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De miris proprietatibus curvae elasticae sub aequatione y=intxxdx/V(1-x^4)

Leonhard Euler

DE MIRIS PROPRIETATIBVS CVRVAE ELASTICAE

fub aequatione $y = \int \frac{xx\,dx}{\sqrt{(1 - x^4)}}$ contentae.

Auctore

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§. 1.

Tab II. Fig. 1. Sit EGF lamina elaftica, quae ope funiculi terminis E et F alligati incuruetur in curuam elafticam EGF, tum vero, fi funiculus eo vsque conftringatur, donec anguli in E et F fiant recti, ea curua elaftica oritur, quae vocari folet rectangula et in aequatione contenta $y = \int \frac{xx\,dx}{\sqrt{(1 - x^4)}}$, cuius nonnullas proprietates prorfus fingulares et admirandas hic fum commemoraturus.

Fig. 2. §. 2. Sit igitur CAC' talis curua elaftica, rectae CC', quae funiculum refert, vtrinque normaliter infiftens, et euidens eft rectam AD, ad punctum medium D inter vtrumque terminum C et C' perpendiculariter ductam, fore curuae diametrum, et punctum A eius quafi verticem referre. Tum vero fi ex C ad rectam CC' erigatur perpendiculum CB, quod tanquam axem hic fpectabimus, in eoque capiamus abfciffam $CP = x$ et vocemus applicatam $PM = y$, pofita altitudine $AD = AB = 1$, erit,

vti

vti constat, $dy = \frac{xxdx}{\sqrt{(1-x^4)}}$; vnde si arcus curuae C M ponatur $= s$, fiet $ds = \frac{dx}{\sqrt{(1-x^4)}}$; atque tam ex natura rei quam ex hac aequatione intelligere licet totam hanc curvam constare ex infinitis portionibus C A C', C' A' C'', C'' A'' C''', etc. inter se similibus et aequalibus super recta C C''' vtrinque in infinitum producta constitutus, vnde etiam tota haec curua infinitos habebit diametros A D, A' D', A'' D'', etc. totidemque vertices A, A', A'', A''', etc. tam dextrorsum quam sinistrorsum. Puncta autem C, C', C'', C''', etc., quoniam circa eorum singula curva similiter alternatim protenditur, centra vocari poterunt. Quemadmodum autem singularum harum portionum altitudines A D, A' D', A'' D'', etc. vnitate designamus, ponamus semilatitudinem cuiusque portionis C D $=$ A B $= a$; ipsos vero arcus C A $=$ C' A $=$ C' A' $=$ etc. $= c$, et quomodo hae binae quantitates a et c se ad vnitatem seu altitudinem A D habeant deinceps accuratius inuestigabimus.

§. 3. His de quantitatibus ad hanc curuam pertinentibus notatis quantitates variabiles P M $= y$ et C M $= s$, ad abscissam C P $= x$ referamus, vnde statim patet, tam y quam s fore functiones infinitiformes eiusdem abscissae C P $= x$. Cum enim applicata P M vtrinque in infinitum producta curuam secet in infinitis punctis M, M', M'', M''', etc. applicata y infinitos recipiet valores, scilicet P M, P M', P M'', P M''', P M'''', etc qui ex principali P M $= y$ et quantitate constante A B $=$ C D $= a$ erunt P M $= y$;

$$PM' = 2a - y;\; PM'' = 4a + y;\; PM''' = 6a - y;$$
$$PM'''' = 8a + y;\; PM''''' = 10a - y;$$

 qui

qui omnes valores in his generalibus formis continentur:

$$4ia+y \text{ et } (4i+2)a-y,$$

vbi littera i omnes numeros integros tam positiuos quam negatiuos denotare potest. Simili modo eidem abscissae $CP=x$ respondebunt infiniti arcus curuae, qui erunt

$$CM=s;\ CAM'=2c-s;\ CAA'M''=4c+s;$$
$$CAA'A''M'''=6c-s;$$

qui omnes etiam in his geminis formulis continentur:

$$4ic+s,\ (4i+2)c-s$$

sumendo pro i successiue omnes numeros tam positiuos quam negatiuos.

Tab. II. Fig. 3. §. 4. Sufficiet igitur solam huius curuae portionem CMA considerasse, quoniam reliquae omnes ei sunt aequales, pro qua posuimus $CB=AD=1$, $AB=CD=a$, et arcum $CMA=c$ Tum vero pro puncto indefinito M si vocentur coordinatae $CP=x$, $PM=y$ et arcus $CM=s$ erit

$$dy=\frac{xx\,dx}{\sqrt{(1-x^4)}} \text{ et } ds=\frac{dx}{\sqrt{(1-x^4)}}.$$

His positis ad curuam in M ducamus normalem MN basi CD productae occurrentem in N Hinc si ducatur ad basin perpendiculum $MQ=x$, ob $CQ=y$ erit interuallum $QN=\frac{x\,dx}{dy}=\frac{\sqrt{(1-x^4)}}{x}$ et ipsa normalis $MN=\frac{x\,ds}{dy}=\frac{1}{x}$, ita vt rectangulum $MQ.MN$ sit $=1=AD^2$. Hinc si vocetur angulus $CNM=\Phi$, qui metitur amplitudinem arcus CM, erit $\sin.\Phi=xx$,

$$\cos.\Phi=\sqrt{(1-x^4)} \text{ et } \tang.\Phi=\frac{xx}{\sqrt{(1-x^4)}}.$$

§. 5.

§. 5. Quaeramus nunc etiam radium osculi curvae in puncto M, qui ſit M O, hunc in finem faciamus $\frac{dy}{dx} = p = \frac{xx}{\sqrt{(1 - x^4)}}$, vnde fit $\sqrt{(1 + pp)} = \frac{1}{\sqrt{(1 - x^4)}}$, hinc porro fiat $\frac{p}{\sqrt{(1 + pp)}} = xx = q$, eritque vti conſtat radius osculi $= \frac{dx}{dq} = \frac{1}{2x}$; ſicque erit $MO = \frac{1}{2x}$, ideoque $MO = \frac{1}{2} MN$, ita vt centrum curuaturae cadat in punctum medium normalis MN; ex quo patet radium osculi MO reciproce eſſe proportionalem interuallo $MQ = x$, quae eſt proprietas, quam natura elaſticae poſtulat. Cum enim vis laminam in puncto C tendens directionem habeat MN, eius momentum reſpectu puncti M erit vi multiplicatae per $QM = x$ aequale, cui per naturam elaſticitatis radius oſculi in M reciproce debet eſſe proportionalis. Manifeſtum igitur eſt radium osculi in ipſo puncto C eſſe infinitum; in altero autem termino $A = \frac{1}{2} = \frac{1}{2} AD$: ſicque in hoc puncto A curuatura erit maxima.

§. 6. Nunc etiam videamus, quomodo ex data abſciſſa $CP = x$ tam applicata $PM = y$, quam ipſe arcus $CM = s$ proxime per ſeries infinitas exprimi queat, id quod duplici modo praeſtari poteſt. Prior maxime obvius in eo conſiſtit vt formula $\frac{1}{\sqrt{(1 - x^4)}} = (1 - x^4)^{-\frac{1}{2}}$ in ſeriem reſoluatur, quae erit:

$$1 + \frac{1}{2} x^4 + \frac{1}{2} \cdot \frac{3}{4} x^8 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} x^{12} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} x^{16} + \text{etc.}$$

vnde per integrationem colligitur:

$$PM = y = \frac{1}{3} x^3 + \frac{1}{2} \cdot \frac{1}{7} x^7 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{11} x^{11} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{15} x^{15} \text{ etc.}$$

tum vero etiam arcus:

$$CM = s = x + \frac{1}{2} \cdot \frac{1}{5} x^5 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{9} x^9 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{13} x^{13} + \text{etc.}$$

Hinc

Hinc igitur patet, ſi abſciſſa x fuerit valde parua, tum fore proxime $y = \frac{1}{3} x^3$ et $s = x$. Verum ſi capiamus $x = 1$, per ſeries ambae quantitates a et c ita exprimentur, vt ſit

$$a = \tfrac{1}{3} + \tfrac{1}{2} \cdot \tfrac{1}{7} + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{11} + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} \cdot \tfrac{1}{15} + \text{etc.}$$

$$c = 1 + \tfrac{1}{2} \cdot \tfrac{1}{5} + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{9} + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} \cdot \tfrac{1}{13} + \text{etc.}$$

Hae autem ſeries nimis lente conuergunt, quam vt inde valores litterarum a et c ſatis exacte definiri queant.

