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Dilucidationes in capita postrema calculi mei differentalis de functionibus inexplicabilibus

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DILUCIDATIONES
IN CAPITA POSTREMA CALCULI MEI DIFFERENTIALIS
DE FUNCTIONIBUS INEXPLICABILIBUS.

AUCTORE

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§. 1. Cum hoc argumentum, utpote in Analysi prorsus novum, neutiquam satis dilucide sit pertractatum, constitui hic idem majori studio retractare, atque omnia momenta, quibus innititur, ex primis principiis repetere; ubi plurimum juvabit idonea signa in calculum introduxisse. Ita si proposita fuerit series quaecunque, ejus terminos, indicibus 1, 2, 3, 4, etc. respondentes, his signis repraesentabo: (1), (2), (3), (4), etc., hincque terminus generalis hujus seriei, indici indefinito x respondens, mihi erit (x), qui ergo pro quavis serie certa erit functio ipsius x, quam penitus cognitam assumo, ita scilicet comparatam, ut ejus valores non solum pro numeris integris, loco x assumtis, sed etiam pro fractis, atque adeo surdis, exhiberi queant.

§. 2. Denotet porro $\Sigma : x$ terminum summatorium ejusdem seriei, qui exprimat summam terminorum, a primo incipiendo, usque ad terminum (x), ita ut sit:

$$\Sigma : x = (1) + (2) + (3) + (4) + \ldots\ldots + (x),$$

cujus ergo omnes valores, quoties x fuerit numerus integer positivus, ex ipsa serie actu exhiberi poterunt, siquidem erit ut sequitur:

$$\Sigma : 1 = (1)$$
$$\Sigma : 2 = (1) + (2)$$
$$\Sigma : 3 = (1) + (2) + (3)$$
$$\Sigma : 4 = (1) + (2) + (3) + (4)$$
$$\text{etc.} \qquad \text{etc.}$$

Cujusmodi autem valores eadem formula $\Sigma : x$ sit acceptura, quando ipsi x valores fracti, vel adeo surdi, sive positivi, sive negativi, tribuantur, hinc neutiquam apparet; unde istos valores ad peculiare functionum genus, quas *inexplicabiles* vocavi, refero. Quemadmodum igitur tales functiones per formulas analyticas determinatas exprimi queant, hic imprimis sum investigaturus.

§. 3. Totum autem hoc negotium commodissime expedietur per differentias continuas ex serie proposita derivatas, dum scilicet quilibet terminus a sequente subtrahitur, quo pacto orietur series primarum differentiarum, ex qua porro simili modo differentiae secundae, tertiae, quartae, etc. formabuntur. Omnes autem has differentias sequentibus characteribus indicabo.

Different. I.	Different. II.	Different. III.	
$(2) - (1) = \Delta 1$	$\Delta 2 - \Delta 1 = \Delta^2 1$	$\Delta^2 2 - \Delta^2 1 = \Delta^3 1$	
$(3) - (2) = \Delta 2$	$\Delta 3 - \Delta 2 = \Delta^2 2$	$\Delta^2 3 - \Delta^2 2 = \Delta^3 2$	etc.
$(4) - (3) = \Delta 3$	$\Delta 4 - \Delta 3 = \Delta^2 3$	$\Delta^2 4 - \Delta^2 3 = \Delta^3 3$	
$(5) - (4) = \Delta 4$	$\Delta 5 - \Delta 4 = \Delta^2 4$	$\Delta^2 5 - \Delta^2 4 = \Delta^3 4$	
etc.	etc.	etc.	

§. 4. His characteribus constitutis singuli seriei termini ex primo (1) ejusque differentiis: $\Delta 1$, $\Delta^2 1$, $\Delta^3 1$, $\Delta^4 1$, etc. exprimi poterunt. Cum enim sit $(2) = (1) + \Delta 1$ et $\Delta 2 = \Delta 1 + \Delta^2 1$, ob $(3) = (2) + \Delta 2$, erit $(3) = (1) + 2\Delta 1 + \Delta^2 1$. Hinc jam fluit ista aequalitas: $\Delta 3 = \Delta 1 + 2\Delta^2 1 + \Delta^3 1$. Quia nunc $(4) = (3) + \Delta 3$, habebimus: $(4) = (1) + 3\Delta 1 + 3\Delta^2 1 + \Delta^3 1$, inde porro sequitur $\Delta 4 = \Delta 1 + 3\Delta^2 1 + 3\Delta^3 1 + \Delta^4 1$. Ob $(5) = (4) + \Delta 4$, erit $(5) = (1) + 4\Delta 1 + 6\Delta^2 1 + 4\Delta^3 1 + \Delta^4 1$, et ita porro. Ex ipsa formatione harum formularum manifestum est, hic eosdem coëfficientes, qui in potestate binomiali habentur, cujus exponens est unitate minor quam index termini propositi, occurrere. Ita erit:

$$(n) = 1 + \frac{n-1}{1}\Delta 1 + \frac{n-1}{1}\cdot\frac{n-2}{2}\Delta^2 1 + \frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\Delta^3 1 + \text{etc.}$$

§. 5. Quod si hic numerum n unitate augeamus, habebimus: $(n+1) = 1 + \frac{n}{1}\Delta 1 + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 1 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 1 + \text{etc.}$ Cum jam haec postrema expressio exhibeat terminum, qui a primo n gradibus est remotus, simili modo terminus qui a secundo per totidem gradus est promotus $(n+2)$, ex secundo, ejusque differentiis, determinatur: erit enim:

$$(n+2) = 2 + \frac{n}{1}\Delta 2 + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 2 + \text{etc.}$$

Eodem modo evidens est fore protinus:

$$(n+3) = 3 + \frac{n}{1}\Delta 3 + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 3 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 3 + \text{etc.}$$

$$(n+4) = 4 + \frac{n}{1}\Delta 4 + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 4 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 4 + \text{etc.}$$

§. 6. Hinc ergo patet, ipsum seriei notrae terminum generalem (x) ex primo, ejusque differentiis, hoc modo definiri:

$$(x) = (1) + \frac{x-1}{1}\Delta 1 + \frac{x-1}{1}\cdot\frac{x-2}{2}\Delta^2 1 + \frac{x-1}{1}\cdot\frac{x-2}{2}\cdot\frac{x-3}{3}\Delta^3 1 + \text{etc.}$$

unde terminus ultimum sequens $(x+1)$ manifesto erit:

$$(x+1) = (1) + \frac{x}{1}\Delta 1 + \frac{x}{1}\cdot\frac{x-1}{2}\cdot\Delta^2 1 + \frac{x}{1}\cdot\frac{x-1}{2}\cdot\frac{x-2}{3}\Delta^3 1 + \text{etc.}$$

quae expressio cum in sequentibus frequentissime occurrat, brevitatis gratia introducamus sequentes characteres:

$$\frac{x}{1} = x,$$

$$\frac{x}{1}\cdot\frac{x-1}{2} = x',$$

$$\frac{x}{1}\cdot\frac{x-1}{2}\cdot\frac{x-2}{3} = x'',$$

$$\frac{x}{1}\cdot\frac{x-1}{2}\cdot\frac{x-2}{3}\cdot\frac{x-3}{4} = x''',$$

etc.

quibus adhibitis habebimus sequentes aequationes:

$$(x+1) = (1) + x\Delta 1 + x'\Delta^2 1 + x''\Delta^3 1 + \text{etc.}$$
$$(x+2) = (2) + x\Delta 2 + x'\Delta^2 2 + x''\Delta^3 2 + \text{etc.}$$
$$(x+3) = (3) + x\Delta 3 + x'\Delta^2 3 + x''\Delta^3 3 + \text{etc.}$$
$$(x+4) = (4) + x\Delta 4 + x'\Delta^2 4 + x''\Delta^3 4 + \text{etc.}$$
$$- \quad - \quad - \quad - \quad - \quad -$$
$$- \quad - \quad - \quad - \quad - \quad -$$
$$(x+n) = (n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + \text{etc.}$$

