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De summatione serierum, in quibus terminorum signa alternantur

Leonhard Euler

DE SVMMATIONE SERIERVM IN QVIBVS TERMINORVM SIGNA ALTERNANTVR.

Auctore
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Cum olim essem perscrutatus quemadmodum ex dato termino generali cuiusque seriei eius summam definiri conveniat, casus quo termini seriei signis alternantibus + et — sunt affecti, non parum molestiae facessebat, ac demum post longas ambages mihi licuit ad formulam satis simplicem pertingere. Hac re igitur accuratius perpensa modum inueni qui directe ad istas formulas perducit, quem igitur hoc loco exponere constitui, quandoquidem aptus videtur hanc partem Analyseos vlterius perficiendi.

Problema I.

Sit X *functio quaecunque ipsius* x, *quae, dum loco* x *successiue scribuntur valores* $x+1$, $x+2$, $x+3$, *etc. induat hos valores*: X', X'', X''', *etc. propositaque sit ista series infinita*: $X - X' + X'' - X''' + X'''' -$ etc. *in infinitum* $= S$, *eius summam* S *inuestigare.*

Solutio

Solutio.

§. 1. Cum igitur S quoque ſit certa functio ipſius x, abeat ea in S', ſi loco x ſcribatur $x + 1$, ac perſpicuum eſt fore $S' = X' - X'' + X''' - X'''' + X''''' -$ etc. in infinitum, cui ergo ſeriei ſi propoſita addatur, orietur iſta aequatio $S + S' = X$, ex qua valorem functionis quaeſitae S inueſtigari oportet.

§. 2. Quoniam igitur functio S' naſcitur ex functione S, dum loco x ſcribitur $x + 1$, ex natura differentialium erit

$$S' = S + \frac{\partial S}{\partial x} + \frac{\partial \partial S}{2\, \partial x^2} + \frac{\partial^3 S}{6\, \partial x^3} + \frac{\partial^4 S}{24\, \partial x^4} + \frac{\partial^5 S}{120\, \partial x^5} + \text{etc.}$$

vnde nobis reſoluenda proponitur iſta aequatio:

$$2S + \frac{\partial S}{\partial x} + \frac{\partial \partial S}{2\, \partial x^2} + \frac{\partial^3 S}{6\, \partial x^3} + \frac{\partial^4 S}{24\, \partial x^4} + = X,$$

vbi euidens eſt valorem ipſius S per ſeriem infinitam expreſſum iri, cuius primus terminus ſit $S = \frac{1}{2} X$; ipſam vero hanc ſeriem huiusmodi formam eſſe habituram:

$$S = \tfrac{1}{2} X + \frac{\alpha\, \partial X}{\partial x} + \frac{\beta\, \partial \partial X}{\partial x^2} + \frac{\gamma\, \partial^3 X}{\partial x^3} + \frac{\delta\, \partial^4 X}{\partial x^4} + \text{etc.}$$

§. 3. Subſtituamus igitur hanc ſeriem in noſtra aequatione, et pro eius ſingulis partibus erit vt ſequitur:

$$\begin{array}{llllllll}
2S & = X & + \frac{2\alpha\, \partial X}{\partial x} & + \frac{2\beta\, \partial \partial X}{\partial x^2} & + \frac{2\gamma\, \partial^3 X}{\partial x^3} & + \frac{2\delta\, \partial^4 X}{\partial x^4} & + \frac{2\varepsilon\, \partial^5 X}{\partial x^5} & + \frac{2\zeta\, \partial^6 X}{\partial x^6} \\
\frac{\partial S}{\partial x} & & \frac{1}{2} \ldots & + \alpha \ldots & + \beta \ldots & + \gamma \ldots & + \delta \ldots & + \varepsilon \ldots \\
\frac{\partial \partial S}{2\, \partial x^2} = & & & + \frac{1}{4} \ldots & + \frac{1}{2}\alpha \ldots & + \frac{1}{2}\beta \ldots & + \frac{1}{2}\gamma \ldots & + \frac{1}{2}\delta \ldots \\
\frac{\partial^3 S}{6\, \partial x^3} = & & & & + \frac{1}{12} \ldots & + \frac{1}{6}\alpha \ldots & + \frac{1}{6}\beta \ldots & + \frac{1}{6}\gamma \ldots \\
\frac{\partial^4 S}{24\, \partial x^4} = & & & & & + \frac{1}{48} \ldots & + \frac{1}{24}\beta \ldots & + \frac{1}{24}\beta \ldots \\
\frac{\partial^5 S}{120\, \partial x^5} = & & & & & & + \frac{1}{240} \ldots & + \frac{1}{120}\alpha \ldots \\
\frac{\partial^6 S}{720\, \partial x^6} = & & & & & & & + \frac{1}{1440} \ldots \\
\text{etc.} & & & & & & & \text{etc.}
\end{array}$$

qua-

quarum ferierum fumma quia aequari debet functioni X, hinc fequentes orientur aequalitates:

$$2\alpha + \tfrac{1}{2} = 0$$
$$2\beta + \alpha + \tfrac{1}{4} = 0$$
$$2\gamma + \beta + \tfrac{1}{2}\alpha + \tfrac{1}{12} = 0$$
$$2\delta + \gamma + \tfrac{1}{2}\beta + \tfrac{1}{6}\alpha + \tfrac{1}{48} = 0$$
$$2\varepsilon + \delta + \tfrac{1}{2}\gamma + \tfrac{1}{6}\beta + \tfrac{1}{24}\alpha + \tfrac{1}{240} = 0$$
$$2\zeta + \varepsilon + \tfrac{1}{2}\delta + \tfrac{1}{6}\gamma + \tfrac{1}{24}\beta + \tfrac{1}{120}\alpha + \tfrac{1}{1440} = 0.$$

etc.

§. 4. Quanquam hae formulae iam fufficiunt ad valores fingularum litterarum α, β, γ, δ, etc. tamen hic labor nimis fieret moleftus propter continuo plures fractiones in vnam fummam colligendas, praecipue autem quoniam, vt mox videbimus, harum litterarum alternae fponte in nihilum abeunt; quamobrem aliam viam inire conueniet veros valores harum litterarum expeditius determinandi, quae in hoc confiftit, vt euoluamus fequentis feriei fummationem:

$$s = \tfrac{1}{2} + \alpha t + \beta tt + \gamma t^3 + \delta t^4 + \text{etc.}$$

Quod fi enim huius feriei fummam s affignare valuerimus, ex ea viciffim valores fingulorum coefficientium α, β, γ, δ, etc. inueftigare licebit; vbi probe notetur, hos coefficientes prorfus conuenire cum iis qui in praecedentem aequationem ingrediuntur.

§. 5. Hac iam ferie conftituta ex inuentis relationibus inter litteras α, β, γ, δ, etc. fequentes formemus feries:

$2s = 1$

$$\begin{array}{rl}
2s = & 1 + 2\alpha t + 2\beta t^2 + 2\gamma t^3 + 2\delta t^4 + 2\varepsilon t^5 + 2\zeta t^6 + 2\eta t^7 + 2\iota t^8 + \text{etc.} \\
st = & \tfrac{1}{2}t + \alpha\ldots + \beta\ldots + \gamma\ldots + \delta\ldots + \varepsilon\ldots + \zeta\ldots + \eta\ldots \\
\tfrac{1}{2}stt = & + \tfrac{1}{4}\ldots + \tfrac{1}{2}\alpha\ldots + \tfrac{1}{2}\beta\ldots + \tfrac{1}{2}\gamma\ldots + \tfrac{1}{2}\delta\ldots + \tfrac{1}{2}\varepsilon\ldots + \tfrac{1}{2}\zeta\ldots \\
\tfrac{1}{6}st^3 = & + \tfrac{1}{12}\ldots + \tfrac{1}{6}\alpha\ldots + \tfrac{1}{6}\beta\ldots + \tfrac{1}{6}\gamma\ldots + \tfrac{1}{6}\delta\ldots + \tfrac{1}{6}\varepsilon\ldots \\
\tfrac{1}{24}st^4 = & + \tfrac{1}{48}\ldots + \tfrac{1}{24}\alpha\ldots + \tfrac{1}{24}\beta\ldots + \tfrac{1}{24}\gamma\ldots + \tfrac{1}{24}\delta\ldots \\
\tfrac{1}{120}st^5 = & + \tfrac{1}{240}\ldots + \tfrac{1}{120}\alpha\ldots + \tfrac{1}{120}\beta\ldots + \tfrac{1}{120}\gamma\ldots \\
\tfrac{1}{720}st^6 = & + \tfrac{1}{1440}\ldots + \tfrac{1}{720}\alpha\ldots + \tfrac{1}{720}\beta\ldots \\
\text{etc.} & \qquad\qquad\qquad\qquad \text{etc.}
\end{array}$$