§. 7. Alter modus non adeo obuius in eo conſiſtit, vt ſtatuatur

$$y = \int \frac{xx\,dx}{\sqrt{(1 - x^4)}} = u \sqrt{(1 - x^4)},$$

ſumtis igitur differentialibus erit

$$xx\,dx = du\,(1 - x^4) - 2 u x^3 dx, \text{ ſiue}$$

$$\frac{du}{dx}(1 - x^4) - 2 u x^3 - xx = 0.$$

Fingatur nunc iſta ſeries:

$$u = \alpha x^3 + \beta x^7 + \gamma x^{11} + \delta x^{15} + \varepsilon x^{19} + \text{etc.}$$

quandoquidem iam nouimus, ſi x fuerit valde paruum, fieri debere $y = \frac{1}{3} x^3$, ideoque etiam $u = \frac{1}{3} x^3$; deinde ex forma aequationis manifeſtum eſt, in ſerie exponentes ipſius x continuo quaternario creſcere debere. Hac igitur ſerie ſubſtituta fiat ſequens euolutio:

$$\begin{array}{rl}
\frac{du}{dx} = & 3\alpha xx + 7\beta x^6 + 11\gamma x^{10} + 15\delta x^{14} + 19\varepsilon x^{18} + \text{etc.} \\
-\frac{x^4 du}{dx} = & - 3\alpha x^6 - 7\beta x^{10} - 11\gamma x^{14} - 15\delta x^{18} - \text{etc.} \\
-2ux^3 = & - 2\alpha x^6 - 2\beta x^{10} - 2\gamma x^{14} - 2\delta x^{18} - \text{etc.} \\
-xx = & -xx
\end{array}$$

Singulis igitur membris ad nihilum redactis fiet

$$\alpha = \tfrac{1}{3};\ \beta = \tfrac{1.5}{3.7};\ \gamma = \tfrac{1.5.9}{3.7.11};\ \delta = \tfrac{1.5.9.13}{3.7.11.15};\ \text{etc.}$$

quamobrem habebimus:

$$y = \left(\tfrac{1}{3}x^3 + \tfrac{1.5}{3.7}x^7 + \tfrac{1.5.9}{3.7.11}x^{11} + \tfrac{1.5.9.13}{3.7.11.15}x^{15} + \text{etc.}\right)\sqrt{(1 - x^4)}.$$

§. 8. Simili modo si statuamus

$$s = \int \frac{dx}{\sqrt{(1 - x^4)}} = v\sqrt{(1 - x^4)},$$

peruenietur ad hanc aequationem:

$$\frac{dv}{dx}(1 - x^4) - 2vx^3 - 1 = 0,$$

vbi iam statuamus

$$v = \alpha x + \beta x^5 + \gamma x^9 + \delta x^{13} + \varepsilon x^{17} + \zeta x^{21} + \text{etc.}$$

cuius euolutio ita repraesentetur:

$$\begin{array}{llllll}
\frac{dv}{dx} = \alpha & + 5\beta x^4 & + 9\gamma x^8 & + 13\delta x^{12} & + 17\varepsilon x^{16} & + \text{etc.} \\
-\frac{x^4 dv}{dx} = & -\alpha & -5\beta & -9\gamma & -13\delta & -\text{etc.} \\
-2vx^3 = & -2\alpha & -2\beta & -2\gamma & -2\delta & -\text{etc.} \\
-1 = -1 & & & & &
\end{array}$$

Hinc reperiuntur coefficientes

$$\alpha = 1;\ \beta = \tfrac{3}{5};\ \gamma = \tfrac{3.7}{5.9};\ \delta = \tfrac{3.7.11}{5.9.13};\ \varepsilon = \tfrac{3.7.11.15}{5.9.13.17}\ \text{etc.}$$

vnde colligitur fore

$$s = \left(x + \tfrac{3}{5}x^5 + \tfrac{3.7}{5.9}x^9 + \tfrac{3.7.11}{5.9.13}x^{13} + \text{etc.}\right)\sqrt{(1 - x^4)}.$$

His autem seriebus plane ad valores litterarum a et c eruendos vti non licet: facto enim $x = 1$ formula $\sqrt{(1 - x^4)}$ euanescit; tum autem ipsae series in infinitum excrescunt.

§. 9. Pro litteris autem a et c cognoscendis alias adhiberi conueniet methodos inde petendas, quod integralia harum formularum: $\int \frac{xxdx}{\sqrt{(1 - x^4)}}$ et $\int \frac{dx}{\sqrt{(1 - x^4)}}$, pro eo tan-

tum

tum casu quaeruntur, quo post integrationem fit $x = 1$. Hunc in finem formula $\frac{1}{\sqrt{(1-x^4)}}$ ita repraesentetur:

$\frac{(1+xx)^{-\frac{1}{2}}}{\sqrt{(1-xx)}}$ et numerator $(1+xx)^{-\frac{1}{2}}$ in seriem convertatur, quae erit

$$1 - \frac{1}{2}xx + \frac{1}{2}\cdot\frac{3}{4}x^4 - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}x^8 - \text{etc.}$$

ita vt loco $\frac{1}{\sqrt{(1-x^4)}}$ scripturi simus hanc seriem:

$$\frac{1}{\sqrt{(1-xx)}}\left(1 - \frac{1}{2}xx + \frac{1}{2}\cdot\frac{3}{4}x^4 - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}x^8 - \text{etc.}\right)$$

quo facto tam pro y quam pro s sequentes formulae integrandae occurrent:

$$\int\frac{dx}{\sqrt{(1-xx)}},\ \int\frac{xx\,dx}{\sqrt{(1-xx)}},\ \int\frac{x^4\,dx}{\sqrt{(1-xx)}},\ \text{etc.}$$

§. 10. Harum autem formularum integralia hic non in genere requiruntur, sed tantum pro casu quo post integrationem ponitur $x = 1$. Hoc autem casu novimus, si $1:\pi$ denotet rationem diametri ad peripheriam, esse

$$\int\frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2},\qquad \int\frac{xx\,dx}{\sqrt{(1-xx)}} = \frac{1}{2}\cdot\frac{\pi}{2},$$

$$\int\frac{x^4\,dx}{\sqrt{(1-xx)}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2},\ \int\frac{x^6\,dx}{\sqrt{(1-xx)}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{2},$$

et ita porro, quibus valoribus substitutis primo ex formula

$$y = \int\frac{xx\,dx}{\sqrt{(1-x^4)}} = \int\frac{xx\,dx\,(1+xx)^{-\frac{1}{2}}}{\sqrt{(1-xx)}},$$

colligimus fore

$$a = \frac{\pi}{2}\left(\frac{1}{2} - \frac{1^2}{2^2}\cdot\frac{3}{4} + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5}{6} - \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7}{8} + \text{etc.}\right)$$

ex altera autem formula $s = \int\frac{dx}{\sqrt{(1-x^4)}}$, colligitur longitudo totius arcus

CA =

$$CA = c = \frac{\pi}{2}\left(1 - \frac{1^2}{2^2} + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2} - \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2} + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2} - \text{etc.}\right)$$

Verum etiam hae feries non fatis funt aptae pro veris valoribus quantitatum a et c cognofcendis.

