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Commentatio de curvis tractoriis

Leonhard Euler

COMMENTATIO
DE
CVRVIS TRACTORIIS.

Auctore

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§. 1.

Quae olim a Geometris de curuis tractoriis sunt inuestigata, quanquam ad doctrinam motus pertinere videntur, tamen nullo modo ad Mechanicam referri possunt: eiusmodi enim hypothesi innituntur, quae veris principiis motus manifesto refragatur. Nihilo vero minus, admissa ista hypothesi, si res tantum geometrice consideretur, quae super hoc argumento sunt inuenta omni attentione digna sunt putanda, atque adeo ab experientia vix aberrare solent. Quamobrem haud inutile fore arbitror, totum hoc negotium accuratius perscrutari et secundum vera motus principia diiudicare.

§. 2. Considerari autem solet via, quam corpusculum super plano horizontali describit, dum ope fili secundum lineam siue rectam siue curuam protrahitur; atque haec quaestio ita ad Geometriam reuocari solet, vt curua descripta perpetuo a directione fili tangatur, atque adeo omnes tangentes istius curuae descriptae vsque ad lineam, iuxta quam

 filum

filum protrahitur, productae, vbique eiusdem ſint longitudinis. Vt autem talis motus eueniat, auctores probe monuerunt, planum, ſuper quo iſte motus producitur, neutiquam politum, ſed ſatis eſſe debere aſperum; tum vero etiam neceſſe eſſe, vt filum lente promoueatur, quandoquidem, niſi hae conditiones obſeruentur, curua deſcripta plurimum a calculo eſſet diſcrepatura.

Tab. I. Fig. 1. §. 3. Ita ſi corpuſculo C alligatum ſit filum $CA = a$, cuius terminus A iuxta lineam rectam AB protrahitur, corpuſculum in linea quadam curua CY promouebitur, cuius tangentes YT e ſingulis punctis ad rectam AB productae vbique longitudini fili a aequentur; vnde ſi pro puncto Y vocetur abſciſſa $AX = x$ et applicata $XY = y$, elementum vero curuae $Yy = \partial s$, erit $-\partial y : \partial s = y : a$, ideoque $y\,\partial s = -a\,\partial y$ et $\partial s = -\frac{a\,\partial y}{y}$, vnde integrando ſtatim colligitur arcus curuae $Cy = s = -a\,l\,y + C$. Quare ſi initio filum CA ad rectam AB fuerit normale, tum erat $y = a$ et $s = 0$, ex quo colligitur $s = a\,l\,\frac{a}{y}$. Vt autem aequatio inter coordinatas eruatur, loco ∂s ſcribatur eius valor $\surd(\partial x^2 + \partial y^2)$, et ſumtis quadratis erit $yy\,\partial x^2 + yy\,\partial y^2 = aa\,\partial y^2$, vnde deducitur $\partial x = -\frac{\partial y \surd(aa - yy)}{y}$, pro cuius integratione faciamus $\surd(aa - yy) = v$, eritque $yy = aa - vv$, hinc $\frac{\partial y}{y} = -\frac{v\,\partial v}{aa - vv}$, ergo

$$\partial x = \frac{vv\,\partial v}{aa - vv} = -\partial v + \frac{aa\,\partial v}{aa - vv},$$

conſequenter

$$x = C - v + \tfrac{1}{2}a\,l\,\frac{a+v}{a-v} = C - \surd(aa - yy) + \tfrac{1}{2}a\,l\,\frac{a + \surd(aa - yy)}{a - \surd(aa - yy)},$$

et quia caſu $x = 0$ fieri debet $y = a$, fiet

$$x = \tfrac{1}{2}a\,l\,\frac{a + \surd(aa - yy)}{a - \surd(aa - yy)} - \surd(aa - yy), \text{ ſiue}$$

$$x = a\,l\,\frac{a + \surd(aa - yy)}{y} - \surd(aa - yy).$$

Vnde

Vnde patet, corpuſculum non ante ad rectam AB peruenire quam percurſo ſpatio infinito.

§. 4. Conſideremus nunc quoque caſum, quo filum iuxta lineam curuam quamcunque A T protrahitur. Ita ſi Y ſit punctum in Tractoria, eiusque tangens vsque ad curuam datam in T ducatur, recta YT perpetuo aequetur longitudini fili $= a$. Referatur curua data ad axem AB, ad quem ex T demittatur perpendiculum TU, ſitque $AU = u$ et $UT = t$, atque ob curuam datam dabitur aequatio inter t et u. Nunc vero ex puncto Tractoriae Y ad eundem axem ducatur normalis YX, ſitque $AX = x$ et $XY = y$ et arcus Tractoriae $= s$. Hinc cum YT curuam tangat, ducta ex T axi normali TS, ob $YT = a$, erit $\partial s : \partial x = a : TS$ et $\partial s : -\partial y = a : YS$, vnde fit $TS = (u - x) = \frac{a \partial x}{\partial s}$ et $SY = y - t = -\frac{a \partial y}{\partial s}$. Ponamus nunc $\partial y = p \partial x$, erit $\partial s = \partial x \sqrt{(1 + pp)}$, hincque fiet $u - x = \frac{a}{\sqrt{(1+pp)}}$ et $t - y = \frac{ap}{\sqrt{(1+pp)}}$. Ex his igitur formulis, ſi curua tractoria eſſet cognita, facile determinaretur curua AT, iuxta quam filum produci debet. Tab. I. Fig. 2.

§. 5. Vt autem ex data aequatione inter t et u inueſtigemus aequationem inter x et y, calculus ita inſtituatur. Ex binis formulis inuentis: $u = x + \frac{a}{\sqrt{(1+pp)}}$ et $t = y + \frac{ap}{\sqrt{(1+pp)}}$, habebimus differentiando

$$\text{I. } \partial u = \partial x - \frac{ap\partial p}{(1+pp)^{\frac{3}{2}}} \text{ et II. } \partial t = p\,\partial x + \frac{a\partial p}{(1+pp)^{\frac{3}{2}}},$$

vnde $II - I \times p$ praebet $\partial t - p\partial u = \frac{a\partial p}{\sqrt{(1+pp)}}$, ex qua, conceſſa aequationum differentialium reſolutione, quantitas variabilis p definietur per coordinatas datas t et u; ita vt t ſpectari poſſit tanquam certa functio ipſius u, quia t per u dari aſſumitur.

 Porro

Porro haec combinatio: I. + II. p dat $\partial u + p\partial t = \partial x(1+pp)$, vnde colligimus $\partial x = \frac{\partial u + p\partial t}{1+pp}$, hincque porro $\partial y = \frac{p(\partial u + p\partial t)}{1+pp}$, ſicque etiam x et y per eandem variabilem u determinabuntur.

§. 6. Hic quidem aſſumere ſumus coacti, reſolutionem aequationis differentialis $\frac{a\partial p}{\sqrt{(1+pp)}} + p\partial u = \partial t$ eſſe in poteſtate, quod tamen pauciſſimis tantum caſibus exſequi licet. Viciſſim igitur, ſi curuam tractoriam tanquam iam cognitam ſpectemus, quandoquidem eius deſcriptio mechanica datur, ipſam hanc aequationem differentialem reſoluere licebit. Atque adeo iam olim hoc modo conſtructionem aequationis *Riccatianae* exhibui.

