



1786

Speculationes super formula integrali $\int (x^n dx) / \sqrt{aa - 2bx + cxx}$, ubi simul egregiae observationes circa fractiones continuas occurrunt

Leonhard Euler

Follow this and additional works at: <https://scholarlycommons.pacific.edu/euler-works>

Record Created:

2018-09-25

Recommended Citation

Euler, Leonhard, "Speculationes super formula integrali $\int (x^n dx) / \sqrt{aa - 2bx + cxx}$, ubi simul egregiae observationes circa fractiones continuas occurrunt" (1786). *Euler Archive - All Works*. 606.
<https://scholarlycommons.pacific.edu/euler-works/606>

This Article is brought to you for free and open access by the Euler Archive at Scholarly Commons. It has been accepted for inclusion in Euler Archive - All Works by an authorized administrator of Scholarly Commons. For more information, please contact mgibney@pacific.edu.

SPECVLATIONES
SVPER
FORMVLA INTEGRALI

$$\int \frac{x^n dx}{\sqrt{(aa - 2bx + cxx)}}$$

VBI SIMVL EGREGIAE OBSERVATIONES CIRCA
FRACTIONES CONTINVAS OCCVRRVNT.

Auctore

L. E V L E R O.

§. I.

Incipiamus a casu simplicissimo, quo $n = 0$, et quaeramus integrale formulae $\frac{dx}{\sqrt{aa - 2bx + cxx}}$, quae posito $x = \frac{b+z}{c}$ transit in hanc: $\frac{dz}{\sqrt{(aacc - bbc + czz)}}$, vbi duo casus distingui conuenit, prout c fuerit vel quantitas positua vel negatiua. Sit igitur primo $c = +ff$ et formula nostra fiet $\frac{dz}{f\sqrt{(aaff - bb + zz)}}$, cuius integrale est $\frac{1}{f} \int \frac{z + \sqrt{(aaff - bb + zz)}}{c}$, ideoque erit nostrum integrale

$$\frac{1}{\sqrt{c}} \int \frac{cx - b + \sqrt{(aac - 2bcx + cxx)}}{c}$$

quod ergo, ita sumtum vt enanescat posito $x = 0$, euadet

$$\frac{1}{\sqrt{c}} \int \frac{cx - b + \sqrt{c(aa - 2bx + cxx)}}{-b + a\sqrt{c}}$$

At vero si c fuerit quantitas negatiua, puta $c = -gg$, formula differentialis per z expressa erit $\frac{dz}{g\sqrt{(aagg + bb - zz)}}$

cu-

cuius integrale est $\frac{1}{g} A \sin. \frac{x}{\sqrt{(aag + bb)}} + C$; quare integrale ita sumtum vt euanescat posito $x = 0$ fiet

$$= \frac{1}{g} A \sin. \frac{cx - b}{\sqrt{(aagg + bb)}} + \frac{1}{g} A \sin. \frac{b}{\sqrt{(aagg + bb)}}.$$

§. 2. Denotet nunc Π valorem formulae integralis $\int \frac{dx}{\sqrt{(aa - 2bx + cxx)}}$ ita sumtum vt euanescat posito $x = 0$, siue c fuerit quantitas positua siue negatiua, ac si sit $c = +ff$ erit vti vidimus

$$\Pi = \frac{1}{f} \int \frac{ffx - b + \sqrt{aa - 2bx + ffx}}{af - b};$$

altero vero casu, quo $c = -gg$, erit

$$\Pi = -\frac{1}{g} A \sin. \frac{g gx + b}{\sqrt{(aagg + bb)}} + \frac{1}{g} A \sin. \frac{b}{\sqrt{(aagg + bb)}}$$

siue ambobus arcubus contractis habebimus

$$\Pi = \frac{1}{g} A \sin. \frac{bg \sqrt{(aa - 2bx - g gx)} - abg - ag^2 x}{aagg + bb}.$$

Quoniam igitur mox ostendemus integrationem formulae generalis $\int \frac{x^n dx}{\sqrt{(aa - 2bx + cxx)}}$ semper reduci posse ad casum $n = 0$, si modo fuerit n numerus integer posituius, omnia haec integralia per istum valorem Π exprimi poterunt.

§. 3. Iam post integrationem quantitati variabili x eiusmodi valorem constantem tribuamus, quo formula irrationalis $\sqrt{(aa - 2bx + cxx)}$ ad nihilum redigatur, id quod fit si sumatur $x = \frac{b \pm \sqrt{(bb - aac)}}{c}$, ideoque duobus casibus. Ponamus pro utroque casu functionem Π abire in Δ , ita vt casu $c = +ff$ sit

$$\Delta = \frac{1}{f} \int \frac{\sqrt{(bb - aaff)}}{af - b} = \frac{1}{f} \int \sqrt{\frac{b + af}{b - af}};$$

pro altero autem casu, quo $c = -gg$

$$\Delta = \frac{1}{g} A \sin. \frac{+ag \sqrt{(bb + aagg)}}{aagg + bb} = \frac{1}{g} A \sin. \frac{ag}{\sqrt{(bb + aagg)}}.$$

Hos

Hos autem valores Δ in sequentibus casibus, quibus ipsa formula radicalis $\sqrt{aa - 2bx + cxx}$ evanescit, potissimum sumus contemplaturi.