§. 7. Deinde etiam summas quotcunque terminorum nostrae seriei ex solo termino primo, ejusque differentiis, determinari poterit, quemadmodum sequens tabula declarat:

$$\Sigma : 1 = (1)$$
$$\text{add. } (2) = (1) + \Delta 1$$
$$\Sigma : 2 = 2\,(1) + \Delta 1$$
$$(3) = (1) + 2\,\Delta 1 + \Delta^2 1$$
$$\Sigma : 3 = 3\,(1) + 3\,\Delta 1 + \Delta^2 1$$
$$(4) = (1) + 3\,\Delta 1 + 3\,\Delta^2 1 + \Delta^3 1$$
$$\Sigma : 4 = 4\,(1) + 6\,\Delta 1 + 4\,\Delta^2 1 + \Delta^3 1$$
$$(5) = (1) + 4\,\Delta 1 + 6\,\Delta^2 1 + 4\,\Delta^3 1 + \Delta^4 1$$
$$\Sigma : 5 = 5\,(1) + 10\,\Delta 1 + 10\,\Delta^2 1 + 5\,\Delta^3 1 + \Delta^4 1,$$

etc.

Hic iterum evidens est coëfficientes eosdem esse, qui in potestate binominali ejusdem ordinis occurrunt.

§. 8. In usum igitur vocatis characteribus modo ante stabilitis, ipsum terminum summatorium nostrae seriei $\Sigma : x$ exprimere valemus: erit enim

$$\Sigma : x = x(1) + x' \Delta 1 + x'' \Delta^2 1 + x''' \Delta^3 1 + \text{etc.}$$

quae forma jam ita est comparata, ut loco x non solum numeros integros, sed etiam fractos, imo surdos quoscunque tam positivos quam negativos accipere liceat, quibus casibus utique ista expressio in infinitum progredietur, nisi forte series proposita deducat tandem ad differentias evanescentes, cujusmodi series algebraicae vocari solent, quibus ergo casibus non ad functiones inexplicabiles pervenitur. Interim tamen ista expressio, pro termino summatorio inventa, quando in infinitum porrigitur, nihil adjumenti offert, quando differentiationes, vel etiam summationes, sunt instituendae; quamobrem in id erit incumbendum, quemadmodum, saltem pro certis casibus, terminus summatorius inventus in alias formas transfundi queat, quae neque differentiationi neque integrationi refragentur, atque huc pertinent omnia subsidia, quae in Calculo differentiali fusius exposui, et quarum inventio non parum erat abstrusa. Sequenti autem modo totum hoc negotium facile conficietur.

§. 9. Ad expressionem pro termino summatorio $\Sigma : x$ modo ante inventam addantur plures formulae sub hac specie contentae:

$$(n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + \text{etc.} \ldots - (x + n),$$

quarum summae, cum sint nihilo aequales, omnes quotcunque fuerint, cum $\Sigma : x$ junctum sumtae, nihilominus terminum summatorium expriment. Sumantur ergo pro n successive omnes numeri 1, 2, 3, 4, etc. et tota expressio secundum columnas verticales, singulis valoribus x, x', x'', etc. respondentes, sequenti modo disponantur:

Expressio generalis pro termino summatorio.

$$
\begin{array}{l}
\quad x\,(1) + x'\Delta\,1 + x''\Delta^2 1 + x'''\Delta^3 1 + \text{etc.} \\
(1) + x\Delta 1 + x'\Delta^2 1 + x''\Delta^3 1 + x'''\Delta^4 1 + \ldots\ldots - (x + 1) \\
(2) + x\Delta 2 + x'\Delta^2 2 + x''\Delta^3 2 + x'''\Delta^4 2 + \ldots\ldots - (x + 2) \\
(3) + x\Delta 3 + x'\Delta^2 3 + x''\Delta^3 3 + x'''\Delta^4 3 + \ldots\ldots - (x + 3) \\
- \quad - \quad - \quad - \quad - \quad - \quad - \\
- \quad - \quad - \quad - \quad - \quad - \quad - \\
(n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + \Delta'''\Delta^4 n + \ldots\ldots - (x + n).
\end{array}
$$

§. 10. Etsi veritas hujus expressionis nulli amplius dubio est obnoxia, tamen non parum juvabit, eam ex ipsa forma confirmasse. Colligantur nimirum in unam summam singulae columnae verticales; ac primae quidem summa erit:

$$(1) + (2) + (3) + (4) + \ldots + (n) = \Sigma : n.$$

Secunda columna dat:

$$x\,(1 + \Delta 1 + \Delta 2 + \Delta 3 + \ldots + \Delta n).$$

Cum autem sit $\Delta 1 = (2) - (1);$

$$\Delta 2 = (3) - (2);$$
$$\Delta 3 = (4) - (3);$$
etc.

tota haec summa contrahetur in $x \cdot (n + 1)$. Simili modo tertiae columnae summa erit:

$$x' (\Delta 1 + \Delta^2 1 + \Delta^2 2 + \Delta^2 3 + \Delta^2 4 + \ldots . \Delta^2 n);$$

et quia $\Delta^2 1 = \Delta 2 - \Delta 1;\ \Delta^2 2 = \Delta 3 - \Delta 2 \ldots \Delta^2 n = \Delta(n+1) \Delta n,$ illa summa contrahitur in $x' \Delta (n + 1)$. Eodem modo patet fore quartae columnae summam $x'' \Delta^2 (n + 1)$ et quintae $= x''' \Delta^3 (n + 1)$, et ita porro. Ultimae vero columnae subtrahendae summa est:

$$(x + 1) + (x + 2) + (x + 3) + \ldots . . (x + n) = \Sigma : (x + n) - \Sigma : x.$$

§. 11. Summa igitur omnium columnarum verticalium mediarum, praeter primam et ultimam, est ut vidimus:

$$x (n + 1) + x' \Delta (n + 1) + x'' \Delta^2 (n + 1) + x''' \Delta^3 (n + 1).$$

Cum autem sit:

$$x (1) + x' \Delta 1 + x'' \Delta^2 1 + x''' \Delta^3 1 + \text{etc.} = \Sigma : x,$$

singulis terminis numero n auctis erit summa nostrae seriei: $x (n + 1) + x' \Delta (n + 1) + x'' \Delta^2 (n + 1) + \text{etc.} = \Sigma : (x + n) - \Sigma : n,$ consequenter omnium plane columnarum summa praeter ultimam est $= \Sigma : (x + n)$: unde si summa ultimae columnae, quae est $\Sigma : (x + n) - \Sigma : x$ subtrahatur, remanebit summa totius expressionis $= \Sigma : x$, hoc est terminus summatorius quaesitus.

§. 12. Maxime hic mirum videbitur, quod valorem formulae $\Sigma : x$, quae serie satis simplici exprimitur, per congeriem innumerabilium serierum expressum et involutum dederimus; verum mox summus usus hujus formae complicatissimae patebit, quando numerum serierum horizontalium adeo in infinitum continuaverimus, quod fiet, si pro n numerum infinitum accipiamus, quemadmodum nunc clarius explicabimus.

§. 13. Denotante igitur n numerum infinite magnum, summa secundae columnae verticalis, quae est $x(n+1)$, continebit terminum seriei nostrae infinitesimum qui ergo si evanescat, multo magis summae sequentium columnarum verticalium evanescent; quamobrem hoc casu sufficiet solam primam columnam cum ultima in calculo retinuisse. Sin autem termini infinitesimi non evanescant, sed tamen inter se fuerint aequales, tum tertiam columnam, cum sequentibus, abjicere licebit. Porro autem si demum differentiae secundae infinitesimae evanescant, tres priores columnas verticales in calculo retineri debebunt; similique modo quatuor, si tertiae demum infinitesimae evanescant. Secundum hoc igitur serierum discrimen ipsas series in sequentes species distribuemus.

