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De motu globi circa axem obliquum quemcunque gyrantis et super plano horizontali incedentis

Leonhard Euler

DE MOTV GLOBI CIRCA AXEM OBLIQVVM QVEMCVNQVE GYRANTIS ET SVPER PLANO HORIZONTALI INCEDENTIS.

Auctore
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Cum in omnibus quae hactenus de motu globorum super planis sunt tradita alius motus gyratorius non sit consideratus, nisi qui fiat circa axem ad motus directionem normalem; quaestio superest maxime ardua: quomodo globus, cui circa axem quemcunque obliquum fuerit impressus motus gyratorius, super plano sit progressurus; quoniam principia, ex quibus huiusmodi motus determinari oportet, neutiquam adhuc satis sunt euoluta, vt ad quosuis casus, qui occurrere possunt, applicari queant. Primus equidem haec principia in Tractatu meo de motu corporum solidorum seu rigidorum ista principia in lucem produxi, indeque plurima motus Phoenomena explicaui, quae principiis mechanicis vulgaribus prorsus erant inaccessa. Quin etiam in vltimo huius libri capite hoc ipsum argumentum de globo super plano horizontali incedente, dum interea circa axem obliquum quemcunque gyratur, omnio studio sum perscrutatus. Verum quia hic liber in paucorum manibus versatur, ac ista ipsa tractatio plerisque Geometris etiamnunc videtur incognita; haud abs re fore arbitror, si totum hoc argumentum hic denuo in lucem protraxero, prouti in loco memorato est pertractatum, vbi quidem nonnullas dilucidationes, si opus fuerit visum, adiungam, quo vniuersa Theoria globorum super plano horizontali vtcunque propulsorum completa reddatur. In hac autem inuestigatione imprimis ad effectum frictionis est respiciendum, quandoquidem remota frictione globus perpetuo eundem motum tam progressiuum quam gyratorium, sine vlla alteratione, esset conseruaturus. Frictionem autem eodem modo in calculum introducam, quo hactenus a Geometris tractari est solita. Quanquam enim forte omnia frictionis symptomata nondum fuerint penitus perspecta: tamen ista tractatio inde nullam mutationem pati est censenda; quoniam praecipuum negotium hic in euolutione abstrusissimorum principiorum mechanicae sublimioris et in integratione plurium formularum differentialium alias difficillimarum absoluitur.

Problema I.

§. 1. *Si globus super plano horizontali vtcunque tam motu progressiuo quam gyratorio moueatur, determinare celeritatem et directionem, qua punctum contactus radit superficiem horizontalem.*

Solutio.

Tab. IV. Fig. 1. Sit I centrum globi, simulque eius centrum inertiae eiusque radius sit $= f$, et contactus fiat in puncto imo T, motus

motus autem globi ita sit comparatus, vt centrum inertiae moueatur secundum directionem P I R celeritate $= v$, simul vero gyretur circa axem quemcunque I O celeritate angulari $= \varepsilon$, in eum sensum, vt punctum T circa O incedat per arculum $T t$, ac pro positione puncti O statuamus angulum $P T O = \theta$ et arcum $T O = s$, vbi quidem arcus ita sumo, quasi radius globi esset $= 1$. Ducatur T V ipsi P I R parallela, ac si motus gyratorius abesset, punctum contactus T rasurum esset planum horizontale celeritate v in directione T V. Deinde si globus solo motu gyratorio ferretur, punctum T per $T t$ moueretur celeritate $f \varepsilon$ sin. $T O = f \varepsilon$ sin. s, cuius directio cum sit horizontalis, in plano per rectam $T \Theta$ referatur, ita vt sit angulus $S T \Theta = P T t = \theta - 90°$, ob $O T t$ rectum. Erit ergo $V T \theta = 270° - \theta$. Capiantur rectae $T V = v$ et $T \Theta = f \varepsilon$ sin. s, et quia punctum T his duobus motibus coniunctim mouetur, eius verus motus fiet secundum rectam T F, diagonalem parallelogrammi $T V F \Theta$. Ex F ad T V ducta normali F H, erit $V H = f \varepsilon$ sin. s sin. θ et $F H = - f \varepsilon$ sin. s cos. θ, vnde fit $F H = v - f \varepsilon$ sin. s sin. θ atque celeritas radens

$$T F = \sqrt{(v v - 2 f \varepsilon v \text{ sin. } s \text{ sin. } \theta + f f \varepsilon \varepsilon \text{ sin. } s^2)} \text{ et}$$
$$\text{tang. } V T F = \frac{- f \varepsilon \text{ sin. } s \text{ cos. } \theta}{v - f \varepsilon \text{ sin. } s \text{ sin. } \theta}.$$

Ducatur ex centro I ipsi T F parallela I Q, erit arcus T Q quadrans, et angulus $R T Q = V T F$; quare si I Q sit directioni, secundum quam punctum T radit, parallela, erit

$$\text{tang. } P T Q = \frac{f \varepsilon \text{ sin. } s \text{ cos. } \theta}{v - f \varepsilon \text{ sin. } s \text{ sin. } \theta},$$

ac posita celeritate radente

$\sqrt{(v v - 2 f \varepsilon v \sin. s \sin. \theta + f f \varepsilon^2 \sin. s^2)} = u$, erit
$\sin. PTQ = - \frac{f \varepsilon \sin. s \cos. \theta}{u}$ et
$\cos. PTQ = \frac{f \varepsilon \sin. s \sin. \theta - v}{u}$.

Corollarium 1.

§. 2. Fieri ergo poteſt, vt celeritas radens ideoque et attritus euaneſcat, quo caſu hae duae aequationes locum habere debent: altera $\varepsilon \sin. s \cos. \theta = 0$, altera vero $v = f \varepsilon \sin. s \sin. \theta$. Vnde ſtatim patet, ſi nullus adſit motus progreſſiuus, ſeu $v = 0$, nullum attritum affore ſi $\sin. s = 0$, hoc eſt ſi globus circa axem verticalem ZT gyretur.

Corollarium 2.

§. 3. Deinde motus globi ab attritu erit liber, ſi fuerit primo $\cos. \theta = 0$, ſeu angulus PTO rectus: deinde celeritas progreſſiua v ad angularem ε hanc relationem tenere debet, vt ſit $v = f \varepsilon \sin. s$, ſeu $TV = T\Theta$, et angulus $ST\Theta = 0$.

Corollarium 3.

§. 4. Quando ergo globus huiusmodi motum eſt conſecutus, quia ſublato omni attritu etiam nulla adeſt frictio, globus eundem motum conſtanter conſeruabit, ſiquidem axis gyrationis IO habeat proprietatem axis principalis.

Scholion 1.

§. 5. Quemadmodum hic frictionem conſtituimus, ea non obſtat, quominus globus ſuper plano horizontali

motum

motum suum intemeratum conseruari possit, quod tamen minime fieri obseruamus, cum globus super tabula tali motu latus mox omnem motum amittat, cuius rei caussa resistentiae aëris tribui nequit. Verum hic primum animaduerto, experimenta Theoriae nunquam perfectissime congruere: veluti dum casu hic tractato assumsimus, contactum vnico fieri puncto, id semper in praxi secus euenit. Interim tamen si arcus T O est quadrans et $PTO = 90°$, existente $v = f\varepsilon$, etsi contactus non fiat in vnico puncto, tamen attritus euanescit, ideoque haec motus extinctio frictioni neutiquam adscribi potest. Ex quo concludere debemus, praeter frictionem, vti hic eam definiuimus, aliud adhuc dari motus impedimentum, dum corpora super superficiebus incedunt, a frictione probe distinguendum, cuius ratio vtcunque fuerit comparata, eius effectus potius seorsim inuestigari conuenit, quam frictionis indolem hic stabilitam immutari. Et quemadmodum hic a resistentia aëris mentem abstrahimus ita etiam licebit hoc obstaculum frictionem concomitans a praesenti argumento seiungere.

Scholion 2.

§. 6. Corpora hic sphaerica considero, in quorum centro situm sit ipsorum centrum inertiae I, quod ergo ipsum etiam in plano horizontali moueatur, et puncto contactus T perpetuo verticaliter immineat; ex quo patet, pressionem in contactu semper fore ponderi corporis M aequalem. Si ergo corpus ex materia vniformi constaret, omnes eius diametri proprietate axium principalium gauderent. Sed concipiamus materiae distributionem

vtcun-

vtcunque inaequabilem, ita tamen vt centrum inertiae cadat in centrum figurae: hocque pacto necesse erit in globo ternos axes principales considerari, qui ex centro I pertingant in puncta A, B, C, quadrantibus a se inuicem distantia, quorum respectu sint momenta inertiae Maa, Mbb, Mcc. Quanquam autem deinceps bina vel omnia haec momenta inter se aequalia statuemus, tamen conueniet tria huiusmodi puncta in superficie fixa notasse, quo ex eorum relatione ad spatium absolutum facilius motus globi definiri possit. Constitutis autem in globis his tribus punctis A, B, C, quoniam motus gyrationis circa O, quem in plagam Tt dirigi assumimus, sensum habet CBA, contrarium ei quem supra statuimus, in applicatione formularum generalium ad hunc casum celeritatem angularem ϑ vt negatiuam spectare debemus.

Problema II.

§. 7. *Si globus super plano horizontali vtcunque moueatur, definire vires, quibus sollicitatur, earumque momenta respectu ternorum axium principalium globi.*

Solutio.

Tab. IV. Fig. 2. Inclusus concipiatur globus Sphaerae vel fixae vel cum eo parem motum progressiuum habenti, in qua Z sit punctum verticale eiusque oppositum T punctum contactus, DE vero sit diameter horizontalis ad certam mundi plagam tendens et DPQE circulus maximus horizontalis. Nunc autem elapso tempore t moueatur globus motu progressiuo secundum directionem PI celeritate $= v$, ponaturque arcus DP, seu angulus $DZP = \varphi$; axes autem prin-

principales nunc sint in A, B, C. Tum vero globus iam gyretur circa axem IO, celeritate angulari $= \varepsilon$ in sensum ACB, sitque pro situ puncti O angulus PTO seu PZO $= \theta$ et arcus ZO $= s$. Etsi enim ante arcum TO posuimus $= s$, quia tantum eius sinus in computum intrat, perinde est. Erit ergo angulus DZO $= \theta + \Phi$ et EZO $= 180° - \theta - \Phi$. Deinde a punctis A, B, C tam ad O quam ad Z arcus circulorum maximorum ducti concipiantur, sintque hi arcus AO $= \alpha$, BO $= \beta$, CO $= \gamma$; ZA $= l$, ZB $= m$, ZC $= n$ et anguli EZA $= \lambda$, EZB $= \mu$, EZC $= \nu$. In praecedente autem probleмate ostendimus, punctum contactus T planum subiectum radere secundum directionem radio IQ parallelam celeritate $= \sqrt{(vv - 2f\varepsilon v \,\text{sin.}\, s \,\text{sin.}\, \theta + ff\varepsilon\varepsilon \,\text{sin.}\, s^2)}$, foreque

$$\text{tang. PTQ} = \text{tang. PZQ} = \frac{f\varepsilon \,\text{sin.}\, s \,\text{cos.}\, \theta}{v - f\varepsilon \,\text{sin.}\, s \,\text{sin.}\, \theta},$$

denotante f radium globi. Cum igitur pressio in T sit $= M$, frictio erit $= \delta M$, quae puncto T est applicata secundum directionem ipsi QI parallelam. Hac ergo vi resoluta secundum directiones axium principalium IA, IB, IC, prodibunt ternae vires, quae in puncto T applicatae sunt concipiendae, ex quibus porro colliguntur sequentia momenta:

Respectu axis IA in sensum BC:

$$-\delta M f \,\text{cos.}\, CQ \,\text{cos.}\, BT + \delta M f \,\text{cos.}\, BQ \,\text{cos.}\, CT = P.$$

Respectu axis IB in sensum CA:

$$-\delta M f \,\text{cos.}\, AQ \,\text{cos.}\, CT + \delta M f \,\text{cos.}\, CQ \,\text{cos.}\, AT = Q.$$

Respectu axis IC in sensum AB:

$$-\delta M f \,\text{cos.}\, BQ \,\text{cos.}\, AT + \delta M f \,\text{cos.}\, AQ \,\text{cos.}\, BT = R,$$

erunt

erunt ergo haec tria momenta:

$$P = \delta M f (\text{cof. } m \text{ cof. } CQ - \text{cof. } n \text{ cof. } BQ)$$
$$Q = \delta M f (\text{cof. } n \text{ cof. } AQ - \text{cof. } l \text{ cof. } CQ)$$
$$R = \delta M f (\text{cof. } l \text{ cof. } BQ - \text{cof. } m \text{ cof. } AQ).$$

Pro puncto autem Q ponamus angulum $PZQ = \xi$, vt fit

$$\text{tang. } \xi = \frac{f \vartheta \text{ fin. } s \text{ cof. } \theta}{v - f \vartheta \text{ fin. } s \text{ fin. } \theta}$$

et pofita celeritate radente

$$\sqrt{(vv - 2 f v \vartheta \text{ fin. } s \text{ fin. } \theta + ff \vartheta\vartheta \text{ fin. } \theta^2} = u, \text{ erit}$$
$$\text{fin. } \xi = \frac{-f \vartheta \text{ fin. } s \text{ cof. } \theta}{u} \text{ et cof. } \xi = \frac{f \vartheta \text{ fin. } s \text{ fin. } \theta - v}{u}.$$

Fit ergo $DZQ = \Phi + \xi$ et $EZQ = 180° - \Phi - \xi$, hinc $AZQ = 180° - \xi - \Phi - \lambda$; $BZQ = \mu + \xi + \Phi - 180°$; ergo

$$\text{cof. } AQ = -\text{cof. } (\xi + \Phi + \lambda) \text{ fin. } l$$
$$\text{cof. } BQ = -\text{cof. } (\xi + \Phi + \mu) \text{ fin. } m$$
$$\text{cof. } CQ = -\text{cof. } (\xi + \Phi + \nu) \text{ fin. } n.$$

Ex relatione igitur, quae inter angulos λ, μ et ν intercedit, concludemus momenta virium:

$$P = \delta M f \text{ fin. } l \times \text{fin. } (\lambda + \Phi + \xi)$$
$$Q = \delta M f \text{ fin. } m \times \text{fin. } (\mu + \Phi + \xi)$$
$$R = \delta M f \text{ fin. } n \times \text{fin. } (\nu + \Phi + \xi).$$

Pro-

Problema III.

