s}{\cos. \frac{s}{r}} - \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right) s \,\mathrm{tang.} \frac{s}{r} - \frac{1}{g}(rr'' + r'r')$$
$$+ \frac{1}{2g} \cdot \frac{r'r'ss}{rr} + \frac{1}{2g \cos. \frac{s}{r}} \Gamma : t$$

sicque per tensionem formula $\int P \partial s$ ita exprimitur, vt sit

$$\int P \partial s = -T \cos. \frac{s}{r} - \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right) s \sin. \frac{s}{r} - \frac{1}{g}(rr'' + r'r') \cos. \frac{s}{r}$$
$$+ \frac{1}{2g} \cos. \frac{s}{r} \cdot \frac{r'r'ss}{rr} + \frac{1}{2g} \Gamma : t$$

vnde differentiando, si ponamus $\partial T = T' \partial s$ quandoquidem hic sola quantitas s variabilis assumitur, fiet

$$P = -T' \cos. \frac{s}{r} + \frac{T}{r} \sin. \frac{s}{r} - \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right) \sin. \frac{s}{r} - \frac{s}{2gr}\left(r'' + \frac{2r'r'}{r}\right) \cos. \frac{s}{r}$$
$$+ \frac{1}{gr}(rr'' + r'r') \sin. \frac{s}{r} - \frac{1}{2gr} \sin. \frac{s}{r} \cdot \frac{r'r'ss}{rr} + \frac{r'r's}{grr} \cos. \frac{s}{r}$$

quae manifesto reducitur ad hanc

$$P = -T' \cos. \frac{s}{r} + \frac{T}{r} \sin. \frac{s}{r} + \frac{r''}{2g} \sin. \frac{s}{r} - \frac{s}{2gr}\left(r'' + \frac{2r'r'}{r}\right) \cos. \frac{s}{r}$$
$$- \frac{1}{2gr} \sin. \frac{s}{r} \cdot \frac{r'r'ss}{rr} + \frac{r'r's}{grr} \cos. \frac{s}{r}.$$

Simili modo, quia ex prima aequatione est

$$\int P \partial s = \cot. \frac{s}{r} \int Q \partial s - \frac{1}{2g \sin. \frac{s}{r}} r'' s \cot. \frac{s}{r}$$
$$- \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right) \frac{s}{\sin. \frac{s}{r}} + \frac{1}{2g} \Gamma : t - \frac{1}{2g} \cot. \frac{s}{r} \Delta : t,$$

qui valor in expressione tensionis substitutus praebet

$$T = -\frac{\int Q \partial s}{\sin. \frac{s}{r}} + \frac{1}{2g \sin. \frac{s}{r}} r'' s + \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right) s \cot. \frac{s}{r}$$
$$- \frac{1}{g}(rr'' + r'r') + \frac{1}{2g} \cdot \frac{r'r'ss}{rr} + \frac{1}{2g \sin. \frac{s}{r}} \Delta : t,$$

inde porro colligitur

$$\int Q\partial s = -T\,\text{fin.}\,\frac{s}{r} + \frac{r''s}{2g} + \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)\text{cof.}\,\frac{s}{r}$$
$$-\frac{1}{g}(rr'' + r'r')\,\text{fin.}\,\frac{s}{r} + \frac{\text{fin.}\,\frac{s}{r}.\frac{r'r'ss}{rr}}{2g} + \frac{1}{2g}\Delta t$$

vnde tandem differentiando elicitur Q

$$Q = -T'\,\text{fin.}\,\frac{s}{r} - \frac{T}{r}\,\text{cof.}\,\frac{s}{r} + \frac{r''}{2g} + \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)\text{cof.}\,\frac{s}{r}$$
$$-\frac{s}{2gr}\left(r'' + \frac{2r'r'}{r}\right)\text{fin.}\,\frac{s}{r} - \frac{1}{gr}(rr'' + r'r')\,\text{cof.}\,\frac{s}{r}$$
$$+\frac{1}{2gr}\,\text{cof.}\,\frac{s}{r}.\frac{r'r'ss}{rr} + \frac{1}{g}\,\frac{r'r's}{rr}.\text{fin.}\,\frac{s}{r};\ \text{fiue}$$

$$Q = -T'\,\text{fin.}\,\frac{s}{r} - \frac{T}{r}\,\text{cof.}\,\frac{s}{r} + \frac{r''}{2g} - \frac{rr''}{2gr}\,\text{cof.}\,\frac{s}{r}$$
$$-\frac{sr''}{2gr}\,\text{fin.}\,\frac{s}{r} + \frac{r'r'ss}{2gr^3}\,\text{cof.}\,\frac{s}{r}.$$

§. 22. Hoc igitur modo ambas litteras incognitas P et Q per tenfionem definiuimus, vbi notari meretur has litteras defignare vires acceleratrices filo in puncto y applicatas. Quoniam enim elementi $Yy = \partial s$ maffa quoque exprimitur per ∂s, vires motrices vtique erunt $P\partial s$ et $Q\partial s$, prouti fupra asfumfimus. Non folum autem ipfas has vires P et Q per tenfionem expreffimus, fed etiam formulas integrales $\int P\partial s$ et $\int Q\partial s$.