Hae igitur ſeries in vnam ſummam collectae ob relationes ſupra §. 3. aſſignatas praebebunt hanc aequationem:

$$s\left(2 + t + \tfrac{1}{2}tt + \tfrac{1}{6}t^3 + \tfrac{1}{24}t^4 + \tfrac{1}{120}t^5 + \tfrac{1}{720}t^6 + \text{etc.}\right) = 1.$$

§. 6. Cum igitur, denotante e numerum cuius logarithmus hyperbolicus $= 1$, ſit $e^t = 1 + t + \frac{1}{2}tt + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \text{etc.}$ euidens eſt aequationem inuentam reduci ad hanc formam finitam: $s(1 + e^t) = 1$, vnde totum negotium huc redit, vt valor litterae s per ſeriem exprimatur, cuius ſinguli termini ſecundum poteſtates litterae t progrediantur; tum enim ſemper coefficientes iſtius ſeriei cum ſupra aſſumtis α, β, γ, δ congruant neceſſe eſt. Quamobrem in hoc nobis erit incumbendum, quemadmodum iſtam aequationem $s(1 + e^t) = 1$ aptiſſime in ſeriem infinitam conuertamus.

§. 7. Ante omnia igitur hanc aequationem a quantitate exponentiali e^t liberemus, et cum ſit $e^t = \frac{1}{s} - 1$, erit $t = l\frac{1-s}{s}$, hincque differentiando $\partial t = \frac{-\partial s}{s(1-s)}$. Ponamus hic $s = \frac{1}{2} + v$, et iſta aequatio fiet

$$\partial t = \frac{-\partial v}{(\frac{1}{2} + v)(\frac{1}{2} - v)} = \frac{+\partial v}{vv - \frac{1}{4}}.$$

 Nunc

Nunc autem v aequabitur isti seriei:

$$\alpha t + \beta tt + \gamma t^3 + \delta t^4 + \text{etc.}$$

cuius coefficientes quaerimus.

§. 8. Aequationi inuentae tribuamus hanc formam: $vv - \frac{1}{4} = \frac{\partial v}{\partial t}$, ex qua facile intelligitur, cum primus terminus seriei pro v inuestigandae debeat esse αt, sequentes terminos tantum per potestates impares ipsius t esse ascensuros, quam ob rem pro v constituamus sequentem seriem:

$$v = At + Bt^3 + Ct^5 + Dt^7 + Et^9 + \text{etc.}$$

eritque hinc

$$\frac{\partial v}{\partial t} = A + 3Btt + 5Ct^4 + 7Dt^6 + 9Et^8 + 11Ft^{10} + 13Gt^{12} + \text{etc.}$$

pro parte vero aequationis nostrae sinistra erit

$$\begin{aligned} vv - \tfrac{1}{4} = -\tfrac{1}{4} + AAtt &+ 2ABt^4 + 2ACt^6 + 2ADt^8 + 2AEt^{10} + 2AFt^{12} + \text{etc.} \\ &+ BB \quad + 2BC \quad + 2BD \quad + 2BE \quad + \text{etc.} \\ &\qquad\qquad\qquad + CC \quad + 2CD \quad + \text{etc.} \end{aligned}$$

ex quarum seriērum aequalitate statim concluditur fore: $A = -\frac{1}{4} = \alpha$, tum vero reliqui termini praebebunt has relationes:

$$3B = AA,$$
$$5C = 2AB,$$
$$7D = 2AC + BB,$$
$$9E = 2AD + 2BC,$$
$$11F = 2AE + 2BD + CC,$$
$$13G = 2AF + 2BE + 2CD,$$
$$\text{etc.}$$

vnde patet, cum valor ipsius A sit negatiuus $= -\frac{1}{4}$, reliquarum valores alternatim fore positiuos et negatiuos.

§. 9.

§. 9. Hac iam ſerie cum primum inuenta comparata colligitur fore:

$$\alpha = A,\ \beta = 0,\ \gamma = B,\ \delta = 0,\ \varepsilon = C,\ \zeta = 0,\ \eta = D, \text{ etc.}$$

ita vt alternae litterarum graecarum ſponte euaneſcant, vt iam ſupra innuimus, reliquarum vero determinatio per has nouas formulas multo facilius et promptius expediatur quam per relationes initio inuentas. Ante enim verbi gratia valores ipſius ε per quinque fractiones colligere oportebat, dum nunc littera C illi aequalis vnico membro exprimitur. His igitur nouis litteris A, B, C, D introductis ſummatio ſeriei propoſitae ita contrahetur vt ſit

$$S = \tfrac{1}{2} X + \frac{A\,\partial X}{\partial x} + \frac{B\,\partial^3 X}{\partial x^3} + \frac{C\,\partial^5 X}{\partial x^5} + \frac{D\,\partial^7 X}{\partial x^7} + \text{etc.}$$

§. 10. Quo autem inueſtigatio harum litterarum A, B, C, D, etc. facilior reddatur, quoniam $A = -\frac{1}{4}$ et ſequentium litterarum valores euadunt alternatim poſitiui et negatiui, denuo nouas litteras in calculum introducamus, ponendo

$$A = -\frac{\mathfrak{A}}{4},\ B = +\frac{\mathfrak{B}}{4^2},\ C = -\frac{\mathfrak{C}}{4^3},\ D = +\frac{\mathfrak{D}}{4^4},\ E = -\frac{\mathfrak{E}}{4^5}, \text{ etc.}$$

et nunc determinationes harum nouarum litterarum ſequenti modo ſe habebunt.

$$\mathfrak{A} = 1,\qquad \mathfrak{E} = \frac{2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}}{9},$$

$$\mathfrak{B} = \frac{\mathfrak{A}\mathfrak{A}}{3},\qquad \mathfrak{F} = \frac{2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}}{11},$$

$$\mathfrak{C} = \frac{2\mathfrak{A}\mathfrak{B}}{5},\qquad \mathfrak{G} = \frac{2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{E} + 2\mathfrak{C}\mathfrak{D}}{13},$$

$$\mathfrak{D} = \frac{2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}}{7},\qquad \mathfrak{H} = \frac{2\mathfrak{A}\mathfrak{G} + 2\mathfrak{B}\mathfrak{F} + 2\mathfrak{C}\mathfrak{E} + \mathfrak{D}\mathfrak{D}}{15},$$

etc.

atque ex his litteris ſummatio noſtra ita ſe habebit:

$$S = \tfrac{1}{2} X - \frac{\mathfrak{A}\,\partial X}{4\,\partial x^2} + \frac{\mathfrak{B}\,\partial^3 X}{4^2\,\partial x^3} - \frac{\mathfrak{C}\,\partial^5 X}{4^3\,\partial x^5} + \frac{\mathfrak{D}\,\partial^7 X}{4^4\,\partial x^7} - \text{etc.}$$

§. 11. Harum igitur litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. valores numerice euoluamus et calculo non admodum molesto expedito reperiemus sequentes valores:

$$\mathfrak{A}=1,\ \mathfrak{B}=\tfrac{1}{3},\ \mathfrak{C}=\tfrac{2}{3.5},\ \mathfrak{D}=\tfrac{17}{3^2.5.7},\ \mathfrak{E}=\tfrac{62}{3^2.5.7.9},$$
$$\mathfrak{F}=\tfrac{1382}{3^4.5^2.7.11},\ \mathfrak{G}=\tfrac{21844}{3^5.5^2.7.11.13}.$$

Vbi numerator penultimi termini $1382=2.691$ commonefacere potest, hos numeros in arcto nexu cum numeris Bernoullianis dictis consistere.