§. 8. *Si motum gyratorium ad quoduis tempus vt datum spectemus, definire motum progressiuum globi.*

Solutio.

Tab. IV. Fig. 3.

Quia centrum globi in plano horizontali moue- tur, descripserit id tempore t lineam GI, quae referatur ad directionem GX superiori directioni fixae DE paral- lelam, ductaque IX ad GX normali, sint coordinatae GX $= X$, XY $= Y$. Per I ducatur recta DE ipsi GX parallela, quae erit ipsa diameter DE (fig. 2). Ducatur IP, ita vt sit angulus DIP $=$ EIR $= \Phi$, et centrum I per hypothesin progreditur in directione IR celeritate $= v$, ita vt sit celeritas secundum GX $= v \operatorname{cos.} \Phi$ et celeritas secundum XI $= v \operatorname{fin.} \Phi$, ideoque $dX = v\,dt \operatorname{cof.} \Phi$ et $dY = v\,dt \operatorname{fin.} \Phi$. Ducatur recta QIS, ita vt IQ sit di- rectioni, qua punctum contactus radit, parallela, erit angulus EIQ $=$ DIS $= 180° - \xi - \Phi$; (est enim aequalis angulo EZQ in praecedente figura) vnde globus sollicitari cen- sendus est vi $= \delta M$ in directione IS. Hinc ergo oritur vis secundum ID $= -\delta M \operatorname{cof.} (\xi + \Phi)$ et vis secundum XI $= \delta M \operatorname{fin.} (\xi + \Phi)$, ex quibus colligitur

$$\frac{d.\, v \operatorname{cof.} \Phi}{2g\,dt} = \frac{dv \operatorname{cof.} \Phi - v\,d\Phi \operatorname{fin.} \Phi}{2g\,dt} = \delta \operatorname{cof.} (\xi + \Phi)$$

$$\frac{d.\, v \operatorname{fin.} \Phi}{2g\,dt} = \frac{dv \operatorname{fin.} \Phi + v\,d\Phi \operatorname{cof.} \Phi}{2g\,dt} = \delta \operatorname{fin.} (\xi + \Phi)$$

hincque porro

$$\frac{dv}{2g\,dt} = \delta \operatorname{cof.} \xi \text{ et } \frac{v\,d\Phi}{2g\,dt} = \delta \operatorname{fin.} \xi,$$

ita

ita vt ſit

$$\frac{v\, d\, \Phi}{d\, v} = \text{tang.}\, \xi = \frac{f\, \upsilon\, \text{ſin.}\, s\, \text{ſin.}\, \theta}{v - f\, \upsilon\, \text{ſin.}\, s\, \text{ſin.}\, \theta}.$$

Problema IV.

§. 9. *Definito motu progreſſiuo globi, determinare eius motum gyratorium.*

Solutio.

Tab. IV. Fig. 2. Spectetur nunc centrum globi I vt quiescens, et maneant omnes denominationes in Problemate II adhibitae, ſintque $M\,a\,a$, $M\,b\,b$, $M\,c\,c$ momenta inertiae reſpectu axium principalium I A, I B, I C, quae primo vt inaequalia conſideremus. Quoniam vero hic celeritatem angularem υ vt negatiuam ſpectare debemus, quia tendit in ſenſum A C B, ſi ponamus $\upsilon\,\text{coſ.}\,\alpha = x$, $\upsilon\,\text{coſ.}\,\beta = y$, et $\upsilon\,\text{coſ.}\,\gamma = z$, in formulis generalibus has litteras x, y, z negatiue ſumi oportet, quo facto ex § 810. Theoriae meae motus corporum ſolidorum deducuntur hae aequationes motum determinantes:

$$d\,x + \frac{bb-cc}{a\,a}\, y\, z\, d\,t + \frac{2\delta f g}{a\,a}\, d\,t\, \text{ſin.}\, l\, \text{ſin.}\, (\lambda + \Phi + \xi) = 0;$$

$$d\,y + \frac{cc-aa}{b\,b}\, x\, z\, d\,t + \frac{2\delta f g}{b\,b}\, d\,t\, \text{ſin.}\, m\, \text{ſin.}\, (\mu + \Phi + \xi) = 0;$$

$$d\,z + \frac{aa-bb}{c\,c}\, x\, y\, d\,t + \frac{2\delta f g}{c\,c}\, d\,t\, \text{ſin.}\, n\, \text{ſin.}\, (\nu + \Phi + \xi) = 0;$$

$$d\,l\, \text{ſin.}\, l = d\,t\, (z\, \text{coſ.}\, m - y\, \text{coſ.}\, n);$$

$$d\,m\, \text{ſin.}\, m = d\,t\, (x\, \text{coſ.}\, n - z\, \text{coſ.}\, l);$$

$$d\,n\, \text{ſin.}\, n = d\,t\, (y\, \text{coſ.}\, l - x\, \text{coſ.}\, m);$$

$$d\,\lambda\, \text{ſin.}\, l^2 = d\,t\, (y\, \text{coſ.}\, m + z\, \text{coſ.}\, n);$$

$$d\,\mu\, \text{ſin.}\, m^2 = d\,t\, (z\, \text{coſ.}\, n + x\, \text{coſ.}\, l);$$

$$d\,\nu\, \text{ſin.}\, n^2 = d\,t\, (x\, \text{coſ.}\, l + y\, \text{coſ.}\, m).$$

Tum

Tum vero ex motu progressiuo habemus

$$d v = 2\delta g d t \operatorname{cos.} \xi,$$
$$v d \Phi = 2\delta g d t \operatorname{sin.} \xi \text{ et}$$
$$\operatorname{tang.} \xi = \frac{f s \operatorname{sin.} s \operatorname{cos.} \theta}{v - f s \operatorname{sin.} s \operatorname{cos.} \theta},$$

vbi est $PZO = \theta$ et $ZO = s$. Cum ergo sit angulus $EZO = 180° - \theta - \Phi$, erit $AZO = 180° - \lambda - \theta - \Phi$: hincque

$$\operatorname{cos.} \alpha = \operatorname{cos.} l \operatorname{cos.} s - \operatorname{sin.} l \operatorname{sin.} s \operatorname{cos.} (\lambda + \theta + \Phi)$$
$$\operatorname{cos.} \beta = \operatorname{cos.} m \operatorname{cos.} s - \operatorname{sin.} m \operatorname{sin.} s \operatorname{cos.} (\mu + \theta + \Phi)$$
$$\operatorname{cos.} \gamma = \operatorname{cos.} n \operatorname{cos.} s - \operatorname{sin.} n \operatorname{sin.} s \operatorname{cos.} (\nu + \theta + \Phi),$$

existente

$$\operatorname{cos.} s = \operatorname{cos.} l \operatorname{cos.} \alpha + \operatorname{cos.} m \operatorname{cos.} \beta + \operatorname{cos.} n \operatorname{cos.} \gamma,$$

vnde sequitur fore

$$\left.\begin{aligned} &+ \operatorname{sin.} l \operatorname{cos.} l \operatorname{cos.} (\lambda + \theta + \Phi) \\ &+ \operatorname{sin.} m \operatorname{cos.} m \operatorname{cos.} (\mu + \theta + \Phi) \\ &+ \operatorname{sin.} n \operatorname{cos.} n \operatorname{cos.} (\nu + \theta + \Phi) \end{aligned}\right\} = 0.$$

Ponamus $s \operatorname{cos.} s = p$ et $s \operatorname{sin.} s = q$ ita vt sit

$$\operatorname{tang.} \xi = \frac{f q \operatorname{cos.} \theta}{v - f q \operatorname{sin.} \theta} = \frac{v\, d\Phi}{d v},$$

eritque

$$x = p \operatorname{cos.} l - q \operatorname{sin.} l \operatorname{cos.} (\lambda + \theta + \Phi),$$
$$y = p \operatorname{cos.} m - q \operatorname{sin.} m \operatorname{cos.} (\mu + \theta + \Phi),$$
$$z = p \operatorname{cos.} n - q \operatorname{sin.} n \operatorname{cos.} (\nu + \theta + \Phi),$$

ex quibus valoribus fit

$$d l = q d t \operatorname{sin.} (\lambda + \theta + \Phi);$$
$$d m = q d t \operatorname{sin.} (\mu + \theta + \Phi);$$
$$d n = q d t \operatorname{sin.} (\nu + \theta + \Phi);$$
$$d \lambda = p d t + q d t \operatorname{cot.} l \operatorname{cos.} (\lambda + \theta + \Phi);$$
$$d \mu = p d t + q d t \operatorname{cot.} m \operatorname{cos.} (\mu + \theta + \Phi);$$
$$d \nu = p d t + q d t \operatorname{cot.} n \operatorname{cos.} (\nu + \theta + \Phi);$$

P 3 inde-

indeque porro

$$d x = d p \operatorname{cos.} l - d q \operatorname{sin.} l \operatorname{cos.} (\lambda + \theta + \Phi) + q (d \theta + d \Phi) \operatorname{sin.} l \operatorname{sin.} (\lambda + \theta + \Phi),$$

$$d y = d p \operatorname{cos.} m - d q \operatorname{sin.} m \operatorname{cos.} (\mu + \theta + \Phi) + q (d \theta + d \Phi) \operatorname{sin.} m \operatorname{sin.} (\mu + \theta + \Phi),$$

$$d z = d p \operatorname{cos.} n - d q \operatorname{sin.} n \operatorname{cos.} (\nu + \theta + \Phi) + q (d \theta + d \Phi) \operatorname{sin.} n \operatorname{sin.} (\nu + \theta + \Phi),$$

At sine subsidio harum substitutionum ex aequationibus ternis primis cum in genere sit

$$\operatorname{sin.} l \operatorname{cos.} l \operatorname{sin.} (\lambda + A) + \operatorname{sin.} m \operatorname{cos.} m (\mu + A) + \operatorname{sin.} n \operatorname{cos.} n \operatorname{sin.} (\nu + A) = 0,$$

elicimus hanc aequationem:

$$\left. \begin{array}{l} a a d x \operatorname{cos.} l + b b d y \operatorname{cos.} m + c c d z \operatorname{cos.} n \\ - a a x d l \operatorname{sin.} \lambda - b b y d m \operatorname{sin.} m - c c z d n \operatorname{sin.} n \end{array} \right\} = 0,$$

cuius integrale est

$$a a x \operatorname{cos.} l + b b y \operatorname{cos.} m + c c z \operatorname{cos.} n = C,$$

quae aequatio, adhibitis substitutionibus, abit in hanc:

$$\left. \begin{array}{l} p (a a \operatorname{cos.} l^2 + b b \operatorname{cos.} m^2 + c c \operatorname{cos.} n^2) \\ - q a a \operatorname{sin.} l \operatorname{cos.} l \operatorname{cos.} (A + \theta + \Phi) \\ - q b b \operatorname{sin.} m \operatorname{cos.} m \operatorname{cos.} (\mu + \theta + \Phi) \\ - q c c \operatorname{sin.} n \operatorname{cos.} n \operatorname{cos.} (\nu + \theta + \Phi) \end{array} \right\} = \text{Const.}$$

Deinde etiam per reductiones §. 934. Theoriae meae traditas pro vi viua colligitur haec aequatio differentialis:

$$a a x d x + b b y d y + c c z d z = 2 \delta f g q d t \operatorname{sin.} (\xi - \theta).$$

Scholion.