Species prima serierum quarum termini infinitesimi evanescunt.

§. 14. Quoties igitur talis series proponatur, pro ejus termino summatorio sufficiet terminos primae et ultimae columnae verticalis in calculo retinuisse, sicque nanciscemur pro termino summatorio sequentem expressionem:

$$\Sigma : x = \begin{cases} (1) + (2) + (3) + (4) + \text{etc.} \\ -(x+1)-(x+2)-(x+3)-(x+4)-\text{etc.} \end{cases}$$

quae quidem in infinitum excurrit, atque eo magis convergit, quo minor fuerit index x, quandoquidem, si is evanescat, tota series in nihilum abibit, sive erit $\Sigma : 0 = 0$, id quod cum rei natura egregie congruit; quando enim numerus terminorum addendorum est nullus, etiam summa necessario debet esse nulla.

§. 15. Quando autem index x est numerus praemagnus, haec series utique parum converget; verum semper licebit hujusmodi casus ad indices minores reducere. Cum enim sit $\Sigma : (x+1) = \Sigma : x + (x+1)$, simili modo erit $\Sigma : (x+2) = \Sigma : x + (x+1) + (x+2)$, atque adeo in genere, denotante i numerum integrum:

$$\Sigma : (x+i) = \Sigma : x + (x+1) + (x+2) \ldots\ldots + (x+i).$$

Quamobrem si summa $x+i$ terminorum desideretur, sufficiet summam x terminorum, hoc est $\Sigma : x$, investigasse, hocque modo omnes hujusmodi quaestiones reduci poterunt ad casus, ubi index x est adeo unitate minor, quo casu series pro $\Sigma : x$ ante data vehementer converget.

§. 16. Talis reductio imprimis est necessaria, quando index x est numerus negativus. Cum enim sit:
$\Sigma : x = \Sigma (x - 1) + (x)$, erit $\Sigma : (x - 1) = \Sigma : (x) - (x)$; eodemque modo, $\Sigma : (x - 2) = \Sigma : x - (x) - (x - 1)$ et $\Sigma : (x - 3) = \Sigma : x - (x) - (x - 1) - (x - 2)$, et in genere $\Sigma : (x - i) = \Sigma : x - (x) - (x - 1) - \ldots\ldots (x - i + 1)$; hocque modo, quantumvis numerus negativus $x - i$ fuerit magnus, resolutio semper ad $\Sigma : x$ reduci potest, ita ut sit $x < 1$.

Exemplum.

§. 17. Proposita sit haec series harmonica:

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \ldots\ldots \tfrac{1}{x} = \Sigma : x,$$

cujus summa x terminorum desideretur, ubi pro x numeros quoscunque, praeter integros positivos, accipere liceat, siquidem pro casibus, quibus x est numerus integer positivus, tota res nulla difficultate laborat. Hoc igitur casu ex forma ante data erit:

$$\Sigma : x = \begin{cases} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} \\ - \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4} - \text{etc.} \end{cases}$$

quae duae series in hanc unicam contrahentur:

$$\Sigma : x = \frac{x}{x+1} + \frac{x}{2(x+2)} + \frac{x}{3(x+3)} + \frac{x}{4(x+4)} + \text{etc.}$$

cujus seriei summa per se constat, quoties x fuerit numerus integer positivus. Ita erit:

Si $x = 1$	$1 = \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} +$ etc.
- $x = 2$	$1 + \frac{1}{2} = \frac{2}{1 \cdot 3} + \frac{2}{2 \cdot 4} + \frac{2}{3 \cdot 5} + \frac{2}{4 \cdot 6} +$ etc.
- $x = 3$	$1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{1 \cdot 4} + \frac{3}{2 \cdot 5} + \frac{3}{3 \cdot 6} + \frac{3}{4 \cdot 7} +$ etc.
- $x = 4$	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{4}{1 \cdot 5} + \frac{4}{2 \cdot 6} + \frac{4}{3 \cdot 7} + \frac{4}{4 \cdot 8} +$ etc.
etc.	etc.

quae quidem series sunt notissimae.

§. 18. Quo haec clarius intelligantur, construamus curvam, cujus abscissae $o\,x = x$ respondeat applicata: Tab. I. Fig. 1.

$$x\,y = y = \Sigma : x,$$

ita ut, sumtis super axe $o\,x$ intervallis unitate aequalibus $0,1$; $1,2$; $2,3$; $3,4$; etc. applicatae futurae sint:

$$1 \ldots (1) = 1,$$
$$2 \ldots (2) = 1 + \tfrac{1}{2},$$
$$3 \ldots (3) = 1 + \tfrac{1}{2} + \tfrac{1}{3},$$
$$4 \ldots (4) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4},$$

atque aequatio inter binas coordinatas erit:

$$y = \frac{x}{x+1} + \frac{x}{2(x+2)} + \frac{x}{3(x+3)} + \frac{x}{4(x+4)} + \text{etc.}$$

ex qua ergo aequatione omnes applicatae intermediae definiri poterunt, atque adeo sufficiet pro x valores unitate minores accepisse. Ita si applicata $\frac{1}{2} \ldots (\frac{1}{2})$, abscissae $o \ldots \frac{1}{2} = \frac{1}{2}$ respondens, desideretur, reperietur:

$$\tfrac{1}{2} \cdot \cdot (\tfrac{1}{2}) = \tfrac{1}{3} + \tfrac{1}{2 \cdot 5} + \tfrac{1}{3 \cdot 7} + \tfrac{1}{4 \cdot 9} + \tfrac{1}{5 \cdot 11} + \text{etc.}$$

cujus seriei summa per logarithmos assignari poterit, hoc modo. Formetur haec series:

$$y = \frac{t^3}{1 \cdot 3} + \frac{t^5}{2 \cdot 5} + \frac{t^7}{3 \cdot 7} + \frac{t^9}{4 \cdot 9} + \text{etc.}$$

quae ergo series, sumto $t = 1$, dabit valorem quaesitum; at vero differentiando habebimus:

$$\frac{\partial y}{\partial t} = \frac{t^2}{1} + \frac{t^4}{2} + \frac{t^6}{3} + \frac{t^8}{4} + \text{etc.}$$

et denuo differentiando:

$$\frac{\partial \partial y}{2 \partial t^2} = t + t^3 + t^5 + t^7 + \text{etc.} = \frac{t}{1 - tt}.$$

Hinc ergo vicissim erit $\frac{\partial y}{2 \partial t} = \int \frac{t \partial t}{1 - tt}$ et $y = 2 \int \partial t \int \frac{t \partial t}{1 - tt}$, quae duplex integratio reducitur more solito ad unicam, quo facto erit $y = 2 t \int \frac{t \partial t}{1 - tt} - 2 \int \frac{tt \partial t}{1 - tt}$. Quia autem post integrationem statui debet $t = 1$, erit:

$$y = 2 \int \frac{t \partial t}{1 - tt} - 2 \int \frac{tt \partial t}{1 - tt} = 2 \int \frac{t \partial t}{1 + t};$$

quamobrem integrando fiet $y = 2 t - 2 l (t + 1)$, ideoque nostro casu $y = 2 - 2 l 2$, cujus valor proxime verus est 0,61370564.