§. 7. Vt hanc aequationem ab irrationalitate liberemus, faciamus $p = \frac{zz-1}{2z}$, vt fiat $\frac{\partial p}{\sqrt{(1+pp)}} = \frac{\partial z}{z}$, et noſtra aequatio differentialis erit $\frac{a\partial z}{z} + \frac{(zz-1)\partial u}{2z} = \partial t$, ſiue

$$a\partial z + \tfrac{1}{2}(zz-1)\partial u = z\partial t,$$

quam ergo ſemper per motum tractorium conſtruere licet, qualiscunque functio quantitas t fuerit ipſius u. Inuento valore literae z erit

$$x = \int \frac{4zz\partial u + 2z(zz-1)\partial t}{(1+zz)^2} \text{ et}$$

$$y = \int \frac{(2z\partial u + (zz-1)\partial t)(zz-1)}{(1+zz)^2}.$$

Euidens autem eſt, in hac aequatione formulam illam *Riccatianam* latiſſimo ſenſu acceptam contineri. Si enim ſtatuamus $z = e^{\frac{t}{a}}$, erit $\partial z = e^{\frac{t}{a}}\partial v + e^{\frac{t}{a}}\frac{v\partial t}{a}$, et aequatio noſtra hanc induet formam:

$$a e^{\frac{t}{a}}\partial v + \tfrac{1}{2}e^{\frac{2t}{a}}vv\partial u = \tfrac{1}{2}\partial u, \text{ ſiue}$$

$$a\partial v + \tfrac{1}{2}e^{\frac{t}{a}}vv\partial u = \tfrac{1}{2}e^{-\frac{t}{a}}\partial u,$$

vnde

vnde cum $e^{\frac{f}{a}}$ ſemper ſit certa functio ipſius u, quae ponatur $= U$, conſtrui poterit haec aequatio differentialis:

$$a \partial v + \tfrac{1}{2} v v U \partial u = \frac{\frac{1}{2} \partial u}{U}.$$

§. 8. Hanc igitur ob cauſſam ſi curua, iuxta quam filum protrahitur, pro lubitu accipiatur, determinatio Tractoriae plerumque vires Analyſeos ſuperat. At ſi filum iuxta peripheriam circuli protrahatur, cuius centrum ſit in C, et radius $AC = c$, ſingulari fortuna euenit, vt Tractoria definiri poſſit. Inceperit enim iſte motus, dum corpuſculum erat in B et filum $BA = a$ ad circulum erat normale; nunc autem corpuſculum peruenerit in Z, vbi recta tangens ZT circulo in T occurrat, ita vt ſit $ZT = a$. Iam ducta recta CZ vocetur angulus $ACZ = \omega$ et $CZ = z$, ita vt pro Tractoria inueniienda ſit aequatio inter rectam z et angulum ω, quae quidem inueſtigatio, niſi artificium adhibeatur, in calculos non parum moleſtos induceret. Tab. I. Fig. 3.

§. 9. Ad has difficultates euitandas in calculum introducamus angulum $CZT = \Phi$; ſic enim conſideratio trianguli CZT, cuius latera ſunt $CZ = z$, $ZT = a$ et $CT = c$, ſtatim praebet $cc = aa + zz - 2az \operatorname{cof.} \Phi$, vnde deducitur $z = a \operatorname{cof.} \Phi \pm \surd (cc - aa \operatorname{fin.} \Phi^2)$, vbi ſignum ambiguum ad ſitum puncti z reſpicit, prouti id fuerit vel extra circulum vel intra circulum. Quia autem in figura punctum z extra circulum ſitum repraeſentatur, valebit ſignum ſuperius, eritque $z = a \operatorname{cof.} \Phi + \surd (cc - aa \operatorname{fin.} \Phi^2)$. Praeterea hinc ſimul innoteſcunt anguli ZCT et ZTC; erit enim $\operatorname{fin.} ZCT = \frac{a \operatorname{fin.} \Phi}{c}$ et $\operatorname{fin.} ZTC = \frac{z \operatorname{fin.} \Phi}{c}$. Nunc quia recta ZT eſt tangens Tractoriae in Z, ducatur recta proxima $Cz = z + \partial z$, et ex Z deſcripto arculo zs, in triangulo Zzs erit

Zs

$Zs = -\partial z$, et ob angulum $Zcz = \partial\omega$ erit $zs = z\partial\omega$, vnde ſtatim colligitur tang. sZz, hoc eſt tang. $\Phi = \frac{z\partial\omega}{-\partial z}$, hincque porro $\frac{\partial z}{z} = -\frac{\partial\omega}{tang.\Phi}$, ſiue $\partial\omega = -\frac{\partial z}{z}$ tang. Φ, ſicque angulus ω per z et Φ definitur. Iam vero relationem inter z et Φ inuenimus. Praeterea vero cum ipſum Tractoriae elementum Zz, quod vocemus $= \partial s$, ſit $\partial s = -\frac{\partial z}{cof.\Phi}$, hinc longitudo Tractoriae concluditur $BZ = s = -\int\frac{\partial z}{cof.\Phi}$.

§. 10. Cum igitur inuenerimus

$$z = a\,\mathrm{cof.}\,\Phi + \sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}, \text{ erit}$$

$$\partial z = -a\partial\Phi\,\mathrm{fin.}\,\Phi - \frac{aa\partial\Phi\,\mathrm{fin.}\,\Phi\,\mathrm{cof.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}}$$

$$- \frac{a\partial\Phi\,\mathrm{fin.}\,\Phi(\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)} + a\,\mathrm{cof.}\,\Phi)}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}},$$

quae manifeſto reducitur ad hanc formam $\frac{-az\partial\Phi\,\mathrm{fin.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}}$, ita vt ſit $\frac{\partial z}{z} = -\frac{a\partial\Phi\,\mathrm{fin.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}}$. Quamobrem angulus ω ita determinabitur, vt ſit $\partial\omega = \frac{a\partial\Phi\,\mathrm{fin.}\,\Phi\,\mathrm{tang.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}}$; tum vero erit etiam

$$\partial s = \frac{az\partial\Phi\,\mathrm{tang.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}} = \frac{aa\partial\Phi\,\mathrm{fin.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}} + a\partial\Phi\,\mathrm{tang.}\,\Phi,$$

vnde integrando prodit

$$s = -al\,\mathrm{cof.}\,\Phi + aa\int\frac{\partial\Phi\,\mathrm{fin.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}}.$$

§. 11. Totum ergo negotium reducitur ad has formulas integrales; $\int\frac{\partial\Phi\,\mathrm{fin.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}}$ et $\int\frac{\partial\Phi\,\mathrm{fin.}\,\Phi\,\mathrm{tang.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}}$. Quod ad priorem attinet, quia $-\partial\Phi\,\mathrm{fin.}\,\Phi$ eſt differentiale ipſius cof. Φ, ponamus cof. $\Phi = v$, et haec formula transformabitur in hanc:

$$\int\frac{\partial\Phi\,\mathrm{fin.}\,\Phi}{\sqrt{(cc - aa\,\mathrm{fin.}\,\Phi^2)}} = -\int\frac{\partial v}{\sqrt{(cc - aa(1 - vv))}},$$

cuius integrale eſt

$$-\frac{1}{a}l\left(\frac{av + \sqrt{(bb + aavv)}}{b}\right) = -\frac{1}{a}l\frac{av + \sqrt{(cc - aa + aavv)}}{\sqrt{(cc - aa)}},$$

vnde

vnde restituto valore cos. Φ loco v reperietur tandem

$$s = C - a\,l\,\text{cos.}\,\Phi - a\,l\,[a\,\text{cos.}\,\Phi + \surd(cc - aa\,\text{sin.}\,\Phi^2)].$$

Vbi ad constantem definiendam notetur, initio fuisse tam $s = 0$ quam $\Phi = 0$: erit igitur $C = a\,l\,(a + c)$, hinc fit

$$s = a\,l\,\frac{a + c}{\text{cos.}\,\Phi\,(a\,\text{cos.}\,\Phi + \surd(cc - aa\,\text{sin.}\,\Phi^2))},$$

vnde patet, rectificationem huius Tractoriae per solos logarithmos expediri.

§. 12. Praecipuum autem negotium versatur in integratione formulae $\omega = a\int\frac{\partial\Phi\,\text{sin.}\,\Phi\,\text{tang.}\,\Phi}{\surd(cc - aa\,\text{sin.}\,\Phi^2)}$, quae commodissime tractabitur si statuamus $\surd(cc - aa\,\text{sin.}\,\Phi^2) = x\,\text{sin.}\,\Phi$, vt fiat $\omega = a\int\frac{\partial\Phi\,\text{sin.}\,\Phi}{x\,\text{cos.}\,\Phi}$. Verum inde habebitur

$$cc - aa\,\text{sin.}\,\Phi^2 = xx\,\text{sin.}\,\Phi^2,\ \text{hincque}$$

$$\text{sin.}\,\Phi^2 = \frac{cc}{aa + xx}\ \text{et}\ \text{cos.}\,\Phi^2 = \frac{aa - cc + xx}{aa + xx}.$$

Sumtis logarithmis erit

$$2\,l\,\text{cos.}\,\Phi = l\,(aa - cc + xx) - l\,(aa + xx),$$

vnde differentiando fiet

$$\frac{\partial\Phi\,\text{sin.}\,\Phi}{\text{cos.}\,\Phi} = \frac{-x\,\partial x}{aa - cc + xx} + \frac{x\,\partial x}{aa + xx},$$

quo valore substituto prodit

$$\omega = a\int\frac{\partial x}{aa + xx} - a\int\frac{\partial x}{aa - cc + xx},$$

vbi pars prior manifesto fit

$$= \text{A tang.}\,\frac{x}{a} = \text{A tang.}\,\frac{\surd(cc - aa\,\text{sin.}\,\Phi^2)}{a\,\text{sin.}\,\Phi}.$$

Pro parte autem posteriore tres casus considerari conuenit, prouti fuerit vel $a > c$, vel $a < c$, vel $a = c$, quos singulos igitur percurramus.

