1786

Methodus facilis omnia symptomata linearum curvarum non in eodem plano sitarum investigandi

Leonhard Euler

METHODVS FACILIS
OMNIA SYMPTOMATA
LINEARVM CVRVARVM
NON IN EODEM PLANO SITARVM
INVESTIGANDI.

Dissertatio prior.

Auctore

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§. 1.

Symptomata linearum curuarum, quae non totae in eodem plano iacent, et a Geometris lineae duplicis curuaturae vocari solent, iam pridem quidem definita reperiuntur: verum analysis, qua ea sunt inuestigata, figuris tantopere perplexis innituntur, vt non solum summam attentionem, sed etiam maximam circumspectionem requirant, ne repraesentatio quantitatum differentialium, atque adeo differentialium secundi gradus, imaginationem confun-

 dat,

dat et in errores ſeducat. Quamobrem ſaepe et multum mecum cogitaui, annon eadem ſymptomata methodo faciliori ex primis elementis deriuari queant, ita vt non opus ſit figuras tantopere complicatas et propemodum inextricabiles conſiderare. His omnibus difficultatibus probe perpenſis tandem perſpexi, totum negotium ſatis commode ad Trigonometriam ſphaericam reuocari atque adeo multo plenius tractari poſſe, quam quidem methodo vulgari fieri licet.

§. 2. Ante autem indolem harum linearum curuarum, in genere ſaltem, more ſolito conſiderari conueniet, quam totam quaeſtionem ad doctrinam ſphaericam transferre queamus. Conſtitutis igitur pro lubitu ternis axibus inter ſe normalibus O A, O B, O C, in puncto fixo
Tab. I. Fig. 5. O concurrentibus, ſit Z punctum quodcunque talis curvae, cuius ſitus determinetur per ternas coordinatas

$$OX = x,\ XY = y,\ YZ = z,$$

axibus illis parallelas, inter quas duabus aequationibus opus erit ad indolem curuae propoſitae exprimendam, quoniam pro quauis abſciſſa $OX = x$, vtraque reliquarum y et z aſſignari debet, vt punctum Z determinatum locum obtineat. Omnes autem quaeſtiones, quae circa tales curuas proponi poſſunt, in genere huc redeunt, vt primo pro quouis puncto Z poſitio tangentis reſpectu ternorum axium definiatur. Deinde vero etiam bina curuae elementa contigua conſiderari debent, quae, quatenus non in directum iacent, certum planum determinabunt, cuius poſitionem reſpectu axium, vel reſpectu ternorum planorum A O B, B O C et C O A inueſtigari oportet. Denique etiam neceſſe

cesse est, vt radius curuaturae pro talibus binis elementis exploretur, eiusque non solum quantitas, sed etiam positio respectu ternorum axium assignetur. Euidens autem est, has posteriores determinationes differentialia secundi gradus inuoluere, quorum repraesentatio in figura taediosam attentionem postulare solet.

§. 3. Ante omnia autem conueniet calculum ita exsequi, vt nulli ternorum axium prae reliquis vlla praerogatiua tribuatur, atque omnia pari modo ad vnumquemque referri queant. Hunc in finem ipsum arcum curuae, quem vocemus $= s$, tanquam praecipuam variabilem in calculum introducamus, ad eamque omnes reliquas variabiles reuocemus; ita vt omnes tanquam functiones ipsius s spectari queant. Hunc in finem statuamus statim ab initio $dx = p\,ds$, $dy = q\,ds$ et $dz = r\,ds$, vnde cum sit $ds^2 = dx^2 + dy^2 + dz^2$, erit $pp + qq + rr = 1$, ideoque differentiando $p\,dp + q\,dq + r\,dr = 0$. Tum vero nihil impedit, quo minus elementum ds pro constante accipiamus, quandoquidem aequaliter ad singulos axes refertur, vnde nanciscemur differentialia secundi gradus $ddx = dp\,ds$; $ddy = dq\,ds$ et $ddz = dr\,ds$, sicque hoc modo differentialia secunda penitus euitabimus.

§. 4. Consideremus nunc seorsim elementum curvae $Zz = ds$, quod quomodo ad ternos nostros axes se habeat videamus. Hunc in finem ex puncto Z ducamus rectas Zp, Zq et Zr, axibus parallelas, et ipsum elementum Zz erit diagonalis parallelepipedi à ternis lateribus Tab. I. Fig. 6.

$Zp = dx = p\,ds$, $Zq = dy = q\,ds$ et $Zr = dz = r\,ds$,

formati; vnde patet ipſum elementum Zz ad directionem Zp ita inclinari, vt anguli zZp coſinus ſit $= p$, anguli vero zZq coſinus $= q$ et anguli zZr coſinus $= r$, vnde etiam horum angulorum ſinus definire licebit, ita vt hinc nacturi ſimus

$$\text{cof. } zZp = p; \ \text{ſin. } zZp = \sqrt{(1 - pp)} = \sqrt{(qq + rr)};$$
$$\text{cof. } zZq = q; \ \text{ſin. } zZq = \sqrt{(1 - qq)} = \sqrt{(pp + rr)};$$
$$\text{cof. } zZr = r; \ \text{ſin. } zZr = \sqrt{(1 - rr)} = \sqrt{(pp + qq)}.$$

Hae ergo formulae ſimul monſtrabunt poſitionem tangentis curuae in Z reſpectu ternorum axium. Scilicet iſta tangens in puncto Z inclinabitur ad axem O A angulo cuius coſ. $= p$ et ſin. $= \sqrt{(qq + rr)}$; ad axem O B angulo cuius coſinus $= q$ et ſinus $= \sqrt{(pp + rr)}$; ad axem O C angulo cuius coſinus $= r$ et ſinus $= \sqrt{(pp + qq)}$. Sicque iam primo requiſito circa poſitionem tangentium curuae reſpectu axium principalium ſatisfecimus, neque adhuc opus fuerat ad doctrinam ſphaericam confugere.