§. 19. Inventa jam applicata abscissae $\frac{1}{2}$ respondente, scilicet $\Sigma : \frac{1}{2} = 2 - 2 l 2$, ex ea sequentes per formulas supra datas facile derivantur, scilicet:

$$\Sigma : (1 + \tfrac{1}{2}) = \tfrac{2}{3} + \Sigma \tfrac{1}{2},$$

$$\Sigma : (2 + \tfrac{1}{2}) = \tfrac{2}{3} + \tfrac{2}{5} + \Sigma : \tfrac{1}{2},$$

$$\Sigma : (3 + \tfrac{1}{2}) = \tfrac{2}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \Sigma : \tfrac{1}{2},$$

etc. etc.

Quin etiam praecedentes applicatae, in figura non expressae, ex formula $\Sigma:(x-i)$, quam invenimus, scil.: ex $\Sigma:(x-i)=\Sigma:x-(x)-(x-1)-(x-2)\ldots-(x-i+1)$, deduci poterunt. Quia igitur nostro casu $x=\frac{1}{2}$, erit applicata:

$$\Sigma:(-\tfrac{1}{2})=\Sigma:\tfrac{1}{2}-2=-2l2;$$

erit scilicet negativa. Sumto autem $x=-1$, ea fit infinita. Infinita vero etiam evadet casibus $x=-2$, $x=-3$, $x=-4$, etc. Intra autem haec intervalla erit:

$$\Sigma:-(1+\tfrac{1}{2})=\Sigma:\tfrac{1}{2}-2+2,$$
$$\Sigma:-(2+\tfrac{1}{2})=\Sigma:\tfrac{1}{2}-2+2-\tfrac{2}{3},$$
$$\Sigma:-(3+\tfrac{1}{2})=\Sigma:\tfrac{1}{2}-2+2-\tfrac{2}{3}+\tfrac{2}{5},$$

etc. etc.

§. 20. Differentiemus nunc seriem pro applicata y inventam, fietque: $\frac{\partial y}{\partial x}=\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\text{etc.}$; quae ergo series exprimit tangentem anguli, sub quo elementum curvae in y ad axem inclinatur; unde patet pro abscissa infinita hanc inclinationem fore nullam, sive tractum curvae in infinito axi esse parallelum. Tum vero, sumto $x=0$, innotescet inclinatio curvae ad ipsum initium $=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}=\frac{\pi\pi}{6}=1{,}644$, ideoque angulus $=58^\circ.\,42'$. Tum vero, sumto $x=1$, erit:

$$\frac{\partial y}{\partial x}=\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}=\frac{\pi\pi}{6}-1=0{,}644,$$

ubi ergo inclinatio erit $=32^\circ.\,48'$, hincque ulterius continuando inclinatio continuo decrescet.

§. 21. Retrogrediendo vero ad abscissas negativas supra vidimus, casibus, quibus $x = -1$, vel $x = -2$, vel $x = -3$, applicatas fieri infinite magnas, et totidem curvae assymptotas constituere. Nunc vero videbimus, iisdem locis fieri $\frac{\partial y}{\partial x} = \infty$, ibique inclinationem curvae esse 90°, sive tangentes ad axem fore perpendiculares. Praeterea, quoniam series pro $\frac{\partial y}{\partial x}$ inventa semper habet summam positivam, sequitur omnes partes curvae dextrorsum semper ascendere, contra vero, sinistrorsum, descendere.

§. 22. Quin etiam poterimus integrationem adhibere atque aream curvae ab initio usque ad applicatum xy assignare. Ex prima enim forma, ad quam sumus perducti immediate, manifesto fiet:

$$\int y\,\partial x = \left\{ \begin{array}{l} x + \frac{1}{2}x + \frac{1}{3}x + \text{etc.} \\ -l(1+x) - l(2+x) - l(3+x) - \text{etc.} \end{array} \right\} + \text{Const.}$$

quae constans ita debet determinari, ut casu $x = 0$ tota area evanescat; unde illa rite ita exprimetur:

$$\int y\,\partial x = \left\{ \begin{array}{l} x + \frac{1}{2}x + \frac{1}{3}x + \text{etc.} \\ -l(1+x) - l(1+\frac{1}{2}x) - l(1+\frac{1}{3}x) - \text{etc.} \end{array} \right.$$

Cum igitur sit $l\left(1+\frac{x}{n}\right) = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} + \text{etc.}$ superior expressio per series sequentes exprimi poterit:

$$\int y\partial x = \begin{cases} \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^6}{6} - \text{etc.} \\ + \frac{x^2}{2\cdot 4} - \frac{x^3}{3\cdot 8} + \frac{x^4}{4\cdot 16} - \frac{x^5}{5\cdot 32} + \frac{x^6}{6\cdot 64} - \text{etc.} \\ + \frac{x^2}{2\cdot 9} - \frac{x^3}{3\cdot 27} + \frac{x^4}{4\cdot 81} - \frac{x^5}{5\cdot 243} + \frac{x^6}{6\cdot 729} - \text{etc.} \\ + \frac{x^2}{2\cdot 16} - \frac{x^3}{3\cdot 64} + \frac{x^4}{4\cdot 256} - \frac{x^5}{5\cdot 1024} + \frac{x^6}{6\cdot 4096} - \text{etc.} \\ \text{etc.} \end{cases}$$

§. 23. Quod si jam has series verticaliter colligamus, habebimus:

$$\int y\partial x = \begin{cases} \tfrac{1}{2}x^2(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.}) = +0{,}822467 \cdot x^2, \\ -\tfrac{1}{3}x^3(1 + \tfrac{1}{8} + \tfrac{1}{27} + \tfrac{1}{64} + \tfrac{1}{125} + \text{etc.}) = -0{,}400685 \cdot x^3, \\ +\tfrac{1}{4}x^4(1 + \tfrac{1}{16} + \tfrac{1}{81} + \tfrac{1}{256} + \tfrac{1}{625} + \text{etc.}) = +0{,}270581 \cdot x^4, \\ -\tfrac{1}{5}x^5(1 + \tfrac{1}{32} + \tfrac{1}{243} + \tfrac{1}{1024} + \tfrac{1}{3125} + \text{etc.}) = -0{,}207385 \cdot x^5, \\ \text{etc.} \end{cases}$$

Ponamus nunc $x = 1$, ut prodeat area 0 1 (1), et quia fractiones decimales hic datae parum convergunt, notetur, seriei cujuscunque, ubi signa alternantur, scil.:

$$s = a - b + c - \partial + e - \text{etc.}$$

summam per differentias continuas ita exprimi, ut sit:

$$s = \tfrac{1}{2}a - \tfrac{1}{4}\Delta a + \tfrac{1}{8}\Delta^2 a - \tfrac{1}{16}\Delta^3 a + \text{etc.}$$

cujus ergo regulae ope calculus sequenti modo institui poterit:

		$-\Delta$	$+\Delta^2$	$-\Delta^3$	$+\Delta^4$	$-\Delta^5$	$+\Delta^6$	$-\Delta^7$	$+\Delta^8$	
a)	0,822467	0,421782								
b)	0,400685	0,130104	0,291678	0,224770						
c)	0,270581	0,063196	0,066908	0,041540	0,183230	0,154737				
d)	0,207385	0,037828	0,025368	0,013047	0.028493	0,020801	0,133936	0,118072		
e)	0,169557	0,025507	0,012321	0,005355	0,007692	0,004937	0,015864	0,012508	0,105564	etc.
f)	0.144050	0,018541	0,006966	0,002600	0,002755	0,001581	0,003356			
g)	0,125509	0,014175	0,004366	0,001426	0,001174					
h)	0,111334	0,011235	0,002940							
i)	0,100099									