§. 11. Supereft autem adhuc alia methodus eosdem valores per producta ex infinitis factoribus exprimendi, cuius rationem, quamquam a me iam dudum fufius eft explicata, hic fequenti modo fuccincte exponam. Confideretur haec formula: $z = x^n \sqrt{(1 - x^4)}$ et cum fit

$$dz = n x^{n-1} \sqrt{(1-x^4)} - \frac{2x^{n+3}\,dx}{\sqrt{(1-x^4)}} = \frac{n x^{n-1}\,dx - (n+2) x^{n+3}\,dx}{\sqrt{(1-x^4)}},$$

hinc viciffim integrando erit

$$x^n \sqrt{(1-x^4)} = n\int \frac{x^{n-1}\,dx}{\sqrt{(1-x^4)}} - (n+2)\int \frac{x^{n+3}\,dx}{\sqrt{(1-x^4)}},$$

quare fi haec integralia tantum defiderentur pro cafu $x = 1$, fiet

$$\int \frac{x^{n-1}\,dx}{\sqrt{(1-x^4)}} = \frac{n+2}{n}\int \frac{x^{n+3}\,dx}{\sqrt{(1-x^4)}}.$$

Simili modo erit

$$\int \frac{x^{n+3}\,dx}{\sqrt{(1-x^4)}} = \frac{n+6}{n+4}\int \frac{x^{n+7}\,dx}{\sqrt{(1-x^4)}} \text{ et}$$

$$\int \frac{x^{n+7}\,dx}{\sqrt{(1-x^4)}} = \frac{n+10}{n+8}\int \frac{x^{n+11}\,dx}{\sqrt{(1-x^4)}} \text{ etc.}$$

Quod fi ergo hoc modo in infinitum afcendamus, erit

$$\int \frac{x^{n-1}\,dx}{\sqrt{(1-x^4)}} = \frac{n+2}{n}\cdot\frac{n+6}{n+4}\cdot\frac{n+10}{n+8}\cdot\frac{n+14}{n+12}\cdot\ldots\int \frac{x^{n+\infty}\,dx}{\sqrt{(1-x^4)}}.$$

§. 12. Substituamus nunc successiue pro n numeros 1, 2, 3, 4, ac prodibunt sequentes quatuor reductiones ad producta infinita, casu scilicet $x = 1$.

$$\text{I. } \int \frac{dx}{\sqrt{(1 - x^4)}} = \frac{3}{1} \cdot \frac{7}{5} \cdot \frac{11}{9} \cdot \frac{15}{13} \cdot \frac{19}{17} \cdot \cdot \cdot \cdot \cdot \int \frac{x^{1+\infty}\, dx}{\sqrt{(1 - x^4)}} = c.$$

$$\text{II. } \int \frac{x\, dx}{\sqrt{(1 - x^4)}} = \frac{4}{2} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{16}{14} \cdot \frac{20}{18} \cdot \cdot \cdot \cdot \cdot \int \frac{x^{2+\infty}\, dx}{\sqrt{(1 - x^4)}} = \frac{\pi}{4}.$$

$$\text{III. } \int \frac{x x\, dx}{\sqrt{(1 - x^4)}} = \frac{5}{3} \cdot \frac{9}{7} \cdot \frac{13}{11} \cdot \frac{17}{15} \cdot \frac{21}{19} \cdot \cdot \cdot \cdot \cdot \int \frac{x^{3+\infty}\, dx}{\sqrt{(1 - x^4)}} = a.$$

$$\text{IV. } \int \frac{x^3\, dx}{\sqrt{(1 - x^4)}} = \frac{6}{4} \cdot \frac{10}{8} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{22}{20} \cdot \cdot \cdot \cdot \cdot \int \frac{x^{4+\infty}\, dx}{\sqrt{(1 - x^4)}} = \frac{1}{2}.$$

§. 13. Hic iam probe notandum est postremas formulas integrales inter se omnes esse aequales. Cum enim in genere sit

$$\int \frac{x^{n-1}\, dx}{\sqrt{(1 - x^4)}} = \frac{n+2}{n} \int \frac{x^{n+3}\, dx}{\sqrt{(1 - x^4)}}$$

sumto $n = \infty$ erit

$$\int \frac{x^{\infty - 1}\, dx}{\sqrt{(1 - x^4)}} = \int \frac{x^{\infty + 3}\, dx}{\sqrt{(1 - x^4)}}.$$

Quod si ergo harum quatuor formularum quamlibet per aliam diuidamus, postremi factores integrales se mutuo tollunt eritque

$$\frac{\text{I}}{\text{II}} = \frac{4c}{\pi} = \frac{2 \cdot 3}{1 \cdot 4} \cdot \frac{6 \cdot 7}{5 \cdot 8} \cdot \frac{10 \cdot 11}{9 \cdot 12} \cdot \frac{14 \cdot 15}{13 \cdot 16} \cdot \frac{18 \cdot 19}{17 \cdot 20} \cdot \text{ etc.}$$

$$\frac{\text{I}}{\text{III}} = \frac{c}{a} = \frac{3 \cdot 3}{1 \cdot 5} \cdot \frac{7 \cdot 7}{5 \cdot 9} \cdot \frac{11 \cdot 11}{9 \cdot 13} \cdot \frac{15 \cdot 15}{13 \cdot 17} \cdot \frac{19 \cdot 19}{17 \cdot 21} \cdot \text{ etc.}$$

$$\frac{\text{I}}{\text{IV}} = 2c = \frac{3 \cdot 4}{1 \cdot 6} \cdot \frac{7 \cdot 8}{5 \cdot 10} \cdot \frac{11 \cdot 12}{9 \cdot 14} \cdot \frac{15 \cdot 16}{13 \cdot 18} \cdot \frac{19 \cdot 20}{17 \cdot 22} \cdot \text{ etc.}$$

$$\frac{II}{III.} = \frac{\pi}{4a} = \frac{3.4}{2.5} \cdot \frac{7.8}{6.9} \cdot \frac{11.12}{10.13} \cdot \frac{15.16}{14.17} \cdot \frac{19.20}{18.21} \cdot \text{etc.}$$
$$\frac{II}{IV.} = \frac{\pi}{2} = \frac{4.4}{2.6} \cdot \frac{8.8}{6.10} \cdot \frac{12.12}{10.14} \cdot \frac{16.16}{14.18} \cdot \frac{20.20}{18.22} \cdot \text{etc.}$$
$$\frac{III}{IV.} = 2a = \frac{4.5}{3.6} \cdot \frac{8.9}{7.10} \cdot \frac{12.13}{11.14} \cdot \frac{16.17}{15.18} \cdot \frac{20.21}{19.22} \cdot \text{etc.}$$

§. 14. Hae iam expreſſiones multo ſunt aptiores ad veros valores litterarum a et c proxime definiendos. Pro valore autem ipſius c inueniendo formula $\frac{I}{II.}$ maxime videtur idonea, unde fit

$$\frac{4c}{\pi} = \frac{1.3}{2.1} \cdot \frac{3.7}{4.5} \cdot \frac{5.11}{6.9} \cdot \frac{7.15}{8.13} \cdot \frac{9.19}{10.17} \text{ etc. ſiue}$$
$$\frac{4c}{\pi} = \frac{3}{2} \cdot \frac{21}{20} \cdot \frac{55}{54} \cdot \frac{105}{104} \cdot \frac{171}{170} \cdot \text{etc.}$$

quae pro faciliori calculo ita poteſt exhiberi:

$$\frac{4c}{\pi} = (1 + \tfrac{1}{2})(1 + \tfrac{1}{20})(1 + \tfrac{1}{54})(1 + \tfrac{1}{104})(1 + \tfrac{1}{170}) \text{ etc.}$$