§. 4. Nunc ad sequentem casum progressuri, consideremus formulam $s = \sqrt{aa - 2bx + cxx} - a$, ut scilicet evanescat facto $x = 0$, et quoniam est

$$ds = \frac{-bdx + cxdx}{\sqrt{aa - 2bx + cxx}}$$

erit vicissim integrando

$$c \int \frac{xdx}{\sqrt{aa - 2bx + cxx}} = b \int \frac{dx}{\sqrt{aa - 2bx + cxx}} + s$$

vnde colligimus

$$\int \frac{xdx}{\sqrt{aa - 2bx + cxx}} = \frac{b}{c} \Pi + \frac{\sqrt{aa - 2bx + cxx} - a}{c},$$

quare si post integrationem statuamus $x = \frac{b \pm \sqrt{bb - aac}}{c}$ quippe quibus casibus fit $\sqrt{aa - 2bx + cxx} = 0$ et $\Pi = \Delta$; fiet

$$\int \frac{xdx}{\sqrt{aa - 2bx + cxx}} = \frac{b}{c} \Delta - \frac{a}{c}.$$

§. 5. Sumamus porro $s = x \sqrt{aa - 2bx + cxx}$ fiet $ds = \frac{aadx - 2bx dx + 2cxdx}{\sqrt{aa - 2bx + cxx}}$, vnde vicissim integrando colligitur

$$2c \int \frac{xx dx}{\sqrt{aa - 2bx + cxx}} = 3b \int \frac{xdx}{\sqrt{aa - 2bx + cxx}} - a \int \frac{dx}{\sqrt{aa - 2bx + cxx}} + s$$

vnde statim pro casu $\sqrt{aa - 2bx + cxx} = 0$ deducimus

$$\int \frac{xx dx}{\sqrt{aa - 2bx + cxx}} = \frac{(3bb - aac)}{2cc} \Delta - \frac{3ab}{2cc}.$$

§. 6. Iam ad altiores potestates ascensuri statuamus

mus $s = xx \sqrt{(aa - 2bx + cxx)}$, et quia hinc fit

$$ds = \frac{2ax dx - 2bx dx + 2cx dx}{\sqrt{(aa - 2bx + cxx)}}, \text{ erit}$$

$$3c \int \frac{x^3 dx}{\sqrt{(aa - 2bx + cxx)}} = 5b \int \frac{xx dx}{\sqrt{(aa - 2bx + cxx)}} - 2aa \int \frac{xdx}{\sqrt{(aa - 2bx + cxx)}} + s,$$

hincque porro pro casu quo post integrationem statuitur

$x = \frac{b + \sqrt{(bb - cxx)}}{c}$ habebitur

$$\begin{aligned} \int \frac{x^3 dx}{\sqrt{(aa - 2bx + cxx)}} &= \left(\frac{5b^3 - 3abc}{2c^3} \right) \Delta - \frac{15abb}{6c^3} + \frac{2x^4}{3c} \\ &= \left(\frac{5b^3}{2c^3} - \frac{3aab}{2cc} \right) \Delta - \frac{5abb}{2c^3} + \frac{2x^4}{3cc} \end{aligned}$$

§. 7. Simili modo fit $s = x^2 \sqrt{(aa - 2bx + cxx)}$, et quia hinc fit

$$ds = \frac{2ax dx - 2bx dx + 2cx dx}{\sqrt{(aa - 2bx + cxx)}}$$

erit vicissim integrando

$$4c \int \frac{x^4 dx}{\sqrt{(aa - 2bx + cxx)}} = 7b \int \frac{x^3 dx}{\sqrt{(aa - 2bx + cxx)}} - 3aa \int \frac{xx dx}{\sqrt{(aa - 2bx + cxx)}} + s;$$

tum igitur pro casu quo fit $\sqrt{(aa - 2bx + cxx)} = 0$, habebimus

$$\int \frac{x^4 dx}{\sqrt{(aa - 2bx + cxx)}} = \left(\frac{35b^4}{8c^4} - \frac{15aab^3}{4c^3} + \frac{5a^4}{8cc} \right) \Delta - \frac{55ab^3}{8c^4} + \frac{55a^3b}{24c^3}$$