§. 24. Harum columnarum, quarum prima ex calculi differentialis cap. VI. part. II. pag. 456 est desumta, numeri supremi referunt terminum primum a, cum suis differentiis continuis; secundi vero descendendo referunt terminum b cum suis differentiis; tertii terminum c cum suis differentiis. Quia nunc supremi termini parum convergunt, duos primos $a-b$ actu colligamus, eritque $a-b=0{,}421782$: sequentium vero $c - \partial + e - f + \text{etc.}$ summam:

$$= \tfrac{1}{2}c - \tfrac{1}{4}\Delta c + \tfrac{1}{8}\Delta^2 c - \tfrac{1}{16}\Delta^3 + \text{etc.}$$

secundum datam legem computemus, eritque:

$$
\begin{aligned}
\tfrac{1}{2}c &= 0{,}135290 \\
-\tfrac{1}{4}\Delta c &= 0{,}015799 \\
+\tfrac{1}{8}\Delta^2 c &= 0{,}003171 \\
-\tfrac{1}{16}\Delta^3 c &= 0{,}000815 \\
+\tfrac{1}{32}\Delta^4 c &= 0{,}000220 \\
-\tfrac{1}{64}\Delta^5 c &= 0{,}000077 \\
+\tfrac{1}{128}\Delta^6 c &= 0{,}000026 \\
-\int eqq &= 0{,}000010 \\
\text{Summa} &= 0{,}155408 \\
a - b &= 0{,}421782 \\
\text{Area} &= 0{,}577190
\end{aligned}
$$

Spero autem, fusiorem evolutionem hujus lineae curvae satis memorabilis nemini fore ingratam, praecipue cum aequatio pro hac curva pertineat ad functiones inexplicabiles, atque idcirco ista ad casum specialiorem digressio a nostro scopo haud aliena sit estimanda.

Species secunda serierum quarum differentiae infinitesimae primae evanescunt.

§. 25. Ad hanc ergo speciem pertinent omnes series, quarum termini infinitesimi inter se sunt aequales. Ut ergo terminum summatorium harum serierum $\Sigma : x$ exprimamus, nihil aliud opus est, nisi ut ad expressionem praecedentis speciei insuper termini secundae columnae verticalis formae generalis §. 9. exhibitae adjungantur, cujus quidem terminus supremus seorsim erit exhibendus; et quia columnae singulae horizontales jam tribus terminis constant, terminus summatorius quaesitus $\Sigma : x$ sequenti serie triplicata definietur:

$$\Sigma : x = \left\{ \begin{array}{l} \quad\;\; + (1) \;-\; (2) \;+\; (3) \;-\; (4) \\ x(1) + x\Delta 1 + x\Delta 2 + x\Delta 3 + x\Delta 4 \\ \quad\;\; -(x+1) - (x+2) - (x+3) - (x+4) \end{array} \right\} \text{etc.}$$

quae forma, ob $\Delta 1 = (2) - (1)$; $\Delta 2 = (3) - (2)$; $\Delta 3 = (4) - (3)$; etc. transfundetur in hanc:

$$\Sigma : x = \left\{ \begin{array}{l} \quad\;\; + \overline{1-x}(1) + \overline{1-x}(2) + \overline{1-x}(3) \\ x(1) + \quad x(2) + \quad x(3) + \quad x(4) \\ \quad\;\; - (x+1) - (x+2) - (x+3) \end{array} \right\} \text{etc.}$$

quae series eo magis convergit, quo minor x accipiatur. Supra autem docuimus, omnes casus semper eo reduci posse ubi x sit fractio unitate minor.

§. 26. Consideremus primo casum simplicissimum, quo omnes seriei termini sunt inter se aequales, scilicet: $(x) = a$: per se enim patet, ejus terminum summatorium esse ax, quem eundem valorem nostra expressio statim declarabit. Erit enim $\Sigma : x = xa$.

§. 27. Nunc consideretur casus quo $(x) = \frac{x+1}{x}$, ita ut nostra series sit $\Sigma : x = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots . \frac{x+1}{x}$, cujus termini infinitesimi omnes unitati aequantur. Nostra igitur formula nobis dabit:

$$\Sigma : x = \left\{ \begin{array}{l} +\overline{1-x}.\frac{2}{1} + \overline{1-x}.\frac{3}{2} + \overline{1-x}.\frac{4}{3} \\ 2x + \quad x.\frac{3}{2} + \quad x.\frac{4}{3} + \quad x.\frac{5}{4} \\ \quad - \frac{(x+2)}{x+1} - \frac{(x+3)}{x+2} - \frac{(x+4)}{x+3} \end{array} \right\} \text{etc.}$$

unde patet, sumto $x=1$, fore $\Sigma : x = \frac{2}{1}$; at sumto $x=2$ fiet:

$$\Sigma : x = \left\{ \begin{array}{l} \quad - 1.\frac{2}{1} - 1.\frac{3}{2} - 1.\frac{4}{3} \\ 4 + 2.\frac{3}{2} + 2.\frac{4}{3} + 2.\frac{5}{4} \\ \quad - \frac{4}{3} \quad - \frac{5}{4} \quad - \frac{6}{5} \end{array} \right\} \text{etc.} = 4 - \frac{2}{1} + \frac{3}{2}.$$

§. 28. Iste vero casus facile reduci potest ad speciem praecedentem. Cum enim terminus generalis sit $(x) = \frac{x+1}{x}$, is in partes resolutus dabit $(x) = 1 + \frac{1}{x}$; quamobrem duae formentur series, prior scilicet ex termino generali 1, altera vero ex termino generali $\frac{1}{x}$, haeque duae series junctim sumtae dabunt summam quaesitam $\Sigma : x$; erit scilicet:

$$\Sigma : x = \left\{ \begin{array}{l} 1 + 1 + 1 + 1 + \ldots . x \\ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots . \frac{1}{x}. \end{array} \right.$$

Jam superioris seriei summa est x, inferior vero per speciem primam evolvi potest, indeque habebitur:

$$\Sigma : x = \begin{cases} x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} \\ -\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4} - \text{etc.} \end{cases}$$

quae expressio multo est simplicior praecedente, nihilo vero minus eundem valorem exhibet. Ita si sumatur $x = \frac{1}{2}$, prior expressio nobis dabit:

$$\Sigma : x = \left\{ \begin{matrix} + \frac{1}{2} \cdot \frac{2}{1} + \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{4}{3} + \frac{1}{2} \cdot \frac{5}{4} \\ 1 + \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{4}{3} + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot \frac{6}{5} \\ - \frac{5}{4} - \frac{7}{5} - \frac{9}{7} - \frac{11}{9} \end{matrix} \right\} \text{etc.}$$

terminisque secundum ordinem collectis fiet:

$$\Sigma : \tfrac{1}{2} = 1 + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 12} + \frac{1}{7 \cdot 24} + \frac{1}{9 \cdot 40} + \frac{1}{11 \cdot 60} + \text{etc.}$$

cujus ordo clarius patescet ex sequenti forma:

$$\Sigma : \tfrac{1}{2} = 1 \; \frac{1}{1 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 5 \cdot 6} + \frac{1}{3 \cdot 7 \cdot 8} + \frac{1}{4 \cdot 9 \cdot 10} + \frac{1}{5 \cdot 11 \cdot 12} + \text{etc.}$$

Altera vero expressio dat hanc seriem:

$$\Sigma : \tfrac{1}{2} = \begin{cases} \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} \\ - \frac{2}{3} - \frac{2}{5} - \frac{2}{7} - \frac{2}{9} - \text{etc.} \end{cases}$$

quae, collectis terminis dabit:

$$\Sigma : \tfrac{1}{2} = \tfrac{1}{2} + \tfrac{1}{3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 7} + \frac{1}{4 \cdot 9} + \text{etc.}$$

§. 29. Ex hoc exemplo apparet, seriem ex specie secunda deductam magis convergere quam posteriorem ex specie prima derivatam; quare operae pretium erit prioris seriei convergentiam attentius considerare. Quilibet scilicet hujus seriei terminus oritur ex his tribus partibus: $\frac{1}{2} \cdot \frac{n+1}{n} + \frac{1}{2} \cdot \frac{n+2}{n+1} - \frac{2n+3}{2n+1}$, quae cum se mutuo proxime destruant, summa duorum priorum aequalis erit tertiae, unde sequitur haec formula satis memorabilis: $\frac{n+1}{n}+\frac{n+2}{n+1}=\frac{2(2n+3)}{2n+1}$, quod eo propius ad veritatem accedit, quo major fuerit numerus n. Hinc utrinque subtrahendo 2, erit proxime $\frac{1}{n}+\frac{1}{n+1}=\frac{4}{2n+1}$.