§. 12. Designemus igitur numeros istos Bernoullianos litteris latinis minusculis a, b, c, d, etc. ita vt sit

$$a=1,\ b=\tfrac{1}{3},\ c=\tfrac{1}{3},\ d=\tfrac{3}{5},\ e=\tfrac{5}{3},\ f=\tfrac{691}{105},\ g=\tfrac{35}{1},\ h=\tfrac{3617}{15},$$

quemadmodum hos numeros in Introductione mea in Analysin Infinitorum, pag. 131. exhibui, atque examine instituto valores nostrarum litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. sequenti modo exprimi poterunt:

$$\mathfrak{A}=\frac{2^1(2^2-1)}{2.3}.a \qquad \mathfrak{F}=\frac{2^{11}(2^{12}-1)}{2\,.\,.\,.\,.\,13}.f$$
$$\mathfrak{B}=\frac{2^3(2^4-1)}{2.3.4.5}.b \qquad \mathfrak{G}=\frac{2^{13}(2^{14}-1)}{2\,.\,.\,.\,.\,15}.g$$
$$\mathfrak{C}=\frac{2^5(2^6-1)}{2\,.\,.\,.\,.\,7}.c \qquad \mathfrak{H}=\frac{2^{15}(2^{16}-1)}{2\,.\,.\,.\,.\,17}.h$$
$$\mathfrak{D}=\frac{2^7(2^8-1)}{2\,.\,.\,.\,.\,9}.d \qquad \mathfrak{I}=\frac{2^{17}(2^{18}-1)}{2\,.\,.\,.\,.\,19}.i$$
$$\mathfrak{E}=\frac{2^9(2^{10}-1)}{2\,.\,.\,.\,.\,11}.e \qquad \mathfrak{K}=\frac{2^{19}(2^{20}-1)}{2\,.\,.\,.\,.\,21}.k$$

etc.

§. 13. In gratiam eorum, quibus non vacat istos numeros Bernoullianos ex mea Introductione depromere, eos hic, quousque equidem eos sum prosecutus, hic subiungam:

$a=1$,

$a = 1,$ $i = \frac{43867}{21},$

$b = \frac{1}{3},$ $k = \frac{1222277}{55},$

$c = \frac{1}{3},$ $l = \frac{854513}{3},$

$d = \frac{3}{5},$ $m = \frac{1181820455}{273},$

$e = \frac{5}{3},$ $n = \frac{76977927}{1},$

$f = \frac{691}{105},$ $o = \frac{23749461029}{15},$

$g = \frac{35}{1},$ $p = \frac{8615841276005}{231},$

$h = \frac{3617}{15},$ $q = \frac{84802531453387}{85},$

$r = \frac{90219075042845}{3}.$

§. 14. His igitur numeris Bernoullianis in fubfidium vocatis fumma noftrae feriei propofitae

$$S = X - X' + X'' - X''' + X'''' - \text{etc.}$$

in infinitum fequenti modo exprimetur:

$$S = \tfrac{1}{2}X - \frac{(2^2-1)}{2.3}\cdot\frac{a}{2}\cdot\frac{\partial X}{\partial x} + \frac{(2^4-1)}{2.3.4.5}\cdot\frac{b}{2}\cdot\frac{\partial^3 X}{\partial x^3} - \frac{(2^6-1)}{2\ldots7}\cdot\frac{c}{2}\cdot\frac{\partial^5 X}{\partial x^5}$$
$$+ \frac{(2^8-1)}{2\ldots9}\cdot\frac{d}{2}\cdot\frac{\partial^7 X}{\partial x^7} - \frac{(2^{10}-1)}{2\ldots11}\cdot\frac{e}{2}\cdot\frac{\partial^9 X}{\partial x^9} + \frac{(2^{12}-1)}{2\ldots13}\cdot\frac{f}{2}\cdot\frac{\partial^{11} X}{\partial x^{11}} - \text{etc.}$$

ficque Problemati noftro penitus fatisfecimus.

Alia folutio Problematis propofiti.

§. 15. Cum fumma quaefita S fit functio ipfius x, abeat ea in T, fi loco x fcribatur $x + \frac{1}{2}$, atque viciffim ex hac functione T obtinebitur ipfa fumma S, fi loco x fcribatur $x - \frac{1}{2}$, ita vt, quando inuenerimus valorem litterae T, ex eo etiam ipfa fumma quaefita S innotefcat. Tum vero manifeftum eft, fi in hac functione T loco x fcribatur $x + \frac{1}{2}$, tum prodituram effe valorem litterae S'. Hinc igitur ex natura differentialium habebimus

 S =

$$S = T - \frac{\partial T}{2.\partial x} + \frac{\partial\partial T}{4.2\,\partial x^2} - \frac{\partial^3 T}{8.6\,\partial x^3} + \frac{\partial^4 T}{16.24\,\partial x^4} - \text{etc.}$$

$$S' = T + \frac{\partial T}{2.\partial x} + \frac{\partial\partial T}{4.2\,\partial x^2} + \frac{\partial^3 T}{8.6\,\partial x^3} + \frac{\partial^4 T}{16.24\,\partial x^4} + \text{etc.}$$

Quare cum ſolutio problematis contineatur in hac aequatione: $S + S' = X$; his valoribus ſubſtitutis emergit iſta aequatio:

$$T + \frac{\partial\partial T}{4.2\,\partial x^2} + \frac{\partial^4 T}{16.24\,\partial x^4} + \frac{\partial^6 T}{64.720\,\partial x^6} + \text{etc.} = \tfrac{1}{2} X.$$

§. 16. Hinc ſtatim manifeſtum eſt ſeriei pro T aſſumendae hanc formam tribui debere:

$$T = \tfrac{1}{2} X + \frac{\alpha\,\partial\partial X}{\partial x^2} + \frac{\beta\,\partial^4 X}{\partial x^4} + \frac{\gamma\,\partial^6 X}{\partial x^6} + \text{etc.};$$

hoc igitur valore in noſtram aequationem introducto habebimus

$$T = \tfrac{1}{2} X + \frac{\alpha\,\partial\partial X}{\partial x^2} + \frac{\beta\,\partial^4 X}{\partial x^4} + \frac{\gamma\,\partial^6 X}{\partial x^6} + \frac{\delta\,\partial^8 X}{\partial x^8} + \frac{\varepsilon\,\partial^{10} X}{\partial x^{10}} + \frac{\zeta\,\partial^{12} X}{\partial x^{12}} + \text{etc.}$$

$$\frac{\partial\partial T}{4.2\,\partial x^2} = \frac{1}{2.4.2} + \frac{\alpha}{4.2} + \frac{\beta}{4.2} + \frac{\gamma}{4.2} + \frac{\delta}{4.2} + \frac{\varepsilon}{4.2} + \text{etc.}$$

$$\frac{\partial^4 T}{16.24\,\partial x^4} = \qquad + \frac{1}{2.16.24} + \frac{\alpha}{16.24} + \frac{\beta}{16.24} + \frac{\gamma}{16.24} + \frac{\delta}{16.24} + \text{etc.}$$

$$\frac{\partial^6 T}{64.720\,\partial x^6} = \qquad\qquad + \frac{1}{2.64.720} + \frac{\alpha}{64.720} + \frac{\beta}{64.720} + \frac{\gamma}{64.720} + \text{etc.}$$

etc. etc.