§. 10. Ad reductiones hic factas intelligendas ex formulis traditis, vbi angulos μ et ν per l, λ, m, n exprespressimus, notari conuenit fieri

$$\cos.(\mu+\theta+\Phi) = -\frac{\cos. l \cos. m \cos.(\lambda+\theta+\Phi) + \cos. n \sin.(\lambda+\theta+\Phi)}{\sin. l \sin. m},$$

$$\cos.(\nu+\theta+\Phi) = -\frac{\cos. l \cos. n \cos.(\lambda+\theta+\Phi) - \cos. m \sin.(\lambda+\theta+\Phi)}{\sin. l \sin. m},$$

$$\sin.(\mu+\theta+\Phi) = -\frac{\cos. l \cos. m \sin.(\lambda+\theta+\Phi) - \cos. n \cos.(\lambda+\theta+\Phi)}{\sin. l \sin. m},$$

$$\sin.(\nu+\theta+\Phi) = -\frac{\cos. l \cos. n \sin.(\lambda+\theta+\Phi) + \cos. m \cos.(\lambda+\theta+\Phi)}{\sin. l \sin. n}.$$

Ac simili modo anguli $\mu+\Phi+\xi$ et $\nu+\Phi+\xi$ ad angulum $\lambda+\Phi+\xi$ reuocari possunt. Deinde etiam pro sequentibus reductionibus haec forma imprimis est notanda:

$$\sin.(\mu+B)\cos.(\nu+C) - \sin.(\nu+B)\cos.(\mu+C),$$

quae ob

$$\sin. M \cos. N = \tfrac{1}{2}\sin.(M+N) + \tfrac{1}{2}\sin.(M-N),$$

reducitur ad

$$\sin.(\mu-\nu)\cos.(B-C);$$

hocque modo reductionem pro aliis formulis instituendo, reperiemus:

$$\sin.(\mu+B)\cos.(\nu+C) - \sin.(\nu+B)\cos.(\mu+C)$$
$$= \sin.(\mu-\nu)\cos.(B-C),$$

$$\sin.(\mu+B)\sin.(\nu+C) - \sin.(\nu+B)\sin.(\mu+C)$$
$$= -\sin.(\mu-\nu)\sin.(B-C),$$

$$\cos.(\mu+B)\cos.(\nu+C) - \cos.(\nu+B)\cos.(\mu+C)$$
$$= -\sin.(\mu-\nu)\sin.(B-C),$$

vbi sin $(\mu-\nu)$ per formulas vsurpatas datur: est enim $\sin.(\mu-\nu) = \frac{\cos. e}{\sin. m \sin. n}$.

Problema V.

§. 11. *Si globus ex materia vniformi constet, vel saltem ita fuerit comparatus, vt omnia momenta inertiae sint*

inter

inter se aequalia, eique initio impressus fuerit motus quicunque, determinare eius continuationem.

Solutio.

Cum hic sit $aa = bb = cc$, seu momentum inertiae respectu omnium diametrorum $= Maa$, prima aequatio integrata praebet $aap =$ Const. vnde p erit quantitas constans. Statuatur ergo $p = b$, et ternae aequationes priores hanc induent formam:

$$\text{I.}\ -dq\,\text{cos.}(\lambda+\theta+\varphi)+q(d\theta+d\varphi)\,\text{sin.}(\lambda+\theta+\varphi)$$
$$+\tfrac{2\delta fg}{aa}\,dt\,\text{sin.}(\lambda+\varphi+\xi)=0,$$

$$\text{II.}\ -dq\,\text{cos.}(\mu+\theta+\varphi)+q(d\theta+d\varphi)\,\text{sin.}(\mu+\theta+\varphi)$$
$$+\tfrac{2\delta fg}{aa}\,dt\,\text{sin.}(\mu+\varphi+\xi)=0,$$

$$\text{III.}\ -dq\,\text{cos.}(\nu+\theta+\varphi)+q(d\theta+d\varphi)\,\text{sin.}(\nu+\theta+\varphi)$$
$$+\tfrac{2\delta fg}{aa}\,dt\,\text{sin.}(\nu+\varphi+\xi)=0,$$

quarum autem sufficit binas consideraffe, quia iam inde nata est conclusio $p = b$. Iam per superiores reductiones binae posteriores aequationes ita combinentur:

$$\text{II. cos.}(\nu+\theta+\varphi)-\text{III. cos.}(\mu+\theta+\varphi)$$

quae combinatio praebet

$$q(d\theta+d\varphi)\,\text{sin.}(\mu-\nu)+\tfrac{2\delta fg}{aa}\,dt\,\text{sin.}(\mu-\nu)\,\text{cos.}(\xi-\theta)=0,$$

seu

$$q(d\theta+d\varphi)+\tfrac{2\delta fg}{aa}\,dt\,\text{cos.}(\xi-\theta)=0.$$

Deinde combinatio

$$\text{II. sin.}(\nu+\theta+\varphi)-\text{III. sin.}(\mu+\theta+\varphi)\ \text{dat}$$

$$d q \sin.(\mu - \nu) - \tfrac{2\delta f g}{aa} d t \sin.(\mu - \nu) \sin.(\xi - \theta) = 0$$

seu

$$d q = \tfrac{2\delta f g}{aa} d t \sin.(\xi - \theta),$$

qui valor in vltima aequatione pro viribus viuis substitutus praebet

$$x\,d x + y\,d y + z\,d z = q\,d q, \text{ hincque}$$

$$x x + y y + z z = \varrho\varrho = \text{const.} + q q = \text{const.} + \varrho\varrho \sin. s^2$$

ita vt sit $\varrho\varrho \cos. s^2$ quantitas constans, vti iam inuenimus, ob $\varrho \cos s = p = h$. Hinc istas habemus aequationes a litteris l, m, n, λ, μ, ν immunes:

$$\text{I. } q(d\theta + d\varphi) + \tfrac{2\delta f g}{aa} d t \cos.(\xi - \theta) = 0,$$

$$\text{II. } d q = \tfrac{2\delta f g}{aa} d t \sin.(\xi - \theta) = 0,$$

$$\text{III. } d v = 2\delta g\, d t \cos. \xi,$$

$$\text{IV. } v\, d\varphi = 2\delta g\, d t \sin. \xi,$$

quibus adiungatur haec finita: $\operatorname{tang.} \xi = \frac{f q \cos. \theta}{v - f q \sin. \theta}$, quae in hanc transformata:

$$v \sin. \xi - f q \cos.(\xi - \theta) = 0,$$

differentietur, prodibitque

$$d v \sin. \xi + v\, d\xi \cos. \xi - f\, d q \cos.(\xi - \theta) + f q\, d\xi \sin.(\xi - \theta)$$
$$- f q\, d\theta \sin.(\xi - \theta) = 0.$$

Iam combinatio:

$$\text{I. } \sin.(\xi - \theta) + \text{II. } \cos.(\xi - \theta) \text{ dat}$$

$$q(d\theta + d\varphi) \sin. \xi - \theta) + d q \cos.(\xi - \theta) = 0,$$

quae aequatio per f multiplicata illi addatur, fietque

$$d v \sin. \xi + v\, d\xi \cos. \xi + f q(d\xi + d\varphi) \sin.(\xi - \theta) = 0.$$

 Porro

Porro ob $\frac{dv}{v\,d\phi} = \frac{\text{cos}.\xi}{\text{sin}.\xi}$ erit

$$v\,(d\phi + d\xi)\,\text{cos}\,\xi + fq\,(d\phi + d\xi)\,\text{sin}.\,(\xi - \theta) = 0,\ \text{seu}$$
$$(d\phi + d\xi)\,(v\,\text{cos}.\,\xi + fq\,\text{sin}.\,(\xi - \theta)) = 0,$$

quorum factorum finitus: $v\,\text{cos}.\,\xi + fq\,\text{sin}.\,(\xi - \theta)$, euanescere nequit ob

$$v\,\text{sin}.\,\xi - fq\,\text{cos}.\,(\xi - \theta) = 0,$$

sequeretur enim inde $v\,\text{cos}.\,\theta = 0$ et $fq\,\text{cos}.\,\theta = 0$, quod non nisi casu $\theta = 90°$ locum habet. Relinquitur ergo vt sit $d\phi + d\xi = 0$, ideoque $\phi + \xi$ constans. Hoc impetrato reliqua non difficulter expedientur. Ad integrationes autem determinandas pro statu initiali $t = 0$, ponamus fuisse celeritatem progressiuam $v = e$, $\phi = 0$, $PZO = \theta = \mathfrak{h}$; $ZO = s = \mathfrak{f}$ et celeritatem angularem $\varepsilon = \varepsilon$ in sensum ACB; hinc erit $p = b = \varepsilon\,\text{cos}.\,\mathfrak{f}$ et $q = \varepsilon\,\text{sin}.\,\mathfrak{f}$; porro

$$\text{tang}\,\xi = \frac{\varepsilon f\,\text{sin}.\,\mathfrak{f}\,\text{cos}.\,\mathfrak{h}}{e - \varepsilon f\,\text{sin}.\,\mathfrak{f}\,\text{sin}.\,\mathfrak{h}}.$$

Statuatur

$$\frac{\varepsilon f\,\text{sin}.\,\mathfrak{f}\,\text{cos}.\,\mathfrak{h}}{e - \varepsilon f\,\text{sin}.\,\mathfrak{f}\,\text{sin}.\,\mathfrak{h}} = \text{tang}.\,\zeta,$$

vt fuerit initio $\xi = \zeta$, ac perpetuo erit $\xi + \phi = \zeta$, ita vt angulus $DZQ = \zeta$ maneat constans. Quare cum sit $\xi = \zeta - \phi$ erit

$$v\,\text{sin}.\,(\zeta - \phi) = fq\,\text{cos}.\,(\zeta - \theta - \phi).$$

Supra autem inuenimus:

$$\frac{d.\,v\,\text{cos}.\,\phi}{2g\,dt} = \delta\,\text{cos}.\,(\xi + \phi) = \delta\,\text{cos}.\,\zeta\ \text{et}$$
$$\frac{d.\,v\,\text{sin}.\,\phi}{2g\,dt} = \delta\,\text{sin}.\,(\xi + \phi) = \delta\,\text{sin}.\,\zeta,$$

vnde integrando colligimus:

$$v\,\text{cos}.\,\phi = e + 2\delta g t\,\text{cos}.\,\zeta\ \text{et}\ v\,\text{sin}.\,\phi = 2\delta g t\,\text{sin}.\,\zeta,$$

hinc-

hincque

$$v = \sqrt{(ee + 4\delta eg t \operatorname{cof.}\zeta + 4\delta\delta gg tt)},$$

$$\operatorname{tang.}\phi = \frac{2\delta g t \operatorname{fin.}\zeta}{e + 2\delta g t \operatorname{cof.}\zeta}, \text{ atque}$$

$$\operatorname{tang.}(\zeta - \phi) = \frac{e \operatorname{fin.}\zeta}{e \operatorname{cof.}\zeta + 2\delta g t} = \frac{f q \operatorname{cof.}\theta}{v - f q \operatorname{fin.}\theta} = \operatorname{tang.}\xi.$$

Deinde ob $d\Phi = -d\xi$ binae priores aequationes abeunt in

$$\text{I. } q(d\xi - d\theta) = \frac{2\delta f g}{aa} dt \operatorname{cof.}(\xi - \theta).$$

$$\text{II. } dq = \frac{2\delta f g}{aa} dt \operatorname{fin.}(\xi - \theta),$$

quarum haec per illam diuifa dat

$$\frac{dq}{q(d\xi - d\theta)} = \frac{\operatorname{fin.}(\xi - \theta)}{\operatorname{cof.}(\xi - \theta)},$$

qua integrata prodit $q \operatorname{cof.}(\xi - \theta) = C$, ideoque

$$q \operatorname{cof.}(\xi - \theta) = \varepsilon \operatorname{fin.} \mathfrak{f} \operatorname{cof.}(\zeta - \mathfrak{h}),$$

vnde valor ipfius q in prima fubftitutus praebet:

$$\frac{\varepsilon (d\xi - d\theta) \operatorname{fin.}\mathfrak{f} \operatorname{cof.}(\zeta - \mathfrak{h})}{\operatorname{cof.}(\xi - \theta)^2} = \frac{2\delta f g}{aa} dt,$$

et integrando

$$\varepsilon \operatorname{fin.}\mathfrak{f} \operatorname{cof.}(\zeta - \mathfrak{h}) \operatorname{tang.}(\xi - \theta) = C + \frac{2\delta f g}{aa} t,$$

vbi $C = \varepsilon \operatorname{fin.}\mathfrak{f} \operatorname{fin.}(\theta - \mathfrak{h})$, at

$$\operatorname{tang.}(\xi - \theta) = \operatorname{tang.}(\zeta - \phi - \theta) = \frac{\operatorname{tang.}(\zeta - \Phi) - \operatorname{tang.}\theta}{1 + \operatorname{tang.}(\zeta - \Phi)\operatorname{tang.}\theta} \text{ et}$$

$$\operatorname{tang.}\theta = \frac{\operatorname{tang.}\xi - \operatorname{tang.}(\xi - \theta)}{1 + \operatorname{tang.}\xi \operatorname{tang.}(\xi - \theta)}.$$

Sed per hypothefin eft $\varepsilon \operatorname{fin.}\mathfrak{f} = \frac{e \operatorname{fin.}\zeta}{f \operatorname{cof.}(\zeta - \mathfrak{h})}$, vnde fit

$$\operatorname{tang.}(\xi - \theta) = \operatorname{tang.}(\zeta - \mathfrak{h}) + \frac{2\delta f f g t}{e a a \operatorname{fin.}\zeta} \text{ et}$$

$$\operatorname{tang.}\xi = \frac{e \operatorname{fin.}\zeta}{e \operatorname{cof.}\zeta + 2\delta g t},$$

hincque angulus θ facile determinatur: indeque $q = \frac{e \operatorname{fin.}\zeta}{f \operatorname{cof.}(\xi - \theta)}$.