§. 30. Talis autem reductio ad speciem primam semper locum habere potest, quando series proposita tandem ad valorem finitum convergit; verum si seriei termini tandem in infinitum crescant, haec reductio non amplius locum habere potest, ideoque necessario ad speciem secundam erit recurrendum. Talis est casus quo $(x)=\sqrt{x}$; denotante enim n numerum infinitum bini termini infinitesimi contigui erunt $\sqrt{n}$ et $\sqrt{n+1}$, quorum differentia est $\frac{1}{2\sqrt{n}}$, ideoque evanescens. Hoc ergo casu series nostra erit:

$$\Sigma:x=\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\ldots\ldots\sqrt{x}.$$

Hinc ergo per praecepta data habebimus hanc expressionem:

$$\Sigma:x=\left\{\begin{array}{llll} +\overline{1-x}\sqrt{1} & +\overline{1-x}\sqrt{2} & +\overline{1-x}\sqrt{3} & \\ x & +x\sqrt{2} & +x\sqrt{3} & +x\sqrt{4} \\ -\sqrt{x+1} & -\sqrt{x+2} & -\sqrt{x+3} & \end{array}\right\}\text{ etc.}$$

quae series quantopere convergat videamus casu $x=\frac{1}{2}$, eritque:

$$\Sigma:\tfrac{1}{2}=\left\{\begin{array}{l} +\tfrac{1}{2}\sqrt{1}+\tfrac{1}{2}\sqrt{2}+\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\sqrt{4} \\ \tfrac{1}{2}+\tfrac{1}{2}\sqrt{2}+\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\sqrt{4}+\tfrac{1}{2}\sqrt{5} \\ -\sqrt{\tfrac{3}{2}}-\sqrt{\tfrac{5}{2}}-\sqrt{\tfrac{7}{2}}-\sqrt{\tfrac{9}{2}} \end{array}\right\}\text{ etc.}$$

et collectis terminis quicunque erit $\frac{1}{2}\sqrt{n}+\frac{1}{2}\sqrt{n+1}-\sqrt{\frac{2n+1}{2}}$, quod eo propius ad nihilum accedere debet, quo major fuerit numerus n, quocirca proxime erit $\sqrt{n}+\sqrt{n+1}=\sqrt{2(2n+1)}$. Sumtis enim quadratis habebimus $2n+1+2\sqrt{n(n+1)}=2(2n+1)$, ideoque $2\sqrt{n(n+1)}=2n+1$. Sumtis denuo quadratis fiet $4nn+4n=4nn+4n+1$, quae ratio utique proxime ad aequalitatem accedit. Ceterum hic notari meretur, veros valores pro fractionibus loco x assumtis tantopere esse transcendentes, ut nullis plane formulis analyticis exprimi queant. Quin etiam quilibet valor pro x assumtus ad peculiare transcendentium genus pertinebit.

§. 31. Antequam hanc speciem deseramus, adjungamus adhuc insigne Theorema circa convergentiam formularum multo generalius eo quod modo ante attulimus.

Theorema.

Sequens aequalitas: $(\beta-\alpha)\sqrt[\mu]{n^{\nu}}+\alpha\sqrt[\mu]{(n+1)^{\nu}}=\beta\sqrt[\mu]{(n+\frac{\alpha}{\beta})^{\nu}}$, *eo propius ad veritatem accedet, quo major sumatur numerus* n, *simulque quo minor fuerit fractio* $\frac{\alpha}{\beta}$, *si modo exponens* $\frac{\nu}{\mu}$ *unitate fuerit minor. At vero sumto* ν *negativo, ista aequalitas:*

$$\frac{(\beta-\alpha)}{\sqrt[\mu]{n^{\nu}}}+\frac{\alpha}{\sqrt[\mu]{(n+1)^{\nu}}}=\frac{\beta}{\sqrt[\mu]{(n+\frac{\alpha}{\beta})^{\nu}}}$$

sine posteriore conditione ad veritatem eo propius accedet, quo major fuerit numerus n *et quo minor fuerit fractio* $\frac{\alpha}{\beta}$. *Quin etiam sub iisdem conditionibus pro-*

xime per logarithmos erit tam:

$$(\beta-\alpha)\, l n + \alpha\, l\,(n+1) = \beta\, l\,(n+\tfrac{\alpha}{\beta}),$$

quam $\frac{\beta-\alpha}{l n} + \frac{\alpha}{l(n+1)} = \frac{\beta}{l\,(n+\frac{\alpha}{\beta})}.$

Demonstratio.

§. 32. Sequitur hoc theorema ex solutione generali pro hac specie data, cujus terminus quicunque consistit his partibus: $\overline{1 + x\,(n) + x\,(n+1) - (n+x)}$, atque eo minor evadit, quo major sumatur numerus n, existente x fractione unitate minore. Quod si jam ponamus $x = \frac{\alpha}{\beta}$ et $(x) = \sqrt[\mu]{x^\nu}$, ideoque etiam $(n) = \sqrt[\mu]{n^\nu}$, necesse est ut sit $\frac{\mu}{\nu} < 1$, quia alioquin termini infinitesimi non haberent differentias evanescentes. Hae autem substitutiones praebent formulas priores in theoremate datas. Quando vero fractio $\frac{\mu}{\nu}$ negativa accipitur, tum series proposita adeo in specie prima continebitur, siquidem ipsi termini infinitesimi in nihilum abeunt.

§. 33. Quo vis hujus theorematis intelligatur, notasse juvabit, has formulas quatuor casibus exacte cum veritate convenire, quorum primus est: si $\alpha = 0$; secundus, quo $\alpha = \beta$; tertius, quo $\nu = 0$; quartus denique locum habet si pro n accipiatur numerus infinitus. Praeterea vero datur casus quintus, quo in forma priore est $\mu = \nu$, sive $\sqrt[\mu]{n^\nu} = n$.

Species tertia serierum quarum differentiae demum infinitesimae secundae evanescunt.

§. 34. Hoc igitur eveniet, quoties ipsi termini infinitesimi progressionem arithmeticam constituunt; formula igitur pro $\Sigma : x$ ante in superiore specie inventa ad hunc casum accommodabitur, si insuper singuli termini columnae tertiae verticalis adjungantur. Hoc modo terminus summatorius sequenti modo exprimetur:

$$\Sigma : x = \begin{cases} \quad + (1) + (2) + (3) \ldots\ldots + (n) \\ \quad x(1) + x\Delta 1 + x\Delta 2 + x\Delta 3 \ldots\ldots + x\Delta n \\ + x'\Delta 1 + x'\Delta^2 1 + x'\Delta^2 2 + x'\Delta^2 3 \ldots\ldots + x'\Delta^2 n \\ \quad - (x+1) - (x+2) - (x+3) \ldots\ldots - (x+n). \end{cases}$$

§. 35. Transmutemus nunc hanc expressionem in formam ad usum magis accommodatam, ac primo quidem loco x' scribamus ejus valorem $\frac{xx-x}{2}$; tum vero ob $\Delta n = (n+1) - (n)$ et $\Delta^2 n = (n+2) - 2(n+1) + (n)$, his valoribus substitutis postrema columna praecedentis formulae abibit in hanc formam:

$$(n) + \overline{x(n+1)} + \overline{\tfrac{xx-x}{2}}(n+2)$$
$$- x(n) - \overline{xx-x}\,(n+1)$$
$$+ \frac{xx-x}{2}(n),$$

qui termini collecti praebent:

$$\frac{xx-3x+2}{2}(n) - \overline{xx-2x}\,(n+1) + \overline{\tfrac{xx-x}{2}}(n+2).$$

Ponamus igitur brev. gr. $\frac{xx-3x+2}{2}=p$; $xx-2x=q$ et $\frac{xx-x}{2}=r$, sicque terminus summatorius quaesitus sequenti forma exprimetur:

$$\Sigma : x = \begin{cases} \frac{3x-xx}{2}(1)+\frac{xx-x}{2}(2) \\ +p(1)-q(2)+r(3)-(x+1) \\ +p(2)-q(3)+r(4)-(x+2) \\ +p(3)-q(4)+r(5)-(x+3), \\ \quad\text{etc.} \end{cases}$$

quae series jam vehementer converget.