§. 8. Quo autem ordo in his formulis melius explorari possit, singulas exhibeamus per factores, quemadmodum ordine oriuntur, sine vlla abbreviacione, atque hęc modo formulę integrales inuentę ita repręsententur:

$$\int \frac{dx}{\sqrt{(aa - 2bx + cxx)}} = \Delta$$

$$\int \frac{xx dx}{\sqrt{(aa - 2bx + cxx)}} = \frac{b}{c} \Delta - \frac{a}{c}$$

$$\int \frac{xx dx}{\sqrt{(aa-2bx+cx^2)}} = \left(\frac{1.3bb}{1.2cc} - \frac{aa}{1.2c} \right) \Delta - \frac{1.3.ab}{1.2.cc}$$

$$\int \frac{x^2 dx}{\sqrt{(aa-2bx+cx^2)}} = \left(\frac{1.3.5b^2}{1.2.3c^2} - \frac{1.3.3aab}{1.2.3cc} \right) \Delta - \frac{1.3.5abb}{1.2.3c^2} + \frac{1.2.2a^2}{1.2.3cc}$$

$$\int \frac{x^4 dx}{\sqrt{(aa-2bx+cx^2)}} = \left(\frac{1.3.5.7b^4}{1.2.3.4c^4} - \frac{1.3.5.6aab}{1.2.3.4c^4} + \frac{1.3.3a^4}{1.2.3.4cc} \right) \Delta - \frac{1.3.5.7ab^2}{1.2.3.4c^4} + \frac{1.5.11a^2b}{1.2.3.4c^2}$$

§. 9. Instituamus nunc in genere istam euolutionem, sumendo $s = x^n \sqrt{(aa - 2bx + cx^2)}$ et quia hinc fit

$$ds = \frac{naax^{n-1} dx - (2n+1)bx^n dx + (n+1)cx^{n+1} dx}{\sqrt{(aa - 2bx + cx^2)}}$$

inde vicissim integrando colligitur

$$(n+1)c \int \frac{x^{n+1} dx}{\sqrt{(aa - 2bx + cx^2)}} = (2n+1)b \int \frac{x^n dx}{\sqrt{(aa - 2bx + cx^2)}} - naa \int \frac{x^{n-1} dx}{\sqrt{(aa - 2bx + cx^2)}} + x^n \sqrt{(aa - 2bx + cx^2)}$$

Quod si vero iam ante elicuerimus

$$\int \frac{x^{n-1} dx}{\sqrt{(aa - 2bx + cx^2)}} = M \Delta - \mathfrak{M} \text{ et}$$

$$\int \frac{x^n dx}{\sqrt{(aa - 2bx + cx^2)}} = N \Delta - \mathfrak{N}$$

ita vt hae duae formulae sint cognitae, sequens ex iis ita determinabitur, vt fit

$$\int \frac{x^{n+1} dx}{\sqrt{(aa - 2bx + cx^2)}} = \left(\frac{(2n+1)bN}{(n+1)c} - \frac{naaM}{(n+1)c} \right) \Delta - \frac{(2n+1)b\mathfrak{N}}{(n+1)c} + \frac{naa\mathfrak{M}}{(n+1)c}$$

Hoc

Hoc igitur modo has integrationes quovsque libuerit continuare licet, dum ex binis quibusque sequens ope huius regulae formatur, ita ut omnia haec integralia vel a logarithmis vel ab arcubus circularibus pendeant, prouti coefficientis c fuerit vel positivus vel negativus. Manifestum autem est istos valores assignari non posse, nisi exponens n fuerit numerus integer positivus.

§. 10. Ex forma integrali modo inventa, si post integrationem statuatur $x = \frac{b + \sqrt{bb - aax}}{c}$, unde fit $s = 0$, erit

$$naa \int \frac{x^{n-1} dx}{\sqrt{(aa - 2bx + cxx)}} = (2n+1) b \int \frac{x^n dx}{\sqrt{(aa - 2bx + cxx)}} - (n+1) c \int \frac{x^{n+1} dx}{\sqrt{(aa - 2bx + cxx)}}$$

unde si brevitatis gratia ponamus

$$\int \frac{x^{n-1} dx}{\sqrt{(aa - 2bx + cxx)}} = P, \int \frac{x^n dx}{\sqrt{(aa - 2bx + cxx)}} = Q,$$

$$\int \frac{x^{n+1} dx}{\sqrt{(aa - 2bx + cxx)}} = R, \int \frac{x^{n+2} dx}{\sqrt{(aa - 2bx + cxx)}} = S, \text{ etc}$$

hae quantitates P, Q, R, S ita a se inuicem pendent ut fit

$$naaP = (2n+1)bQ - (n+1)cR;$$

$$(n+1)aaQ = (2n+3)bR - (n+2)cS;$$

$$(n+2)aaR = (2n+5)bS - (n+3)cT;$$

$$(n+3)aaS = (2n+7)bT - (n+4)cU;$$

$$(n+4)aaT = (2n+9)bU - (n+5)cW;$$

etc.

etc.