Verum hic notari oportet, cum ſit

$$\text{tang.}\ \zeta = \frac{e f\ \text{ſin.}\ \mathfrak{f}\ \text{coſ.}\ \mathfrak{h}}{e - \varepsilon f\ \text{ſin.}\ \mathfrak{f}\ \text{ſin.}\ \mathfrak{h}},$$

eſſe vt ſupra de angulo ξ oſtendimus

$$\text{ſin.}\ \zeta = \frac{-e f\ \text{ſin.}\ \mathfrak{f}\ \text{coſ.}\ \mathfrak{h}}{\sqrt{(e e - 2 e \varepsilon f\ \text{ſin.}\ \mathfrak{f}\ \text{ſin.}\ \mathfrak{h} + \varepsilon \varepsilon f f\ \text{ſin.}\ \mathfrak{f}^2)}}\ \text{et}$$

$$\text{coſ.}\ \zeta = \frac{-e + \varepsilon f\ \text{ſin.}\ \mathfrak{f}\ \text{ſin.}\ \mathfrak{h}}{\sqrt{(e e - 2 \varepsilon e f\ \text{ſin.}\ \mathfrak{f}\ \text{ſin.}\ \mathfrak{h} + \varepsilon \varepsilon f f\ \text{ſin.}\ \mathfrak{f}^2)}},\ \text{vnde}$$

$$\text{coſ.}\ (\zeta - \mathfrak{h}) = \frac{-e\ \text{coſ.}\ \mathfrak{h}}{\sqrt{(e e - 2 e \varepsilon f\ \text{ſin.}\ \mathfrak{f}\ \text{ſin.}\ \mathfrak{h} + \varepsilon \varepsilon f f\ \text{ſin.}\ \mathfrak{f}^2)}}.$$

His inuentis cum ſit $\text{ʊ}\ \text{coſ.}\ s = \varepsilon\ \text{coſ.}\ \mathfrak{f}$ et $\text{ʊ}\ \text{ſin.}\ s = q$, erit

$$\text{ʊ} = \sqrt{(q q + \varepsilon \varepsilon\ \text{coſ.}\ \mathfrak{f}^2)}\ \text{et tang.}\ s = \frac{q}{\varepsilon\ \text{coſ.}\ \mathfrak{f}}.$$

Sicque tam motus progreſſiuus, quam ad quoduis tempus axis gyrationis O cum celeritate angulari ʊ poterit aſſignari, id quod ad motus cognitionem ſufficit. Determinatio autem ſitus punctorum A, B, C ad quoduis tempus nimis eſt ardua, quam vt eam perficere liceat.

Corollarium 1.

§. 12. Cum ſit celeritas angularis $\text{ʊ} = \frac{\varepsilon\ \text{coſ.}\ \mathfrak{f}}{\text{coſ.}\ s}$, ſeu coſinui arcus S O reciproce proportionalis; ſequitur ſi polus gyrationis O initio fuerit in ſuperiori hemiſphaerio DZE, eum nunquam in inferius peruenire poſſe; in tranſitu enim per circulum horizontalem D E prodiret celeritas angularis ʊ infinita.

Corollarium 2.

§. 13. Ob eandem rationem, ſi polus gyrationis O initio fuerit in hemiſphaerio inferiori D T E, is nunquam in ſuperius aſcendet; ſin autem initio fuerit in ipſo

ipſo circulo horizontali DE, perpetuo in eodem manebit: ſcilicet ſi initio axis gyrationis fuerit horizontalis, perpetuo horizontalis manebit.

Corollarium 3.

§. 14. Si fuerit initio angulus DZO = $\mathfrak{h}$ rectus, fiet ſin. $\zeta = 0$ et ob

$$\text{tang.}\,(\zeta - \mathfrak{h}) = \frac{e f\,\text{ſin.}\,\mathfrak{f} - e\,\text{ſin.}\,\mathfrak{h}}{e\,\text{coſ.}\,\mathfrak{h}},$$

erit etiam $\xi - \theta$ rectus. Sed ob tang. $\xi = \frac{e\,\text{ſin.}\,\zeta}{e\,\text{coſ.}\,\zeta + 2\delta g t}$ angulus ξ euaneſcit, vnde angulus θ = PZO prodit rectus. Simulatque igitur angulus PZO factus fuerit rectus, perpetuo rectus manebit.

Corollarium 4.

§. 15. Memorabilis eſt etiam proprietas, quod angulus $\xi + \varphi$, ſeu DZQ, et in fig. 3 angulus DIQ ſit conſtans. Recta enim QIS ſibi perpetuo manebit parallela, et quia globus in motu progreſſiuo ſollicitatur vi conſtante δM ſecundum eandem directionem IS, curua ab eo deſcripta GI parabola ſit neceſſe eſt.

Scholion 1.

§. 16. Hic autem motus globi, vti noſtris formulis eſt definitus, diutius non durat, quam reuera frictio adeſt, ſeu planum horizontale in puncto contactus T raditur. Si enim eueniat vt raſio ceſſet, ſeu celeritas radens in T euaneſcat, ſubito frictio euaneſcit, formulaeque inuentae non amplius locum habent. Tum igitur globus motu tam progreſſiuo quam gyratorio vniformiter in directum progredietur, neque axis gyrationis vllam amplius mutationem patietur. Ac si statim initio motus globo impressus ita fuerit comparatus, vt frictio fuerit nulla, quod euenit si tam $\varepsilon f \,\text{sin.}\, \mathfrak{f} \,\text{cos.}\, \mathfrak{h} = 0$, quam $e = \varepsilon f \,\text{sin.}\, \mathfrak{f} \,\text{sin.}\, \mathfrak{h}$, tum etiam globus nullam frictionem sentiet, et statim ab initio motum progressiuum uniformiter in directum prosequetur, simulque vniformiter circa eundem axem gyrabitur. Verum si corpori ab initio alius motus quicunque fuerit impressus, semper aliquo tempore elapso eo reducetur, vt frictio euanescat, indeque motum suum vniformiter prosequetur, quod memorabile temporis punctum in sequenti problemate inuestigabimus.

Scholion 2.

Tab. IV. Fig. 2. §. 17. Quae in solutione problematis elicuimus, huc redeunt: ex motu primum impresso habemus celeritatem motus progressiui $= e$, secundum directionem DI: ac si gyretur circa axem IO celeritate angulari ε in sensum ACB, seu $ZETD$, qui sensus *antrorsum tendens* dici solet, fueritque arcus $ZO = \mathfrak{f}$ et angulus $DZO = \mathfrak{h}$: tum vero radius globi sit $= f$ eiusque momentum inertiae $= Maa$ respectu omnium diametrorum, existente M eius massa: ex his datis colligitur celeritas radens in puncto contactus $= \sqrt{(ee - 2\varepsilon e f \,\text{sin.}\, \mathfrak{f} \,\text{sin.}\, \mathfrak{h} + \varepsilon\varepsilon f f \,\text{sin.}\, \mathfrak{f}^2)}$, quae si ponatur $= k$, quaeratur angulus $DZQ = \zeta$, vt sit

$$\text{sin.}\, \zeta = \frac{-\varepsilon f \,\text{sin.}\, \mathfrak{f} \,\text{cos.}\, \mathfrak{h}}{k} \text{ et } \text{cos.}\, \zeta = \frac{\varepsilon f \,\text{sin.}\, \mathfrak{f} \,\text{sin.}\, \mathfrak{h} - e}{k},$$

eritque IQ directio motus radentis. Tum si elapso tempore t globi centrum proferatur celeritate v secundum directionem PI, et gyret celeritate angulari $= \mathfrak{v}$ in sensum

sum ZETD circa polum O, ponaturque $DZP = \phi$, $PZO = \theta$ et $ZO = s$: inuenimus primo

$$\text{tang.}\,\phi = \frac{2\delta g t \,\text{sin.}\,\zeta}{e + 2\delta g t \,\text{cos.}\,\zeta}$$

et celeritatem centri $= \sqrt{(ee + 4\delta e g t \,\text{cos.}\,\zeta + 4\delta\delta g g t t)}$, at celeritas radens etiamnunc fiet in directione IQ, existente $DZQ = \zeta$; vnde posito $PZQ = \xi$ erit

$$\text{tang.}\,\xi = \frac{e \,\text{sin.}\,\zeta}{e \,\text{cos.}\,\zeta + 2\delta g t}.$$ Porro est

$$\text{tang.}\,(\xi - \theta) = \text{tang.}\,(\zeta - \mathfrak{h}) + \frac{2\delta f f g t}{e a a \,\text{sin.}\,\zeta},$$

existente

$$\text{tang.}\,(\zeta - \mathfrak{h}) = \frac{\varepsilon f \,\text{sin.}\,\mathfrak{f} - e \,\text{sin.}\,\mathfrak{h}}{e \,\text{cos.}\,\mathfrak{h}},$$

vnde angulus θ innotescit, hincque ob $DZO = \phi + \theta = \zeta - \xi + \theta$, concluditur

$$\text{tang.}\,DZO = \text{tang.}\,(\phi + \theta) = \frac{\varepsilon a a k \,\text{sin.}\,\mathfrak{f} \,\text{sin.}\,\mathfrak{h} + 2\delta f g t (e - \varepsilon) f \,\text{sin.}\,\mathfrak{f} \,\text{sin.}\,\mathfrak{h}}{\varepsilon a a k \,\text{sin.}\,\mathfrak{f} \,\text{cos.}\,\mathfrak{h} - 2\delta \varepsilon f f g t \,\text{sin.}\,\mathfrak{f} \,\text{cos.}\,\mathfrak{h}}.$$

Atque ex his tandem nacti sumus $\mathfrak{z} \,\text{cos.}\, s = \varepsilon \,\text{cos.}\,\mathfrak{f}$ et $\mathfrak{z} \,\text{sin.}\, s = \frac{e \,\text{sin.}\,\zeta}{f \,\text{cos.}\,(\xi - \theta)}$. Denique pro celeritate radente secundum IQ, ea est $\sqrt{(vv - 2\mathfrak{z} v f \,\text{sin.}\, s \,\text{sin.}\,\theta + \mathfrak{z}\mathfrak{z} f f \,\text{sin.}\, s^2)}$; quae si vocetur $= w$, supra ostendimus esse

$$\text{sin.}\,\xi = -\frac{\mathfrak{z} f \,\text{sin.}\, s \,\text{cos.}\,\theta}{w} \quad \text{et} \quad \text{cos.}\,\xi = \frac{\mathfrak{z} f \,\text{sin.}\, s \,\text{sin.}\,\theta - v}{w},$$

vnde $\mathfrak{z}$ et s definiuntur. Sed pro situ punctorum A, B, C in globo fixorum ad quoduis tempus determinando formulae adeo fiunt intricatae, vt nihil inde concludi queat. Interim si pro puncto A vocetur $ZA = l$ et $EZA = \lambda$, ad has binas aequationes totum negotium reducitur:

$$\text{I. } dl = dt\left(\varepsilon \sin.\mathfrak{f} \sin.(\mathfrak{h}+\lambda) - \frac{2\delta f g t}{aa}\cos.(\zeta+\lambda)\right),$$

$$\text{II. } d\lambda \sin.l = \varepsilon dt \cos.\mathfrak{f} \sin.l + \varepsilon dt \cos.l \sin.\mathfrak{f} \cos.(\mathfrak{h}+\lambda) + \frac{2\delta f g t}{aa}\sin.(\zeta+\lambda),$$

quarum resolutio vereor ne frustra suscipiatur. Cum autem ad quoduis tempus axem gyrationis cum celeritate angulari assignare valeamus, quod ad motus cognitionem, qualis vulgo desideratur, sufficere potest, eo magis mirum videtur, quod motus singulorum globi punctorum quasi vires analyseos superet. Multo minus igitur de motu globorum, in quibus momenta inertiae non sunt aequalia, quicquam definire licebit.