Casus I.

$a > c.$

§. 13. Sit igitur primo $a > c$, ponaturque $aa - cc = bb$, eritque

$$\int \frac{a \partial x}{aa - cc + xx} = \int \frac{a \partial x}{bb + xx} = \frac{a}{b} \int \frac{b \partial x}{bb + xx},$$

cuius integrale est

$$\frac{a}{b} \text{ A tang. } \frac{x}{b} = \frac{a}{b} \text{ A tang. } \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{b \sin. \Phi},$$

quocirca pro hoc casu habebimus

$$\omega = \text{A tang.} \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{a \sin. \Phi} - \frac{a}{\sqrt{(aa - cc)}} \text{ A tang.} \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{\sin. \Phi \sqrt{(aa - cc)}} + C.$$

Pro constante C autem determinanda notetur, initio fieri tam $\omega = 0$ quam $\Phi = 0$, vnde concluditur $C = \frac{\pi}{2} \left(\frac{a - \sqrt{(aa - cc)}}{\sqrt{(aa - cc)}} \right)$, quo valore inducto erit

$$\omega = \frac{\pi}{2} \left(\frac{a - \sqrt{(aa - cc)}}{\sqrt{(aa - cc)}} \right) + \text{A tang.} \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{a \sin. \Phi} - \frac{a}{\sqrt{(aa - cc)}} \text{ A tg.} \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{\sin. \Phi \sqrt{(aa - cc)}},$$

qui valor etiam ita referri potest:

$$\omega = \frac{a}{\sqrt{(aa - cc)}} \text{ A tang.} \frac{\sin. \Phi \sqrt{(aa - cc)}}{\sqrt{(cc - aa \sin. \Phi^2)}} - \text{A tang.} \frac{a \sin. \Phi}{\sqrt{(cc - aa \sin. \Phi^2)}}.$$

Hoc igitur casu sin. Φ non vltra terminum $\frac{c}{a}$ augeri potest; quando autem fit sin. $\Phi = \frac{c}{a}$, tum fit angulus

$$\omega = \left(\frac{a}{\sqrt{(aa - cc)}} - 1 \right) 90^\circ$$

et distantia $z = \sqrt{(aa - cc)}$.

§. 14. Hoc igitur casu angulus ω per solos arcus circulares, ideoque etiam per angulos definitur; vnde si modo hi anguli rationem teneant rationalem inter se, id quod euenit quoties $\frac{a}{\sqrt{(aa - cc)}}$ fuerit numerus rationalis, angulum ω geometrice definire licebit, sicque ipsa curua tractoria euadet algebraica, siue eius natura per aequationem algebraicam exprimi poterit. Haec igitur circumstantia vtique meretur, vt exemplo illustretur.

Exem-

Exemplum.

§. 15. Euoluamus igitur casum quo $\frac{a}{\sqrt{(aa-cc)}} = 2$, siue $c = \frac{a\sqrt{3}}{2}$: sic enim fiet $\sqrt{(aa-cc)} = \frac{1}{2}a$, hincque porro

$$\omega = 2\,\text{A tang.}\,\frac{\text{fin.}\,\Phi}{\sqrt{(3-4\,\text{fin.}\,\Phi^2)}} - \text{A tang.}\,\frac{2\,\text{fin.}\,\Phi}{\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}.$$

Cum igitur in genere sit $2\,\text{A tang.}\,t = \text{A tang.}\,\frac{2t}{1-tt}$, nostro autem casu sit $t = \frac{\text{fin.}\,\Phi}{\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}$, erit

$$2\,\text{A tang.}\,\frac{\text{fin.}\,\Phi}{\sqrt{(3-4\,\text{fin.}\,\Phi^2)}} = \text{A tang.}\,\frac{2\,\text{fin.}\,\Phi\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}{3-5\,\text{fin.}\,\Phi^2},$$

ideoque erit

$$\omega = \text{A tang.}\,\frac{2\,\text{fin.}\,\Phi\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}{3-5\,\text{fin.}\,\Phi^2} - \text{A tang.}\,\frac{2\,\text{fin.}\,\Phi}{\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}.$$

Cum porro sit $\text{A tang.}\,p - \text{A tang.}\,q = \frac{p-q}{1+pq}$, quia nostro casu est

$$p = \frac{2\,\text{fin.}\,\Phi\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}{3-5\,\text{fin.}\,\Phi^2} \text{ et } q = \frac{2\,\text{fin.}\,\Phi}{\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}, \text{ erit}$$

$$p-q = \frac{2\,\text{fin.}\,\Phi^3}{(3-5\,\text{fin.}\,\Phi^2)\sqrt{(3-4\,\text{fin.}\,\Phi^2)}} \text{ et } 1+pq = \frac{3-\text{fin.}\,\Phi^2}{3-5\,\text{fin.}\,\Phi^2},$$

consequenter obtinebimus

$$\omega = \text{A tang.}\,\frac{2\,\text{fin.}\,\Phi^3}{(3-\text{fin.}\,\Phi^2)\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}, \text{ ideoque}$$

$$\text{tang.}\,\omega = \frac{2\,\text{fin.}\,\Phi^3}{(3-\text{fin.}\,\Phi^2)\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}.$$

Hoc igitur modo ex assumto angulo Φ colligitur angulus ω.

§. 16. Porro igitur cum pro hoc exemplo sit

$$z = a\,\text{cof.}\,\Phi + \tfrac{1}{2}a\sqrt{(3-4\,\text{fin.}\,\Phi^2)},$$

si ex puncto Z ad rectam CB ducatur normalis ZX, et pro Tractoria vocenter coordinatae $CX = x$ et $XZ = y$, fiet $x = z\,\text{cof.}\,\omega$ et $y = z\,\text{fin.}\,\omega$, sicque tam x quam y per eundem angulum Φ determinabitur. Ex tangente autem anguli ω concluditur

$$\text{fin.}\,\omega = \frac{2\,\text{fin.}\,\Phi^3}{3\,\text{cof.}\,\Phi^2\sqrt{3}} \text{ et } \text{cof.}\,\omega = \frac{(3-\text{fin.}\,\Phi^2)\sqrt{(3-4\,\text{fin.}\,\Phi^2)}}{3\,\text{cof.}\,\Phi^2\sqrt{3}}.$$

 Quod

Quodsi autem hinc ipsum angulum Φ eliminare vellemus, aequatio inter x et y sine dubio ad plures dimensiones assurgeret. Interim tamen constructio geometrica huius curuae non nimis est prolixa.

§. 17. Ad has formulas simpliciores reddendas statuatur $\sqrt{(3 - 4\,\text{sin.}\,\Phi^2)} = 2u\,\text{sin.}\,\Phi$, vt fiat $z = a\,\text{cos.}\,\Phi + a\,u\,\text{sin.}\,\Phi$, et tang. $\omega = \frac{\text{sin.}\,\Phi^2}{u(3 - \text{sin.}\,\Phi^2)}$; tum autem erit sin. $\Phi^2 = \frac{3}{4(1 + uu)}$, vnde fit tang. $\omega = \frac{1}{3 + 4uu}$. Deinde vero ob cos. $\Phi^2 = \frac{1 + 4uu}{4(1 + uu)}$ fiet $z = \frac{\sqrt{(1 + 4uu)} + u\sqrt{3}}{2\sqrt{(1 + uu)}}$. Ponatur porro $\frac{u\sqrt{3}}{\sqrt{(1 + 4uu)}} = \text{cos.}\,\theta$, erit sin. $\theta = \sqrt{\frac{1 + uu}{1 + 4uu}}$, vnde fit $\frac{z}{a} = \frac{1 + \text{cos.}\,\theta}{2\,\text{sin.}\,\theta} = \frac{1}{2}\,\text{cot.}\,\frac{1}{2}\theta$, deinde vero ob $uu = \frac{\text{cos.}\,\theta^2}{3 - 4\,\text{cos.}\,\theta^2}$ erit tang. $\omega = \frac{3 - 4\,\text{cos.}\,\theta^2}{9 - 8\,\text{cos.}\,\theta^2} = \frac{4\,\text{sin.}\,\theta^2 - 1}{1 + 8\,\text{sin.}\,\theta^2}$.