I 2

Ex

Ex his relationibus deducuntur sequentes determinationes:

$$\begin{aligned} \frac{P}{Q} &= \frac{(2n+1)b}{naa} - \frac{(n+1)c}{naaQ:R}; \\ \frac{Q}{R} &= \frac{(2n+3)b}{(n+1)aa} - \frac{(n+2)c}{(n+1)aaR:S}; \\ \frac{R}{S} &= \frac{(2n+5)b}{(n+2)aa} - \frac{(n+3)c}{(n+2)aaS:T}; \\ \frac{S}{T} &= \frac{(2n+7)b}{(n+3)aa} - \frac{(n+4)c}{(n+3)aaT:U}; \\ &\text{etc.} \qquad \qquad \text{etc.} \end{aligned}$$

hinc igitur patet, singulas has fractiones $\frac{P}{Q}$, $\frac{Q}{R}$, $\frac{R}{S}$, etc. per sequentes satis commode determinari.

§. 11. Quod si iam in qualibet harum expressionum valores modo exhibiti successive substituuntur, pro fractione $\frac{P}{Q}$ impetrabimus fractionem continuam in infinitum progredientem, quae erit:

$$naa \frac{P}{Q} = \frac{(2n+1)b - (n+1)^2aac}{(2n+3)b - (n+2)^2aac} \frac{(2n+3)b - (n+2)^2aac}{(2n+5)b - (n+3)^2aac} \frac{(2n+5)b - (n+3)^2aac}{(2n+7)b - (n+4)^2aac} \frac{(2n+7)b - (n+4)^2aac}{(2n+9)b - \text{etc.}}$$

ficque peruenimus ad fractionem continuam satis concinam et ordine perspicuo progredientem, cuius igitur valor semper vel per logarithmos (si fuerit $c > 0$), vel per arcus circulares (si fuerit $c < 0$) exprimi potest.

§. 12. Sumamus nunc $n = 1$ ac fiet

$$\begin{aligned} P &= \int \frac{dx}{\sqrt{(aa - 2bx + cxx)}} = \Delta \text{ et} \\ Q &= \int \frac{x dx}{\sqrt{(aa - 2bx + cxx)}} = \frac{b}{c} \Delta - \frac{a}{c}, \end{aligned}$$

qui casus nobis suppeditat sequentem fractionem continuam:

aac

$$\frac{\frac{aac\Delta}{b\Delta-a} = 3b-4aac}{5b-9aac} \\ \frac{7b-16aac}{9b-25aac} \\ 11b - \text{etc.}$$

quae ob elegantiam omni attentione digna est censenda. Hic autem notasse iuuabit, si c fuerit numerus negatiuus tum omnes numeratores in hac fractione euadere positiuos.

§. 12. Fractio autem haec continua capite quasi trunca videtur; vnde si superne ei adiungatur membrum $b - aac$, ea adhuc concinnior eiusque valor simplicior reddetur. Si enim ista fractio breuitatis gratia designetur littera S , ita vt sit $S = \frac{aac\Delta}{b\Delta-a}$, adiecto isto membro eius valor erit $b - \frac{aac}{S} = \frac{a}{\Delta}$; sicque habebimus

$$\frac{\frac{a}{\Delta} = b - aac}{3b-4aac} \\ \frac{5b-9aac}{7b-16aac} \\ \frac{9b-25aac}{11b - \text{etc.}}$$

quae expressio eo magis est memorabilis, quod nulla adhuc via patet, qua talis fractionis continuae valor a priori inueniri potest.

§. 13. Euoluamus nunc seorsim binos casus supra memoratos, et quos sollicite a se inuicem distingui conuenit.

nit. Sit igitur primo $c = ff$, atque supra inuenimus fore

$$\Delta = \frac{1}{f} l \frac{\sqrt{(bb - aaff)}}{b - af},$$

vbi signum radicale ambigüe accipi potest. Ante omnia igitur necesse est, vt sit $bb > aaff$, quia alioquin haec expressio euaderet imaginaria; duo ergo casus se offerunt, prouti b fuerit quantitas siue positiua siue negatiua. Priore casu, quo $b > 0$, atque adeo $b > af$, euidens est signo radicali tribui debere signum $-$, vt fiat

$$\Delta = \frac{1}{f} l \frac{\sqrt{(bb - aaff)}}{b - af} = \frac{1}{f} l \frac{b + af}{b - af},$$

et iam habebimus istam summationem

$$\frac{2af}{l \frac{b+af}{b-af}} = b - aaff$$

$$3b - 4aaff$$

$$5b - 9aaff$$

$$7b - 16aaff$$

$$9b - \text{etc.}$$

vnde cum sit $\frac{b+af}{b-af} > 1$, patet valorem huius expressionis fore positium.

§. 14. Sin autem fuerit b numerus negatiuus, siue si loco b scribatur $-b$, etiamnunc esse debet $b > af$; tum autem erit $\Delta = \frac{1}{f} l \frac{b-af}{b+af}$, qui ergo logarithmus erit negatiuus, siue $\Delta = -\frac{1}{f} l \frac{b+af}{b-af}$, vnde obtinebitur sequens aequatio:

*) 71 (

$$\frac{-2af}{\frac{b+af}{b-af}} = -b - aaff$$

$$\frac{-3b - 4aaff}{-5b - 9aaff}$$

$$\frac{-7b - 16aaff}{-9b - \text{etc.}}$$

$$\frac{-9b - \text{etc.}}$$

sive mutatis signis

$$\frac{2af}{\frac{b+af}{b-af}} = b + aaff$$

$$\frac{-3b + 4aaff}{5b + 9aaff}$$

$$\frac{-7b + 16aaff}{9b + \text{etc.}}$$

$$\frac{9b + \text{etc.}}$$

cuius ergo fractionis continuae summa aequalis est illi, quam in §. precedente inuenimus. Ista autem aequalitas harum duarum expressionum calculum facienti mox fiet manifesta.