Problema VI.

§. 18. *Si globo, cuius omnia momenta inertiae sunt inter se aequalia, motus quicunque fuerit impressus, assignare temporis punctum, vbi celeritas radens, ideoque et frictio euanescit, indeque globus motu vniformi progredi pergit.*

Solutio.

Supra §. 2. vidimus, vt attritus euanescat, has duas conditiones requiri: alteram $\mathfrak{v} \sin. s \cos. \theta = 0$, alteram $v = f \mathfrak{v} \sin. s \sin. \theta$, seu in expressione

$$\text{tang. } \xi = \frac{f \mathfrak{v} \sin. s \cos. \theta}{v - f \mathfrak{v} \sin. s \sin. \theta},$$

tam numeratorem quam denominatorem simul euanescere debere. Cum autem inuenerimus

$$\text{tang. } \xi = \frac{e \sin. \zeta}{e \cos. \zeta + 2\delta g t},$$

vbi numerator $e \sin. \zeta$ est constans, si in illa forma numerator euanescat, positio $\cos. \theta = 0$ tempus quaesitum declarabit.

Verum

Verum idem luculentius determinabimus, si ad quoduis tempus elapsum t celeritatem radentem w inuestigemus. Cum igitur ex valore §. 17. inuento: sin. $\xi = -\frac{8 f \sin. s \cos. \theta}{w}$ habeamus

$$w = -\frac{8 f \sin. s \cos. \theta}{\sin. \xi},$$

quae expressio ob $8 \sin. s = \frac{e \sin. \zeta}{f \cos. (\xi - \theta)}$ abit in hanc:

$$w = \frac{-e \sin. \zeta \cos. \theta}{\sin. \xi \cos. (\xi - \theta)}:$$

atque ob $\theta = \xi - (\xi - \theta)$ in hanc:

$$w = -e \sin. \zeta (\cot. \xi + \tang. (\xi - \theta));$$

si hic pro tang. ξ et tang. $(\xi - \theta)$ valores supra inuentos substituamus, reperiemus:

$$w = -(e \cos. \zeta + 2 \delta g t + e \sin. \zeta \tang. (\zeta - \mathfrak{h}) + \frac{2 \delta f f g t}{a a}).$$

At vero est

$$\cos. \zeta + \sin. \zeta \tang. (\zeta - \mathfrak{h}) = \frac{\cos. \mathfrak{h}}{\cos. (\zeta - \mathfrak{h})} \text{ et}$$
$$\cos. (\zeta - \mathfrak{h}) = -\frac{e \cos. \mathfrak{h}}{k}; \text{ vnde fit}$$
$$e \cos. \zeta + e \sin. \zeta \tang. (\zeta - \mathfrak{h}) = -k,$$

vbi k denotat celeritatem radentem initialem. Quamobrem elapso tempore t habebimus celeritatem radentem

$$w = k - 2 \delta g (1 + \frac{ff}{aa}) t,$$

ita vt ea labente tempore vniformiter decrescat, tandem ergo certe euanescat, id quod eueniet elapso tempore $t = \frac{a a k}{2 \delta g (a a + f f)}$, eritque tum cos. $\theta = 0$ et $\theta = 90° = PZO$. Quod ergo cum euenerit, videamus quomodo reliquae motus determinationes se sint habiturae, et quoniam

$2\delta g t = \frac{aak}{aa+ff}$ erit

$\text{tang.}\,\Phi = \frac{aak\,\text{fin.}\,\zeta}{e(aa+ff)+aak\,\text{cof.}\,\zeta}$ et

$\text{tang.}\,\xi = \frac{e(aa+ff)\,\text{tang.}\,\zeta}{e(aa+ff)+aak}$,

hinc fit $\mathfrak{v}\,\text{fin.}\,s = \frac{e\,\text{fin.}\,\zeta}{f\,\text{fin.}\,\xi}$. Cum autem fit

$v = \sqrt{(ee + \frac{2aaek\,\text{cof.}\,\zeta}{aa+ff} + \frac{a^4kk}{(aa+ff)^2})}$ erit

$\text{fin.}\,\Phi = \frac{aak\,\text{fin.}\,\zeta}{(aa+ff)v}$, $\text{cof.}\,\Phi = \frac{e(aa+ff)+aak\,\text{cof.}\,\zeta}{(aa+ff)v}$,

atque $\text{fin.}\,\xi = \frac{e\,\text{fin.}\,\zeta}{v}$ ideoque $\mathfrak{v}\,\text{fin.}\,s = \frac{v}{f}$. Porro quia eſt $\mathfrak{v}\,\text{cof.}\,s = \varepsilon\,\text{cof.}\,\mathfrak{f}$, erit $\text{tang.}\,s = \frac{v}{\varepsilon f\,\text{cof.}\,\mathfrak{f}}$ et

$\mathfrak{v} = \sqrt{(\frac{vv}{ff} + \varepsilon\varepsilon\,\text{cof.}\,\mathfrak{f}^2)}$,

ſiue ſubſtituto valore v:

$$\mathfrak{v} = \frac{\sqrt{(eeff + 2\varepsilon eaaf\,\text{fin.}\,\mathfrak{f}\,\text{fin.}\,\mathfrak{h} + \varepsilon\varepsilon a^4\,\text{fin.}\,\mathfrak{f}^2 + \varepsilon\varepsilon(aa+ff)^2\,\text{cof.}\,\mathfrak{f}^2)}}{aa+ff}$$

ob $kk = ee - 2\varepsilon ef\,\text{fin.}\,\mathfrak{f}\,\text{fin.}\,\mathfrak{h} + \varepsilon\varepsilon ff\,\text{fin.}\,\mathfrak{f}^2$.

Corollarium 1.

§. 19. Quo maior ergo initio fuerit celeritas radens k, eo diutius motus durat, antequam ceſſante frictione ad vniformitatem redigatur. Ac ſi globus conſtet ex materia homogenea, fit $aa = \frac{2}{5}ff$, ideoque motus vniformitas incipit elapſo tempore $t = \frac{k}{7\delta g}$ min. ſec., hinc in hypotheſi $\delta = \frac{1}{3}$ fit $t = \frac{3k}{7g}$, exiſtente $g = 15\frac{5}{8}$ pedum Rhenanorum.

Corollarium 2.

§. 20. Vt centrum globi eodem tempore ad quietem redigatur, ſtatus initialis ita comparatus eſſe debet, vt ſit $\text{cof.}\,\zeta = -1$ et $e = \frac{aak}{aa+ff}$. Fit ergo $\text{fin.}\,\mathfrak{h} = 1$ et $k = e - \varepsilon f\,\text{fin.}\,\mathfrak{f}\,\text{fin.}\,\mathfrak{h} = e - \varepsilon f\,\text{fin.}\,\mathfrak{f}$

hinc-

hincque ε fin. $\mathfrak{f} = \frac{e f}{a a}$. Porro ob $v = 0$, fit $s = 0$ et $\mathfrak{s} = \varepsilon$ cof. $\mathfrak{f}$, qua celeritate angulari iam globus circa axem verticalem quiefcentem gyrabitur, elapfo ab initio tempore $t = \frac{e}{2 \delta g}$ min. fec.

Corollarium 3.

§. 21. Hoc autem cafu, quo initio eft $\mathfrak{h} = 90°$ et $\varepsilon = \frac{-e f}{a a \, \text{fin.} \mathfrak{f}}$ fit $\zeta = 180°$; $\Phi = 0$; $\xi = 180°$; $\theta = 90°$; $v = e - 2 \delta g t$; tum vero habebimus $\mathfrak{s}$ cof. $s = \frac{-e f \, \text{cof.} \mathfrak{f}}{a a \, \text{fin.} \mathfrak{f}}$; $\mathfrak{s}$ fin. $s = -\frac{e f}{a a}\left(1 - \frac{2 \delta g t}{e}\right)$; hincque

$$\text{tang.}\, s = \left(1 - \frac{2 \delta g t}{e}\right) \text{tang.}\, \mathfrak{f} \text{ et}$$

$$\mathfrak{s} = \frac{-e}{a a \, \text{fin.} \mathfrak{f}} \sqrt{\left(1 - \frac{4 \delta g t}{e} \text{fin.}\, \mathfrak{f} + \frac{4 \delta \delta g g t t}{e e} \text{fin.}\, \mathfrak{f}^2\right)}.$$

At initio erat celeritas radens $k = e\left(1 + \frac{f f}{a a}\right)$, elapfo autem tempore t ea eft $w = \left(1 + \frac{f f}{a a}\right)(e - 2 \delta g t)$, ficque pofito $t = \frac{e}{2 \delta g}$ fimul fit $w = 0$, $v = 0$ et $s = 0$, vt ante.

Corollarium 4.

§. 22. Ne valor $\mathfrak{s}$ fin. $s = \frac{e \, \text{fin.} \zeta}{f \, \text{cof.} (\xi - \theta)}$ indefinitus videatur, quod fit fi numerator ac denominator euanefcant, feu $\zeta = 0$, conueniet loco fin. ζ et cof. $(\xi - \theta)$ valores ex fuperioribus fubftitui, atque hinc reperietur:

$$\mathfrak{s} \,\text{fin.}\, s = \sqrt{\left(\varepsilon \varepsilon \,\text{fin.}\, \mathfrak{f}^2 - \frac{4 \delta \varepsilon f g t \,\text{fin.}\, \mathfrak{f} (\varepsilon f \,\text{fin.}\, \mathfrak{f} - e \,\text{fin.}\, \mathfrak{h})}{a a k} + \frac{4 \delta \delta f f g g t t}{a^4}\right)},$$

vnde ob $\mathfrak{s}$ cof. $s = \varepsilon$ cof. $\mathfrak{f}$ prodit

$$\mathfrak{s}\mathfrak{s} = \varepsilon \varepsilon - \frac{4 \delta \varepsilon f g t \,\text{fin.}\, \mathfrak{f} (\varepsilon f \,\text{fin.}\, \mathfrak{f} - e \,\text{fin.}\, \mathfrak{h})}{a a k} + \frac{4 \delta \delta f f g g t t}{a^4}.$$

Corollarium 5.

§. 23. Cum sit vis viua globi $= M(vv + aa\vartheta\vartheta)$, erat ea initio $= M(ee + eeaa)$; elapso autem tempore t ea erit $= M(ee + \varepsilon\varepsilon aa - 4\delta gkt + 4(1 + \frac{ff}{aa})\delta\delta ggtt)$. At elapso tempore $t = \frac{aak}{2\delta g(aa + ff)}$, vis viua fiet

$$\frac{M(eeff + 2\varepsilon eaaf \sin \mathfrak{f} \sin \mathfrak{h} + \varepsilon\varepsilon aa(aa + ff \cos \mathfrak{f}^2))}{aa + ff},$$

cuius defectus ab initiali est

$$\frac{Maa(ee - 2\varepsilon ef \sin \mathfrak{f} \sin \mathfrak{h} + \varepsilon\varepsilon ff \sin \mathfrak{f}^2)}{aa + ff} = \frac{Maakk}{aa + ff},$$

ita vt ista vis viua sit

$$M(ee + \varepsilon\varepsilon aa - \frac{aakk}{aa + ff}).$$

Scholion.

Tab. IV. Fig. 2. §. 24. Ex his ergo formulis totus globi motus assignari potest, quicunque motus ei initio fuerit impressus. Interim tamen hae formulae non parum sunt complexae, vnde ad clariorem explicationem haud abs re erit casus quosdam magis notabiles euoluere. Cuiusmodi sunt, vti iam supra innuimus, duo potissimum: alter quo arcus ZO initio erat quadrans; alter vero quo angulus $DZO = \mathfrak{h}$ erat rectus: vtrumque igitur seorsim explicemus.

Problema VII.

§. 25. *Si globo, in quo omnia momenta inertiae sunt aequalia, initio motus gyratorius circa axem horizontalem fuerit impressus, praeter motum progressiuum definire continuationem motus.*

Solutio.

Solutio.