Casus II.

$a < c$.

§. 18. Sit iam $a < c$, ponaturque $cc = aa + bb$, eritque

$$\omega = \text{A tang.}\,\frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}}{a\,\text{sin.}\,\Phi} - a\int\frac{\partial x}{xx - bb}.$$

Est vero

$$\int\frac{a\,\partial x}{xx - bb} = \frac{a}{b}\int\frac{b\,\partial x}{xx - bb} = \frac{a}{2b}\,l\,\frac{x - b}{x + b}.$$

Cum igitur sit $x = \frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}}{\text{sin.}\,\Phi}$ et $b = \sqrt{(cc - aa)}$, hinc colligitur

$$\omega = C + \text{A tang.}\,\frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}}{a\,\text{sin.}\,\Phi} - \frac{a}{2\sqrt{(cc - aa)}}\,l\,\frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)} - \text{sin.}\,\Phi\sqrt{(cc - aa)}}{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)} + \text{sin.}\,\Phi\sqrt{(cc - aa)}},$$

vbi quia initio fieri debet tam $\Phi = 0$ quam $\omega = 0$, erit constans $C = -\frac{\pi}{2}$, vnde fit

$$\omega = \frac{a}{2\sqrt{(cc - aa)}}\,l\,\frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)} + \text{sin.}\,\Phi\sqrt{(cc - aa)}}{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)} - \text{sin.}\,\Phi\sqrt{(cc - aa)}} - \text{A tang.}\,\frac{a\,\text{sin.}\,\Phi}{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}}.$$

Manet autem vt ante $z = a\,\text{cos.}\,\Phi + \sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}$, vnde patet, has curuas semper esse transcendentes. Ceterum quia hic

hic $c > a$, euidens eſt, angulum Φ a o vsque ad 90° increſcere poſſe, cum primo caſu, vbi erat $c < a$, angulus Φ eo vsque tantum creſcere poterat, quoad fiat ſin. $\Phi = \frac{c}{a}$.

Caſus III.

$c = a$.

§. 19. Poſito autem $c = a$ ſtatim fit $z = 2a$ coſ. Φ et $\omega = a\int\frac{\partial x}{aa+xx} - \int\frac{a\,\partial x}{xx}$, ideoque

$$\omega = \text{A tang.}\,\frac{x}{a} + \frac{a}{x} + C = \text{A tang.}\,\frac{\text{coſ.}\,\Phi}{\text{ſin.}\,\Phi} + \text{tang.}\,\Phi + C.$$

Hoc ergo modo determinata conſtante prodit $\omega = \text{tang.}\,\Phi - \Phi$; vnde intelligitur, ſi angulus Φ increſcat vsque ad 90°, tum fore angulum $\omega = \infty$, ſcilicet hoc caſu filum per infinitas reuolutiones in circulo protrahi poterit. Tum autem denique fiet $z = 0$; vnde patet, confectis infinitis reuolutionibus corpuſculum tandem in ipſum centrum circuli peruenire, ibique in quiete eſſe permanſurum.

§. 20. Ceterum pro ſecundo caſu ſingulare phaenomenon ſeſe exſerit. Statim enim primae aequationi $aa + zz - 2az$ coſ. $\Phi = cc$ ſatisfieri manifeſtum eſt, ſi fuerit $\Phi = 90°$ et $z = \sqrt{(cc - aa)}$; tum autem angulus ω plane non determinatur; quia fit $\partial\omega = \frac{0}{0}$, et hoc caſu ipſa curua tractoria erit circulus etiam centro C radio $cc - aa$ deſcriptus: huius enim tangentes, ad circulum A C B productae, aequabuntur longitudini fili a; atque ad hunc caſum omnes reliqui motus poſt infinitas reuolutiones reducentur, ita vt hae Tractoriae tandem in circulum abeant. Neque tamen ex hac ſolutione ipſam formam harum Tractoriarum ſatis commode cognoſcere licet, vnde aliam ſolutionem ſubiungamus ad hunc ſcopum magis accommodatam.

Alia methodus
Tractorias ex circulo natas determinandi.

§. 21. Maneant denominationes ante adhibitae, scilicet longitudo fili $BA = ZT = a$, radius circuli $CA = CT = c$, distantia $CZ = z$, angulus $ACZ = \omega$ et angulus $CZT = \Phi$, vnde fit vt ante $\partial\omega = -\frac{\partial z}{z}$ tang. Φ. Nunc autem insuper vocemus angulum $ZCT = \theta$, ad quem omnia elementa curvae reuocemus. Tandem etiam fit angulus $ACT = \omega + \theta = \psi$; quandoquidem hoc modo statim innotescet punctum T, quousque filum iam est protractum.

§. 22. His positis ex T ad rectam CZ agatur normalis TP, et ex triangulo CTP erit $TP = c$ fin. θ et $CP = c$ cof. θ: at ex triangulo ZTP erit $TP = a$ fin. Φ et $ZP = a$ cof. Φ, vnde statim colligitur $z = a$ cof. $\Phi + c$ cof. θ; tum vero c fin. $\theta = a$ fin. Φ, vnde fin. $\Phi = \frac{c}{a}$ fin. θ, cof. $\Phi = \frac{\sqrt{(aa - cc\,\text{fin.}\,\theta^2)}}{a}$ et tang. $\Phi = \frac{c\,\text{fin.}\,\theta}{\sqrt{(aa - cc\,\text{fin.}\,\theta^2)}}$. Differentiemus nunc binas illas aequationes, et prodibit

I. $-\partial z = a\,\partial\Phi$ fin. $\Phi + c\,\partial\theta$ fin. θ et

II. $0 = a\,\partial\Phi$ cof. $\Phi - a\,\partial\theta$ cof. θ,

vnde combinatio: I. cof. Φ — II. fin. Φ praebet $-\partial z$ cof. $\Phi = c\,\partial\theta$ fin. θ cof. $\Phi + c\,\partial\theta$ cof. θ fin. $\Phi = c\,\partial\theta$ fin. $(\theta + \Phi)$. At vero ex triangulo CAT habetur CT : fin. $\theta = z$: fin. $(\theta + \Phi)$, ideoque fin. $(\theta + \Phi) = \frac{z\,\text{fin.}\,\theta}{a}$: hoc ergo valore adhibito fiet $-\partial z$ cof. $\Phi = \frac{cz\,\partial\theta\,\text{fin.}\,\theta}{a}$, ideoque $-\frac{\partial z}{z} = \frac{c\,\partial\theta\,\text{fin.}\,\theta}{a\,\text{cof.}\,\Phi}$.