At vero quantitas a commodiſſime definietur ſiue ex hac forma:

$$\frac{\pi}{4a} = \frac{2.3}{1.5} \cdot \frac{4.7}{3.9} \cdot \frac{6.11}{5.13} \cdot \frac{8.15}{7.17} \cdot \frac{10.19}{9.21} \cdot \text{etc. ſiue}$$
$$\frac{\pi}{4a} = \frac{6}{5} \cdot \frac{28}{27} \cdot \frac{66}{65} \cdot \frac{120}{119} \cdot \frac{190}{189} \cdot \text{etc.}$$

quae commodi ergo ita repraeſentetur:

$$\frac{\pi}{4a} = (1 + \tfrac{1}{1.5})(1 + \tfrac{1}{3.9})(1 + \tfrac{1}{5.13})(1 + \tfrac{1}{7.17})(1 + \tfrac{1}{9.21}) \text{ etc.}$$

vel etiam pari ſucceſſu definietur quantitas a ex formula $\frac{III}{IV}$, quae dat

$$2a = \frac{2.5}{3.3} \cdot \frac{4.9}{5.7} \cdot \frac{6.13}{7.11} \cdot \frac{8.17}{9.15} \cdot \frac{10.21}{11.19} \text{ etc. ſiue}$$
$$2a = \frac{10}{9} \cdot \frac{36}{35} \cdot \frac{78}{77} \cdot \frac{136}{135} \cdot \frac{210}{209} \cdot \text{etc. ſiue}$$
$$2a = (1 + \tfrac{1}{3.3})(1 + \tfrac{1}{5.7})(1 + \tfrac{1}{7.11})(1 + \tfrac{1}{9.15})(1 + \tfrac{1}{11.19}) \text{ etc.}$$

Interim tamen ſatis taedioſo calculo opus foret, ſi valores harum litterarum vsque ad partem millioneſimam

 vnita-

vnitatis iuftas exquirere vellemus: verum infra, cum proprietates magis abfconditas huius curuae detexerimus, fatis prompte hos valores exhibere licebit.

§. 15. At vero pro eodem fcopo feries pro a et $\mathfrak{b}$ fupra §. 10. inuentae optimo cum fucceffu vfurpari poffunt, quanquam ipfi termini parum decrefcunt, propterea quod in iftis feriebus figna + et − alternantur. Hinc enim infigne fubfidium nafcitur ad fummas harum ferierum proxime inueniendas. Si enim habeatur huiusmodi feries:

$$A - A' + A'' - A''' + A'''' - A''''' \text{ etc.}$$

cuius termini A, A', A'', A''' continuo fiant minores, tum inde formetur feries differentiarum

$$A - A' = B,\ A' - A'' = B',\ A'' - A''' = B'' \text{ etc.}$$

hincque porro feries differentiarum fecundarum

$$B - B' = C,\ B' - B'' = C',\ B'' - B''' = C'' \text{ etc.}$$

ficque hoc modo continuo differentiae capiantur, tum fumma feriei propofitae femper erit

$$\tfrac{A}{2} + \tfrac{B}{4} + \tfrac{C}{8} + \tfrac{D}{16} + \tfrac{E}{32} + \text{ etc.}$$

§. 16. Quo nunc hanc regulam ad feries §. 10. applicemus, euoluamus in fractionibus decimalibus fingulos terminos qui ibi occurrunt.

$\frac{1}{2} = 0,500000$	$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7}{8} = 0,085450$
$\frac{1^2}{2^2} = 0,250000$	$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2} = 0,074769$
$\frac{1^2}{2^2}\cdot\frac{3}{4} = 0,187500$	$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\frac{9}{10} = 0,067292$
$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2} = 0,140625$	$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\frac{9^2}{10^2} = 0,060563$
$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5}{6} = 0,117188$	$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\frac{9^2}{10^2}\cdot\frac{11}{12} = 0,055516$
$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2} = 0,097657$	$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\frac{9^2}{10^2}\cdot\frac{11^2}{12^2} = 0,050890$

$$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\frac{9^2}{10^2}\cdot\frac{11^2}{12^2}\cdot\frac{13}{14}\cdot\cdot\cdot\cdot\cdot = 0,047255$$

$$\frac{1^2}{2^2}\cdot\cdot\cdot\cdot\cdot\cdot\frac{13^2}{14^2}\cdot\cdot\cdot\cdot\cdot = 0,043880$$

$$\frac{1^2}{2^2}\cdot\cdot\cdot\cdot\cdot\cdot\frac{13^2}{14^2}\cdot\frac{15}{16}\cdot\cdot\cdot\cdot = 0,041138$$

$$\frac{1^2}{2^2}\cdot\cdot\cdot\cdot\cdot\cdot\cdot\frac{15^2}{16^2}\cdot\cdot\cdot\cdot = 0,038567$$

$$\frac{1^2}{2^2}\cdot\cdot\cdot\cdot\cdot\cdot\cdot\frac{15^2}{16^2}\cdot\frac{17}{18}\cdot\cdot\cdot = 0,036424$$

$$\frac{1^2}{2^2}\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\frac{17^2}{18^2}\cdot\cdot\cdot = 0,034400$$

$$\frac{1^2}{2^2}\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\frac{17^2}{18^2}\cdot\frac{19}{20}\cdot\cdot = 0,032700$$

$$\frac{1^2}{2^2}\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\frac{19^2}{20^2}\cdot = 0,031065$$

§. 17. His praeparatis calculum instituamus pro valore litterae c inueniendo, et cum esset

$$\frac{2c}{\pi} = 1 - \frac{1^2}{2^2} + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2} - \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2} + \text{etc.}$$

binis primis terminis ad sinistram translatis erit

$$\frac{2c}{\pi} - \frac{3}{4} = \frac{1^2}{2^2}\cdot\frac{3^2}{4^2} - \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2} + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\ \text{etc.}$$

Nunc singuli huius seriei termini sibi inuicem subscribantur iisque subiungantur series differentiarum litteris B, C, D etc. insignitarum hoc modo:

A	B	C	D
0, 140025			
0, 097657	0, 042968		
0, 074769	0, 022888	0, 020080	
0, 060563	0, 014206	0, 008682	0, 011398
0, 050890	0, 009673	0, 004533	0, 004149
0, 043880	0, 007010	0, 002663	0, 001870
0, 038567	0, 005313	0, 001697	0, 000966
0, 034400	0, 004167	0, 001146	0, 000551
0, 031065	0, 003335	0, 000832	0, 000314

E	F	G	H
0, 007249	0, 004970	0, 003595	0, 002709
0, 002279	0, 001375	0, 000886	0, 000575
0, 000904	0, 000489	0, 000311	
0, 000415	0, 000178		
0, 000237			

§. 18. Hinc igitur summa nostrae seriei sequenti modo colligetur:

$\frac{1}{2}$ A = 0, 070312	0, 084503
$\frac{1}{4}$ B = 0, 010742	$\frac{1}{64}$ F = 0, 000078
$\frac{1}{8}$ C = 0, 002510	$\frac{1}{128}$ G = 0, 000028
$\frac{1}{16}$ D = 0, 000712	$\frac{1}{256}$ H = 0, 000011
$\frac{1}{32}$ E = 0, 000227	pro reliquis 000007
0, 084503	0, 084627
	adde $\frac{3}{4}$ = 0, 750000
	erit $\frac{2c}{\pi}$ = 0, 834627

Hinc ergo erit $c = \pi . 0,417314 = 1,311031$.