§. 15. Eodem modo euoluamus casum quo $c = -gg$, pro quo supra inuenimus $\Delta = \frac{1}{g} A \sin. \frac{ag}{\sqrt{bb+aaagg}}$, qui valor per cosinum expressus dabit $\Delta = \frac{1}{g} A \cos. \frac{ag}{\sqrt{bb+aaagg}}$, unde patet, per tangentem istum valorem adhuc fore simpliciorum: fit scilicet $\Delta = \frac{1}{g} A \tan. \frac{ag}{b}$, quam ob rem pro hoc casu prodit ista summatio

ag

$$\frac{a g}{\text{A tang. } \frac{a g}{b}} = b + \frac{a a g g}{3 b + 4 a a g g} \\ \frac{5 b + 9 a a g g}{7 b + 16 a a g g} \\ \frac{9 b + \text{etc.}}$$

vbi nulla amplius limitatione est opus.

De fractionibus continuis a logarithmis pendentibus.

§. 16. Perpendamus nunc etiam aliquos casus speciales, in vtraque forma contentos, et quoniam iam obseruauimus, binas formas in §§. 13. et 14. inter se congruere, vtamur priori, qua erat

$$\frac{2 a f}{\frac{b+a f}{b-a f}} = b - \frac{a a f f}{3 b - 4 a a f f} \\ \frac{5 b - 9 a a f f}{7 b - \text{etc.}}$$

ac primo consideremus casum, quo $b = a f$, quippe quo euadit summa fractionis

$$\frac{2 a f}{\frac{b+a f}{b-a f}} = 0 = b - \frac{b b}{3 b - 4 b b} \\ \frac{5 b - 9 b b}{7 b - \text{etc.}}$$

quae

quae per reductionem facile mutatur in hanc:

$$0 = 1 - \frac{1}{3 - \frac{4}{5 - \frac{9}{7 - \frac{16}{9 - \text{etc.}}}}}$$

§. 17. In ista igitur forma nihilo aequali necesse est ut denominator primae fractionis sit = 1 ideoque

$$1 = 3 - \frac{4}{5 - \frac{9}{7 - \text{etc.}}} \quad \text{siue } 0 = 2 - \frac{4}{5 - \frac{9}{7 - \text{etc.}}}$$

Hic igitur ob eandem rationem necesse est ut prior denominator fiat = 2, ita ut

$$2 = 5 - \frac{9}{7 - \frac{16}{9 - \text{etc.}}} \quad \text{siue } 0 = 3 - \frac{9}{7 - \frac{16}{9 - \text{etc.}}}$$

Hic iterum primus denominator debet esse = 3 ideoque

$$3 = 7 - \frac{16}{9 - \frac{25}{11 - \text{etc.}}} \quad \text{siue } 0 = 4 - \frac{16}{9 - \frac{25}{11 - \text{etc.}}}$$

Denuo igitur primus denominator esse debet = 4, ita ut $4 = 9 - \frac{25}{11 - \text{etc.}}$ atque hoc modo patet, istam relationem

eodem ordine in infinitum locum habere, in quo ipso criterium veritatis huius aequationis est situm.

§. 18. Quoniam in hac forma numerus b maior esse debet quam af , statuamus nunc $b = 2af$, et nanciscemur sequentem summationem:

$$\frac{2af}{12} = 2af - \frac{aaff}{6af - 4aaff} \\ \frac{aaff}{10af - 9aaff} \\ \frac{aaff}{14af - \text{etc.}}$$

quae reducitur ad hanc formam mere numericam

$$\frac{2}{12} = 2 - \frac{1}{6 - 4} \\ \frac{1}{10 - 9} \\ \frac{1}{14 - 16} \\ \frac{1}{18 - \text{etc.}}$$

§. 19. Simili modo omnes litterae ex calculo expelli possunt, si pro b accipiatur multipulum ipsius af . Sit enim in genere $b = naf$, ac prodit

$$\frac{2af}{1 \frac{n+1}{n-2}} = naf - \frac{aaff}{3naf - 4aaff} \\ \frac{aaff}{5naf - 9aaff} \\ \frac{aaff}{7naf - \text{etc.}}$$

quae fractio reducitur ad formam sequentem:

$$\frac{2}{1 \frac{n+1}{n-2}} = n - 1 \\ \frac{1}{3n - 4} \\ \frac{1}{5n - 9} \\ \frac{1}{7n - \text{etc.}}$$

vnde

vnde intelligitur, quemadmodum omnes logarithmos per fractiones continuas exprimi conueniat.

