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De transformatione seriei divergentis $1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + etc.$ in fractionem continuam

Leonhard Euler

DE TRANSFORMATIONE SERIEI DIVERGENTIS

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + m(m+n)(m+2n)(m+3n)x^4 \text{ etc.}$$

IN FRACTIONEM CONTINVAM.

Auctore
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§. 1.

Cum olim indolem huiusmodi ſerierum diuergentium eſſem perſcrutatus, et veram ſummam ſeriei hypergeometricae

$$1 - 1 + 2 - 6 + 24 - 120 + 720 - \text{ etc.}$$

aſſignauiſſem ope transformationis in fractionem continuam, mentionem quoque feci iſtius ſeriei multo latius patentis:

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + m(m+n)(m+2n)(m+3n)x^4 - \text{ etc.}$$

cuius ſummam inueneram aequari huic fractioni continuae:

$$\cfrac{1}{1 + \cfrac{mx}{1 + \cfrac{nx}{1 + \cfrac{(m+n)x}{1 + \cfrac{2nx}{1 + \cfrac{(m+2n)x}{1 + \text{ etc.}}}}}}}$$

cuius

cuius rei veritatem ex conuerſione aequationis Riccatianae in fractionem continuam deduxeram. Cum autem haec demonſtratio nimis longe petita videri queat, eandem reductionem hic ex principiis ſimplicioribus ſum traditurus.

§. 2. Primo autem iſtam ſeriem generalem in formam concinniorem contrahi conueniet ponendo $mx = a$ et $nx = b$, vt propoſita ſit iſta ſeries infinita:

$$1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

Praeterea vero vt ſequentes reſolutiones commodius peragi queant, neque tot clauſulis ſit opus, ſtatuam vt ſequitur:

$$a = A,\ a+b = B,\ a+2b = C,\ a+3b = D, \text{ etc.}$$

ſicque habebitur iſta ſeries:

$$1 - A + AB - ABC + ABCD - \text{etc.}$$

cuius ſummam quaeſitam deſignemus littera S, ita vt ſit

$$S = 1 - A + AB - ABC + ABCD - \text{etc.}$$

hinc porro

$$\frac{1}{S} = \frac{1}{1 - A + AB - ABC + ABCD - etc.}.$$

§. 3. Cum igitur ſit $\frac{1}{S} > 1$, poſtrema aequatio reducatur ad hanc formam:

$$\frac{1}{S} = 1 + \frac{A - AB + ABC - ABCD + etc.}{1 - A + AB - ABC + ABCD\ etc.}.$$

Nunc autem ponamus $\frac{1}{S} = 1 + \frac{A}{P}$, eritque

$$P = \frac{1 - A + AB - ABC + ABCD\ etc.}{1 - B + BC - BCD + BCDE\ etc.},$$

quae expreſſio cum iterum vnitatem ſuperet, ob $B - A = b$, $C - A = 2b$, $D - A = 3b$, etc. ea dabit

$$P = 1 + \frac{b - 2bB + 3bBC - 4bBCD + etc.}{1 - B + BC - BCD + BCDE - etc.}.$$

Ponatur ergo $P = 1 + \frac{b}{Q}$ eritque

Q =

$$Q = \frac{1 - B + BC - BCD + BCDE - etc.}{1 - 2B + 3BC - 4BCD + etc.},$$

vnde deducimus

$$Q = 1 + \frac{B - 2BC + 3BCD - 4BCDE\ etc.}{1 - 2B + 3BC - 4BCD + etc.}.$$

Hanc ob rem ponamus nunc $Q = 1 + \frac{B}{R}$, ac prodibit

$$R = \frac{1 - 2B + 3BC - 4BCD + etc.}{1 - 2C + 3CD - 4CDE + etc.}.$$

§. 4. Hic ergo tam in numeratore quam in denominatore iidem coefficientes occurrunt, at litterae maiufculae in denominatore vno gradu funt promotae. Cum igitur fit $C - B = b$, $D - B = 2b$, $E - B = 3b$, etc. fiet

$$R = 1 + \frac{2b - 2.3bC + 3.4bCD + 4.5bCDE - etc.}{1 - 2C + 3CD - 4CDE + 5CDEF - etc.}.$$