§. 23. Ex hoc igitur valore nanciscimur $\partial\omega = \frac{c\,\partial\theta\,\text{fin.}\,\theta\,\text{fin.}\,\Phi}{a\,\text{cof.}\,\Phi^2}$; erat autem fin. $\Phi = \frac{c}{a}$ fin. θ et cof. $\Phi^2 = \frac{aa - cc\,\text{fin.}\,\theta^2}{aa}$, vnde rationationaliter angulum Φ ex calculo elidimus; prodibit enim

$$\partial\omega = \frac{cc\,\partial\theta\,\sin.\theta^2}{aa - cc\,\sin.\theta^2} = -\partial\theta + \frac{aa\,\partial\theta}{aa - cc\,\sin.\theta^2},$$

vnde cum fit $\partial\omega + \partial\theta = \partial\psi$, erit

$$\partial\psi = \frac{aa\,\partial\theta}{aa - cc\,\sin.\theta^2} + \frac{aa\,\partial\theta}{aa\,\cos.\theta^2 + (aa - cc)\sin.\theta^2}.$$

§. 24. Hinc euoluamus primo casum quo $a > c$, ac ponamus breuitatis gratia $aa - cc = bb$, vt habeamus $\partial\psi = \frac{aa\,\partial\theta}{aa\,\cos.\theta^2 + bb\,\sin.\theta^2}$, pro cuius integrali inueniendo ponamus $\frac{b\,\sin.\theta}{a\,\cos.\theta} = t$, eritque $\partial t = \frac{ab\,\partial\theta}{aa\,\cos.\theta^2}$; tum vero etiam $1 + tt = \frac{aa\,\cos.\theta^2 + bb\,\sin.\theta^2}{aa\,\cos.\theta^2}$, ideoque $\frac{\partial t}{1+tt} = \frac{ab\,\partial\theta}{aa\,\cos.\theta^2 + bb\,\sin.\theta^2} = \frac{b\,\partial\psi}{a}$, hinc integrando $\frac{b\psi}{a} = \text{A tang.}\, t$, quamobrem hinc angulus $ACT = \psi$ ita fuccincte exprimitur, vt fit

$$\psi = \frac{a}{b}\,\text{A tang.}\,\frac{b\,\sin.\theta}{b\,\cos.\theta}.$$

§. 25. Pro hoc ergo cafu, quo $aa - cc = bb$, ex folo angulo θ omnia elementa, quae ad curuam pertinent, fequenti modo fatis concinne exprimuntur: 1.) Pro angulo Φ inuenimus $\sin.\Phi = \frac{c}{a}\sin.\theta$. 2.) Diftantia $CZ = z = a\cos.\Phi + c\cos.\theta$, fiue $z = \sqrt{aa - cc\,\sin.\theta^2} + \cos.\theta$. Pro angulo $ACT = \psi$, prodiit $\psi = \frac{a}{b}\,\text{A tang.}\,\frac{b\,\sin.\theta}{a\,\cos.\theta}$, fiue $\psi = \frac{a}{b}\,\text{A tang.}\,\frac{b}{a}\,\text{tang.}\,\theta$, ita vt fit $\frac{b\psi}{a} = \text{A tang.}\,\frac{b}{a}\,\text{tang.}\,\theta$ et hinc $\text{tang.}\,\frac{b\psi}{a} = \frac{b}{a}\,\text{tang.}\,\theta$. Nunc igitur facile erit pro angulo θ valores continuo maiores fubftituere, indeque pro fingulis tam diftantiam z quam angulum ψ affignare. Hinc autem ftatim patet, fumto $\theta = 0$ fore 1.) $\Phi = 0$. 2.) $z = a + c$. 3.) $\psi = 0$.

§. 26.

§. 26. Hae igitur formulae imprimis idoneae ſunt ad curuam conſtruendam, ac fere ſufficiet angulos θ continuo per 90° vel ſaltem per 45° creſcentes aſſumere. Quod ſi enim breuitatis gratia angulos α, β, γ ita capiamus, vt ſit $\text{ſin.}\,\alpha = \frac{c}{a\sqrt{2}}$, $\text{contang.}\,\beta = \frac{b}{a}$ et $\text{ſin.}\,\gamma = \frac{c}{a}$, omnes valores ad curuam conſtruendam neceſſarii in ſequenti tabella exhibentur.

θ	Φ	z	ψ
0°	0°	$a + c$	0
45	α	$a\,\text{coſ.}\,\alpha + \frac{c}{\sqrt{2}}$	$\frac{a}{b}\,\beta$
90	γ	$a\,\text{coſ.}\,\gamma$	$\frac{a}{b}\,90°$
135	α	$a\,\text{coſ.}\,\alpha - \frac{c}{\sqrt{2}}$	$\frac{a}{b}(180 - \beta)$
180	0	$a - c$	$\frac{a}{b}\,180$
225	$-\alpha$	$a\,\text{coſ.}\,\alpha - \frac{c}{\sqrt{2}}$	$\frac{a}{b}(180 + \beta)$
270	$-\gamma$	$a\,\text{coſ.}\,\gamma$	$\frac{a}{b}\,270$
315	$-\alpha$	$a\,\text{coſ.}\,\alpha + \frac{c}{\sqrt{2}}$	$\frac{a}{b}(360 - \beta)$
360	0	$a + c$	$\frac{a}{b}\,360$
405	α	$a\,\text{coſ.}\,\alpha + \frac{c}{\sqrt{2}}$	$\frac{a}{b}(360 + \beta)$
450	γ	$a\,\text{coſ.}\,\gamma$	$\frac{a}{b}\,450$
495	α	$a\,\text{coſ.}\,\alpha - \frac{c}{\sqrt{2}}$	$\frac{a}{b}(540 - \beta)$
540	0	$a - c$	$\frac{a}{b}\,540$
585	$-\alpha$	$a\,\text{coſ.}\,\alpha - \frac{c}{\sqrt{2}}$	$\frac{a}{b}(540 + \beta)$
630	$-\gamma$	$a\,\text{coſ.}\,\gamma$	$\frac{a}{b}\,630$
675	$-\alpha$	$a\,\text{coſ.}\,\alpha + \frac{c}{\sqrt{2}}$	$\frac{a}{b}(720 - \beta)$
720	0	$a + c$	$\frac{a}{b}\,720$

Vnde patet, quo maior fuerit fractio $\frac{a}{b}$, tum numerum reuolutionum anguli ψ eo magis multiplicari pro iisdem angulis θ;

Ac

ac ſi fuerit $b = 0$, ideoque $a = 0$, qui erat tertius caſus, tum numerum reuolutionum anguli ψ iam fieri infinitum, dum angulus θ tantum vsque ad 90° augetur.

§. 27. Sin autem fuerit $aa < cc$, ponamus $cc - aa = bb$, tum erit $\partial\psi = \frac{aa\partial\theta}{aa\,cof.\theta^2 - bb\,fin.\theta^2}$. Ponatur $\frac{b\,fin.\theta}{a\,cof.\theta} = u$, eritque $\partial u = \frac{ab\partial\theta}{aa\,cof.\theta^2}$ et $1 - uu = \frac{aa\,cof.\theta^2 - bb\,fin.\theta^2}{aa\,cof.\theta^2}$, vnde fit

$$\frac{\partial u}{1 - uu} = \frac{ab\partial\theta}{aa\,cof.\theta^2 - bb\,fin.\theta^2} = \frac{b\partial\psi}{a},$$

hincque integrando colligitur $\frac{b\psi}{a} = \frac{1}{2}l\frac{1+u}{1-u}$, ex quo adipiscimur $\psi = \frac{a}{2b}\,l\,\frac{a\,cof.\theta + b\,fin.\theta}{a\,cof.\theta - b\,fin.\theta}$; vbi patet, quia valorem ipſius u non vltra vnitatem augere licet, angulum θ nunquam maiorem euadere poſſe, quam donec fiat tang. $\theta = \frac{a}{b}$, quippe quo caſu angulus ψ iam in infinitum increſcit; atque hinc ſimul intelligitur, ſi fuerit $b = 0$, ſiue $a = c$, tum ob $\partial\psi = \frac{\partial\theta}{cof.\theta^2}$, fore $\psi =$ tang. θ, qui erat tertius caſus ante commemoratus.

§. 28. Quoniam igitur, ſi filum corpuſculo alligatum per peripheriam circuli circumducitur, Tractoria ſemper aſſignari et conſtrui poteſt, videamus cuiusmodi forma Riccatianae ſimilis huic caſui reſpondeat.