§. 19. Simili modo computabitur interuallum AB = CD = a. Erat autem

$$\frac{2a}{\pi} = \frac{1}{2} - \frac{1^2}{2^2} \cdot \frac{3}{4} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5}{6} - \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7}{8} + \text{etc.}$$

vbi bini primi termini

$$\frac{1}{2} - \frac{3}{16} = \frac{5}{16} = 0,312500$$

dant ad alteram partem translati

$$\frac{2a}{\pi} - 0,312500 = \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5}{6} - \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7}{8} + \text{etc.}$$

vnde calculus sequenti modo expediatur:

A

A	B	C	D
0, 117188			
0, 085450	0, 031738		
0, 067292	0, 018158	0, 013580	
0, 055516	0, 011776	0, 006382	0, 007198
0, 047255	0, 008261	0, 003515	0, 002867
0, 041138	0, 00611[illegible]	0, 002144	0, 001371
0, 036424	0, 004714	0, 001403	0, 000741
0, 032700	0, 003724	0, 000990	0, 000413

E	F	G	H
0, 004331			
0, 001496	0, 002835		
0, 000630	0, 000866	0, 001969	
0, 000328	0, 000302	0, 000564	0, 001405

§. 20. Hinc igitur ſeriei ſumma colligitur

$\frac{1}{2}A = 0,058594$
$\frac{1}{4}B = 0,007934$
$\frac{1}{8}C = 0,001697$
$\frac{1}{16}D = 0,000450$
$\frac{1}{32}E = 0,000135$
$\overline{\quad 0,068810}$

$0,068810$
$\frac{1}{64}F = 0,000044$
$\frac{1}{128}G = 0,000015$
$\frac{1}{256}H = 0,000005$
$\overline{\quad 0,068874}$
adde $\frac{5}{16} = 0,312500$
et prodit $\frac{2a}{\pi} = 0,381374$

hinc ergo $a = \pi . 0,190687 = 0,599061$.

§. 20. His valoribus quantitatum a et c proxime veris inuentis, quos autem deinceps adhuc accuratius definire docebo, progredior ad illas proprietates huius curuae magis

magis abstrusas, quas sum pollicitus demonstrandas, quippe quas per solitas calculi operationes vix ac ne vix quidem eruere licet, et quae propterea profundioris indaginis merito sunt censendae. Ac primo quidem hic eam insignem relationem, quae inter ternas principales dimensiones huius curuae, scilicet altitudinem $BC = AD$, et inter latitudinem $AB = CD$ atque ipsam curuae longitudinem AMC intercedit, et quam iam pridem detexi, hic accuratius exponam, et sequenti Theoremate complectar.

Theorema I.

§. 21. In curua elastica rectangula AMC cuius vertex est A et centrum alternationis C, ternae dimensiones principales, quae sunt: 1) altitudo $BC = AD$; 2) latitudo $AB = CD$; ac 3) longitudo arcus AMC, ita a se inuicem pendent, vt rectangulum ex latitudine AB in longitudinem arcus AMC aequale sit areae circuli circa diametrum altitudinis BC descripti, siue positis vt fecimus $BC = AD = 1$, $AB = CD = a$ et arcu $AMC = c$ erit $ac = \frac{\pi}{4}$.

Demonstratio.

§. 22. Insignis ista proprietas deducitur ex formulis quas supra per producta in infinitum excurrentia expressimus (§. 13) quarum prima dabat

$$\frac{4c}{\pi} = \frac{2.3}{1.4} \cdot \frac{6.7}{5.8} \cdot \frac{10.11}{9.12} \cdot \frac{14.15}{13.16} \text{ etc} \quad \text{vltima vero}$$

$$2a = \frac{4.5}{3.6} \cdot \frac{8.9}{7.10} \cdot \frac{12.13}{11.14} \cdot \frac{16.17}{15.18} \text{ etc.}$$

Quod si iam in priore expressione primus factor simplex $\frac{2}{1}$ seorsim exhibeatur, ex reliquis autem sequentibus bini inter

inter ſe combinentur habebitur:

$$\frac{4c}{\pi} = \frac{2}{1} \cdot \frac{3.6}{4.5} \cdot \frac{7.10}{8.9} \cdot \frac{11.14}{12.13} \cdot \frac{15.18}{16.17} \cdot \text{etc.}$$

Quod ſi ergo haec expreſſio per alteram multiplicetur, omnes factores praeter primum manifeſto ſe mutuo tollunt, ita vt proditurum ſit $\frac{8ac}{\pi} = 2$, vnde ſit $ac = \frac{\pi}{4}$, quae eſt ipſa illa proprietas quam demonſtrare oportebat.

§. 23. Etſi haec veritas modo prorſus ſingulari ex contemplatione infiniti eſt concluſa: tamen deinceps obſeruaui, eandem quoque per operationes calculi magis conſuetas elici poſſe. Quaeramus enim in genere pro quouis curuae puncto indefinito M productum ex applicata $PM = y$ et arcu $CM = s$, ſitque hoc productum $P = ys$, erit $dP = y\,ds + s\,dy$, hincque iterum integrando

$$P = \int y\,ds + \int s\,dy,$$

quas ambas formulas ſeorſim euoluamus. Pro priori initio oſtendimus eſſe

$$y = \frac{1}{3}x^3 + \frac{1.1}{2.7}x^7 + \frac{1.3.1}{2.4.11}x^{11} + \frac{1.3.5.1}{2.4.6.15}x^{15} + \text{etc.}$$

quae ſeries ducta in $ds = \frac{dx}{\sqrt{(1-x^4)}}$, per ſingulos terminos ita integretur, vt poſt integrationem ſtatuatur $x = 1$, quippe in quo verſatur caſus noſtri theorematis.

§. 24 Pro hac autem inueſtigatione habebimus

$$\int \frac{x^3\,dx}{\sqrt{(1-x^4)}} = \frac{1}{2} - \frac{1}{2}\sqrt{(1-x^4)} = \frac{1}{2},$$

poſito $x = 1$; tum vero in genere vidimus eſſe (§. 11.)

$$\int \frac{x^{n+3}\,dx}{\sqrt{(1-x^4)}} = \frac{n}{n+2}\int \frac{x^{n-1}\,dx}{\sqrt{(1-x^4)}},$$

 vnde

vnde deducimus

$$\int \frac{x^7\, dx}{\sqrt{(1 - x^4)}} = \frac{2}{3} \cdot \frac{1}{2}$$

$$\int \frac{x^{11}\, dx}{\sqrt{(1 - x^4)}} = \frac{8}{10} \cdot \frac{4}{6} \cdot \frac{1}{2} = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{1}{2}$$

$$\int \frac{x^{15}\, dx}{\sqrt{(1 - x^4)}} = \frac{12}{14} \cdot \frac{8}{10} \cdot \frac{4}{6} \cdot \frac{1}{2} = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{1}{2}.$$

Hinc igitur pro noſtro caſu, quo $x = 1$, erit

$$\int y\, ds = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{7} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{11} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{1}{2} \cdot$$
$$+ \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{15} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{1}{19} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{1}{2}, \text{ etc.}$$

quae ſeries reducitur ad ſequentem formam:

$$\int y\, ds = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3.7} + \frac{1}{5.11} + \frac{1}{7.15} + \frac{1}{9.19} + \text{etc.}\right).$$