Quod fi ergo ponamus $R = 1 + \frac{2b}{S}$, erit

$$S = \frac{1 - 2C + 3CD - 4CDE + etc.}{1 - 3C + 6CD - 10CDE + etc.},$$

vbi in denominatore manifefto occurrunt numeri trigonales, quae expreffio reducitur ad hanc:

$$S = 1 + \frac{C - 3CD + 6CDE - 10CDEF + etc.}{1 - 3C + 6CD - 10CDE + etc.}.$$

Quod fi ergo ftatuamus $S = 1 + \frac{C}{T}$, erit

$$T = \frac{1 - 3C + 6CD - 10CDE + 15CDEF - etc.}{1 - 3D + 6DE - 10DEF + 15DEFG - etc.}.$$

§. 5. Ifta forma ob $D - C = b$, $E - C = 2b$, $F - C = 3b$, etc. abit in hanc:

$$T = 1 + \frac{3b - 2.6bD + 3.10bDE - 4.15bDEF + etc.}{1 - 3D + 6DE - 10DEF + 15DEFG - etc.}.$$

Ponamus $T = 1 + \frac{3b}{U}$, vt fiat

$$U = \frac{1 - 3D + 6DE - 10DEF + 15DEFG - etc.}{1 - 4D + 10DE - 20DEF + 35DEFG - etc.},$$

vbi in denominatore reperiuntur numeri pyramidales primi fiue fummae trigonalium, hincque nancifcimur:

U =

$$U = 1 + \frac{D - 4DE + 10DEF - 20DEFG + etc.}{1 - 4D + 10DE - 20DEF + 35DEFG - etc.},$$

vbi iam ſupra et infra occurrunt numeri pyramidales. Statuatur porro $U = 1 + \frac{D}{V}$ fietque

$$V = \frac{1 - 4D + 10DE - 20DEF + 35DEFG - etc.}{1 - 4E + 10EF - 20EFG + 35EFGH - etc.}.$$

§. 6. Hinc calculum vt ſupra proſequendo, cum ſit $E - D = b$, $F - D = 2b$, $G - D = 3b$, erit

$$V = 1 + \frac{4b - 2.10bE + 3.20bEF - 4.35bEFG + etc.}{1 - 4E + 10EF - 20EFG + 35EFGH + etc.}.$$

Sit $V = 1 + \frac{4b}{X}$, vt fiat

$$X = \frac{1 - 4E + 10EF - 20EFG + 35EFGH - etc.}{1 - 5E + 15EF - 35EFG + 70EFGH - etc.},$$

quae expreſſio reducitur ad hanc:

$$X = 1 + \frac{E - 5EF + 15EFG - 35EFGH + etc.}{1 - 5E + 15EF - 35EFG + etc.}.$$

Sit $X = 1 + \frac{E}{Y}$ eritque

$$Y = \frac{1 - 5E + 15EF - 35EFG + 70EFGH - etc.}{1 - 5F + 15FG - 35FGH + 70FGHI - etc.}.$$

§. 7. Cum igitur ſit $F - E = b$, $G - E = 2b$, $H - E = 3b$, etc. erit

$$Y = 1 + \frac{5b - 2.15b.F + 3.35bFG - 4.70bFGH + etc.}{1 - 5F + 15FG - 35FGH + 70FGHI - etc.}.$$

Sit nunc $Y = 1 + \frac{5b}{Z}$, vt fiat

$$Z = \frac{1 - 5F + 15FG - 35FGH + 70FGHI - etc.}{1 - 10F + 21FG - 56FGH + 126FGHI - etc.}.$$

Cum igitur initio poſuerimus $\frac{1}{s} = 1 + \frac{A}{P}$, erit ſumma quaeſita $S = \frac{1}{1 + \frac{A}{P}}$; tum vero factae ſunt ſequentes poſitiones:

$$P = 1 + \frac{b}{Q},\ Q = 1 + \frac{B}{R},\ R = 1 + \frac{2b}{S},\ S = 1 + \frac{C}{T},\ T = 1 + \frac{3b}{U},$$

$$U = 1 + \frac{D}{V},\ V = 1 + \frac{4b}{X},\ X = 1 + \frac{E}{Y},\ Y = 1 + \frac{5b}{Z},\ etc.$$

quibus

quibus valoribus ordine ſubſtitutis oritur iſta fractio continua:

$$S = \cfrac{1}{1+\cfrac{A}{1+\cfrac{b}{1+\cfrac{B}{1+\cfrac{2b}{1+\cfrac{C}{1+\cfrac{3b}{1+\cfrac{D}{1+\cfrac{4b}{1+\text{etc.}}}}}}}}}}$$