Cum initio axis gyrationis fuerit horizontalis, erit $f = ZO = 90^\circ$. Denotante ergo e celeritatem progressiuam secundum directionem DIE, et ε celeritatem angularem circa axem IO, in sensum ZETD, sit pro puncto O angulus $DZO = \mathfrak{h}$, manente f radio globi et Maa momento inertiae. Ex his erat initio celeritas radens

$$k = \sqrt{(ee - 2\varepsilon ef \sin.\mathfrak{h} + \varepsilon\varepsilon ff)}$$

et pro eius directione IQ angulus $DZQ = \zeta$, vt sit

$$\sin.\zeta = \frac{-\varepsilon f \cos.\mathfrak{h}}{k} \text{ et } \cos.\zeta = \frac{\varepsilon f \sin.\mathfrak{h} - e}{k}.$$

His pro statu initiali constitutis, elapso tempore t centrum globi descripserit viam GI, vt iam sit in I, vbi eius celeritas secundum IR erit

Tab IV. Fig. 3.

$$v = \sqrt{(ee + \frac{4\delta egt(\varepsilon f \sin.\mathfrak{h} - e)}{k} + 4\delta\delta ggtt)},$$

vnde positis coordinatis $GX = X$, et $XI = Y$, ob

$$\text{tang.}\, EIR = \text{tang.}\,\Phi = \frac{-2\delta\varepsilon fgt \cos.\mathfrak{h}}{ek + 2\delta gt(\varepsilon f \sin.\mathfrak{h} - e)}$$

erit

$$dX = \varepsilon dt + \frac{2\delta gtdt}{k}(\varepsilon f \sin.\mathfrak{h} - e) \text{ et}$$

$$dY = \frac{-2\delta\varepsilon fgtdt \cos.\mathfrak{h}}{k}, \text{ ideoque}$$

$$GX = X = \varepsilon t + \frac{\delta gtt}{k}(\varepsilon f \sin.\mathfrak{h} - e) \text{ et}$$

$$XI = Y = -\frac{\delta\varepsilon fgtt}{k} \cos.\mathfrak{h}.$$

Tum vero pro motu gyratorio, qui iam fiat in sensum ZETD celeritate angulari $= \mathfrak{s}$ circa polum O, existente $ZO = s$, $PZO = \theta$ et $DZQ = \Phi + \xi$, vbi IQ refert directionem celeritatis radentis, quia constanter est $\Phi + \xi = \zeta$, seu directio IQ constans, erit Fig. 2.

tang.

$$\text{tang.}\, \xi = \frac{-\varepsilon f \cos.\mathfrak{h}}{\varepsilon f \sin.\mathfrak{h} - ee + 2\delta gkt} \text{ et}$$

$$\text{tang.}\, (\xi - \theta) = \frac{2f - e \sin.\mathfrak{h}}{e \cos.\mathfrak{h}} - \frac{2\delta fgkt}{\varepsilon eaa \cos.\mathfrak{h}},$$

vnde ambo anguli ξ et θ definiuntur. Vel erit

$$\text{tang.}\, (\Phi + \theta) = \frac{\varepsilon aak \sin.\mathfrak{h} + 2\delta fgt(e - \varepsilon f \sin.\mathfrak{h})}{\varepsilon aak \cos.\mathfrak{h} - 2\delta \varepsilon ffgt \cos.\mathfrak{h}}.$$

Celeritas autem radens ſecundum directionem IQ eſt

$$w = k - 2\delta g \left(1 + \frac{ff}{aa}\right) t.$$

Tum vero ob $\upsilon \cos. s = 0$ erit arcus $ZO = s$ quadrans et

$$\upsilon = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta \varepsilon fgt(\varepsilon f - e \sin.\mathfrak{h})}{aak} + \frac{4\delta\delta ffggtt}{a^4}\right)}.$$

Hic autem motus inaequabilis tantum durabit per tempus $t = \frac{aak}{2\delta g(aa + ff)}$, quo elapſo eſt $s = 90^\circ$,

$$\text{tang.}\, \Phi = \frac{-\varepsilon aaf \cos.\mathfrak{h}}{e(aa + ff) + aa(\varepsilon f \sin.\mathfrak{h} - e)} = \frac{-\varepsilon aa \cos.\mathfrak{h}}{ef + \varepsilon aa \sin.\mathfrak{h}}$$

$$= \sqrt{\left(ee + \frac{2aae(\varepsilon f \sin.\mathfrak{h} - e)}{aa + ff} + \frac{a^4 kk}{(aa + ff)^2}\right)};$$

$$\upsilon = \frac{v}{f} = \frac{\sqrt{(eeff + 2\varepsilon eaaf \sin.\mathfrak{h} + \varepsilon\varepsilon a^4)}}{aa + ff}$$

ſubſtituto pro kk valore. Tum autem fit angulus $\theta = 90^\circ$ et $\sin.\, \xi = \frac{e \sin.\zeta}{v}$.

Corollarium I.

§. 26. Si initio fuerit angulus $DZO = \mathfrak{f} = 0$, erit $k = \sqrt{(ee + \varepsilon\varepsilon ff)}$: pro angulo $DZQ = \zeta$ fit $\sin.\zeta = -\frac{\varepsilon f}{k}$ et $\cos.\zeta = \frac{-e}{k}$; tum vero poſt tempus t prodit

$$v = \sqrt{\left(ee - \frac{4\delta eegt}{k} + 4\delta\delta ggtt\right)}$$

$$\text{tang.}\, \Phi = \frac{-2\delta \varepsilon ft}{e(k - 2\delta gt)};$$

$$X = t\left(1 - \frac{\delta gt}{k}\right)\varepsilon;$$

$$Y = \frac{-\delta \varepsilon fgtt}{k}.$$

tang.

$$\text{tang.}\, \xi = \frac{\varepsilon e f}{e e - 2\delta g k t};$$
$$\text{tang.}\, (\xi - \theta) = \frac{\varepsilon f}{e} - \frac{2\delta f g k t}{\varepsilon e a a};$$
$$\text{tang.}\, (\phi = \theta) = \frac{2\delta e f g t}{\varepsilon a a k - 2\delta e f f g t}:$$
$$\vartheta = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon\varepsilon f f g t}{a a k} + \frac{4\delta\delta f f g g t t}{a^4}\right)} \text{ et}$$
$$w = k - 2\delta g \left(1 + \frac{ff}{aa}\right) t.$$

Elapso autem tempore $t = \frac{a a k}{2\delta g (a a + f f)}$ erit tang. $\phi = \frac{\varepsilon a a}{e f}$;

$$v = \frac{f\sqrt{(e e f f + \varepsilon\varepsilon a^4)}}{aa + ff} = f\vartheta;\ \theta = 90^\circ \text{ et}$$
$$\text{tang.}\, \xi = \frac{\varepsilon e f (a a + f f)}{ee(aa + ff) - (a a k k)} = \frac{\varepsilon e (a a + f f)}{f(e e - \varepsilon\varepsilon a a)}.$$

Corollarium 2.

§. 27 Si angulus DZO = f esset = 180°, eaedem formulae motum indicabunt sumta celeritate angulari ϑ negatiua, seu motu gyratorio in contrarium verso. At si sit $\varepsilon = 0$, seu globo solus motus progressiuus fuerit impressus, fit $k = e$, $\zeta = 180^\circ$, $v = e - 2\delta g t$; $\phi = 0$, $X = t(e - \delta g t)$, $Y = 0$, $\xi = 180^\circ$; $\theta = 90^\circ$; $\vartheta = \frac{2\delta f g t}{a a}$, et elapso tempore $t = \frac{a a e}{2\delta g (a a + f f)}$ fit $v = \frac{e f f}{a a + f f}$, $\vartheta = \frac{e f}{a a + f f}$ et

$$X = \frac{\varepsilon t (a a + 2 f f)}{2 (a a + f f)} = \frac{a a e e (a a + 2 f f)}{4\delta g (a a + f f)}.$$

Scholion.

§. 27. Casus hic, quo globus initio nullum motum gyratorium est adeptus, in genere valet, neque ad vllam hypothesin angulorum f et h est adstrictus. Tum igitur globus in directum progreditur motu progressiuo retardato, motumque paullatim gyratorium accipiet, donec elapso tempore $t = \frac{a a e}{2\delta g (a a + f f)}$ motum vniformem acquirat, quo deinceps continuo progrediatur. Hinc deducimur

mur ad casum, quo globus initio motum tantum gyratorium acceperit, sine vllo motu progressiuo, cuius euolutio est facilis. Posito enim $e = 0$ erit $k = \varepsilon f \sin. \mathfrak{f}$, hincque fit $\sin. \zeta = -\cos. \mathfrak{h}$ et $\cos. \zeta = \sin. \mathfrak{h}$, ergo $\zeta = \mathfrak{h} - 90^\circ$, vbi pro axe gyrationis initio impressae IO est $ZO = \mathfrak{f}$ et $DZO = \mathfrak{h}$, existente celeritate angulari in sensum $ZETD = \varepsilon$. Elapso ergo tempore t fit $\Phi = \zeta$, scilicet sublato ab angulo $DZO = \mathfrak{h}$ angulo recto PZO, erit PI directio motus progressiui, quem globus acquiret, cuius celeritas erit $v = 2\delta g t$, ideoque tempori proportionalis. Tum vero erit $\tang. \xi = 0$ et $\tang. (\xi - \theta) = \infty$, ergo ob $\Phi + \xi = \zeta = \mathfrak{h} - 90^\circ$ erit $\xi = 0$ et $\theta = 90^\circ$, hinc $DZO = \zeta + 90^\circ = \mathfrak{h}$, ita vt polus gyrationis O in eodem perpetuo circulo verticali reperiatur. Denique ex § 22. est

$$\text{ȣ} \sin. s = \sqrt{\left(e e \sin. \mathfrak{f}^2 - \frac{4\delta \varepsilon f g t \sin. \mathfrak{f}}{a a} + \frac{4\delta\delta f f g g t t}{a^4}\right.} = \varepsilon \sin \mathfrak{f} - \frac{2\delta f g t}{a a})$$

et $\text{ȣ} \cos. s = \varepsilon \cos. \mathfrak{f}$, vnde fit

$$\tang. s = \tang. \mathfrak{f} - \frac{2\delta f g t}{\varepsilon a a \cos. \mathfrak{f}},$$

ita vt arcus ZO diminuatur, nisi fuerit quadrans vel eo maior, et

$$\text{ȣ} = \sqrt{\left(\varepsilon \varepsilon - \frac{4\delta \varepsilon f g t \sin. \mathfrak{f}}{a a} + \frac{4\delta\delta f f g g t t}{a^4}\right)}.$$

Motus autem ad vniformitatem reducetur elapso tempore

$t = \frac{\varepsilon a a f \sin. \mathfrak{f}}{2\delta g (a a + f f)}$; fitque tum

$$\text{ȣ} = \frac{\varepsilon \sqrt{(a^4 \sin. \mathfrak{f}^2 + (a a + f f)^2 \cos. \mathfrak{f}^2)}}{a a + f f};$$

$$v = \frac{\varepsilon a a f \sin. \mathfrak{f}}{a a + f f} \text{ et } \tang. s = \frac{a a \tang. \mathfrak{f}}{a a + f f}.$$

Si ergo fuisset $\mathfrak{f} = 0$, seu globo motus gyratorius circa axem verticalem impressus esset, sine vllo motu progressiuo, eundem motum sine vlla mutatione esset conseruaturus.

Problema

Problema VIII.

§. 28. *Si globo, in quo omnia momenta inertiae sunt aequalia, motus gyratorius fuerit impressus circa axem ad motus progressiui directionem normalem; definire continuationem motus.*

Solutio.

Cum motus progressiui initio impressi directio sit recta DIE, et celeritas $= e$, angulus $DZO = \mathfrak{h}$ est rectus, et sumto $ZO = \mathfrak{f}$ erat O polus circa quem initio globus accepit celeritatem angularem $= \varepsilon$ in sensum ZETD. Habemus ergo $k = \pm (e - f\varepsilon \sin. \mathfrak{f})$, vbi valorem positiuum pro k sumi oportet, ita vt hic duo prodeant casus seorsim euoluendi.