Quia igitur ſumma harum ſerierum aequari debet ipſi $\tfrac{1}{2}X$, hinc naſcentur ſequentes determinationes:

$$\alpha + \frac{1}{2.4.2} = 0,$$

$$\beta + \frac{\alpha}{4.2} + \frac{1}{2.16.24} = 0,$$

$$\gamma + \frac{\beta}{4.2} + \frac{\alpha}{16.24} + \frac{1}{2.64.720} = 0,$$

$$\delta + \frac{\gamma}{2^2.1.2} + \frac{\beta}{2^4.1\ldots4} + \frac{\alpha}{2^6.1\ldots6} + \frac{1}{2.256.5040} = 0,$$

$$\varepsilon + \frac{\delta}{2^2.1.2} + \frac{\gamma}{2^4.1\ldots4} + \frac{\beta}{2^6.1\ldots6} + \frac{\alpha}{2^8.1\ldots8} + \frac{1}{2.2^{10}.1\ldots10} = 0.$$

etc.

§. 17. Quanquam haud difficile foret hinc valores α, β, γ, δ, etc. elicere, ſiquidem prodiret $\alpha = -\frac{1}{16}$ et $\beta = \frac{5}{768}$; tamen

tamen simili modo, quo supra vsi sumus, in aliam legem, qua isti valores progrediuntur, inquiramus. Hunc in finem ponamus

$$s = \tfrac{1}{2} + \alpha tt + \beta t^4 + \gamma t^6 + \delta t^8 + \varepsilon t^{10} + \text{etc.}$$

vnde formemus sequentes series:

$$s = \tfrac{1}{2} + \alpha tt + \beta t^4 + \gamma t^6 + \delta t^8 + \varepsilon t^{10} + \zeta t^{12} + \eta t^{14} + \text{etc.}$$

$$\frac{stt}{2^2.1.2} = \frac{1}{2.2^2.1.2} + \frac{\alpha}{2^2.1.2} + \frac{\beta}{2^2.1.2} + \frac{\gamma}{2^2.1.2} + \frac{\delta}{2^2.1.2} + \frac{\varepsilon}{2^2.1.2} + \frac{\zeta}{2^2.1.2} + \text{etc.}$$

$$\frac{st^4}{2^4.1.4} = \frac{1}{2.2^4.1..4} + \frac{\alpha}{2^4.1..4} + \frac{\beta}{2^4.1...4} + \frac{\gamma}{2^4.1...4} + \frac{\delta}{2^4.1...4} + \frac{\varepsilon}{2^4.1...4} + \text{etc.}$$

$$\frac{st^6}{2^6.1...6} = \frac{1}{2.2^6.1..6} + \frac{\alpha}{2^6.1...6} + \frac{\beta}{2^6.1...6} + \frac{\gamma}{2^6.1...6} + \frac{\delta}{2^6.1...6} + \text{etc.}$$

etc. etc.

Hae igitur series in vnam summam collectae, ob superiores litterarum α, β, γ, etc. determinationes, nobis suppeditabunt hanc aequationem:

$$s\left(1 + \frac{tt}{2^2.1.2} + \frac{t^4}{2^4.1..4} + \frac{t^6}{2^6.1...6} + \frac{t^8}{2^8.1...8} + \text{etc.}\right) = \tfrac{1}{2}.$$

Sicque totum negotium huc est reductum, vt valor litterae s per idoneam seriem secundum potestates ipsius t procedentem exprimatur. Vbi tantum notetur, posito $t = 0$ fieri debere $s = \frac{1}{2}$.

§. 18. Cum iam, denotante e numerum cuius logarithmus hyperbolicus $= 1$, fit

$$e^{\frac{1}{2}t} = 1 + \frac{t}{2^1.1} + \frac{tt}{2^2.1.2} + \frac{t^3}{2^3.1..3} + \frac{t^4}{2^4.1..4} + \frac{t^5}{2^5.1...5} + \text{etc. et}$$

$$e^{-\frac{1}{2}t} = 1 - \frac{t}{2^1.1} + \frac{tt}{2^2.1.2} - \frac{t^3}{2^3.1..3} + \frac{t^4}{2^4.1..4} - \frac{t^5}{2^5.1...5} + \text{etc.}$$

harum duarum serierum semi-summa nobis praebebit

$$\tfrac{1}{2}\left(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}\right) = 1 + \frac{tt}{2^2.1.2} + \frac{t^4}{2^4.1..4} + \frac{t^6}{2^6.1...6} + \text{etc.},$$

hinc

hinc patet nostram aequationem futuram esse $s(e^{\frac{1}{2}t}+e^{-\frac{1}{2}t})=1$, vnde valorem ipsius s per seriem euolui oportet.

§. 19. Ex ista aequatione igitur deducimus statim

$$e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}=\frac{1}{s},$$

quae differentiata et bis sumta praebet

$$e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}=-\frac{2\partial s}{ss\partial t^2},$$

quarum aequalitatum summa dat

$$2e^{\frac{1}{2}t}=\frac{1}{s}-\frac{2\partial s}{ss\partial t};$$

differentia vero

$$2e^{-\frac{1}{2}t}=\frac{1}{s}+\frac{2\partial s}{ss\partial t};$$

harum autem productum praebet

$$4=\frac{1}{ss}-\frac{4\partial s^2}{s^4\partial t^2} \text{ siue } \frac{4\partial s^2}{\partial t^2}=ss-4s^4.$$

Differentietur iam ista aequatio denuo, sumto ∂t constante, ac habebimus $\frac{4\partial\partial s}{\partial t^2}=s-8s^3$, siue $\frac{4\partial\partial s}{\partial t^2}+8s^3-s=0$.

§. 20. Pro hac aequatione resoluenda statuamus vti supra assumsimus

$$s=\frac{1}{2}+\alpha tt+\beta t^4+\gamma t^6+\delta t^8+\text{ etc.}$$

vnde fit

$$\frac{\partial\partial s}{\partial t^2}=1.2\alpha+3.4\beta tt+5.6\gamma t^4+7.8\delta t^6+9.10\varepsilon t^8+\text{ etc.}$$

Deinde ob $2s=1+2\alpha tt+2\beta t^4+2\gamma t^6+2\delta t^8+$ etc. erit cubum sumendo

$$\begin{array}{rl} 8s^3 = 1 + 6\alpha tt + 6\beta t^4 + & 6\gamma t^6 + 6\delta t^8 + 6\varepsilon t^{10} + 6\zeta t^{12} + \text{etc.} \\ & + 12\alpha^2 + 24\alpha\beta + 24\alpha\gamma + 24\alpha\delta + 24\alpha\varepsilon + \text{etc.} \\ & + 8\alpha^3 + 12\beta\beta + 24\beta\gamma + 24\beta\delta + \text{etc.} \\ & + 24\alpha\alpha\beta + 24\alpha\alpha\gamma + 12\gamma\gamma + \text{etc.} \\ & + 24\alpha\beta\beta + 24\alpha\alpha\delta + \text{etc.} \\ & + 48\alpha\beta\gamma + \text{etc.} \\ & + 8\beta^3 + \text{etc.} \end{array}$$

quae series aequalis esse debet $s - \frac{4\partial\partial s}{\partial t^2}$.

§. 21. Haec igitur series aequalis statui debet isti:

$$s = \tfrac{1}{2} + \alpha tt + \beta t^4 + \gamma t^6 + \delta t^8 + \varepsilon t^{10} + \zeta t^{12} + \eta t^{14} + \text{etc.}$$

$$-\frac{4\partial\partial s}{\partial t^2} = -8\alpha - 4.3.4\beta - 4.5.6\gamma - 4.7.8\delta - 4.9.10\varepsilon - 4.11.12\zeta - \text{etc.}$$

vnde deducuntur sequentes determinationes:

$$4.1.2\alpha + \tfrac{1}{2} = 0;$$
$$4.3.4\beta + 5\alpha = 0;$$
$$4.5.6\gamma + 5\beta + 12\alpha^2 = 0;$$
$$4.7.8\delta + 5\gamma + 24\alpha\beta + 8\alpha^3 = 0;$$
$$4.9.10\varepsilon + 5\delta + 24\alpha\gamma + 12\beta\beta + 24\alpha\alpha\beta = 0.$$

etc.