§. 23. Cum autem in formulis pro P et Q inuentis non folum tenfio ipfa T infit fed etiam eius differentiale $\partial T = T'\partial s$, operae pretium erit per combinationem harum formularum fiue T fiue T′ eliminare. Hoc modo reperiemus

$$P\,\text{fin.}\,\frac{s}{r} - Q\,\text{cof.}\,\frac{s}{r} = \frac{T}{r} - \frac{r''}{2g}\,\text{cof.}\,\frac{s}{r} - \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)$$
$$+\frac{1}{gr}(rr'' + r'r') - \frac{r'r'ss}{2gr^3} = \frac{T}{r} - \frac{r''}{2g}\,\text{cof.}\,\frac{s}{r} + \frac{r''}{g} - \frac{r'r'ss}{2gr^3};$$

$$P\,\text{cof.}\,\frac{s}{r} + Q\,\text{fin.}\,\frac{s}{r} = -T' + \frac{r''}{2g}\,\text{fin.}\,\frac{s}{r} - \frac{s}{2gr}\left(r'' + \frac{2r'r'}{r}\right)$$
$$+\frac{r'r's}{grr} = -T' + \frac{r''}{2g}\,\text{fin.}\,\frac{s}{r} - \frac{r''s}{2gr};$$

vbi

vbi notasse iuuabit, exprimere formulam posteriorem P cos. $\frac{s}{r}$ + Q sin. $\frac{s}{r}$ vim tangentialem qua filum in puncto Y sollicitatur, alteram vero formulam P sin. $\frac{s}{r}$ — Q cos. $\frac{s}{r}$ vim normalem eidem puncto applicatam, quarum ergo vtraque ex tensione T, quam quidem pro lubitu fingere licet, perfecte determinabitur. Atque hinc pro ipso fili initio E vbi $s = o$ fiet

$$P \sin. \frac{s}{r} - Q \cos. \frac{s}{r} = \frac{T}{r} = -Q$$

ideoque $Q = -\frac{T}{r}$. Similique modo

$$P \cos. \frac{s}{r} + Q \sin. \frac{s}{r} = -T' = P.$$

§. 24. His formulis euolutis ponamus vim tangentialem acceleratricem secundum directionem Y y agentem $= \Theta$, at vim normalem secundum directionem Y O versus centrum circuli tendentem $= \Pi$, ita vt sit

$$\Theta = P \cos. \frac{s}{r} + Q \sin. \frac{s}{r} \text{ et}$$

$$\Pi = Q \cos. \frac{s}{r} - P \sin. \frac{s}{r}$$

atque valores harum duarum virium erunt

$$\Theta = -T' + \frac{r''}{2g} \sin. \frac{s}{r} - \frac{r''s}{2gr} \text{ et}$$

$$\Pi = -\frac{T}{r} + \frac{r''}{2g} \cos. \frac{s}{r} - \frac{r''}{2g} + \frac{r'r's\,s}{2gr^3}.$$

Nunc igitur cum quaestio in se sit indeterminata, sequentia Problemata specialia percurramus, in quibus ratio virium sollicitantium praescribitur, vt filo motus supra assignatus inducatur.

Problema I.

§. 25. *Definire vires tangentiales ad motum supra descriptum in filo producendum requisitas.*

Solutio.

Cum igitur hic solae vires tangentiales requirantur, vires normales Π euanescent ita vt sit $\Pi = o$, vnde ex postrema aequatione colligitur tenſio:

$$T = \frac{r''r}{2g}\,\text{coſ.}\,\frac{s}{r} - \frac{rr''}{2g} + \frac{r'r'}{2grr}\,ss,$$

cuius differentiale ſumto ſolo s variabili praebet

$$T' = -\frac{r''}{2g}\,\text{ſin.}\,\frac{s}{r} - \frac{r'r's}{grr},$$

quo valore ſubſtituto reperimus vim tangentialem:

$$\Theta = \frac{r''}{g}\,\text{ſin.}\,\frac{s}{r} - \left(\frac{rr'' - 2r'r'}{2grr}\right)s + \frac{r''}{g}\,\text{ſin.}\,\frac{s}{r} - \frac{r''s}{2gr},$$

quae ergo in ipſo termino E vbi $s = 0$ euadit $\Theta = 0$, in fine autem fili ſeu puncto T vbi $s = a$ erit

$$\Theta = \frac{r''}{g}\,\text{ſin.}\,\frac{a}{r} - \frac{(rr'' - 2rr)}{2grr}\,a.$$

Corollarium.