Eodem modo euoluatur altera formula $\int s\, dy$, et cum per ſeriem priorem eſſet

$$s = x + \frac{1}{2} \cdot \frac{1}{5} x^5 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{9} x^9 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{13} x^{13} + \text{etc.}$$

at vero $dy = \frac{xx\, dx}{\sqrt{(1 - x^4)}}$, ſingulis terminis integrandis ope formularum ante datarum pro caſu $x = 1$ reperietur

$$\int s\, dy = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{9} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{13} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{1}{2},$$

quae ſeries contrahitur in ſequentam formam:

$$\int s\, dy = \frac{1}{2}\left(1 + \frac{1}{3.5} + \frac{1}{5.9} + \frac{1}{7.13} + \frac{1}{9.17} + \frac{1}{11.21} + \text{etc.}\right.$$

His igitur duabus ſeriebus coniunctis fiet:

$$P = \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{5.9} + \frac{1}{5.11} + \frac{1}{7.13} + \text{etc.}\right).$$

§. 25. Quod ſi in hac ſerie bini termini ſe inſequentes in vnum contrahantur, obtinebitur ſequens ſeries:

$$P = ys = \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15} + \frac{2}{17.19} + \text{etc.}$$

Quoniam autem porro eſt $\frac{2}{3} = 1 - \frac{1}{3}$ et $\frac{2}{5.7} = \frac{1}{5} - \frac{1}{7}$, $\frac{2}{9.11} = \frac{1}{9} - \frac{1}{11}$, etc. iſta ſeries reſoluitur in hanc formam:

P = 1

$$P = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \text{etc.}$$

quae cum ſit notiſſima ſeries *Leibniziana* cuius ſumma $= \frac{\pi}{4}$, erit $P = ys = \frac{\pi}{4}$, caſu ſcilicet quo $x = 1$. Verum hoc caſu aſſumſi fieri $y = a$ et $s = c$, ſicque etiam hinc apparet eſſe productum $ac = \frac{\pi}{4}$.

Praeparatio
ad ſequentes huius curuae proprietates magis abſtruſas.

§. 26. In diſſertatione, cui titulus: *Plenior explicatio circa comparationem quantitatum in formula integrali* $\int \frac{Z\,dz}{\sqrt{(1 + mzz + nz^4)}}$ *contentarum*, quaeque Parti poſteriori Actorum pro anno 1781 inſerta fuit, oſtendi: ſi $\Pi : z$ denotet valorem huius formulae integralis: $\int \frac{dz\,(\alpha + \beta zz)}{\sqrt{(1 + mzz + nz^4)}}$, ita ſumtum vt euaneſcat poſito $z = 0$, tum plures huius generis quantitates tranſcendentes modo prorſus ſingulari inter ſe comparari poſſe. Scilicet ſi propoſitae fuerint duae huiusmodi formulae: $\Pi : x$ et $\Pi : y$, atque ex litteris x et y ita determinetur tertia z, vt ſit

$$z = \frac{x\sqrt{(1 + myy + ny^4)}}{1 - nxxyy} + \frac{y\sqrt{(1 + mxx + nx^4)}}{1 - nxxyy},$$

vnde fit

$$\sqrt{(1 + mzz + nz^4)} = \left(\frac{mxy + \sqrt{(1 + mxx + nx^4)(1 + myy + ny^4)}(1 + nx^2yy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}\right),$$

tum ſemper erit

$$\Pi : z = \Pi : x + \Pi : y + \beta xyz,$$

ita vt quantitas tranſcendens $\Pi : z$ ſuperet ſummam datarum $\Pi : x$ et $\Pi : y$ quantitate algebraica βxyz.

§. 27. Euidens iam eſt has formulas generales duplici modo ad inſtitutum noſtrum accommodari poſſe,

ſcilicet

scilicet tam ad arcus huius curuae inter se comparandos, quam ad applicatas cuique abscissae z respondentes. Pro vtroque casu autem erit $m = 0$ et $n = -1$, tum vero in numeratore pro arcubus sumi debebit $\alpha = 1$ et $\beta = 0$, at pro applicatis $\alpha = 0$ et $\beta = 1$.

§. 28. Quod si iam littera z denotet abscissam quamcunque in axe C B assumtam, applicatam ei respondentem designemus charactere $\Pi : z$, arcum vero respondentem hoc charactere $\Theta : z$, eritque ex natura nostrae elasticae

$$\Pi : z = \int \frac{z\,z\,d\,z}{\sqrt{(1 - z^4)}} \text{ et } \Theta\, z = \int \frac{d\,z}{\sqrt{(1 - z^4)}},$$

quibus characteribus in sequentibus vtemur. Tum igitur sumto $z = 0$ erit $\Pi : 0 = 0$ et $\Theta : 0 = 0$. Sumto autem $z = 1$ erit $\Pi : 1 = A\,B = a$ et $\Theta : 1 = C\,A = c$. Praeterea vero notari oportet, sumta abscissa z negatiua, tam applicatam quam arcus longitudinem etiam fore negatiuas; sicque erit $\Pi : (-z) = -\Pi : z$, similique modo $\Theta : (-z) = -\Theta : z$. His igitur praemissis duplicem istam comparationem in sequentibus Problematibus ad nostrum institutum accommodabimus.

Problema I.

Tab II. Fig 4. *Propositis in nostra curua elastica binis arcubus* C X *et* C Y, *abscindere arcum* C Z, *qui aequalis sit summae arcuum* C X + C Y.

Solutio.

§. 29. Vocentur abscissae his arcubus respondentes $C\,x = x$; $C\,y = y$ et $C\,z = z$, eruntque applicatae stabi-

ſtabilito ſignandi modo $xX = \Pi : x$, $yY = \Pi : y$ et $zZ = \Pi : z$, ipſi vero arcus $CX = \Theta : x$, $CY = \Theta : y$ et $CZ = \Theta : z$, et quoniam requiritur vt ſit $\Theta : z = \Theta : x + \Theta : y$, regula generalis ſupra allata, quoniam hoc caſu littera $\beta = 0$, pro datis litteris x et y ita definire iubet z, vt ſit

$$z = \frac{x\sqrt{(1-y^4)} + y\sqrt{(1-x^4)}}{1 + xxyy},$$

tum autem erit

$$\sqrt{(1-z)^4} = \frac{(1-xxyy)\sqrt{(1-x^4)(1-y^4)} - 2xy(xx+yy)}{(1+xxyy)^2},$$

vnde patet, quomodo ex binis abſciſſis datis $Cx = x$ et $Cy = y$ quaeſitam z conſtrui oporteat, vt arcus CZ aequalis fiat ſummae arcuum $CX + CY$.

§. 30. Quemadmodum hic ex datis abſciſſis x et y determinauimus abſciſſam z, ita viciſſim, ſi dentur abſciſſae x et z, tertia y ex iis ſimili modo determinabitur. Cum enim hic eſſe debeat $\Theta : y = \Theta : z - \Theta : x$, euidens eſt hic y eodem modo per z et $-x$ definiri, quo ante z per $+x$ et $+y$ expreſſimus. Hinc igitur erit

$$y = \frac{z\sqrt{(1-x^4)} - x\sqrt{(1-z^4)}}{1 + xxzz} \text{ et}$$

$$\sqrt{(1-y^4)} = \frac{(1-xxzz)\sqrt{(1-x^4)(1-z^4)} + 2xz(xx+zz)}{(1+xxzz)^2}.$$

Parique modo ex datis y et z abſciſſa x ita determinabitur, vt ſit

$$x = \frac{z\sqrt{(1-y^4)} - y\sqrt{(1-z^4)}}{1 + yyzz} \text{ et}$$

$$\sqrt{(1-x^4)} = \frac{(1-yyzz)\sqrt{(1-y^4)(1-z^4)} + 2yz(yy+zz)}{(1+yyzz)^2}.$$