§. 22. Quoniam vero hae relationes multo magis sunt complicatae quam eae ad quas primo sumus perducti, istis potius inhaereamus earumque euolutionem sequenti modo subleuemus. Ponamus scilicet

$$\alpha = -\frac{A}{2^3},\ \beta = +\frac{B}{2^5},\ \gamma = -\frac{C}{2^7},\ \delta = +\frac{D}{2^9},\ \varepsilon = -\frac{E}{2^{11}},\ \text{etc.}$$

vnde summatio nostra induet hanc formam:

$$T = \tfrac{1}{2}X - \frac{A}{2^3}\cdot\frac{\partial\partial X}{\partial x^2} + \frac{B}{2^5}\cdot\frac{\partial^4 X}{\partial x^4} - \frac{C}{2^7}\cdot\frac{\partial^6 X}{\partial x^6} + \frac{D}{2^9}\cdot\frac{\partial^8 X}{\partial x^8} - \text{etc.}$$

ac relationes pro his nouis litteris ſequenti modo ſe habebunt:

$$A = \frac{1}{1 \cdot 2};$$
$$B = \frac{A}{1 \cdot 2} - \frac{1}{1 \ldots 4};$$
$$C = \frac{B}{1 \cdot 2} - \frac{A}{1 \ldots 4} + \frac{1}{1 \ldots 6};$$
$$D = \frac{C}{1 \cdot 2} - \frac{B}{1 \ldots 4} + \frac{A}{1 \ldots 6} - \frac{1}{1 \ldots 8};$$
$$E = \frac{D}{1 \cdot 2} - \frac{C}{1 \ldots 4} + \frac{B}{1 \ldots 6} - \frac{A}{1 \ldots 8} + \frac{1}{1 \ldots 10}.$$
etc. etc.

§. 23. Quo calculum iſtarum litterarum magis contrahamus atque adeo totum negotium ad numeros integros reducamus, ponamus porro $A = \frac{a}{1 \cdot 2}$, $B = \frac{b}{1 \ldots 4}$, $C = \frac{c}{1 \ldots 6}$, etc. vt noſtra ſummatio fiat

$$T = \tfrac{1}{2}X - \frac{a}{2^3 \cdot 1 \cdot 2}\,\frac{\partial\partial X}{\partial x^2} + \frac{b}{2^5 \cdot 1 \ldots 4}\,\frac{\partial^4 X}{\partial x^4} - \frac{c}{2^7 \cdot 1 \ldots 6}\,\frac{\partial^6 X}{\partial x^6} + \text{etc.}$$

et nunc iſtae nouae litterae per ſequentes formulas commodiſſime determinabuntur:

$a = \frac{2 \cdot 1}{1 \cdot 2}$, ſiue $a = 1$;

$b = \frac{4 \cdot 3}{1 \cdot 2} a - \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}$,
ſiue $b = 6a - 1 = 5$;

$c = \frac{6 \cdot 5}{1 \cdot 2} b - \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} a + \frac{6 \ldots 1}{1 \ldots 6}$,
ſiue $c = 15b - 15a + 1 = 61$;

$d = \frac{8 \cdot 7}{1 \cdot 2} c - \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} b + \frac{8 \ldots 3}{1 \ldots 6} a - \frac{8 \ldots 1}{1 \ldots 8}$,
ſiue $d = 28c - 70b + 28a - 1 = 1385$;

$e = \frac{10 \cdot 9}{1 \cdot 2} d - \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} c + \frac{10 \ldots 5}{1 \ldots 6} b - \frac{10 \ldots 3}{1 \ldots 8} a + \frac{10 \ldots 1}{1 \ldots 10}$,
ſiue $e = 45d - 210c + 210b - 45a + 1 = 50521$;

$f = \frac{12 \cdot 11}{1 \cdot 2} e - \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} d + \frac{12 \ldots 7}{1 \ldots 6} c - \frac{12 \ldots 5}{1 \ldots 8} b + \frac{12 \ldots 3}{1 \ldots 10} a - \frac{12 \ldots 1}{1 \ldots 12}$;
ſiue $f = 66.e - 495.d + 924.c - 495.b + 66.a - 1$;
etc.

Mani-

Manifeſtum autem eſt coefficientes harum formularum congruere cum iis qui in poteſtatibus binomii occurrunt, ſi modo alterni omittantur.

§. 24. Valoribus igitur harum litterarum a, b, c, d inuentis ſeries ante allata dabit valorem litterae T, qui quouis caſu erit certa functio ipſius x, ex qua, ſi loco x ſcribatur $x - \frac{1}{2}$, orietur ſumma ſeriei propoſitae S. Veluti ſi fuerit $X = x^4$, haecque ſeries ſummanda proponatur:

$$S = x^4 - (x+1)^4 + (x+2)^4 - (x+3)^4 + (x+4)^4 - \text{etc.}$$

ob $\frac{\partial \partial X}{\partial x^2} = 4 \cdot 3\, x\, x$ et $\frac{\partial^4 X}{\partial x^4} = 4 \cdot 3 \cdot 2 \cdot 1$, altiora vero differentialia euaneſcentia, erit

$$T = \tfrac{1}{2} x^4 - \tfrac{3}{4} x\, x + \tfrac{5}{32}, \text{ hincque}$$

$$S = \tfrac{1}{2} (x - \tfrac{1}{2})^4 - \tfrac{3}{4} (x - \tfrac{1}{2})^2 + \tfrac{5}{32}.$$

Hinc ergo ſumto $x = 1$, vt ſeries ſummanda ſit

$$S = 1 - 2^4 + 3^4 - 4^4 + 5^4 - 6^4 + \text{etc.}$$

reperietur $S = 0$, vti aliunde conſtat. Alia exempla non ſubiungimus, quoniam olim iam copioſe ſunt tractata.

Problema II.

Si X vt ante fuerit functio quaecunque ipſius x, ex qua, dum loco x ordine ſcribantur valores $x+1$, $x+2$, $x+3$, etc. naſcantur functiones X', X'', X''', inuenire ſummam huius ſeriei in infinitum excurrentis:

$$n^x X - n^{x+1} X' + n^{x+2} X'' - n^{x+3} X''' + n^{x+4} X'''' - \text{etc.}$$

Solutio.

§. 25. Ponatur huius ſeriei ſumma quaeſita $n^x S$, vt ſit

$$S = X - n X' + n^2 X'' - n^3 X''' + n^4 X'''' - \text{etc.}$$

Hic iam loco x ſcribatur $x+1$, ac reperietur

$$S' = X' - nX'' + n^2 X''' - n^4 X'''' + \text{etc.}$$

quae ſeries ducta in n et priori addita praebet $S + nS' = X$. Quare cum ſit

$$S' = S + \frac{\partial S}{\partial x} + \frac{\partial\partial S}{1.2\,\partial x^2} + \frac{\partial^3 S}{1.2.3\,\partial x^3} + \text{etc.}$$

habebitur iſta aequatio:

$$(1+n)S + \frac{n\,\partial S}{\partial x} + \frac{n\,\partial\partial S}{2\,\partial x^2} + \frac{n\,\partial^3 S}{6\,\partial x^3} + \frac{n\,\partial^4 S}{24\,\partial x^4} + \text{etc.} = X,$$

ex qua valorem litterae S erui oportet.