§. 26. Quia hic r denotat radium circuli ſecundum quem filum elapſo tempore t incuruatur, iam ſupra monuimus r talem eſſe debere functionem ipſius T, quae fiat infinita poſito $P = 0$: conſideremus vnicum caſum.

Exemplum.

§. 27. Sumamus $r = \frac{1}{t}$, erit $r' = -\frac{1}{tt}$ et $r'' = \frac{2}{t^3}$; hinc igitur fiet vis tangentialis quaeſita $\Theta = \frac{2}{gt^3}\,\text{ſin.}\,st$; tenſio autem erit $T = \frac{1}{gt^4}(1 - \text{coſ.}\,st) + \frac{ss}{2gtt}$: hinc igitur ſequentia notari merentur: 1) In ipſo igitur initio vbi $t = 0$ vires tangentiales vbique infinitae requiruntur, vnde etiam tenſio euadet infinita. 2) Elapſo autem quouis tempore pro ſingulis fili punctis vires tangentiales erunt reciproce vt cubus temporis. 3) Pro ipſo autem fili termino E, vbi $s = 0$, tam vis tangentialis Θ quam tenſio euaneſcit, id quod natura rei poſtulat, cum punctum E maneat immotum. 4) Supra vidimus, celeritates puncti Y ſecundum directiones YP et YQ eſſe, priorem

rem $\left(\frac{\partial x}{\partial t}\right) = r' \sin. \frac{s}{r} - \frac{r's}{r} \cos. \frac{s}{r}$. Alteram vero

$$\left(\frac{\partial y}{\partial t}\right) = r'\left(1 - \cos. \frac{s}{r}\right) - \frac{r'}{r} s \sin. \frac{s}{r}.$$

quae ergo hoc casu euadent

$$\left(\frac{\partial y}{\partial t}\right) = \frac{-(1 - \cos. s\,t)}{t\,t} + \frac{1}{t} s \sin. s\,t.$$

$$\left(\frac{\partial x}{\partial t}\right) = -\frac{1}{tt} \sin. s\,t + \frac{1}{t} s \cos. s\,t,$$

quae casu $t = 0$, quo fit $\sin. s\,t = s\,t$ et $\cos. s\,t = 1 - \frac{s\,s\,t\,t}{2}$, erunt $\left(\frac{\partial x}{\partial t}\right) = -\frac{1}{2} s^3 t = 0$ et $\left(\frac{\partial y}{\partial t}\right) = \frac{1}{2} s\,s$, vnde patet, quo hic casus locum habere queat, initio singulis fili punctis Y in directione YQ eiusmodi celeritates imprimi debere, quae sint quadrato arcus $EY = s$ proportionales. Tum vero ipso initio viribus opus esse infinitis, quae deinceps in ratione triplicata temporis decrescent.

Problema II.

§. 28. *Definire vires normales* Π, *ad motum supra descriptum in filo producendum requisitas.*

Solutio.

Hic igitur esse debet $\Theta = 0$, vnde colligimus:

$$T' = \frac{r''}{2g} \sin. \frac{s}{r} - \frac{r''s}{2gr},$$

vnde deducimus integrando:

$$T = -\frac{rr''}{2g} \cos. \frac{s}{r} - \frac{r''ss}{4gr} + f:t,$$

quo valore substituto reperitur vis normalis quaesita

$$\Pi = -\frac{r''}{2g}\left(1 - 2\cos. \frac{s}{r}\right) + \left(\frac{rr'' + 2r'r'}{4gr^3}\right) ss - \frac{1}{r} f:t,$$

vnde pro termino fili E fiet $\Pi = +\frac{r''}{2g} - \frac{1}{r} f:t$ et tensio $T = -\frac{rr''}{2g} + f:t$.

 Exem-

Exemplum.