§. 20. Possent hic pro n numeri fracti accipi, tum autem priores termini in singulis membris prodirent fracti, quas quidem per reductionem ad integros reuocare liceret: verum huiusmodi casus facillime ex forma generali deriuari possunt, scribendo statim $b = n$ et $a f = m$; tum enim habebimus

$$\frac{2 m}{\sqrt{\frac{n+m}{n-m}}} = n - \frac{m m}{n}$$

$$\frac{3 n - 4 m m}{5 n - 9 m m}$$

$$7 n - \text{etc.}$$

vnde si loco m scribatur \sqrt{k} , erit

$$\frac{2 \sqrt{k}}{\sqrt{\frac{n+\sqrt{k}}{n-\sqrt{k}}}} = n - k$$

$$\frac{3 n - 4 k}{5 n - 9 k}$$

$$7 n - \text{etc.}$$

§. 21. Hinc igitur omnium numerorum integrorum logarithmos hyperbolicos per fractiones continuas exprimere poterimus. Propositus igitur sit in genere numerus integer i , ac statuatur $\frac{n+m}{n-m} = i$, erit $\frac{n}{m} = \frac{i+1}{i-1}$. Capiatur ergo $n = i + 1$ et $m = i - 1$, atque habebimus

$$\frac{2(i-1)}{i} = i+1 - \frac{(i-1)^2}{3(i+1) - 4(i-1)^2} - \frac{(i-1)^2}{5(i+1) - 9(i-1)^2} - \frac{(i-1)^2}{7(i+1) - 16(i-1)^2} - \frac{(i-1)^2}{9(i+1) - \text{etc.}}$$

vnde colligimus

$$ii = \frac{2(i-1)}{i+1 - \frac{(i-1)^2}{3(i+1) - 4(i-1)^2} - \frac{(i-1)^2}{5(i+1) - 9(i-1)^2} - \text{etc.}}$$

§. 22. Si huiusmodi fractiones desideremus pro logarithmis numerorum fractorum, statuamus $\frac{n+m}{n-m} = \frac{p}{q}$, vnde fit $n = p+q$ et $m = p-q$, quamobrem habebimus

$$i \frac{p}{q} = \frac{2(p-q)}{1(p+q) - \frac{1(p-q)^2}{3(p+q) - 4(p-q)^2} - \frac{(p-q)^2}{5(p+q) - 9(p-q)^2} - \frac{(p-q)^2}{7(p+q) - \text{etc.}}}$$

quae forma eo magis est notatu digna, quod satis commode adhiberi potest ad logarithmos proxime inuestigandos. Eo magis autem istae fractiones continuae conuergent, quo minor fuerit fractio $\frac{p-q}{p+i}$.

§. 23. Quo hoc exemplo illustremus, fumamus $p = 2$ et $q = 1$, vnde quidem non adeo vehemens conuergentia est expectanda, eritque

$$i2 =$$

$$l_2 = \frac{2}{3-1} = \frac{2}{2} = 1$$

$$\frac{2}{9-4} = \frac{2}{5} = 0,4$$

$$\frac{2}{15-9} = \frac{2}{6} = 0,333333$$

$$21 - \text{etc.}$$

vnde sumendo tantum primum membrum $\frac{2}{3}$, in fractione decimali prodit 0,666666, dum ex tabulis habetur $l_2 = 0,693147$, vbi error iam satis est exiguus. Capiamus iam bina membra priora $\frac{2}{3-1} = \frac{2}{2} = 0,6923$. Sumendo

$$\frac{2}{3-1} = \frac{2}{2} = 1$$

autem tria membra habebimus

$$\frac{2}{3-1} = \frac{2}{2} = 1$$

$$\frac{2}{9-4} = \frac{2}{5} = 0,4$$

$$\frac{2}{15-9} = \frac{2}{6} = 0,333333$$

$$= \frac{262}{378} = 0,693121$$

qui valor a veritate deficit quantitate 0,000026. Multo promptior autem deprehendetur conuergentia, si sumamus $p = 3$ et $q = 2$, vt habeamus

$$l_{\frac{2}{3}} = \frac{2}{5-1} = \frac{2}{4} = 0,5$$

$$\frac{2}{15-4} = \frac{2}{11} = 0,181818$$

$$\frac{2}{25-9} = \frac{2}{16} = 0,125$$

$$35 - \text{etc.}$$

cuius primum membrum dat $\frac{2}{3} = 0,400000$; reuera autem est $l_{\frac{2}{3}} = 0,405465108$. Sumtis autem duobus membris colligitur $l_{\frac{2}{3}} = 0,40540$, vbi error tantum in

$$\frac{2}{5-1} = \frac{2}{4} = 0,5$$

quintam figuram irrepit. Sumantur tria membra

$$\frac{2}{5-1} = \frac{2}{5-\frac{25}{371}} = 0,405464$$

$$\frac{2}{15-4} = \frac{2}{25}$$

vbi error demum in septima figura se manifestat.