Quod ſi ergo loco litterarum A, B, C, D, etc. valores aſſumtos reſtituamus, vt nobis ſit iſta ſeries diuergens:

$$1-a+a(a+b)-a(a+b)(a+2b)+a(a+b)(a+2b)(a+3b)-\text{etc.}$$

eius ſumma exprimetur per ſequentem fractionem continuam:

$$S = \cfrac{1}{1+\cfrac{a}{1+\cfrac{b}{1+\cfrac{a+b}{1+\cfrac{2b}{1+\cfrac{a+2b}{1+\cfrac{3b}{1+\cfrac{a+3b}{1+\cfrac{4b}{1+\text{etc.}}}}}}}}}}$$

quae eſt eadem forma quam olim dederam.

§. 8.

§. 8. Haec transformatio eo magis eſt notatu digna, quod tutiſſimam ac fortaſſe vnicam nobis viam aperit, valorem ſeriei diuergentis vero proxime ſaltem determinandi. Si enim fractio continua more ſolito in fractiones ſimplices reſoluatur 1, $\frac{1}{1+a}$, $\frac{1+b}{1+a+b}$, etc. eae alternatim ſunt maiores et minores quam valor ſeriei diuergentis, et continuo propius ad iſtum valorem accedunt. Tum vero etiam ſingularia olim expoſui artificia, quae multo promptius ad verum valorem deducunt.

§. 9. Praeterea vero etiam notaſſe iuuabit, talem fractionem continuam:

$$1+\cfrac{\alpha}{1+\cfrac{\beta}{1+\cfrac{\gamma}{1+\cfrac{\delta}{1+\text{etc.}}}}}$$

in genere ſatis commode ad dimidium partium numerum redigi poſſe. Poſito enim eius valore $= S$, eum ita repraeſentare licebit:

$$S = 1+\cfrac{\alpha}{1+\cfrac{\beta}{P}}, \quad P = 1+\cfrac{\gamma}{1+\cfrac{\delta}{Q}}, \quad Q = 1+\cfrac{\varepsilon}{1+\cfrac{\zeta}{R}}, \text{ etc.}$$

Iam prima harum formularum erit

$$S = 1+\frac{\alpha P}{P+\beta} = 1+\alpha-\frac{\alpha\beta}{\beta+P},$$

ſecunda deinde formula dat

$$P = 1+\gamma\frac{Q}{Q+\delta} = 1+\gamma-\gamma\frac{\delta}{\delta+Q},$$

 eodem

eodem modo tertia praebet

$$Q = 1 + \frac{\varepsilon R}{R+\zeta} = 1 + \varepsilon - \frac{\varepsilon \zeta}{\zeta + R}, \text{ etc.}$$

Hi igitur valores successiue substituti, producent hanc nouam fractionem continuam:

$$S = 1 + \alpha - \cfrac{\alpha\beta}{1 + \beta + \gamma - \cfrac{\gamma\delta}{1 + \delta + \varepsilon - \cfrac{\varepsilon\zeta}{1 + \zeta + \eta - \cfrac{\eta\theta}{1 + \theta + \iota + \text{etc.}}}}}$$

§. 10. Cum igitur nostro casu series diuergens

$$S = 1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

perducta sit ad istam fractionem continuam:

$$S = \cfrac{1}{1 + \cfrac{a}{1 + \cfrac{b}{1 + \cfrac{a+b}{1 + \cfrac{2b}{1 + \cfrac{a+2b}{1 + \cfrac{3b}{1 + \cfrac{a+3b}{1 + \text{etc.}}}}}}}}}$$

sumamus hic

$$\alpha = a, \ \beta = b, \ \gamma = a+b, \ \delta = 2b, \ \varepsilon = a+2b, \ \text{etc.}$$

eritque

S =

$$S = 1 + a - \cfrac{ab}{1 + a + 2b - \cfrac{2b(a+b)}{1 + a + 4b - \cfrac{3b(a+2b)}{1 + a + 6b - \cfrac{4b(a+3b)}{1 + a + \text{etc.}}}}}$$

Appendix.