§. 26. Statuamus ergo pro S hanc ſeriem:

$$S = \alpha X + \frac{\beta\,\partial X}{\partial x} + \frac{\gamma\,\partial\partial X}{\partial x^2} + \frac{\delta\,\partial^3 X}{\partial x^3} + \text{etc.}$$

et factis ſingulis ſubſtitutionibus obtinebimus:

$$\begin{array}{rcccccc}
(1+n)S & = (1+n)\alpha X & +(1+n)\frac{\beta\,\partial X}{\partial x} & +(1+n)\frac{\gamma\,\partial\partial X}{\partial x^2} & +(1+n)\frac{\delta\,\partial^3 X}{\partial x^3} & +(1+n)\frac{\varepsilon\,\partial^4 X}{\partial x^4} \\
\frac{n\,\partial S}{\partial x} & = & n\alpha & +n\beta & +n\gamma & +n\delta \\
\frac{n\,\partial\partial S}{2\,\partial x^2} & = & & +\tfrac{1}{2}n\alpha & +\tfrac{1}{2}n\beta & +\tfrac{1}{2}n\gamma \\
\frac{n\,\partial^3 S}{6.\,\partial x^3} & = & & & +\tfrac{1}{6}n\alpha & +\tfrac{1}{6}n\beta \\
\frac{n\,\partial^4 S}{24\,\partial x^4} & = & & & & +\tfrac{1}{24}n\alpha \\
\text{etc.} & & & & & \text{etc.}
\end{array}$$

quarum ſerierum ſumma quia aequari debet ipſi X, hinc ſequentes determinationes reſultabunt:

$$(n+1)\alpha = 1;$$
$$(n+1)\beta + n\alpha = 0;$$
$$(n+1)\gamma + n\beta + \tfrac{1}{2}n\alpha = 0;$$
$$(n+1)\delta + n\gamma + \tfrac{1}{2}n\beta + \tfrac{1}{6}n\alpha = 0;$$
$$(n+1)\varepsilon + n\delta + \tfrac{1}{2}n\gamma + \tfrac{1}{6}n\beta + \tfrac{1}{24}n\alpha = 0.$$

etc.

§. 27. Resolutio igitur harum aequalitatum nobis suppeditabit sequentes valores:

$$\alpha = \frac{1}{n+1};$$
$$\beta = -\frac{n}{(n+1)^2};$$
$$\gamma = \frac{n(n-1)}{2(n+1)^3};$$
$$\delta = -\frac{n(nn-4n+1)}{6(n+1)^4}.$$
etc.

Nimis autem molestum foret euolutionem harum formularum vlterius prosequi, quamobrem conueniet, loco horum coefficientium alios in calculum introducere, qui sint

$$\alpha = \frac{A}{n+1}, \; \beta = -\frac{B}{(n+1)^2}, \; \gamma = +\frac{C}{(n+1)^3}, \; \delta = -\frac{D}{(n+1)^4}, \text{ etc.}$$

ita vt series nostra pro S inuenta hanc induat formam:

$$S = \frac{A}{(n+1)}X - \frac{B}{(n+1)^2}\frac{\partial X}{\partial x} + \frac{C}{(n+1)^3}\frac{\partial\partial X}{\partial x^2} - \frac{D}{(n+1)^4}\cdot\frac{\partial^3 X}{\partial x^3} + \text{etc.}$$

§. 28. Nunc igitur istae nouae litterae A, B, C, D etc. per sequentes formulas determinabuntur:

$$A = 1,$$
$$B = nA,$$
$$C = nB - \tfrac{1}{2}n(n+1)A,$$
$$D = nC - \tfrac{1}{2}n(n+1)B + \tfrac{1}{6}n(n+1)^2A,$$
$$E = nD - \tfrac{1}{2}n(n+1)C + \tfrac{1}{6}n(n+1)^2B - \tfrac{1}{24}n(n+1)^3A, \text{ etc.}$$

vnde facilius iam colligentur sequentes valores:

$$A = 1,$$
$$B = n,$$
$$C = \tfrac{1}{2}n(n-1),$$
$$D = \tfrac{1}{6}n(nn-4n+1),$$
$$E = \tfrac{1}{24}n(n^3 - 11nn + 11n - 1).$$

 §. 29.

§. 29. Quo indolem horum numerorum A, B, C, D penitius perscrutemur, contemplemur istam seriem easdem litteras inuoluentem:

$$s = A + Bt + Ctt + Dt^3 + \text{etc.}$$

ex qua secundum relationes ante inuentas formemus sequentes series:

$$\begin{array}{lllll} s = & A & + Bt & + Ctt & + Dt^3 + Et^4 + Ft^5 + \text{etc.} \\ -nst = & & -nA & -nB & -nC - nD - nE - \text{etc.} \\ +\frac{1}{2}n(n+1)stt = & & & \frac{1}{2}n(n+1)A + \frac{1}{2}(n+1)B + \frac{1}{2}n(n+1)C + \frac{1}{2}n(n+1)D + \text{etc.} \\ -\frac{1}{6}n(n+1)^2st^3 = & & & & +\frac{1}{6}n(n+1)^2A - \frac{1}{6}n(n+1)^2B - \frac{1}{6}n(n+1)^2C - \text{etc.} \\ +\frac{1}{24}n(n+1)^3st^4 = & & & & +\frac{1}{24}n(n+1)^3A + \frac{1}{24}n(n+1)^3B + \text{etc.} \end{array}$$

etc.

His igitur seriebus in vnam summam collectis impetrabimus hanc aequationem:

$$s\left(1 - nt + \tfrac{1}{2}n(n+1)tt - \tfrac{1}{6}n(n+1)^2t^3 + \tfrac{1}{24}n(n+1)^3t^4 - \text{etc.}\right) = 1.$$

§. 30. Vt nunc hanc aequationem ad formam finitam reducamus, in subsidium vocemus hanc progressionem:

$$e^{-(n+1)t} = 1 - (n+1)t + \tfrac{1}{2}(n+1)^2tt - \tfrac{1}{6}(n+1)^3t^3 + \tfrac{1}{24}(n+1)^4t^4 - \text{etc.}$$

vnde fit

$$\frac{e^{-(n+1)t} - 1}{n+1} = -t + \tfrac{1}{2}(n+1)tt - \tfrac{1}{6}(n+1)^2t^3 + \tfrac{1}{24}(n+1)^3t^4 - \text{etc.}$$

consequenter

$$\frac{n}{n+1}\left(e^{-(n+1)t} - 1\right) = -nt + \tfrac{1}{2}n(n+1)t^2 - \tfrac{1}{6}n(n+1)^2t^3 + \tfrac{1}{24}n(n+1)^3t^4 - \text{etc.}$$

Hinc igitur nanciscemur sequentem aequationem finitam:

$$s\left(1 + \tfrac{n}{n+1}\left(e^{-(n+1)t} - 1\right)\right) = s\left(\tfrac{1}{n+1} + \tfrac{n}{n+1}e^{-(n+1)t}\right) = 1.$$

Ex hac autem aequatione, si valor ipsius s per seriem eliciatur, ipsa

ipsa series assumta prodire debet, ex qua idcirco nostrae litterae a, b, c, d innotescent. Hinc igitur erit

$$e^{-(n+1)t} = \frac{1+n-s}{ns},$$

ideoque $-(n+1)t = l(1+n-s) - l\,n\,s$ et differentiando

$$-(n+1)\partial t = -\frac{\partial s}{1+n-s} - \frac{\partial s}{s} = -\frac{(1+n)\partial s}{s(1+n-s)};$$

ex qua aequatione colligitur $s(1+n-s) = \frac{\partial s}{\partial t}$.