Casus primus

Sit $e > \varepsilon f \sin. \mathfrak{f}$, erit $k = e - \varepsilon f \sin. \mathfrak{f}$, quae est celeritas radens initio, eiusque directio IQ, ita vt sit $\sin. DQ = 0$ et $\cos. DQ = -1$, ideoque $DQ = \zeta = 180°$, et Q cadat in E globusque a frictione δM secundum directionem ID constanter retrahatur; vnde statim concluditur globi centrum I in eadem recta DE esse mansurum. Elapso ergo tempore t, ob $\cos. \zeta = -1$, fit celeritas centri $v = e - 2\delta g t$, et celeritas radens

$$w = e - \varepsilon f \sin. \mathfrak{f} - 2\delta g \left(1 + \frac{ff}{aa}\right) t;$$

tum vero $\Phi = 0$; $\xi = 180°$ atque $\theta = 0$. Quare pro axe gyrationis praesente IO est $DIO = 90°$, et posito arcu $ZO = s$ et celeritate angulari $= ȣ$ habemus $ȣ \cos. s = \varepsilon \cos. \mathfrak{f}$ et ex (§. 22.)

$$ȣ \sin. s = \varepsilon \sin. \mathfrak{f} + \frac{2\delta f g t}{aa},$$

vnde

vnde colligitur:

$$\text{tang.}\, s = \text{tang.}\, \mathfrak{f} + \frac{2\delta f g t}{\varepsilon a a \cos.\mathfrak{f}} \quad \text{et}$$

$$\vartheta = \sqrt{\left(\varepsilon\varepsilon + \frac{4\delta \varepsilon f g t \sin.\mathfrak{f}}{a a} + \frac{4\delta\delta f f g g t t}{a^4}\right)}.$$

Hoc autem tempore t percurrit centrum I lineam rectam $GX = X = t(e - \delta g t)$. Hic autem motus inaequabilis durabit per tempus $t = \frac{a a (e - \varepsilon f \sin.\mathfrak{f})}{2\delta g (a a + f f)}$, quo elapſo erit ſpatium

$$X = \frac{a a (e - \varepsilon f \sin.\mathfrak{f})(e(a a + f f) + \varepsilon a a f \sin.\mathfrak{f})}{2\delta g (a a + f f)^2}$$

et celeritas $v = \frac{f(\varepsilon a^2 f \sin.\mathfrak{f} + e f)}{a a + f f}$. At pro motu gyratorio fit

$$\text{tang.}\, s = \text{tang.}\, \mathfrak{f} + \frac{f(e - \varepsilon f \sin.f)}{\varepsilon(a a + f f)\cos.\mathfrak{f}} = \frac{e f + \varepsilon a a \sin.\mathfrak{f}}{\varepsilon(a a + f f)\cos.\mathfrak{f}},$$

(exiſtente $DIO = 90°$) et celeritas angularis:

$$\vartheta = \frac{\sqrt{(e e f f + 2\varepsilon e f a a \sin.\mathfrak{f} + \varepsilon\varepsilon a^4 \sin.\mathfrak{f}^2 - \varepsilon\varepsilon(a a + f f)^2 \cos.\mathfrak{f}^2)}}{a a + f f}.$$

Caſus ſecundus

Sit $e < \varepsilon f \sin.\mathfrak{f}$, ſeu $k = \varepsilon f \sin.\mathfrak{f} - e$, quae eſt celeritas radens initio, eiusque directio IQ talis, vt ſit $\sin. DQ = 0$, $\cos. DQ = 1$, ergo $DQ = \zeta = 0$, et Q in D cadat. Globus ergo a frictione δM ſecundum directionem IE conſtanter acceleratur, eiusque centrum I in eadem recta IE progreditur, atque elapſo tempore t erit celeritas $v = e + 2\delta g t$ et celeritas radens

$$w = \varepsilon f \sin.\mathfrak{f} - e - 2\delta g \left(1 + \frac{f f}{a a}\right) t.$$

Tum vero fit $\Phi = 0$ et $\xi = 0$ atque $\theta = 90°$. Quare pro axe gyrationis praeſenti IO eſt $DIO = 90°$ et poſito arcu $ZO = s$ et celeritate angulari $= \vartheta$ habebimus $\vartheta \cos. s = \varepsilon \cos.\mathfrak{f}$ et $\vartheta \sin. s = \varepsilon \sin.\mathfrak{f} - \frac{2\delta f g t}{a a}$, vnde fit

tang.

$$\text{tang.}\, s = \text{tang.}\, f - \frac{2\delta f g t}{\varepsilon a a \,\text{cof.}\, f} \quad \text{et}$$

$$ȣ = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon f g t \,\text{fin.}\, f}{a a} + \frac{4\delta\delta f f g g t t}{a^4}\right)},$$

hoc tempore t centrum globi percurrit lineam rectam $GX = X = t\,(e + \delta g t)$. Hic autem motus inaequabilis durabit tantum per tempus $t = \frac{a a\,(\varepsilon f \,\text{fin.}\, f - e)}{2\delta g\,(a a + f f)}$, quo elapſo erit celeritas $v = \frac{f\,(e f - \varepsilon a a \,\text{fin.}\, f)}{a a + f f}$ et ſpatium

$$X = \frac{a a\,(\varepsilon f \,\text{fin.}\, f - e)\,(e\,(a a + f f) + \varepsilon a a f \,\text{fin.}\, f)}{2\delta g\,(a a + f f)^2}.$$

At pro motu gyratorio reperitur

$$\text{tang.}\, s = \text{tang.}\, ZO = \frac{e f + \varepsilon a a \,\text{fin.}\, f}{\varepsilon\,(a a + f f)\,\text{cof.}\, f},$$

(exiſtente perpetuo $DIO = 90°$), et celeritas angularis

$$ȣ = \frac{\sqrt{(e e f f + 2\varepsilon e a a f \,\text{fin.}\, f + \varepsilon\varepsilon a^4 \,\text{fin.}\, f^2 + \varepsilon\varepsilon\,(a a + f f)^2 \,\text{cof.}\, f^2)}}{a a + f f}.$$

Corollarium 1.

§. 29. Si fuerit $e = \varepsilon f$ ſin. f, globus ſtatim ab initio motum proſequetur vniformem, tam progreſſiuum quam gyratorium, qui caſus limitem conſtituit inter binos tractatus.

Corollarium 2.

§. 30. Ad priorem caſum, quo $e > \varepsilon f$ ſin. f, referendi ſunt ii, quibus ε habet negatiuum valorem, ſeu globo impreſſus fuerit initio motus gyratorius in ſenſum ZDTE. Poſito autem $-\varepsilon$ loco ε, fieri poteſt vt globus reuertatur, antequam ad vniformitatem peruenerit.

Corollarium 3.

§. 31. Caſu hoc quo ε negatiue capitur, ad tempus t habebimus, $\Phi = 0$, $\theta = 0$, $\xi = 180°$, $v = e - 2\delta g t$,

$w =$

$$w = e + \varepsilon f \text{ſin.}\, \mathfrak{f} - 2\delta g\left(1 + \frac{ff}{aa}\right) t;$$

$$\text{tang.}\, s = \text{tang.}\, \mathfrak{f} - \frac{2\delta f g t}{\varepsilon a a \text{ coſ.}\, \mathfrak{f}} \text{ et}$$

$$\vartheta = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta \varepsilon f g t \text{ ſin.}\, \mathfrak{f}}{aa} + \frac{4\delta\delta ff g g t t}{a^4}\right)};$$

at poſt tempus $t = \frac{aa(e + \varepsilon f \text{ ſin.}\, \mathfrak{f})}{2\delta g (aa + ff)}$, percurſo ſpatio

$$X = \frac{aa(e + \varepsilon f \text{ ſin.}\, \mathfrak{f})(e(aa + 2ff) - \varepsilon a^2 f \text{ ſin.}\, \mathfrak{f})}{2\delta g (aa + ff)^2},$$

globi motus vniformitatem attinget, eritque tum

$$v = \frac{f(ef - \varepsilon a a \text{ ſin.}\, \mathfrak{f})}{aa + ff};$$

$$\text{tang.}\, s = \frac{\varepsilon a a \text{ ſin.}\, \mathfrak{f} - ef}{\varepsilon (aa + ff) \text{ coſ.}\, \mathfrak{f}} \text{ et}$$

$$\vartheta = \frac{\sqrt{(eeff - 2\varepsilon e a a f \text{ ſin.}\, \mathfrak{f} + \varepsilon\varepsilon a^4 \text{ ſin.}\, \mathfrak{f}^2 + \varepsilon\varepsilon (aa + ff)^2 \text{ coſ.}\, \mathfrak{f}^2)}}{aa + ff}.$$

Scholion.

§. 32. Caſus hic praecipue eſt memorabilis, quo globo eiusmodi motus imprimi poteſt, vt primo recedat, mox autem iterum reuertatur, quod experimento oſtendi ſolet, dum digito ad globum circa D applicato et deorſum preſſo duplex motus globo imprimitur, alter progreſſiuus in directione DIE, alter gyratorius in ſenſum ZDTE. Sed vt phaenomenon ſuccedat, neceſſe eſt vt celeritas angularis prae progreſſiua certum quendam limitem excedat, quem quo facilius agnoſcamus, calculum ad iſtum caſum accommodemus quo motus gyratorius globo circa axem horizontalem et ad directionem motus progreſſiui normalem imprimitur. Quod ſi ergo e denotet celeritatem progreſſiuam ſecundum directionem DIE, et ε celeritatem angularem retrogyrantem in ſenſum ZDTE, exiſtente f radio globi et Maa eius momento inertiae, frictioneque $= \delta M$; primo globus in directione DIE procedet, et elapſo tempore t eius celeritas ſecundum eandem directionem erit $v = e - 2\delta g t$, confecto ſpatio

X =

$X = t(e - \delta g t)$: tum vero etiamnunc circa eundem axem retrouoluetur celeritate angulari $\vartheta = \varepsilon - \frac{2\delta f g t}{aa}$. Motus autem aequabilis euadit elapſo tempore $t = \frac{aa(e + \varepsilon f)}{2\delta g(aa + ff)}$, eritque tum celeritas progreſſiua $v = \frac{f(ef - \varepsilon a a)}{aa + ff}$ et angularis $\vartheta = \frac{\varepsilon a a - e f}{aa + ff}$. Quare ſi fuerit $\varepsilon > \frac{ef}{aa}$, globus nunc retro mouetur, gyratorio motu adhuc retro vergente: ſin autem fuerit $\varepsilon < \frac{ef}{aa}$, globus adhuc procedit, et gyratio in ſenſum contrarium eſt verſa. Illo caſu globus regredi coepit elapſo tempore $t = \frac{e}{2\delta g}$ et percurſo ſpatio $X = \frac{ee}{4\delta g}$.

Si globus ſit homogeneus, erit $aa = \frac{2}{5}ff$ et ef exprimit celeritatem gyrationis in puncto contactus, quae ſi vocetur $= h$, erit poſt tempus t celeritas progreſſiua $v = e - 2\delta g t$, et gyratoria in puncto contactus, quae ſit $u = h - 5\delta g t$, et ſpatium percurſum $= t(e - \delta g t)$: motus vero aequabilis euadet elapſo tempore $t = \frac{e + h}{7\delta g}$, et confecto ſpatio $= \frac{(6e - h)(e + h)}{49\delta g}$, vbi erit celeritas progreſſiua $v = \frac{5e - 2h}{7}$ et gyratoria $u = \frac{2h - 5e}{7}$. Vt ergo phoenomenon memoratum ſuccedat, debet eſſe initio $h > \frac{5}{2}e$. Sin autem eſſet $h = \frac{5}{2}e$, vterque motus ſimul extingueretur, elapſo ſcilicet tempore $\frac{e}{2\delta g}$ min. ſec. et confecto ſpatio $\frac{ee}{4\delta g}$.

Concluſiones.

Pro determinatione motus, quo globus quomodocunque impulſus ſuper plano horizontali progreditur.

I. *Status quaeſtionis.* Globus hic ita comparatus ſupponitur, vt non ſolum eius centrum grauitatis in ip-

ſum

ſum figurae centrum incidat, ſed etiam omnia momenta inertiae reſpectu cuiusque diametri inter ſe ſint aequalia. Talis globi radius hic ponitur $= f$, eiusque maſſa ſeu pondus $= M$ et momentum inertiae reſpectu axis cuiuscunque per centrum grauitatis tranſeuntis $= Maa$, ita vt, ſi globus ex materia homogenea conſtet, futurum ſit $aa = \frac{2}{5}ff$. Praeterea vero tam ipſum planum horizontale quam tota globi ſuperficies ita aequaliter laeuigata aſſumitur, vt dum globus ſuper plano radendo ingreditur, vbique eandem frictionem patiatur, quae, cum preſſioni ſeu ipſi ponderi globi ſit proportionalis, hic ſtatuitur $= \delta M$.