§. 31. Hinc igitur patet ternas quantitates x, y et z ita inter ſe referri, vt quaelibet per binas reliquas ſimili fere modo determinetur; quamobrem iſtam relatio-

 nem

nem accuratius euoluamus, quo clarius pateat, quomodo a se inuicem pendeant. Ex primis autem valoribus, sumtis quadratis erit

$$zz = \frac{(xx+yy)(1-xxyy)+2xy\sqrt{(1-x^4)(1-y^4)}}{(1+xxyy)^2};$$

ex valore autem formulae $\sqrt{(1-z^4)}$ colligitur:

$$\sqrt{(1-x^4)(1-y^4)} = \frac{(1+xxyy)^2\sqrt{(1-z^4)}+2xy(xx+yy)}{1-xxyy},$$

qui valor si ibi substituatur, orietur haec aequatio:

$$zz(1-xxyy) = xx+yy+2xy\sqrt{(1-z^4)}.$$

Similique modo ex binis reliquis determinationibus fiet

$$yy(1-xxzz) = zz+xx-2xz\sqrt{(1-y^4)} \text{ et}$$
$$xx(1-yyzz) = yy+zz-2yz\sqrt{(1-x^4)}.$$

§. 32. Quod si has aequationes ab omni irrationalitate liberemus, ex singulis eadem resultabit aequatio rationalis, quae erit

$$\left.\begin{cases} +x^4-2xxyy+2x^4yyzz+x^4y^4z^4 \\ +y^4-2xxzz+2xxy^4zz \\ +z^4-2yyzz+2xxyyz^4 \end{cases}\right\} = 0,$$

quaeque etiam ita exhiberi potest:

$$0 = \left\{\begin{matrix} x^4+y^4+z^4-2xxyy-2xxzz-2yyzz \\ +2xxyyzz(xx+yy+zz)+x^4y^4z^4 \end{matrix}\right\},$$

vbi iam manifesto ternae litterae x, y et z aequaliter ingrediuntur; quoniam enim hic litterarum x, y, z tantum quadrata insunt, perinde est siue eae negatiue capiantur, siue positiuae.

§. 33.

§. 33. Quoties ergo ternae abſciſſae $Cx = x$, $Cy = y$ et $Cz = z$, eam inter ſe tenent rationem, quam aſſignauimus, tum arcus CZ ſemper aequabitur ſummae binorum reliquorum CX et CY. Cum igitur hinc ſit $CZ - CY = CX$, erit arcus $YZ = CX$, vnde ſi puncta Y et Z pro lubitu accipiantur, a puncto C ſemper arcus CX abſcindi poterit, qui arcui YZ erit aequalis. Ac viciſſim propoſito arcu CX, a puncto quouis dato Y abſcindi poterit arcus YZ, illi arcui CX aequalis. Sin autem terminus Z vt datus ſpectetur, ab eo retro abſcindi poterit arcus ZY ipſi CX aequalis, quae cum ſint ſatis obuia, ſuperfluum foret pro iis peculiaria problemata conſtituere.

Theorema II.

§. 34. Si ternae abſciſſae $Cx = x$, $Cy = y$, $Cz = z$, ita fuerint aſſumtae, vt arcus CZ aequetur ſummae CX et CY, tum ternae applicatae $xX = \Pi : x$, $yY = \Pi : y$, $zZ = \Pi : z$ ita inter ſe erunt relatae, vt ſit

$$\Pi : z = \Pi : x + \Pi : y + xyz,$$

ſiue erit

$$zZ = xX + yY + \frac{Cx \cdot Cy \cdot Cz}{CB^2}.$$

Demonſtratio.

§. 35. Cum relatio inter formulas $\Pi : x$, $\Pi : y$ et $\Pi : z$ eandem relationem inter abſciſſas x, y et z praebeat quam pro formulis: $\Theta : x$, $\Theta : y$ et $\Theta : z$ aſſignauimus, quoniam pro hoc caſu littera β in forma generali adhibita vnitati aequatur, vi relationis generalis erit

$\Pi : z$

$$\Pi : z = \Pi : x + \Pi : y + x y z,$$

vnde, ad homogeneitatem obſeruandam, quia altitudo C B vnitate eſt definita, ſolidum $x y z$ per eius quadratum dividi oportet, vnde fiet

$$z Z = x X + y Y + \frac{Cx \cdot Cy \cdot Cz}{CB^2}.$$

§. 36. Cum igitur characteres $\Theta : z$ et $\Pi : z$ certas functiones tranſcendentes abſciſſae z denotent, quas conſtat neque per logarithmos neque per arcus circular s exprimi poſſe, quandoquidem per formulas integrales $\int \frac{dz}{\sqrt{(1 - z^4)}}$ et $\int \frac{zz\,dz}{\sqrt{(1 - z^4)}}$ definiuntur, earum valores ſaltem per ſeries infinitas exhibuiſſe iuuabit: erit autem per modum priorem

$$\Theta : z = z + \frac{1}{2} \cdot \frac{1}{5} z^5 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{9} z^9 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{13} z^{13} + \text{etc.}$$

$$\Pi : z = \frac{1}{3} z^3 + \frac{1}{2} \cdot \frac{1}{7} z^7 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{11} z^{11} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{15} z^{15} + \text{etc.}$$

Ex altera autem reſolutione erit ex §. 8.

$$\Theta : z = \left(z + \frac{2}{5} z^5 + \frac{3.7}{5.9} z^9 + \frac{3.7.11}{5.9.13} z^{13} + \text{etc.}\right) \sqrt{(1 - z^4)} \text{ et}$$

$$\Pi : z = \left(\frac{1}{3} z^3 + \frac{1.5.}{3.7} z^7 + \frac{1.5.9}{3.7.11} z^{11} + \frac{1.5.9.13}{3.7.11.15} z^{15} + \text{etc.}\right) \sqrt{(1 - z^4.)}$$

Problema

Elementa principalia noſtrae curuae elaſticae, ſcilicet latitudinem A B = a *et totum arcum* C A = c, *reſpectu altitudinis* C B = 1, *accuratius determinare quam ſupra fieri licuit*

Solutio

§. 37. Hunc in finem accipiatur punctum Z in ipſo vertice curuae A, vt fiat $z = 1$, eritque

$$\Pi : z = AB = a \text{ et } \Theta : z = CA = c,$$

tum

tum igitur erit $\sqrt{(1 - z^4)} = 0$. Nunc quaerantur bini arcus CX et CY, quorum ſumma ſit aequalis arcui $CA = c$. Poſitis ergo eorum abſciſſis $Cx = x$ et $Cy = y$ ex §. 31. erit

$$1 - xx - yy - xxyy = 0,$$

vnde fit $yy = \frac{1 - xx}{1 + xx}$. Quod ſi igitur y hoc modo per x determinetur, tum erit $\Theta : x + \Theta : y = c$; tum vero ob $\Pi : z = a$ erit $a = \Pi : x + \Pi : y + xy$.