§. 31. Statuatur nunc $s = \frac{1}{2}(n+1) + v$, vt fiat

$$v = -\tfrac{1}{2}(n+1)A + Bt + Ctt + Dt^3 + \text{etc.}$$

eritque nostra aequatio $\frac{1}{4}(n+1)^2 - vv = \frac{\partial v}{\partial t}$. Ad calculi igitur compendium ponamus $\frac{1}{2}(n+1) = m$, sitque $A - \frac{1}{2}(n+1) = \Delta$, vt series nostra sit

$$v = \Delta + Bt + Ctt + Dt^3 + Et^4 + Ft^5 + Gt^6 + Ht^7 + \text{etc.}$$

tum vero habebimus:

$$\frac{\partial v}{\partial t} = mm - vv, \text{ siue } \frac{\partial v}{\partial t} + vv = mm.$$

In hac ergo aequatione loco v seriem assumtam substituamus eritque

$$\frac{\partial v}{\partial t} = B + 2Ct + 3Dtt + 4Et^3 + 5Ft^4 + 6Gt^5 + \text{etc.}$$

$$\begin{aligned} vv = \Delta\Delta + 2\Delta B &+ 2\Delta C + 2\Delta D + 2\Delta E + 2\Delta F + \text{etc.} \\ &+ BB + 2BC + 2BD + 2BE + \text{etc.} \\ &\qquad\qquad + CC + 2CD + \text{etc.} \end{aligned}$$

quarum ergo serierum summa debet esse $= mm$, vnde deducuntur sequentes determinationes:

$B + \Delta\Delta = mm;$	hinc $B = mm - \Delta\Delta;$
$2C + 2\Delta B = 0;$	$2C = -2\Delta B;$
$3D + 2\Delta C + BB = 0;$	$3D = -2\Delta C - BB;$
$4E + 2\Delta D + 2BC = 0;$	$4E = -2\Delta D - 2BC,$
$5F + 2\Delta E + 2BD + CC = 0;$	$5F = -2\Delta E - 2BD - CC.$
etc.	etc.

§. 32.

§. 32. Cum iam posuerimus $\Delta = A - \frac{1}{2}(n+1) = A - m$, ob $A = 1$ erit $\Delta = 1 - m = \frac{1-n}{2}$. Retineamus autem litteram m in calculo, existente $m = \frac{1}{2}(n+1)$, ac reperiemus $B = n$, et quia est $-2\Delta = n - 1$, formulae nostrae euadent

$$2C = (n-1)B;$$
$$3D = (n-1)C - BB;$$
$$4E = (n-1)D - 2BC;$$
$$5F = (n-1)E - 2BD - CC;$$
$$6G = (n-1)F - 2BE - 2CD.$$
etc.

haeque formulae ad calculum magis accommodatae videntur quam superiores §. 28. quia hic occurrit minor terminorum numerus atque etiam factores sunt simpliciores. Ex his igitur valores supra inchoatos vlterius prosequemur:

$$A = 1;$$
$$B = n;$$
$$C = \frac{n(n-1)}{1.2};$$
$$D = \frac{n(nn-4n+1)}{1.2.3};$$
$$E = \frac{n(n^3 - 11nn + 11n - 1)}{1.2.3.4};$$
$$F = \frac{n(n^4 - 26n^3 + 66n^2 - 26n + 1)}{1.2.3.4.5};$$
$$G = \frac{n(n^5 - 57n^4 + 302n^3 - 302nn + 57n - 1)}{1 \ldots\ldots 6}.$$

§. 33. Hae expressiones eo magis sunt notatu dignae, quod coefficientes in numeratoribus ad formulas generales reduci possunt; namque coefficientes terminorum secundorum, qui sunt 0, 0, 1, 4, 11, 26, 57, 120, etc. nascuntur ex forma generali $2^{z-1} - z$, coefficientes vero terminorum tertiorum,

rum, qui sunt 0, 0, 0, 1, 11, 66, 302, etc. oriuntur ex formula generali $3^{z-1} - 2^{z-1} z + \frac{z(z-1)}{1.2}$; simili modo terminorum quartorum, qui sunt 0, 0, 0, 0, 1, 26, 302, etc. terminus generalis est

$$4^{z-1} - 3^{z-1} . z + 2^{z-1} . \frac{z(z-1)}{1.2} - \frac{z(z-1)(z-2)}{1.2.3};$$

quintorum vero terminorum coefficientes, qui sunt 0, 0, 0, 0, 0, 1, 57, etc. oriuntur ex forma generali hac:

$$5^{z-1} - 4^{z-1} . z + 3^{z-1} . \frac{z(z-1)}{1.2.} - 2^{z-1} . \frac{z(z-1)(z-2)}{1.2.3} + \frac{z(z-1)(z-2)(z-3)}{1...4},$$

vnde iam satis clarum est, quomodo pro sequentibus terminis formulae generales constitui debeant.

§. 34. Inuentis igitur secundum has regulas valoribus litterarum A, B, C, D, etc. seriei propositae infinitae

$$n^x X - n^{x+1} X' + n^{x+2} X'' - n^{x+3} X''' + \text{etc.}$$

summa erit

$$n^x \left(\frac{A}{n+1} X - \frac{B}{(n+1)^2} \frac{\partial X}{\partial x} + \frac{C}{(n+1)^3} \frac{\partial \partial X}{\partial x^3} - \frac{D}{(n+1)^4} \frac{\partial^3 X}{\partial x^3} + \text{etc.}\right).$$

Ita si fuerit $X = 1$ et series summanda

$$n^x - n^{x+1} + n^{x+2} - n^{x+3} + n^{x+4} - \text{etc.}$$

ob $\frac{\partial X}{\partial x} = 0$, $\frac{\partial \partial X}{\partial x^2} = 0$, erit summa quaesita $= n^x \frac{A}{n+1} = \frac{n^x}{n+1}$.

At si sumatur $X = x$, vt series summanda sit

$$n^x . x - n^{x+1}(x+1) + n^{x+2}(x+2) - n^{x+3}(x+3) + \text{etc.}$$

ob $\frac{\partial X}{\partial x} = 1$, sequentia vero differentialia $= 0$, erit summa quaesita

$$= n^x \left(\frac{A x}{n+1} - \frac{B}{(n+1)^2}\right) = n^x \left(\frac{x}{n+1} - \frac{n}{(n+1)^2}\right).$$

Hinc ergo si sumatur $x = 1$, huius seriei:

$$n - 2n^2 + 3n^3 - 4n^4 + 5n^5 - 6n^6 + \text{etc.}$$

summa erit $= \frac{n}{(n+1)^2}$, cuius fractionis euolutio manifesto producit istam seriem. Plura exempla adiungere superfluum foret, quia hoc argumentum iam alias fusius est tractatum.

Problema III.

Si vt ante X denotet functionem quamcunque ipsius x, *quae loco* x *scribendo successiue* x + 1, x + 2, x + 3, *abeat in* X', X'', X''', *ac proponatur sequens series infinita cum progressione hypergeometrica commista:*

$$1 . 2 . 3 . 4 \ldots x . X$$
$$- 1 . 2 . 3 . 4 \ldots (x+1) X'$$
$$+ 1 . 2 . 3 . 4 \ldots (x+2) X''$$
$$- \text{etc.}$$

eius summam inuestigare.

Solutio.

§. 35. Statuatur ista summa quaesita $= 1 . 2 . 3 \ldots x S$, ita vt tantum functionem S indagari oporteat, eritque

$$S = X - (x+1)X' + (x+1)(x+2)X'' - (x+1)(x+2)(x+3)X''' + \text{etc.}$$

Hinc ergo si loco x vbique scribamus $x+1$, fiet

$$S' = X' - (x+2)X'' + (x+2)(x+3)X''' - (x+2)(x+3)(x+4)X'''' + \text{etc.}$$

quae posterior series per $x+1$ multiplicata ac priori adiecta producet istam aequationem: $S + (x+1)S' = X$, ex qua ergo valorem ipsius S definire oportet.