§. 36. Hinc igitur novum Theorema, simile praecedenti, sed multo latius patens, possumus derivare, ponendo ut ante $x=\frac{\alpha}{\beta}$, $(n)=\sqrt[\mu]{n^\nu}$, ubi jam sufficit ut exponens $\frac{\nu}{\mu}$ binario sit minor; multo magis autem hunc exponentem negativum statuere licebit.

Theorema.

Ista aequalitas: $(\alpha\alpha-3\alpha\beta+2\beta\beta)\sqrt[\mu]{n^\nu}-(2\alpha\alpha-4\alpha\beta)\sqrt[\mu]{(n+1)^\nu}+(\alpha\alpha-\alpha\beta)\sqrt[\mu]{(n+2)^\nu}=2\beta\beta\sqrt[\mu]{(n+\frac{\alpha}{\beta})^\nu}$, *eo propius ad veritatem accedet, quo major capiatur numerus* n *et quo minus fractio* $\frac{\alpha}{\beta}$ *ab unitate discrepet, dummodo* $\frac{\nu}{\mu}$ *binario sit minus. Tum vero, sumto* μ *negativo, erit plerumque multo accuratius:*

$$\frac{\alpha\alpha-3\alpha\beta+2\beta\beta}{\sqrt[\mu]{n^\nu}}-\frac{(2\alpha\alpha-4\alpha\beta)}{\sqrt[\mu]{(n+1)^\nu}}+\frac{(\alpha\alpha-\alpha\beta)}{\sqrt[\mu]{(n+2)^\nu}}=\frac{2\beta\beta}{\sqrt[\mu]{(n+\frac{\alpha}{\beta})^\nu}}.$$

Quin etiam pro formulis radicalibus logarithmi accipi poterunt.

§. 37. Veritas hujus theorematis etiam exacte subsistit his quatuor casibus: 1°) $\alpha = 0$; 2°) $\alpha = \beta$; 3°) $\nu = 0$ et 4°) $n = \infty$. Praeterea vero idem evenit, quando in forma priore est vel $\nu = \mu$ vel $\nu = 2\mu$, ita ut sit $\sqrt[\mu]{n^\nu}$, vel n vel nn. Habemus igitur sex casus, quibus hoc theorema nihil plane a veritate aberrat; unde facile intelligitur etiam reliquis casibus omnibus errorem non esse posse notabilem.

§. 38. Possumus etiam hoc theorema adhuc generalius reddere, loco n scribendo $\frac{n}{c}$ et ubique per debitam potestatem ipsius c multiplicando, quo fractiones tollantur. Sicque prior forma fiet:

$$(\alpha\alpha - 3\alpha\beta + 2\beta\beta)\sqrt[\mu]{n^\nu} - (2\alpha\alpha - 4\alpha\beta)\sqrt[\mu]{(n+c)^\nu}$$
$$+ (\alpha\alpha - \alpha\beta)\sqrt[\mu]{(n+2c)^\nu} = 2\beta\beta\sqrt[\mu]{\left(n+\tfrac{\alpha c}{\beta}\right)^\nu},$$

altera autem forma ab hac non discrepat, nisi quod radicalia in denominatorem ingrediuntur, id quod etiam de logarithmis est intelligendum.

§. 39. Operae pretium erit hoc theorema aliquo exemplo illustrasse. Sumatur igitur $\alpha = 1$ et $\beta = 2$, fientque aequalitates in theoremate exhibitae:

$$3\sqrt[\mu]{n^\nu} + 6\sqrt[\mu]{(n+c)^\nu} - \sqrt[\mu]{(n+2c)^\nu} = 8\sqrt[\mu]{(n+\tfrac{1}{2}c)^\nu}$$

$$\frac{3}{\sqrt[\mu]{n^\nu}} + \frac{6}{\sqrt[\mu]{(n+1)^\nu}} - \frac{1}{\sqrt[\mu]{(n+2)^\nu}} = \frac{8}{\sqrt[\mu]{(n+\tfrac{1}{2})^\nu}}.$$

Applicemus formam priorem ad logarithmos, fietque:

$$3\,ln + 6\,l\,(n+c) - l\,(n+2c) = 8\,l\,(n+\tfrac{1}{2}c).$$

Sit nunc $n = 10$ et $c = 2$, ut prodeat:

$$3\,l\,10 + 6\,l\,12 - l\,14 = 8\,l\,11.$$

Facta igitur evolutione erit:

$$\begin{array}{rlrl} 3\,l\,10 = & 3{,}0000000 & l\,14 = & 1{,}1461280 \\ 6\,l\,12 = & 6{,}4750872 & 8\,l\,11 = & 8{,}3311416 \\ \hline & 9{,}4750872 & = & 9{,}4772696, \end{array}$$

quorum differentia est 0,0021824, quae multo minor prodiisset, si numero n majorem valorem tribuissemus.

§. 40. Circa ipsum autem terminum summatorium seriei propositae imprimis notari convenit, tam differentiationem quam integrationem facile institui posse, sumto scilicet indice x variabili, quemadmodum hoc jam in specie prima fusius est ostensum, ubi ipse terminus summatorius $\Sigma : x$ tanquam applicata cujusdam curvae est consideratus, dum index x referebat abscissam, hocque respectu in calculo differentiali potissimum functiones inexplicabiles sum perscrutatus.

§. 41. Ex formula autem generali, pro termino summatorio $\Sigma:x$ supra data evolvamus hic quoque casum seriei harmonicae, quo est: $\Sigma:x = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{x}$ et quaeramus ejus valorem pro indice $x=\frac{1}{2}$, atque ob $(x)=\frac{1}{x}$, tum vero ob $p=\frac{3}{8}$; $q=-\frac{3}{4}$; $r=-\frac{1}{8}$, habebimus:

$$\Sigma:\tfrac{1}{2} = \left\{ \tfrac{5}{8} - \tfrac{1}{16} \begin{array}{l} +\frac{3}{8}+\frac{3}{16}+\frac{1}{8}+\frac{3}{32} \\ +\frac{3}{8}+\frac{1}{4}+\frac{3}{16}+\frac{3}{20} \\ -\frac{1}{24}-\frac{1}{32}-\frac{1}{40}-\frac{1}{48} \\ -\frac{2}{3}-\frac{2}{5}-\frac{2}{7}-\frac{2}{9} \end{array} \right\} \text{etc.}$$

sive erit:

$$8\,\Sigma:\tfrac{1}{2} = \left\{ \tfrac{9}{2} \begin{array}{l} +\frac{3}{1}+\frac{3}{2}+\frac{3}{3}+\frac{3}{4} \\ +\frac{6}{2}+\frac{6}{3}+\frac{6}{4}+\frac{6}{5} \\ -\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6} \\ -\frac{16}{3}-\frac{16}{5}-\frac{16}{7}-\frac{16}{9} \end{array} \right\} \text{etc.}$$

Contrahamus singulas columnas in unum terminum, eritque:

$$8\,\Sigma:\tfrac{1}{2} = \frac{6}{1\cdot2\cdot3\cdot5}+\frac{6}{2\cdot3\cdot4\cdot5}+\frac{6}{3\cdot4\cdot5\cdot7}+\frac{6}{4\cdot5\cdot6\cdot9}+\text{etc.}$$

quae series utique magis convergit ea, quam specie secunda invenimus.