§. 29. Vt igitur hunc caſum ad figuram ſupra conſideratam accommodemus, rectae B A C normaliter iungamus rectam C D, in eamque tam ex Z quam ex T, perpendicula ZX et TU demittamus, ſitque, vt ſupra poſuimus, C X $= x$ et XZ $= y$; tum vero CU $= u$ et UT $= t$, ſtatuaturque porro $\partial y = p\,\partial x$, quibus poſitis ſupra deducti fuimus ad hanc aequationem: $\frac{a\,\partial p}{\sqrt{(1+pp)}} + p\,\partial u = \partial t$, quae poſito $p = \frac{qq-1}{2q}$ transformatur Tab. I. Fig. 4.

formatur in hanc rationalem: $a\partial q + \frac{1}{2}(qq - 1)\partial u = q\partial t$, siue $a\partial q - q\partial t + \frac{1}{2}qq\partial u = \frac{1}{2}\partial u$. Pro praesente autem casu, ob angulum $ACZ = \omega$ et $CZ = z$, fit $x = z\,\text{sin.}\,\omega$ et $y = z\,\text{cos.}\,\omega$. Deinde ob $CT = t$ et angulum $ACT = \psi$, erit $u = c\,\text{sin.}\,\psi$ et $t = c\,\text{cos.}\,\psi$; praeterea vero habebimus

$$\partial x = \partial z\,\text{sin.}\,\omega + z\partial\omega\,\text{cos.}\,\omega \text{ et}$$

$$\partial y = \partial z\,\text{cos.}\,\omega - z\partial\omega\,\text{sin.}\,\omega, \text{ vnde fit}$$

$$p = \frac{\partial z\,\text{cos.}\,\omega - z\partial\omega\,\text{sin.}\,\omega}{\partial z\,\text{sin.}\,\omega + z\partial\omega\,\text{cos.}\,\omega}.$$

Erat autem $\frac{\partial z}{z} = -\frac{c\partial\theta\,\text{sin.}\,\theta}{a\,\text{cos.}\,\Phi}$, vnde nancifcimur

$$p = \frac{-c\partial\theta\,\text{sin.}\,\theta\,\text{cos.}\,\omega - a\partial\omega\,\text{cos.}\,\Phi\,\text{sin.}\,\omega}{-c\partial\theta\,\text{sin.}\,\theta\,\text{sin.}\,\omega + a\partial\omega\,\text{cos.}\,\Phi\,\text{cos.}\,\omega}.$$

Quia autem repertum est $\partial\omega = \frac{c\partial\theta\,\text{sin.}\,\theta\,\text{sin.}\,\Phi}{a\,\text{cos.}\,\Phi^2}$, erit exclusis differentialibus

$$p = \frac{\text{cos.}\,\omega\,\text{cos.}\,\Phi + \text{sin.}\,\omega\,\text{sin.}\,\Phi}{\text{sin.}\,\omega\,\text{cos.}\,\Phi - \text{cos.}\,\omega\,\text{sin.}\,\Phi} = \frac{\text{cos.}\,(\omega - \Phi)}{\text{sin.}\,(\omega - \Phi)} = \text{cot.}\,(\omega - \Phi),$$

tum vero, ob $q = p + \sqrt{(1 + pp)}$, erit nunc

$$q = \frac{1 + \text{cos.}\,(\omega - \Phi)}{\text{sin.}\,(\omega - \Phi)} = \text{cot.}\,\tfrac{1}{2}(\omega - \Phi).$$

Hocque modo valor quantitatis q satis simpliciter per angulos ω et Φ exprimitur. Deinde vero ex valoribus pro t et u inventis erit $\partial t = -c\partial\psi\,\text{sin.}\,\psi$ et $\partial u = c\partial\psi\,\text{cos.}\,\psi$, sicque formula nostra Riccatiana ita se habebit:

$$a\partial q + cq\partial\psi\,\text{sin.}\,\psi + \tfrac{1}{2}cqq\partial\psi\,\text{cos.}\,\psi = \tfrac{1}{2}c\partial\psi\,\text{cos.}\,\psi,$$

inuoluens duas tantum variabiles q et angulum ψ.

§. 30. Vicissim igitur, quoties occurrit huiusmodi aequatio differentialis resoluenda:

$$a\partial q + cq\partial\psi\,\text{sin.}\,\psi + \tfrac{1}{2}cqq\partial\psi\,\text{cos.}\,\psi = \tfrac{1}{2}c\partial\psi\,\text{cos.}\,\psi,$$

eius resolutio in nostra erit potestate, quandoquidem nouimus fore $q = \text{cot.}\,\tfrac{1}{2}(\omega - \Phi)$; quomodo autem anguli ω et Φ ab angulo ψ pendeant, ex superioribus est manifestum. Primo enim

enim est $\psi = \omega + \theta$; tum vero a fin. $\Phi = c$ fin. θ; denique vero inuenimus $\psi = \int \frac{a a \partial \theta}{a a \cos. \theta^2 + (a a - c c) \sin. \theta^2}$, cuius ope primo ex angulo ψ reperitur angulus θ, hincque porro angulus Φ ex formula fin. $\Phi = \frac{c}{a}$ fin. θ, ac tandem $\omega = \psi - \theta$. Ex his igitur angulus $(\omega - \Phi)$, per quem quantitas q exprimitur, erit $= \psi - \Phi - \theta$. Hunc in finem prolongetur recta Z T in S, et quia angulus C T S $= \theta + \Phi$ et C T U $= \psi$, erit angulus U T S $= \theta + \Phi - \psi$, ita vt iam fit $q = -$ cot. $\frac{1}{2}$ U T S.

§. 31. Quo hanc formulam Riccatianam fimpliciorem reddamus, ponamus $c = 2 n a$, vt prodeat

$$\partial q + 2 n q \partial \psi \sin. \psi + n q q \partial \psi \cos. \psi = n \partial \psi \cos. \psi,$$

quam vt ab angulis liberemus, ponamus cof. $\psi = s$, ita vt fin. $\psi = \surd (1 - s s)$, eritque aequatio

$$\partial q - 2 n q \partial s - \frac{n q q s \partial s}{\surd (1 - s s)} = - \frac{n s \partial s}{\surd (1 - s s)},$$

vel fi ponamus fin. $\psi = r$, prodibit haec forma:

$$\partial q - \frac{2 n q r \partial r}{\surd (1 - r r)} + n q q \partial r = n \partial r.$$

Quod fi ponamus $q = v + \frac{r}{\surd (1 - r r)}$, prodibit ifta aequatio:

$$\partial v + n v v \partial R = n \partial r - \frac{n r r \partial r}{1 - r r} + \frac{2 n r r \partial r}{\surd (1 - r r)} - \frac{\partial r}{(1 - r r)^{\frac{3}{2}}},$$

cuius ergo refolutionem ope noftrae Tractoriae expedire licet.

§. 32. Reducamus eandem aequationem tantum ad ternos terminos, ponendo $q = e^{-2 n \surd (1 - r r)} v$, ac peruenietur ad hanc formam:

$$\partial v + n e^{-2 n \surd (1 - r r)} v v \partial r = n e^{2 n \surd (1 - r r)} \partial r$$

quae porro, ponendo $\surd (1 - r r) = s$, induet hanc formam:

$$\partial v - n e^{-2ns} \frac{v v s \partial s}{\sqrt{(1 - s s)}} + \frac{n e^{+2ns} s \partial s}{\sqrt{(1 - s s)}} = 0.$$

Hae autem formulae ita comparatae videntur, vt per ſolitas methodos haud facile tractari queant.

Animaduerſiones generales in hunc motum tractorium.

§. 33. In hoc motu tractorio aſſumitur, corpuſculum quouis momento ſecundum ipſam fili directionem protrahi, quod quidem per principia mechanica eueniret, ſi corpuſculum quouis momento quieſceret, vel iam motum ſecundum eandem directionem habuiſſet, quod poſterius autem locum habere nequit, quandoquidem directionem motus continuo mutari aſſumimus; vnde patet, iſtam deſcriptionem per motum tractorium locum plane habere non poſſe, niſi quouis momento motus corpuſculo impreſſus ſubito rurſus extinguatur. Quod cum principiis motus directe aduerſetur, manifeſtam eſt talem motum tractorium in natura neutiquam produci poſſe, niſi forte frictio infinite magna ſtatuatur.

§. 34. Vulgo quidem talis motus facile obtineri poſſe videtur, cum, experentia teſte, omnia corpora, quae in ſuperficie plana protrahi ſolent, eo ipſo momento, quo vis trahens ceſſat, ſubito ad quietem redigi cernuntur, quemadmodum currus ab equis protracti, ſimulac vis trahens ceſſat, ſubito ſubſiſtere ſolent; vnde plures philoſophi principiorum motus ignari concludere ſunt conati, omnia corpora niſu eſſe praedita ſeſe ad ſtatum quietis accommodandi. Quam abſurda autem ſit talis opinio nunc quidem non amplius probatione eget.

§. 35.

§. 35. Interim tamen, experientiam consulentes, negare non possumus, quin corpora, super plano tantillum aspero producta, quasi eo ipso momento omnem motum perdant, quo vis trahens cessauerit, quod certe nullo modo euenire posset, si planum perfecte esset politum, vt omnis frictio excluderetur, quippe quo casu corpus adeo motu semel acquisito perpetuo vniformiter esset progressurum; ex quo statim intelligitur, phaenomenon allatum nulli caussae, praeter frictionem adscribi posse.