§. 36. Hic autem pro S talem seriem per differentialia ipsius X procedentem fingere non licet vt supra, propterea quod functio

$$S' = S + \frac{\partial S}{\partial x} + \frac{\partial \partial S}{1.2\, \partial x^2} + \frac{\partial^3 S}{1.2.3\, \partial x^3} + \text{etc.}$$

per factorem variabilem $x+1$ est multiplicata, quamobrem pro S assumamus seriem generalem $p + q + r + s + t +$ etc.

quae

quae ita ſit comparata, vt differentiale cuiusque partis cadat in locum ſequentem. Cum igitur noſtra aequatio ſit

$$(x+2)S+(x+1)\frac{\partial s}{\partial x}+(x+1)\frac{\partial\partial s}{2\partial x^2}+(x+1)\frac{\partial^3 s}{6\partial x^3}+\text{etc.}=X,$$

hic loco S eiusque differentialium ſecundum legem praeſcriptam ſeries aſſumta ſubſtituatur, ac peruenietur ad hanc aequationem:

$$\begin{aligned}X=&(x+2)p+(x+2)q+(x+2)r+(x+2)s+(x+2)t+(x+2)u+\text{etc.}\\&+(x+1)\frac{\partial p}{\partial x}+(x+1)\frac{\partial q}{\partial x}+(x+1)\frac{\partial r}{\partial x}+(x+1)\frac{\partial s}{\partial x}+(x+1)\frac{\partial t}{\partial x}+\text{etc.}\\&+(x+1)\frac{\partial\partial p}{2\partial x^2}+(x+1)\frac{\partial\partial q}{2\partial x^2}+(x+1)\frac{\partial\partial r}{2\partial x^2}+(x+1)\frac{\partial\partial s}{2\partial x^2}+\text{etc}\\&+(x+1)\frac{\partial^3 p}{6\partial x^3}+(x+1)\frac{\partial^3 q}{6\partial x^3}+(x+1)\frac{\partial^3 r}{6\partial x^3}+\text{etc.}\\&\text{etc.}\end{aligned}$$

hicque primum ſtatuatur $X=(x+2)p$, ita vt ſit $p=\frac{X}{x+2}$, tum vero pro reliquis habebuntur ſequentes aequationes:

$$(x+2)q+(x+1)\frac{\partial p}{\partial x}=0,$$

$$(x+2)r+(x+1)\frac{\partial q}{\partial x}+(x+1)\frac{\partial\partial p}{2\partial x^2}=0,$$

$$(x+2)s+(x+1)\frac{\partial r}{\partial x}+(x+1)\frac{\partial\partial q}{2.\partial x^2}+(x+1)\frac{\partial^3 p}{6\partial x^3}=0,$$

$$(x+2)t+(x+1)\frac{\partial s}{\partial x}+(x+1)\frac{\partial\partial r}{2\partial x^2}+(x+1)\frac{\partial^3 q}{6.\partial x^3}+(x+1)\frac{d^4 p}{24\,d x^4}=0.$$

etc.

§. 37. Ex his igitur aequationibus haud difficile erit valores ſingularum litterarum q, r, s, t per praecedentes iam inuentas definire. In genere autem haec euolutio mox ad formulas nimis complicatas perduceret, namque cum ſit $p=\frac{X}{x+2}$, erit $\partial p=\frac{\partial X}{x+2}-\frac{X\partial x}{(x+2)^2}$, vnde colligitur

$$(x+2)q+\frac{(x+1)}{x+2}\frac{\partial X}{\partial x}-\frac{(x+1)X}{(x+2)^2},$$

hincque

$$q=-\frac{(x+1)}{(x+2)^2}\frac{\partial X}{\partial x}+\frac{x+1\,X}{(x+2)^3};$$

cuius

cuius ergo differentiale non ſolum denuo ſumi deberet, ſed etiam differentio-differentiale ipſius p, vt inde deriuetur valor ipſius r. Interim tamen hi valores in genere commodius exprimuntur ſequenti modo:

$$q = -\frac{(x+1)}{(x+2)\partial x}.\partial.p,$$
$$r = -\frac{(x+1)}{(x+2)\partial x}\partial\left(q+\frac{\partial p}{2\partial x}\right),$$
$$s = -\frac{(x+1)}{(x+2)\partial x}\partial\left(r+\frac{\partial q}{2\partial x}+\frac{\partial\partial p}{6\partial x^2}\right),$$
$$t = -\frac{(x+1)}{(x+2)\partial x}\partial\left(s+\frac{\partial r}{2\partial x}+\frac{\partial\partial q}{6\partial x^2}+\frac{\partial^3 p}{24\partial x^3}\right),$$
etc.

§. 38. In genere autem has formulas euoluere non eſt opus, quia quouis caſu propoſito euolutio haud difficulter inſtitui poterit, quod vnico caſu oſtendiſſe ſufficiet. Sumatur igitur $X = 1$ eruntque etiam omnes valores inde deriuati X', X'', etc. vnitati aequales. Ac primo hoc caſu habebitur $p = \frac{1}{x+2}$, cuius ergo differentialia erunt

$$\frac{\partial p}{\partial x} = -\frac{1}{(x+2)^2}, \quad \frac{\partial\partial p}{\partial x^2} = \frac{2}{(x+2)^3}, \quad \frac{\partial^3 p}{\partial x^3} = -\frac{6}{(x+2)^4}, \text{ etc.}$$

hinc igitur primo colligimus $q = +\frac{x+1}{(x+2)^3}$, qui valor reſoluatur in has partes: $q = \frac{1}{(x+2)^2} - \frac{1}{(x+2)^3}$, vnde fiet

$$\frac{\partial q}{\partial x} = -\frac{2}{(x+2)^3} + \frac{3}{(x+2)^4} \text{ et}$$
$$\frac{\partial\partial q}{\partial x^2} = \frac{6}{(x+2)^4} - \frac{12}{(x+2)^5}, \text{ etc.}$$

Ex his igitur porro fit

$$r = -\frac{x+1}{x+2}\left(-\frac{1}{(x+2)^3} + \frac{3}{(x+2)^4}\right).$$

Cum nunc ſit $-\left(\frac{x+1}{x+2}\right) = -1 + \frac{1}{x+2}$, fiet

$$r = +\frac{1}{(x+2)^3} - \frac{4}{(x+2)^4} + \frac{3}{(x+2)^5},$$

vnde fit

$$\frac{\partial r}{\partial x} = -\frac{3}{(x+2)^4} + \frac{16}{(x+2)^5} - \frac{15}{(x+2)^6}$$

ex

ex quo valore colligitur

$$s = -\frac{x+1}{x+2}\left(-\frac{1}{(x+2)^4}+\frac{10}{(x+2)^5}-\frac{15}{(x+2)^6}\right).$$

His igitur valoribus inuentis feriei infinitae

$$\begin{array}{l} 1.\ 2.\ 3.\ 4 \ldots\ldots x \\ -1.\ 2.\ 3.\ 4 \ldots\ldots (x+1) \\ +1.\ 2.\ 3.\ 4 \ldots\ldots (x+2) \\ -1.\ 2.\ 3.\ 4 \ldots\ldots (x+3) \\ \text{etc.} \end{array}$$

fumma erit

$$1.\ 2.\ 3.\ldots\ldots x\,(p+q+r+s+\text{etc.}).$$

§. 39. Sumamus hic pro cafu fpecialiffimo $x=0$, vt fummanda proponatur haec feries hypergeometrica $1-1+2-6+24-120+$ etc., pro qua ergo erit $1\ldots\ldots x=1$, tum vero reperietur

$$p=\tfrac{1}{2},\ q=\tfrac{1}{8},\ r=-\tfrac{1}{32},\ s=-\tfrac{1}{128}.$$

Calculo ergo hucusque producto fumma defiderata prodit

$$=\tfrac{1}{2}+\tfrac{1}{8}-\tfrac{1}{32}-\tfrac{1}{128}=\tfrac{75}{128}=0,5859,$$

quae non multum difcrepat ab ea quam olim omni ftudio elicui.

§. 40. Sumamus nunc $x=1$, vt fummanda fit haec feries $1-2+6-24+120-$ etc., eritque $1\ldots\ldots x=1$, tum vero $p=\frac{1}{3}$, $q=\frac{2}{27}$, $r=0$, $s=-\frac{4}{729}$. Hinc ergo erit noftra fumma $\frac{1}{3}+\frac{2}{27}-\frac{4}{729}=\frac{293}{729}=0,40192$, quae fumma cum praecedente fatis exacte confpirat, quoniam hinc ambae feries iunctae prodeunt $0,9878$: prodire enim deberet vnitas; vnde patet, fi vlterius feriem p, q, r, s effemus profecuti, tum etiam ad veritatem multo propius acceffiffemus.