§. 24. Ob hunc insignem vsu, qui se praeter expectationem obtulit, operae pretium erit talem inuestigationem in genere expedire; atque in hunc finem vtatur formula inter litteras m et n supra §. 20. data, vbi fit

$$l \frac{n+m}{n-m} = \frac{2m}{n-m}$$

$$\frac{3n-4mm}{5n-9mm}$$

$$\frac{7n-16mm}{9n-\text{etc.}}$$

vnde si capiamus tantum primum membrum, fiet prope-
modum $l \frac{n+m}{n-m} = \frac{2m}{n}$; sumtis autem binis prioribus membris.

$\frac{2m}{n-m}$, erit iam propius $l \frac{n+m}{n-m} = \frac{6mn}{5n^2-9mm}$; sumtis ve-

$$\frac{2m}{n-m}$$

$$\frac{3n}{3n}$$

ro tribus membris erit

$$l \frac{n+m}{n-m} = \frac{2m}{n-m}$$

$$\frac{3n-4mm}{5n} = \frac{30mn-4m^2}{15n^2-9mm}$$

§. 25. Non adeo autem operose est has fractiones ulterius continuare: fractionibus enim iam inuentis præfigamus fractionem $\frac{2}{n}$, vt obtineamus hanc fractionum progressionem:

$$\text{I} \quad \text{II} \quad \text{III} \quad \text{IV}$$

$$\frac{2}{n}, \frac{2m}{n}, \frac{6mn}{3nn - mm}, \frac{30mnn - 8m^3}{15n^3 - 9mnn}$$

cuius tam numeratores quam denominatores ex binis præcedentibus, ad similitudinem ferierum recurrentium, formari possunt. Tertia scilicet ex prima et secunda formatur ope huius scalae relationis: $3n, -mm$; quarta vero formatur ex binis præcedentibus ope huius scalae relationis $5n, -4mm$. Pro quinta igitur vtendum erit hac scala: $7n, -9mm$, pro sexta hac: $9n, -16mm$; et ita porro. Hoc igitur modo facile reperitur fractio quinta.

$$\text{V}$$

$$= \frac{210m^3n^3 - 110m^3n}{105n^4 - 90mnn + 9m^4}$$

simili modo

$$\text{VI}$$

$$= \frac{1980m^4n^4 - 1470m^3nn + 128m^5}{145n^5 - 1050mnn^3 + 225m^4n} \text{ etc.}$$

§. 26. Hic autem imprimis notasse iuuabit, has fractiones continuo augeri, et per incrementa continuo minora ad veritatem accedere. Incrementa autem ista egregio ordine procedunt, vti videre hic licet

$$\text{II} - \text{I} = \frac{2m}{n};$$

$$\text{III} - \text{II} = \frac{2m^3}{n(3nn - mm)};$$

$$\text{IV} - \text{III} = \frac{2 \cdot 4 m^5}{(3nn - mm)(15n^3 - 9mnn)};$$

$$\text{V} - \text{IV} = \frac{2 \cdot 4 \cdot 9 \cdot 7}{(15n^4 - 9mnn)(105n^4 - 90mnn + 9m^4)};$$

$$\text{VI} - \text{V} = \frac{2 \cdot 4 \cdot 9 \cdot 16 \cdot m^9}{(105n^4 - 90mnn + 9m^4)(145n^5 - 1050mnn^3 + 225m^4n)}.$$

vnde

vnde patet, quo maior fuerit numerus n prae m , eo citius has differentias tam fieri exiguas, vt sine errore negligi queant.

De fractionibus continuis ab arcubus circularibus pendentibus.

§. 27. Ex § 15. arcus circuli cuius tangens est $\frac{a}{b}$ ita per fractionem exprimetur, vt fit

$$\begin{aligned} \text{A tang. } \frac{a}{b} &= \frac{a}{b} \\ &= \frac{b + a a g}{3 b + 4 a a g} \\ &= \frac{5 b + 9 a a g}{7 b + \text{etc.}} \end{aligned}$$

Ponamus nunc ad similitudinem superiorum formarum $a g = m$ et $b = n$, atque habebimus

$$\begin{aligned} \text{A tang. } \frac{m}{n} &= \frac{m}{n} \\ &= \frac{n + m m}{3 n + 4 m m} \\ &= \frac{5 n + 9 m m}{7 n + \text{etc.}} \end{aligned}$$

quae forma eo citius conuergit, quo maior fuerit numerus n prae m ; vnde patet etiam hanc expressionem cum fructu ad calculum accommodari posse.