Tab. IV. Fig. 1. II. *Status initialis.* Ponamus globum initio in puncto D plano inſiſtere eique motum progreſſiuum ſecundum directionem DO eſſe impreſſum cum ea celeritate, vt globus vno minuto ſecundo ſpatium $= e$ eſſet percurſurus, quae celeritas non tam puncto contactus D quam centro globi impreſſa eſt intelligenda. Tum vero referat circulus ABCD ſectionem verticalem globi ſecundum directionem DO factam, qui ſimul hemiſphaerium globi conuexum nobis obuerſum referat, in quo ſit E polus, circa quem globo motus gyratorius initio fuit impreſſus, cuius celeritas angularis in ſenſum ABCD vergens ſit $= \varepsilon$, ita vt ε deſignet angulum vno minuto ſecundo abſoluendum. Pro ſitu autem huius puncti E ſit B punctum globi ſummum, puncto contactus D diametraliter oppoſitum, vnde per polum E agatur circulus maximus BE et vocetur arcus $BE = \mathfrak{f}$ et angulus $ABE = \mathfrak{h}$; quibus ergo poſitis tota vis viua globo initio impreſſa erit $= M(ee + \varepsilon\varepsilon aa)$. Poſtquam igitur globo talis duplex motus fuerit impreſſus, quaeritur quomodo is deinceps ſit pro-

progressurus; ac primo quidem statim duos casus notasse iuuabit, quibus globus eundem motum impressum perpetuo esset conseruaturus: Alter scilicet casus tum locum habebit, quando celeritas progressiua e fuerit nulla, simulque globus circa axem verticalem BD gyretur, ita vt hoc casu fuerit $BE = f = 0$; quia enim tum nulla adest frictio, globus perpetuo in eodem loco gyrari perget. Alter vero casus tum locum habet, quando axis gyrationis fuerit horizontalis ideoque angulus $\mathfrak{h} = 90°$, simul vero insuper $e = \varepsilon f \sin. f$, quandoquidem hoc casu frictio pariter cessat. Reliquis autem casibus omnibus globus ab initio per aliquod tempus motu inaequabili feretur, dum tam motus progressiuus quam gyratorius continuo variabitur, hocque temporis interuallum repertum est

$$= \frac{aa\sqrt{(ee - 2\varepsilon ef \sin. f \sin. \mathfrak{h} + \varepsilon\varepsilon ff \sin. f^2)}}{2\delta g (aa + ff)},$$

in minutis secundis expressum, siquidem g denotet altitudinem, per quam grauia vno minuto secundo delabuntur. Vnde vniuersus globi motus sponte in duas partes distinguitur, quarum priore motus erit inaequabilis, posteriore vero aequabilis.

III. *Determinatio partis prioris.* Elapsum nunc sit ab initio tempus quodcunque indefinitum $= t$ in minutis secundis expressum, quod autem minus sit quam limes modo assignatus

$$\frac{aa\sqrt{(ee - 2\varepsilon ef \sin. f \sin. \mathfrak{h} + \varepsilon\varepsilon ff \sin. f^2)}}{2\delta g (aa + ff)},$$

hocque tempore tangat globus planum horizontale in puncto T, ex quo ad rectam fixam DO ducatur normalis TX vocenturque coordinatae $DX = X$, $XT = Y$, pro linea curua DT, per quam punctum contactus hucusque processit,

ſit, ita vt centrum grauitatis globi ſimilem viam deſcripſiſſe ſit cenſendum. Tum vero ponatur angulus quo elementum T t ad directionem D O inclinatur $= \Phi$, vt ſit tang. $\Phi = \frac{dY}{dX}$, ipſa autem celeritas, qua centrum globi hoc momento ſecundum T t ingreditur, vocetur tantisper $= v$, eritque $\frac{dX}{dt} = v$ coſ. Φ et $\frac{dY}{dt} = v$ ſin. Φ. Hos autem valores demum ex motu gyratorio, qui nunc globo conuenit, determinari oportet, quos mox exhibebimus, poſtquam ſcilicet motum gyratorium fuerimus contemplati. Hunc in finem ſecetur iterum globus plano in T inſiſtens, plano verticali M Z N T, illi quod in ſtatu initiali conſiderauimus parallelo, ita vt circulus M Z N T hemiſphaerium nunc

Tab IV. Fig. 5. nobis obuerſum repraeſentet, in quo punctum O ſit polus, circa quem globus nunc gyratur in ſenſum M Z N T celeritate angulari $= \varepsilon$. His poſitis iſta motus determinatio ita ſuccincte proponi poterit: Ex elementis ad ſtatum initialem pertinentibus colligatur angulus ζ, vt ſit

$$\text{tang.}\,\zeta = \frac{\varepsilon f \,\text{ſin.}\,\mathfrak{f}\,\text{coſ.}\,\mathfrak{h}}{e - \varepsilon f\,\text{ſin.}\,\mathfrak{f}\,\text{ſin.}\,\mathfrak{h}},$$

Tab. IV. Fig. 4. ex eoque ſtatim pro motu progreſſiou oritur

$$\text{D X} = \text{X} = c\,t + \delta\,g\,t\,t\,\text{coſ.}\,\zeta,$$

$$\text{T X} = \text{Y} = \delta\,g\,t\,t\,\text{ſin.}\,\zeta,$$

vnde colligitur celeritas ſecundum DX $= \frac{dX}{dt} = e + 2\delta g t\,\text{coſ.}\,\zeta$ et celeritas ſecundum X T, ſiue $\frac{dY}{dt} = 2\delta g t\,\text{ſin.}\,\zeta$, hincque fit tang. $\Phi = \frac{2\delta g t\,\text{ſin.}\,\zeta}{e + 2\delta g t\,\text{ſin.}\,\zeta}$, et ipſa celeritas progreſſiua

$$v = \sqrt{(e e + 4\delta e g t\,\text{coſ.}\,\zeta + 4\delta\delta g g t t)}.$$

Pro motu autem gyratorio circa polum O quaeratur angulus η, vt ſit

$$\text{tang.}\,\eta = \text{tang.}\,(\zeta - \mathfrak{h}) + \frac{2\delta f f g t}{e a a\,\text{ſin.}\,\zeta},$$

hinc-

hincque porro quantitas $q = \frac{e \sin. \zeta}{f \cos. \eta \, \xi}$, eritque pro distantia huius poli O a puncto globi summo Z, tang. $ZO = \frac{q}{\varepsilon \cos. \mathfrak{f}}$, ipsa vero celeritas angularis $\vartheta = \sqrt{(qq + \varepsilon\varepsilon \cos. \mathfrak{f}^2)}$, denique erit angulus $MZO = \zeta - \eta$. Sicque omnia quae ad motus determinationem requiruntur sunt definita.

IV. *Determinatio partis posterioris.* Iam notauimus motum aequabilem incipere elapso tempore

$$t = \frac{aa\sqrt{(ee - 2e\varepsilon f \sin. \mathfrak{f} \sin. \mathfrak{h} + \varepsilon\varepsilon ff \sin. \mathfrak{f}^2)}}{2\delta g (aa + ff)}.$$

Quod si ergo hic valor loco t substituatur, coordinatae X et Y dabunt punctum in curua DT, quod sit K, vbi motus aequabilis incipiet. Introducto autem angulo ζ erit tempus illud $t = -\frac{aa\varepsilon f \sin. \mathfrak{f} \cos. \mathfrak{h}}{2\delta g (aa + ff) \sin. \zeta}$, vbi notetur sin. ζ esse negatiuum. Cum igitur corpus vsque ad punctum K peruenerit, erit eius celeritas progressiua in directione DT

Tab IV. Fig. 4.

$$= e - \frac{aa\varepsilon f \sin. \mathfrak{f} \cos. \mathfrak{h} \cos. \zeta}{(aa + ff) \sin. \zeta} = \frac{eff - aa\varepsilon f \sin. \mathfrak{f} \sin. \mathfrak{h}}{aa + ff},$$

et celeritas in directione LK

$$= -\frac{aa\varepsilon f \sin. \mathfrak{f} - e \cos. \mathfrak{h}}{aa + ff}.$$

Pro motu autem gyratorio deinceps sequente habebimus primo

$$\text{tang.}\, \eta = \text{tang.}\, (\zeta - \mathfrak{h}) - \frac{f^3 \varepsilon \sin. \mathfrak{f} \cos. \mathfrak{h}}{e (aa + ff) \sin. \zeta^2},$$

$$\text{tang.}\, \eta = -\frac{e(aa + ff) - aak}{e (aa + ff) \text{tang.}\, \zeta},$$

vnde innotescit pro nostro tempore angulus $MZO = \zeta - \eta$. Porro vero pro eodem tempore, vbi contactus fit in puncto K, celeritas centri inuenta est

$$v = \sqrt{\left(ee + \frac{2aaek \cos. \zeta}{aa + ff} + \frac{a^4 kk}{(aa + ff)^2}\right)},$$

inclinatio autem directionis motus in K ad rectam fixam

DO, quae in genere erat Φ, nunc fiet

$$\text{tang.}\,\Phi = \frac{aak\,\text{sin.}\,\zeta}{e(aa+ff)+aak\,\text{cos.}\,\zeta},$$

vnde cognoscitur motus progressiuus, quo globus post hoc tempus vniformiter progredietur. Pro polo autem gyrationis O iam vidimus esse angulum $MZO = \zeta - \eta$; praeterea vero inuenimus tang. $ZO = \frac{v}{\varepsilon f\,\text{cos.}\,\mathfrak{f}}$, et ipsam celeritatem gyratoriam:

$$\text{ȣ} = \frac{\sqrt{(eeff + 2e\varepsilon aaf\,\text{sin.}\,\mathfrak{f}\,\text{sin.}\,\mathfrak{h} + \varepsilon\varepsilon a^4\,\text{sin.}\,\mathfrak{f}^2 + \varepsilon\varepsilon(aa+ff)^2\,\text{cos.}\,\mathfrak{f}^2)}}{aa+ff}$$

hunc ergo motum gyratorium globus posthac perpetuo conseruabit. Cum autem globus ad hanc vniformitatem peruenerit, erit eius vis viua

$$= \frac{M(eeff + 2\varepsilon eaaf\,\text{sin.}\,\mathfrak{f}\,\text{sin.}\,\mathfrak{h} + \varepsilon\varepsilon aa(aa+ff)\,\text{cos.}\,\mathfrak{f}^2)}{aa+ff},$$

quae deficit ab initiali quantitate

$$\frac{Maa(ee - 2\varepsilon ef\,\text{sin.}\,\mathfrak{f}\,\text{sin.}\,\mathfrak{h} + \varepsilon\varepsilon ff\,\text{sin.}\,\mathfrak{f}^2)}{aa+ff} = \frac{Maakk}{aa+ff},$$

vbi breuitatis gratia posuimus

$$kk = ee - 2\varepsilon ef\,\text{sin.}\,\mathfrak{f}\,\text{sin}\,\mathfrak{h} + \varepsilon\varepsilon ff\,\text{sin.}\,\mathfrak{f}^2.$$

Pro hoc motu vniformi notasse iuuabit fore angulum $MZO = \zeta - \eta = 90^\circ + \Phi$, tum vero $v = f\,\text{ȣ}\,\text{sin.}\,s$, quibus ergo formulis conditio motus aequabilis continetur.

Additamentum.

Praeterea casus hic imprimis notatu dignus videtur quo globo initio nullus plane motus progressiuus fuit impressus, ita vt sit $e = 0$; tum enim erit tang. $\zeta = -\,\text{cot.}\,\mathfrak{h}$ ideoque $\zeta = 90^\circ + \mathfrak{h}$ et $k = \varepsilon f\,\text{sin.}\,\mathfrak{f}$: tum vero pro via descripta erit $X = \delta gtt\,\text{cos}\,\zeta$ et $Y = \delta gtt\,\text{sin.}\,\zeta$; vnde patet, hanc viam esse lineam rectam ad axem DO sub angulo

gulo $= \zeta$ inclinatam. Praeterea vero erit

$$v \cos. \Phi = 2\delta g t \cos. \zeta \text{ et } v \sin. \Phi = 2\delta g t \sin. \zeta,$$

vnde colligitur tang. $\Phi =$ tang. ζ ideoque $\Phi = \zeta = 90° + \mathfrak{h}$, tum vero ipsa celeritas $v = 2\delta g t$. Deinde pro motu gyratorio erit tang. $\eta =$ tang $(\zeta - \mathfrak{h}) + \frac{2\delta f f g t}{e a a \sin. \zeta}$, vbi quia $\zeta - \mathfrak{h} = 90°$, erit tang. $\eta = \infty$, ideoque $\eta = 90°$, consequenter angulus $MZO = \zeta - \eta = \mathfrak{h}$; vnde patet, polum gyrationis O perpetuo in eodem circulo verticali manere. Cum igitur sit

$$\vartheta \cos. s = \varepsilon \cos. f \text{ et } \vartheta \sin. s = \frac{2\delta g t}{e},$$

colligitur

$$\text{tang. } s = \frac{2\delta g t}{\varepsilon f \cos. \mathfrak{f}} \text{ et } \vartheta = \sqrt{\left(\varepsilon \varepsilon \cos. \mathfrak{f} + \frac{4\delta\delta g g f f t t}{a^4}\right)};$$

hincque motus inaequabilis durabit per temporis spatium $t = \frac{\varepsilon a a f \sin. \mathfrak{f}}{2\delta g (a a + f f)}$, quo elapso erit $v = \frac{\varepsilon a a f \sin. \mathfrak{f}}{a a + f f}$, manente $\Phi = \zeta = 90° + \mathfrak{h}$. Porro etiamnunc erit angulus $MZO = \mathfrak{h}$, at tang. $s = \frac{a a}{a a + f f}$ tang. $\mathfrak{f}$ et

$$\vartheta = \frac{\varepsilon \sqrt{(a^4 \sin. \mathfrak{f}^2 + (a a + f f)^2 \cos. \mathfrak{f}^2)}}{a a + f f}.$$