§. 29. Consideremus hic iterum casum quo $r = \frac{1}{t}$, ideoque $r' = -\frac{1}{tt}$ et $r'' = \frac{2}{t^3}$, eritque vis normalis:

$$\Pi = -\frac{1}{g t^3}(1 - 2 \operatorname{cos.} s t) + \frac{1}{g t} s s - t f : t,$$

et tensio

$$T = -\frac{1}{g t^4} \operatorname{cos.} s t - \frac{s s}{2 g t t} + f : t.$$

Hinc igitur pro termino fili E vbi $s = 0$ fiet

$$\Pi = +\frac{1}{g t^3} - t : f : t \text{ et } T = -\frac{1}{g t^4} + f : t,$$

motus autem filo in ipso initio imprimendus erit vt ante

$$\left(\frac{\partial x}{\partial t}\right) = 0 \text{ et } \left(\frac{\partial y}{\partial t}\right) = \frac{1}{2} s s.$$

Corollarium.

§. 30. Hoc igitur problema etiam nunc est indeterminatum, quoniam functio arbitrio nostro relinquitur. Eam igitur ita assumere licebit, vt tensio in ipso fili termino E euanescat, quod ergo fiet si functio $f : t = \frac{1}{g t^4}$, vnde fiet vis normalis:

$$\Pi = \frac{-2}{g t^3}(1 - \operatorname{cos.} s t) + \frac{1}{g t} s s,$$

quae ergo in ipso puncto E euanescit. Hinc igitur patet quo maius euadat tempus t, has vires normales continuo fieri minores.

Problema III.

§. 31. *Inuenire tam vires tangentiales quam normales ad motum propositum fili requisitas, ita vt durante motu tensio fili in singulis punctis perpetuo sit nulla.*

Solutio.

Solutio.

Cum igitur ſit $T = 0$ ideoque etiam $T' = 0$, vires quaeſitae ſequenti modo exprimentur:

$$\Theta = \frac{r''}{2g} \text{ſin.} \frac{s}{r} - \frac{r''s}{2gr} \text{ et}$$

$$\Pi = -\frac{r''}{2g}(1 - \text{coſ.} \frac{s}{r}) + \frac{r'r'}{2gr^3} ss,$$

quae ambae euaneſcunt pro termino fili E vbi fit $s = 0$. Ex his duabus viribus etiam vires initio conſideratae P et Q aſſignari poterunt. Cum enim ſit

$$P = \Theta \text{ coſ.} \frac{s}{r} - \Pi \text{ ſin.} \frac{s}{r} \text{ et}$$

$$Q = \Theta \text{ ſin.} \frac{s}{r} + \Pi \text{ coſ.} \frac{s}{r},$$

hinc colligitur fore

$$P = \frac{r''}{2g} \text{ſin.} \frac{s}{r} - \frac{r''}{2gr} s \text{ coſ.} \frac{s}{r} - \frac{r'r'}{2gr^3} ss \text{ ſin.} \frac{s}{r} \text{ et}$$

$$Q = \frac{r''}{2g}(1 - \text{coſ.} \frac{s}{r}) - \frac{r''}{2gr} s \text{ ſin.} \frac{s}{r} + \frac{r'r'}{2gr^3} ss \text{ coſ.} \frac{s}{r}.$$

Exemplum.

§. 32. Sit iterum $r = \frac{1}{t}$, vt ſit $r' = -\frac{1}{tt}$ et $r'' = \frac{2}{t^3}$, fietque $\Theta = \frac{1}{gt^3} \text{ſin.} st - \frac{s}{gtt}$ et

$$\Pi = -\frac{1}{gt^3}(1 - \text{coſ.} st) + \frac{ss}{2gt},$$

vel loco harum duarum virium applicatae concipi poſſunt ſequentes:

$$P = \frac{1}{gt^3} \text{ſin.} st - \frac{1}{gtt} s \text{ coſ.} st - \frac{1}{2gt} ss \text{ ſin.} st.$$

$$Q = \frac{1}{gt^3}(1 - \text{coſ.} st) - \frac{1}{gtt} s \text{ ſin.} \frac{s}{r} + \frac{1}{2gt} ss \text{ coſ.} \frac{s}{r},$$

ab his ſcilicet viribus filum, quod initio erat in directum extenſum, tandem poſt tempus infinitum quaſi in vnicum punctum conglomerabitur.

Scho-

Scholion.

§. 33. Hinc igitur infinitos casus deducere licet, quibus motus fili, dum a certis viribus continuo sollicitatur, perfecte determinari potest. Atque hi casus maxime sunt memorabiles, cum hactenus nullo plane casu talem motum inuestigare licuerit, ne eo quidem excepto, quo filo nullae plane vires applicatae concipiuntur. Simili autem modo infinitos alios huiusmodi casus euoluere licebit, quibus filum successiue secundum alios atque alios arcus circulares quacunque lege incuruatur; semper enim per theoriam generalem eiusmodi vires assignare licebit, quibus tales motus producentur.

ENODA-