§. 28. Incipiamus a casu quo $m = 1$ et $n = 1$, quo fit

A tang.

$$A \text{ tang. } \frac{\pi}{n} = \frac{\pi}{4} = 1 \frac{1}{1 + \frac{1}{3 + \frac{1}{5 + \frac{1}{7 + \frac{1}{9 + \text{etc.}}}}}}$$

quae quidem fractio non adeo conuergit; attamen videamus, quomodo paulatim ad veritatem accedat, quandoquidem nouimus esse $\frac{\pi}{4} = 0,78539816339$. Ac primum quidem membrum dabit $\frac{\pi}{4} = \frac{1}{1}$, (nimis magnum); duo membra praebent $\frac{\pi}{4} = \frac{1}{1 + \frac{1}{3}} = \frac{3}{4}$, (nimis paruum); tria membra

$$\text{dant } \frac{\pi}{4} = \frac{1}{1 + \frac{1}{3 + \frac{1}{5}}} = \frac{15}{24} = 0,7916, \text{ (nimis magnum). Su-}$$

mantur quatuor membra, vt fiat

$$\frac{\pi}{4} = \frac{1}{1 + \frac{1}{3 + \frac{1}{5 + \frac{1}{7}}}} = \frac{40}{51} = 0,7843$$

vbi error demum in tertia figura deprehenditur. Ceterum haec fractio continua similis fere est illi, quam olim Brouncherus in medium protulit, quae ita se habebat:

$$\frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \text{etc.}}}}}$$

Manifestum autem est nostram fractionem multo magis conuergere; neque minus concinna est censenda.

§. 29. Quo autem fractionem continuam magis conuergentem nanciscamur, statuamus A. tg. $\frac{\pi}{n} = 30^\circ$, cuius tangens cum sit $\frac{1}{\sqrt{3}}$, ne numerus n fiat irrationalis sumamus $m = \sqrt{3}$ et $n = 3$, hinc igitur fiet

$$\frac{\pi}{3} = \sqrt{\frac{3}{3 + \frac{3}{9 + \frac{12}{15 + \frac{27}{21 + \frac{48}{27 + \text{etc.}}}}}}}$$

quae forma reducitur ad sequentem:

$$\frac{\pi}{3\sqrt{3}} = \frac{1}{3 + \frac{1}{3 + \frac{4}{15 + \frac{9}{7 + \frac{16}{27 + \frac{25}{11 + \text{etc.}}}}}}}$$

pro qua euoluenda quaeramus primo proxime valorem $\frac{\pi}{3\sqrt{3}}$, qui est 0,30222998. Nunc vero primum membrum praebet

praebet $\frac{\pi}{6\sqrt{3}} = 0,3333$; duo autem priora praebent

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{3 + \frac{1}{3}} = \frac{3}{10} = 0,3000;$$

tria membra dant.

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{3 + \frac{1}{\frac{3+4}{15}}} = \frac{49}{162} = 0,30247$$

vbi error quartam demum figuram afficit.

§. 30. Multo promptior autem convergentia procurari potest, dum angulum rectum in duas partes secamus, quemadmodum olim ostendi esse $A \text{ tang. } \frac{1}{2} + A \text{ tang. } \frac{1}{3} = A \text{ tang. } 1 = \frac{\pi}{4}$. Sic igitur duas fractiones continuas reperimus, quarum summa dabit valorem ipsius $\frac{\pi}{4}$, quae erunt

$$A \text{ tang. } \frac{1}{2} = \frac{1}{2 + \frac{1}{\frac{6+4}{10+9}}} \quad \text{et} \quad A \text{ tang. } \frac{1}{3} = \frac{1}{3 + \frac{1}{\frac{9+4}{15+9}}} \\ \frac{1}{2 + \frac{1}{\frac{6+4}{10+9}}} \quad \frac{1}{3 + \frac{1}{\frac{9+4}{15+9}}} \\ \frac{1}{14 + \text{etc.}} \quad \frac{1}{21 + \text{etc.}}$$

manifestum autem est has ambas fractiones, et potissimum posteriorem, vehementer convergere.

§. 31. Convertamus vero etiam nostram fractionem continuam generalem in fractiones communes; ac ex primo membro solo reperimus $A \text{ tang. } \frac{m}{n} = \frac{m}{n}$; ex duobus membris prodit $A \text{ tang. } \frac{m}{n} = \frac{2mn}{3nn + mm}$; tria membra praebent

bent A tang. $\frac{m}{n} = \frac{15mn + 4m^2}{15n^2 + 9mm}$. Sumantur quatuor membra, vnde fit A tang. $\frac{m}{n} = \frac{105mn^2 + 55m^3n}{105n^4 + 90mmn + 9m^4}$. Quod si nunc his fractionibus praefigatur vt supra $\frac{0}{1}$, orietur haec progressio:

$$\begin{array}{cccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \\ \frac{0}{1} & \frac{m}{n} & \frac{3mn}{3n^2 + mm} & \frac{15mn + 4m^2}{15n^2 + 9mm} & \frac{105mn^2 + 55m^3n}{105n^4 + 90mmn + 9m^4} & \end{array}$$

cuius singuli termini itidem ex praecedentibus binis secundum certam legem formari possunt, scilicet:

pro III scala relationis est $3n, + m m$

pro IV scala relationis est $5n, + 4 m m$

pro V scala relationis est $7n, + 9 m m$

etc.

etc.