De fractione continua Brouncheriana.

§. 11. Cum olim multum fuissem occupatus in Analysi indaganda, quae *Brouncherum* ad istam singularem fractionem perduxerit, quandoquidem mihi haud probabile est visum, eum per tot ambages, quales a *Wallisio* commemorantur, eo fuisse perductum, tandem mihi quidem satis dilucide ostendisse sum visus, Brouncherum hanc formam ex serie Leibniziana $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} +$ etc. quam magnus *Gregorius*, iam ante inuenerat, deduxisse potius quam ex interpolatione seriei $1, \frac{1}{2}, \frac{1 \cdot 3}{2 \cdot 4}, \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}, \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}$, etc. quemadmodum Wallisius suspicabatur, si quidem consideratio illius seriei per ratiocinium satis planum ad formam Brouncherianam manuducit.

§. 12. Haec obseruatio autem nunc quidem eo maiore attentione digna videtur, postquam Cel. *Dan. Bernoullius* memoriam formae Brouncherianae renouare haud sit dedignatus. Quoniam igitur non ita pridem facilem methodum exposui istam formam ex serie $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ etc. deriuandi, Geometris haud ingratum fore arbitror, si methodum inuersam in medium protulero, cuius ope formulam Brouncherianam vicissim ad seriem Leibnizianam reducere licet.

§. 13. Confiderabo igitur fractionem iftam continuam quafi eius valor nondum effet cognitus, ftatuendo:

$$S = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \text{etc.}}}}}}}$$

quam per partes fequenti modo repraefento:

$$S = \cfrac{1}{1 + \cfrac{1}{-1 + P}}, \quad P = 3 + \frac{9}{-3 + Q}, \quad Q = 5 + \frac{25}{-5 + R}, \quad R = 7 + \frac{49}{-7 + S}, \text{ etc.}$$

Ex his enim partibus debite coniunctis ipfa forma propofita manifefto enafcitur.

§. 14. Singulas igitur has partes feorfim euoluamus, ac prima quidem reducta ad fractionem fimplicem praebet $S = \frac{P-1}{P}$, ideoque $S = 1 - \frac{1}{P}$, fecunda vero erit $\frac{3Q}{Q-3}$, vnde fit $\frac{1}{P} = \frac{Q-3}{3Q}$, fiue $\frac{1}{P} = \frac{1}{3} - \frac{1}{Q}$, fimili modo pars tertia dat $Q = \frac{5R}{R-5}$, ideoque $\frac{1}{Q} = \frac{1}{5} - \frac{1}{R}$; eodem modo ex fequentibus partibus nancifcemur $\frac{1}{R} = \frac{1}{7} - \frac{1}{S}$, $\frac{1}{S} = \frac{1}{9} - \frac{1}{T}$, etc. Quare fi ifti valores fucceffiue fubftituantur, obtinebimus hanc expreffionem:

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

ita ita vt nunc certi fimus effe $S = \frac{\pi}{4}$.

§. 15. Simili modo etiam aliarum huiusmodi fractionum continuarum valorem inueftigare licebit. Veluti fi propofita fuerit

fuerit haec forma:

$$S = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}}$$

ea sequenti modo in membra destribuatur:

$$S = \cfrac{1}{1 + \cfrac{1}{-1 + P}}, \quad P = 2 + \cfrac{4}{-2 + Q}, \quad Q = 3 + \cfrac{9}{-3 + R}, \quad R = 4 + \cfrac{16}{-4 + S}, \text{ etc.}$$

his enim singulis partibus euolutis reperietur:

$$S = 1 - \tfrac{1}{P}, \; \tfrac{1}{P} = \tfrac{1}{2} - \tfrac{1}{Q}, \; \tfrac{1}{Q} = \tfrac{1}{3} - \tfrac{1}{R}, \; \tfrac{1}{R} = \tfrac{1}{4} - \tfrac{1}{5}, \text{ etc.}$$

vnde sequitur fore

$$S = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{ etc.} = l\,2.$$

Haec igitur methodus haud parum in recessu habere videtur.