§. 36. Neque vero etiam hoc modo omnibus difficultatibus occurri potest, dum ex motus principiis certum est, nullum plane motum a frictione, quantumuis fuerit magna, subito, atque eo ipso momento, quo vis trahens cessat, destrui posse, sed ad hoc semper aliquod tempus requiri, quantumuis id fuerit exiguum; ita vt certe affirmare debeamus, nullum plane motum frictione subito ad quietem redigi posse, ac si tale tempus sentiri nequeat, id ita esse exiguum, vt obseruari non possit.

§. 37. Quo igitur omnia dubia, quae in hoc negotio se produnt, clarius diluamus, consideremus corpus, quod super plano horizontali acceperit celeritatem $=c$, ac videamus quanto tempore opus sit, vt iste motus a frictione penitus extinguatur. Fuerit igitur istud corpus eo momento, quo vis sollicitans cessauit, in A, vnde celeritate sua c vlterius progredi conetur. Peruenerit igitur post tempus $=t$ vsque in p, confecto spatio $AP = s$, sitque massa corporis $= M$, et vis frictionis $= F$, celeritas autem in P vocetur $= v$, eritque $\partial v = -\frac{2gF}{M}\partial t$, vnde colligitur $v = C - \frac{2gFt}{M}$. Fiat nunc $v = 0$ ac reperietur tempus, quo hoc euenire potest, $t = \frac{Mc}{2gF}$, quod in minutis secundis exprimetur, si g fuerit altitudo, per quam grauia Tab. I. Fig. 5.

grauia vno minuto ſecundo delabuntur, celeritas autem c per ſpatium vno minuto ſecundo percurrendum exprimatur. Hinc igitur ſi frictio, vt vulgo ſumi ſolet, tertiae parti ponderis M aequetur, vt ſit $F = \frac{1}{3}M$, erit tempus quo motus penitus extinguitur $= \frac{3c}{2g}$, vnde cum propemodum ſit $g = 16$ ped. Londin. et c in iisdem pedibus exprimatur, fiet $t = \frac{3}{32}c$ ped.

§. 38. Plerumque autem in huiusmodi motibus tractoriis celeritas corporibus impreſſa c tam exigua eſſe ſolet, vt tempuſculum ad motus extinctionem requiſitum t ſenſus noſtros effugiat. Si enim celeritati c pes integer tribuatur, tempus iſtud tantum erit $\frac{3}{32}$, ideoque nequidem decima pars minuti ſecundi, quod nemo facile obſeruare poteſt. Verum ſi quis forte tale tempuſculum animaduerti poſſe contendat, probe hic perpendendum, nullam vim trahentem ita ſubito ceſſare poſſe, quemadmodum in hoc calculo ſuppoſuimus, ſed potius paullatim ad nihilum redigi; vnde mirum non eſt ſi hoc tempuſculum plane non obſeruare licet, quoniam motus extinctio iam ante incepit, quam vis trahens ad nihilum fuit perducta.

§. 39. Ex his iam intelligitur, tales curuas, quales hactenus per calculum ſunt definitae, produci non poſſe, niſi ſuper plano horizontali ſatis aſpero; praeterea vero imprimis neceſſe eſſe vt motus, quo filum protrahitur, ſit non ſolum lentiſſimus, ſed etiam per interualla temporis quam minima penitus ſiſtatur et quaſi per ſaltus peragatur. Statim enim ac motus fili fuerit continuus, curua, quam corpuſculum deſcribet, plurimum aberrabit a Tractoria vulgari: cuiusmodi autem curvam ſit deſcripturum, ſi filum motu continuo protrahatur, quaeſtio eſt maxime ardua, cui reſoluendae Analyſis vix ſufficere videtur, ad quod oſtendendum caſum ſaltem ſimpliciſſimum,

quo

quo filum super plano horizontali iuxta lineam rectam vniformiter protrahitur, euoluamus.

De vera curua tractoria, dum filum per lineam rectam vniformiter protrahitur.

§. 40. Protrahatur igitur filum per lineam rectam AD celeritate $= c$, et elapso tempore $= t$ perductum sit vsque in T, dum motus inceperit in puncto A, eritque spatium $AT = ct$, corpusculum autem nunc sit in Y, ita vt fili longitudo sit $TY = a$. Vocemus autem angulum $ATY = \theta$, vnde demisso ex Y perpendiculo YX erit $TX = a \cos\theta$ et $YX = a \sin\theta$, ita vt positis coordinatis $AX = x$ et $XY = y$, sit Tab. I. Fig. 6.

$$x = Ct - a\cos\theta;\quad \partial x = c\,\partial t + a\,\partial\theta \sin\theta,$$

$$y = a \sin\theta\quad;\quad \partial y = a\,\partial\theta \cos\theta.$$

Ponamus autem porro $\frac{\partial y}{\partial x} = \text{tang.}\,\Phi$, ita vt Φ denotet angulum, sub quo elementum curuae descriptae Yy ad axem AB inclinatur, ita vt sit $\text{tang.}\,\Phi = \frac{a\,\partial\theta\cos\theta}{c\,\partial t + a\,\partial\theta \sin\theta}$.

§. 41. Denotet nunc M massam seu pondus corpusculi, et ponatur tensio fili $TY = T$, quae ergo est vis, qua corpusculum a filo protrahitur, quae secundum directiones coordinatarum resoluta praebet vim secundum $AX = T\cos\theta$, et vim secundum $XY = T\sin\theta$, vbi notandum est hanc vim T adhuc esse incognitam. Praeterea vero etiam corpusculum a frictione sollicitatur, cuius vis sit $= F$, quae cum semper directioni motus sit contraria, eius directio erit yY, quae ergo resoluta praebet vim secundum $AX = -F\cos\Phi$ et vim secundum $XY = -F\sin\Phi$. His igitur viribus colligendis sumto elemento temporis ∂t constante principia motus sequentes suppeditant aequationes:

I.)

I.) $\frac{M \partial \partial x}{2 g \partial t^2} = T \text{ cof. } \theta - F \text{ cof. } \Phi.$

II.) $\frac{M \partial \partial y}{2 g \partial t^2} = - T \text{ fin. } \theta - F \text{ fin. } \Phi.$

§. 42. Elidamus hinc ftatim tenfionem fili T, vtpote incognitam, et haec combinatio: I. fin. θ + II. cof. θ dabit hanc aequationem:

$$\frac{M (\partial \partial x \text{ fin. } \theta + \partial \partial y \text{ cof. } \theta)}{2 g \partial t^2} = - F (\text{cof. } \Phi \text{ fin. } \theta + \text{fin. } \Phi \text{ cof. } \theta)$$
$$= - F \text{ fin. } (\Phi + \theta).$$

Statuamus nunc breuitatis gratia $\frac{2gF}{M} = b$; vbi notetur, g exprimere altitudinem lapfus grauium pro vno minuto fecundo, et fractionem $\frac{F}{M}$ vulgo aeftimari $= \frac{1}{3}$; ficque tota quaeftio reducta eft ad refolutionem huius aequationis:

$$\frac{\partial \partial x \text{ fin. } \theta + \partial \partial y \text{ cof. } \theta}{\partial t^2} = - b (\text{fin. } \theta \text{ cof. } \Phi + \text{cof. } \theta \text{ fin. } \Phi).$$

Cum autem fit

$$\partial \partial x = a \partial \partial \theta \text{ fin. } \theta + a \partial \theta^2 \text{ cof. } \theta \text{ et}$$
$$\partial \partial y = a \partial \partial \theta \text{ cof. } \theta - a \partial \theta^2 \text{ fin. } \theta,$$

aequatio refoluenda induet hanc formam:

$$\frac{a \partial \partial \theta}{\partial t^2} + b (\text{fin. } \theta \text{ cof. } \Phi + \text{cof. } \theta \text{ fin. } \Phi) = 0,$$

ex qua angulus Φ facile eliminatur per formulas

$$\text{fin. } \Phi = \frac{a \partial \theta \text{ cof. } \theta}{\sqrt{(c c \partial t^2 + 2 a c \partial t \partial \theta \text{ fin. } \theta + a a \partial \theta^2)}} \text{ et}$$
$$\text{cof. } \Phi = \frac{c \partial t + a \partial \theta \text{ fin. } \theta}{\sqrt{(c c \partial t^2 + 2 a c \partial t \partial \theta \text{ fin. } \theta + a a \partial \theta^2)}}.$$