§. 38. Quo nunc ſeries pro $\Theta : x$ et $\Theta : y$, item pro $\Pi : x$ et $\Pi : y$, maxime conuergentes reddantur, abſciſſas x et y proxime inter ſe aequales accipiamus. Si enim vellemus ſtatuere $y = x$, prodiret

$$x = y = \sqrt{(-1 + \sqrt{2})},$$

qui valor irrationalis minime idoneus foret ad noſtras ſeries euoluendas. Hanc ob rem ſumamus $xx = \frac{1}{2}$, erit $yy = \frac{1}{3}$, ideoque $x = \frac{1}{\sqrt{2}}$ et $y = \frac{1}{\sqrt{3}}$, vnde per priores ſeries fiet

$$\Theta : x = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2.5} \cdot \frac{1}{2^2} + \frac{1.3}{2.4.9} \cdot \frac{1}{2^4} + \frac{1.3.5}{2.4.6.13} \cdot \frac{1}{2^6} + \text{etc.}\right)$$
$$\Pi : x = \frac{1}{2\sqrt{2}}\left(\frac{1}{3} + \frac{1}{2.7} \cdot \frac{1}{2^2} + \frac{1.3}{2.4.11} \cdot \frac{1}{2^4} + \frac{1.3.5}{2.4.6.15} \cdot \frac{1}{2^6} + \text{etc.}\right)$$

Simili vero modo erunt:

$$\Theta : y = \frac{1}{\sqrt{3}}\left(1 + \frac{1}{2.5} \cdot \frac{1}{3^2} + \frac{1.3}{2.4.9} \cdot \frac{1}{3^4} + \frac{1.3.5}{2.4.6.13} \cdot \frac{1}{3^6} + \text{etc.}\right)$$
$$\Pi : y = \frac{1}{3\sqrt{3}}\left(\frac{1}{3} + \frac{1}{2.7} \cdot \frac{1}{3^2} + \frac{1.3}{2.4.11} \cdot \frac{1}{3^4} + \frac{1.3.5}{2.4.6.15} \cdot \frac{1}{3^6} + \text{etc.}\right)$$

§. 39. Hae ſeries manifeſto tantopere conuergunt, vt, qui laborem calculi ſuſcipere voluerit, veros litterarum

a et

a et c valores tam exacte definire queat quam lubuerit; valores autem quas supra assignauimus, iam tam parum a veritate discrepant, vt pro nostro instituto abunde sufficere possint; quandoquidem hic de eo tantum agitur, vt valores inuenti calculum subducendo comprobari queant, quamobrem ad alias insignes proprietates huius curuae progrediamur.

Problema III.

Tab. II. Fig. 5. *Proposito in curua elastica arcu quocunque* P Q, *a puncto dato* R *abscindere arcum* R S, *qui illi arcui* P Q *sit aequalis.*

Solutio.

§. 40. Quoniam igitur in curua quatuor puncta P, Q, R, S consideranda veniunt, sint abscissae illis respondentes $Cp = p$, $Cq = q$, $Cr = r$, $Cs = s$, pro quibus ponamus breuitatis gratia formulas irrationales

$\sqrt{(1-p^4)} = P$, $\sqrt{(1-q^4)} = Q$, $\sqrt{(1-r^4)} = R$ et $\sqrt{(1-s^4)} = S$.

His positis, quoniam arcus R S aequalis esse debet arcui P Q, requiritur vt sit $CS - CR = CQ - CP$, hoc est $\Theta : s - \Theta : r = \Theta : q - \Theta : p$, cui aequationi vt per regulam supra datam satisfaciamus, quaeramus arcum $\Theta : v$, vt sit $\Theta : v = \Theta : q - \Theta : p$, et secundum praecepta superiora esse debet $v = \frac{qP - pQ}{1 + ppqq}$, vnde fit

$$\sqrt{(1 - v^4)} = V = \frac{(1 - ppqq) PQ + 2pq(pp + qq)}{(1 + ppqq)^2}.$$

Hoc iam arcu inuento esse debet $\Pi : s = \Pi : r + \Pi : v$; quare per eadem praecepta fiet $s = \frac{rV + vR}{1 + rrvv}$, hincque porro

$$S = \frac{(1 - rrvv) RV - 2rv(rr + vv)}{(1 + rrvv)^2}.$$

Sub-

Subſtituamus nunc in his formulis valores pro v et V inventos; ac primo erit

$$1 + rrvv = \frac{(1 + ppqq)^2 + rrppQQ + rrqqPP - 2rrpqPQ}{(1 + ppqq)^2},$$

quae aequatio, ſi loco PP et QQ valores ſubſtituantur, ad hanc reducitur:

$$1 + rrvv = \frac{(1 + ppqq)^2 + rr(pp + qq)(1 - ppqq) - 2pqrrPQ}{(1 + ppqq)^2}.$$

At vero pro numeratore erit

$$rV + vR = \frac{r(1 - ppqq)PQ - 2pqr(pp + qq) + (qPR - pQR)(1 + ppqq)}{(1 + ppqq)^2},$$

conſequenter abſciſſa quaeſita CS $= s$ ita erit expreſſa:

$$s = \frac{r(1 - ppqq)PQ + 2pqr(pp + qq) + (qPR - pQR)(1 + ppqq)}{(1 + ppqq)^2 + rr(pp + qq)(1 - ppqq) - 2pqrrPQ}.$$

Quod autem ad valorem litterae S attinet, quia eo in noſtro calculo non indigemus, eius euolutione ſuperſedemus.

§. 41. Hinc igitur videmus, quomodo abſciſſa s per ternas abſciſſas datas p, q et r exprimatur; vbi quidem plurimum abeſt, vt litterae p, q, r in eam aequaliter ingrediantur: cum tamen ex aequatione propoſita

$$\Theta : s = \Theta : r + \Theta : q - \Theta : p$$

intelligatur, iſtas litteras P, Q et R ſimili modo in valorem ipſius s ingredi debere, ſi modo littera p negatiue acciperetur. Neque igitur vllum eſt dubium, quin forma inuenta ita transformari poſſit, vt iſta paritas litterarum p, q et r eluceſcat, id quod tamen neutiquam liquet.

§. 42. Cum autem eſſe debeat

$$\Theta : s = \Theta : r + \Theta : q - \Theta : p,$$

cui-euidens est, manente littera p binas reliquas q et r inter se commutari posse, vnde etiam vera esse debet ista expressio:

$$s = \frac{q(1-pprr)PR + 2pqr(pp+rr) + (rPQ - pQR)(1+pprr)}{(1+pprr)^2 + qq(pp+rr)(1-pprr) - 2prqqPR}.$$

Deinde manente r litterae p et q ita permutari poterunt, si loco q scribatur $-p$ et $-q$ loco p, tum autem erit

$$s = \frac{-p(1-qqrr)QR + 2pqr(qq+rr) + (qPR + rPQ)(1+qqrr)}{(1+qqrr)^2 + pp(qq+rr)(1-qqrr) + 2qrppQR}.$$

Atque hae tres expressiones, quantumuis diuersae videantur, tamen certe eundem valorem exprimunt.

§. 43. Insignis igitur hic occurrit quaestio analytica, quomodo istae tres expressiones tractari debeant, vt perfecta permutabilitas inter ternas litteras p, q, r perspiciatur. Facile quidem intelligitur, si tres istae expressiones in se inuicem multiplicentur, ita vt productum aequetur cubo s^3, tum tam in numeratore quam in denominatore ternas litteras p, q, r, pari modo esse ingressuras; verum tale productum nimis foret perplexum, quam vt vllum vsum habere posset.

Solutio.

§. 44. Quae hactenus de curua elastica rectangula sunt tradita etiam ad omnes curuas elasticas in genere accommodari poterunt. Cum enim pro data abscissa z sit applicata $= \int \frac{dz(\alpha + \beta zz)}{\sqrt{1-(\alpha+\beta zz)^2}}$ et ipse arcus $= \int \frac{dz}{\sqrt{1-(\alpha+\beta zz)^2}}$, praecepta generalia supra tradita pro comparatione harum quantitatum transcendentium simili modo applicari poterunt. Interim tamen hic conditio maxime necessaria pro-

be

be notari debet, qua poſtulatur vt denominator, qui euolutus eſt $V(1 - \alpha\alpha - 2\alpha\beta zz - \beta\beta z^4)$, ad hanc formam: $V(1 + mzz + nz^4)$, reduci queat, quod manifeſto fieri nequit niſi $1 - \alpha\alpha$ fuerit quantitas poſitiua. His igitur caſibus $\alpha\alpha > 1$ omnes comparationes, quas tam inter arcus quam inter applicatas docuimus, ſimili modo ad curvas elaſticas obliquangulas traduci poterunt.