§. 42. Quod si autem terminos non contrahamus, sed eos, qui eundem habent denominatorem, colligamus, omissa serie infima habebimus:

$$8\,\Sigma:\tfrac{1}{2} = \left\{ \tfrac{9}{2}+\tfrac{3}{1}+\tfrac{9}{2} \begin{array}{l} +\;8\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\text{etc.}\right) \\ -16\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\text{etc.}\right), \end{array} \right.$$

sive loco superioris seriei scribendo $16\left(\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\text{etc.}\right)$ habebimus:

$\frac{1}{2}\Sigma : \frac{1}{2} - \frac{3}{4} = -\frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \frac{1}{11} +$ etc. Addamus utrinque $l2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ etc. fiet $\frac{1}{2}\Sigma : \frac{1}{2} - \frac{3}{4} + l2 = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, consequenter $\Sigma : \frac{1}{2} = 2 - 2l2$, qui valor egregie convenit cum eo qui in specie prima est datus.

SUPPLEMENTUM

DE FUNCTIONIBUS INEXPLICABILIBUS FORMAE:

$$\Pi : x = A . B . C . D . E \; . \; . \; . \; . \; . \; X.$$

§. 1. Hic factores A, B, C, D, etc. sunt termini cujusdam seriei, indicibus 1, 2, 3, 4, etc. respondentes, et X terminus indici x respondens; factores autem, qui indicibus sequentibus $x+1$; $x+2$; $x+3$; etc. respondent, per X', X'', X''', etc. designabo. Hinc jam statim patet, fore $\Pi : (x+1) = X' . \Pi : x$ et $\Pi : (x+2) = X' . X'' . \Pi : x$, et ita porro. Praecedentes vero erunt $\Pi : (x-1) = \frac{\Pi : x}{x}$ etc. Unde intelligitur sufficere, dummodo hae formulae pro valoribus ipsius x unitate minoribus assignentur.

§. 2. Quoties fuerit x numerus integer positivus, valores ipsius $\Pi : x$ sponte se produnt. Erit nempe $\Pi : 1 = A$;

$\Pi : 2 = AB$; $\Pi : 3 = ABC$; etc. Quando autem x non est numerus integer positivus, productum, quod charactere $\Pi : x$ designamus, erit functio inexplicabilis ipsius x, nisi forte factores A, B, C, D, etc. ita fuerint comparati, ut praecedentes a sequentibus destruantur, veluti evenit in hac forma: $\Pi : x = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{1}{x+1}$, quandoquidem hic manifesto est $\Pi : x = \frac{1}{x+1}$; vel etiam in hoc exemplo: $\Pi : x = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} \cdot \frac{24}{25} \cdot \ldots \cdot \frac{xx+2x}{(x+1)^2}$. Hinc enim erit: $\Pi : 1 = \frac{3}{2 \cdot 2}$; $\Pi : 2 = \frac{2}{3} = \frac{4}{2 \cdot 3}$; $\Pi : 3 = \frac{5}{8} = \frac{5}{2 \cdot 4}$; $\Pi : 4 = \frac{3}{5} = \frac{6}{2 \cdot 5}$; $\Pi : 5 = \frac{7}{2 \cdot 6}$; etc. unde patet in genere fore $\Pi : x = \frac{x+2}{2(x+1)}$.

§. 3. Casus autem inexplicabiles, sumendis logarithmis, ad praecedentem dissertationem revocabuntur. Erit enim:

$$l\Pi : x = lA + lB + lC + \ldots + lX,$$

quae forma cum supra tractata comparata nobis dabit sequentes valores:

$\Sigma : x = l\Pi : x$; $(1) = lA$; $(2) = lB$; $(3) = lC$; etc. et $(x) = lX$; tum vero erit $(x+1) = lX'$; $(x+2) = lX''$; etc. hocque consensu observato species supra tractatas ad praesentem casum accommodemus.

Species prima,

ubi logarithmi factorum infinitesimorum evanescunt, sive ubi factores infinitesimi unitati aequantur.

§. 4. Cum igitur pro hac prima specie, introductis valoribus modo datis, habeamus:

$$l\Pi : x = \begin{cases} lA + lB + lC + lD \mp \text{etc.} \\ -lX' - lX'' - lX''' - lX^{IV} - \text{etc.} \end{cases}$$

ad numeros ascendendo erit:

$$\Pi : x = \frac{A}{X'} \cdot \frac{B}{X''} \cdot \frac{C}{X'''} \cdot \frac{D}{X^{IV}} \cdot \text{etc.}$$

Hic nulla exempla subjungo, quia jam plura in calculo differentiali sunt evoluta.

Species secunda,

ubi factores infinitesimi inter se fiunt aequales.

§. 5. Tum enim eorum logarithmi etiam inter se erunt aequales, ideoque differentiae primae omnes evanescent. Huc igitur accommodemus formulam supra §. 25. inventam, eritque:

$$l\Pi : x = xlA \left\{ \begin{array}{llll} + \overline{1-x}\,lA & + \overline{1-x}\,lB & + \overline{1-x}\,lC \\ + \quad xlB & + \quad xlC & + \quad xlD \\ - \quad lX' & - \quad lX'' & - \quad lX''' \end{array} \right\} \text{etc.}$$

unde ad numeros ascendendo habebimus:

$$\Pi : x = A^x \cdot \frac{A^{1-x}.B^x}{X'} \cdot \frac{B^{1-x}.C^x}{X''} \cdot \frac{C^{1-x}.D^x}{X'''} \cdot \text{etc.}$$

Species tertia,

ubi termini infinitesimi constituunt progressionem geometricam.

§. 6. Tum enim logarithmi horum terminorum progressionem arithmeticam constituent, cujus ergo differentiae secundae evanescent. Ut jam expressionem supra §. 35.

inventam ad hunc casum accomodemus, notandum est br. gr. positum fuisse $p = \frac{xx - 3x + 2}{2}$, $q = xx - 2x$ et $r = \frac{xx - x}{2}$, unde habebimus:

$$l\Pi : x = \left\{\begin{array}{r} + plA + plB + plC \\ \left(\frac{3x - xx}{2}\right) lA - qlB - qlC - qlD \\ + \frac{xx - x}{2} lB + rlC + rlD + rlE \\ - lX' - lX'' - lX''' \end{array}\right\} \text{etc.}$$

Ponamus autem hic porro compendii causa $\frac{xx - 3x}{2} = m$ et $\frac{xx - x}{2} = n$, atque ad numeros ascendendo habebimus hanc expressionem:

$$\Pi : x = \frac{B^n}{A^m} \cdot \frac{A^p C^r}{B^q X'} \cdot \frac{B^p D^r}{C^q X''} \cdot \frac{C^p E^r}{D^q X'''} \cdot \text{ etc.}$$

§. 7. Hoc modo confido, doctrinam de functionibus inexplicabilibus, quae in Calculo differentiali non satis accurate et luculenter est exposita, fere penitus exhausisse, ita ut nihil amplius desiderari possit; quod eo magis necessarium videbatur, cum hoc argumentum plane sit novum et a nemine adhuc tractatum. Praecipue autem ejus summus usus in interpolatione serierum, atque hinc adeo symptomata linearum curvarum, quarum applicatae per functiones inexplicabiles exprimuntur, investiganda erat.