His enim valoribus fubftitutis habebimus

$$\frac{a \partial \partial \theta}{\partial t^2} + \frac{b (a \partial \theta + c \partial t \text{ fin. } \theta)}{\sqrt{(c c \partial t^2 + 2 a c \partial t \partial \theta \text{ fin. } \theta + a a \partial \theta^2)}} = 0.$$

§. 43. Antequam autem refolutionem huius aequationis fufcipiamus, perpendamus cafum, quo frictio plane euanefcit

cit, ita vt fit $b = 0$, ac motus totus continebitur in hac simplicissima aequatione: $\frac{a \partial \partial \theta}{\partial t^2} = 0$, hinc $\frac{a \partial \theta}{\partial t} =$ const. hoc est celeritas angularis erit constans, quae, quoniam angulus θ continuo minuitur, ponatur $\frac{a \partial \theta}{\partial t} = -f$, vnde fit $a \theta = k - f t$. Hinc si ponamus initio, vbi $t = 0$, filum tenuisse situm A C normalem ad axem, ita vt tum fuerit $\theta = 90^\circ$, erit $k = a \cdot 90^\circ$, ideoque $\theta = 90 - \frac{f}{a} \cdot t$. Denotabit ergo $\frac{f}{a}$ certum angulum, qui sit $= \alpha$, ita vt habeamus $\theta = 90^\circ - \alpha t$, quo inuento habebimus $x = c t - a$ sin. αt et $y = a$ cos. αt, hincque porro $\frac{\partial x}{\partial t} = c - a \alpha$ cos. αt et $\frac{\partial y}{\partial t} = - a \alpha$ sin. αt. Vnde si initio corpusculum in C quieuisse sumamus, tam $\frac{\partial x}{\partial t}$ quam $\frac{\partial y}{\partial t}$ ibi euanuisse necesse est, cui conditioni satisfit si sumatur $\alpha = \frac{c}{a}$, ita vt sit $\theta = 90^\circ - \frac{c t}{a}$, hincque

$$x = c t - a \text{ sin. } \frac{c t}{a} \text{ et } y = a \text{ cos. } \frac{c t}{a}.$$

Ex posteriore fit $\frac{c t}{a} = \text{A cos. } \frac{y}{a}$, quo valore substituto fiet

$$x = a \text{ A cos. } \frac{y}{a} - \sqrt{(a a - y y)},$$

vnde patet hanc curuam fore cycloidem inuersam, a circulo, cuius radius $= a$, sub recta C D axi parallela, voluente descriptam, cuius cuspis in ipso puncto C sit sita.

§. 44. Contemplemur etiam casum oppositum, quo frictio esset infinita, ideoque $b = \infty$, et in nostra aequatione primum membrum prae altero euanescet, eritque $a \partial \theta + c \partial t \text{ sin. } \theta = 0$, vnde fit $c \partial t = - \frac{a \partial \theta}{\text{sin. } \theta}$ et integrando $c t = - a\, l \text{ tang. } \frac{1}{2} \theta + C$. Vnde si pro $t = 0$ fuerit $\theta = 90^\circ$, erit $C = 0$ ideoque $c t = + a\, l \text{ cot. } \frac{1}{2} \theta$, ideoque $x = a\, l \text{ cot. } \frac{1}{2} \theta - a \text{ cos. } \theta$, existente $y = a \text{ sin. } \theta$, ex quibus formulis manifesto deducitur Tractoria vulgaris. Cum enim ob $c \partial t = - \frac{a \partial \theta}{\text{sin. } \theta}$, fit $\partial x = - \frac{a \partial \theta \text{ cos. } \theta^2}{\text{sin. } \theta}$ et $\partial y = a \partial \theta \text{ cos. } \theta$, erit $\frac{\partial y}{\partial x} = - \text{tang. } \theta$, vnde patet ipsum fiſum Y T eſſe tangentem curuae. Ex hoc iam intelligitur, quod ſupra obſeruauimus, Tractorias vulgares tum demum prodire, quando frictio eſt infinite quaſi magna, vel, quod eodem redit, quando vis trahens frictionem quam minime ſuperat.

§. 45. His praemiſſis videamus quomodo aequationem ſupra inuentam tractari conueniat. Ac primo quidem eam ad differentialem primi gradus reduci conueniet, quod fiet ſi ponatur $\partial t = \frac{\partial \theta}{p}$. Quia enim ∂t conſtans eſt aſſumtum, hinc fiet $\partial \partial \theta = \frac{\partial \theta \partial p}{p}$, quibus valoribus ſubſtitutis aequatio noſtra hanc induet formam:

$$\frac{g\, p\, \partial p}{\partial \theta} + \frac{b\,(a\,p + c\,\text{ſin.}\,\theta)}{\sqrt{(c\,c + 2\,a\,c\,p\,\text{ſin.}\,\theta + a\,a\,p\,p)}} = 0,$$

quae autem quomodo ad integrabilitatem perduci queat nullo modo patet.

§. 46. Eam quidem ab irrationalitate liberare haud eſt difficile. Ponatur enim $\frac{a\,p + c\,\text{ſin}\,\theta}{c\,\text{coſ.}\,\theta} = \text{tang.}\,\omega$, ita vt ſit

$$p = \frac{c\,\text{coſ.}\,\theta\,\text{tang.}\,\omega - c\,\text{ſin.}\,\theta}{a},\ \text{vnde ſit}$$

$$a\,p = \frac{c\,\text{ſin.}\,(\omega - \theta)}{\text{coſ.}\,\omega}\ \text{et}$$

$$\partial p = -\tfrac{1}{a}\left(c\,\partial\theta\,\text{ſin.}\,\theta\,\text{tang.}\,\omega - \frac{c\,\partial\omega\,\text{coſ.}\,\theta}{\text{coſ.}\,\omega^2} + c\,\partial\theta\,\text{coſ.}\,\theta\right)$$

$$= + \frac{c\,\partial\omega\,\text{coſ.}\,\theta}{a\,\text{coſ.}\,\omega^2} - \frac{c\,\partial\theta}{a}\,\frac{\text{coſ.}\,(\theta - \omega)}{\text{coſ.}\,\omega},$$

formula autem irrationalis ſequentem induet formam: $\frac{c\,\text{coſ.}\,\theta}{\text{coſ.}\,\theta}$. Subſtituantur igitur iſti valores atque emerget ſequens aequatio:

$$\frac{a\,c\,\partial\omega\,\text{coſ.}\,\theta^2}{a\,\text{coſ.}\,\omega^3} - \frac{c\,c\,\partial\theta\,\text{coſ.}\,\theta\,\text{coſ.}\,(\theta - \omega)}{a\,\text{coſ.}\,\omega^2} + b\,\partial\theta + \frac{b\,\partial\theta\,\text{ſin.}\,\theta\,\text{coſ.}\,\omega}{\text{ſin.}\,(\omega - \theta)} = 0,$$

quae porro transformatur in hanc:

$$c\,c\,\partial\omega\,\text{coſ.}\,\theta - c\,c\,\partial\theta\,\text{coſ.}\,(\omega - \theta)\,\text{coſ.}\,\omega + \frac{a\,b\,\partial\theta\,\text{coſ.}\,\omega^3\,\text{ſin.}\,\omega}{\text{ſin.}\,(\omega - \theta)} = 0.$$

Statu-

Statuatur porro $\frac{ab}{cc} = n$, eritque

$$\partial\omega \, \text{cos.}\, \theta - \partial\theta \, \text{cos.}\, \omega \, \text{cos.}\, (\omega - \theta) + \frac{n \, \partial\theta \, \text{cos.}\, \omega^2 \, \text{sin.}\, \omega}{\text{sin.}\, (\omega - \theta)} = C.$$

Quanquam autem haec aequatio ſatis prodiit concinna tamen haud patet quomodo eam vlterius reſoluere liceat; vnde haec quaeſtio vires analyſeos ſuperare videtur. Multo minus tales quaeſtiones ſuſcipi poterunt, ſi filum per lineam curuam vel etiam motu non vniformi protrahatur. Quamobrem tales quaeſtiones prorſus relinquere cogimur.