Tab. I. Fig. 7. §. 5. Concipiatur nunc ſphaera circa punctum Z deſcripta, quod quidem in figura non apparet, ad cuius ſuperficiem ducti intelligantur terni radii axibus principalibus O A, O B, O C paralleli, qui ſuperficiem ſecent in punctis a, b, c. Hoc modo, ductis circulis maximis ab, ac, bc orietur triangulum ſphaericum abc, cuius ſingula latera erunt quadrantes et anguli a, b, c recti. Tum vero ex centro Z ſecundum tangentem noſtrae cur-

uae

uae in puncto Z educatur radius Zz in superficie designans punctum z, vnde si ad angulos ducantur arcus za, zb, zc, ii metientur inclinationem tangentis Zz, ad ternos axes Za, Zb, Zc, ideoque erit

$$\text{cos.}\, az = p;\ \text{sin.}\, az = \sqrt{(1-pp)} = \sqrt{(qq+rr)};$$
$$\text{cos.}\, bz = q;\ \text{sin.}\, bz = \sqrt{(1-qq)} = \sqrt{(pp+rr)} \text{ et}$$
$$\text{cos.}\, cz = r;\ \text{sin.}\, cz = \sqrt{(1-rr)} = \sqrt{(pp+qq)}.$$

Praeterea vero si arcus az, bz, cz concipiantur producti vsque ad latera opposita, singuli erunt quadrantes, vnde patet, tangentem Zz ad planum aZb siue ad planum AOB (*Fig.* 5.) inclinari sub angulo cuius sinus $= r$; ad planum autem BOC sub angulo cuius sinus $= p$, et ad planum AOC sub angulo cuius sinus $= q$. Hic quidem radium sphaerae tanquam vnitate expressum spectamus, quod tamen non impedit, quo minus deinceps radius ipsi elemento curuae $Zz = ds$ aequalis statuatur.

§. 6. Hoc modo totum nostrum triangulum sphaericum abc diuidetur in tria triangula sphaerica abz, acz et bcz, in quibus dantur terna latera, vnde ex triangulo baz colligetur

$$\text{cos.}\, baz = \frac{\text{cos.}\, bz + \text{cos.}\, ab \,\text{cos.}\, az}{\text{sin.}\, ab \,\text{sin.}\, az} = \frac{q}{\sqrt{(qq+rr)}}.$$

Simili modo ex triangulo caz erit $\text{cos.}\, caz = \frac{r}{\sqrt{(qq+rr)}}$. Quia vero angulus bac est rectus, erit

$$\text{sin.}\, baz = \text{cos.}\, caz = \frac{r}{\sqrt{qq+rr}},$$

vnde sequitur fore

$$\text{tang.}\, baz = \frac{r}{q} \text{ hincque tang.}\, caz = \frac{q}{r}.$$

Eo-

Eodem modo erit

$$\text{tang.}\, a\,b\,z = \frac{r}{p} \quad \text{et} \quad \text{tang.}\, c\,b\,z = \frac{p}{r};$$

$$\text{tang.}\, b\,c\,z = \frac{p}{q} \quad \text{et} \quad \text{tang.}\, a\,c\,z = \frac{q}{p}.$$

§. 7. Quod ſi iam quantitates p, q et r ſuis differentialibus augeamus, perueniemus ad poſitionem ſequentis tangentis reſpectu axium fixorum. Hoc igitur facto transferatur in figura punctum z in punctum z', eritque radius Zz' poſitio ſequentis elementi curuae, ſeu potius ſequens elementum parallelum erit huic radio Zz', et arculus infinite paruus zz' dabit inclinationem binorum elementorum curuae contiguorum, vnde ſtatim colligetur radius curuaturae horum elementorum. Si enim radius oſculi curuae dicatur $= R$, quoniam hinc inclinatio horum elementorum eſt $\frac{ds}{R}$, erit vtique $\frac{ds}{R} = zz'$, ideoque $R = \frac{ds}{zz'}$. Praeterea vero, quia ſequens elementum parallelum eſt radio zz', dum radius Zz refert prius elementum, ambo haec elementa ſita erunt in plano zZz', ſiue particula zz' continuata praebebit circulum maximum cum iſto plano conuenientem, cuius ergo inclinationem ad terna plana principalia per arcus ab, ac, bc determinata aſſignare licebit.

§. 8. Ad haec expedienda vocemus arcum $az = \alpha$, vt ſit cof. $\alpha = p$, eritque arcus $az' = \alpha + d\alpha$, et ducto perpendiculo zs in arcum az', vt fiat $as = az = \alpha$, erit particula $sz' = d\alpha$; quia autem α eſt arcus cuius coſinus $= p$, erit $d\alpha = -\frac{dp}{\sqrt{(1-pp)}} = -\frac{dp}{\sqrt{qq+rr}}$. Simili modo vocemus angulum $baz = \omega$, eritque angulus $baz' = \omega + d\omega$, ideoque angulus elementaris $zaz' = d\omega$; quia vero ω denotat angulum cuius tangens eſt $\frac{r}{q}$, erit $d\omega = \frac{q\,dr - r\,dq}{qq+rr}$, qui ductus in ſinum arcus $az = \sqrt{(qq+rr)}$, praebebit elemen-

mentum $zs = \frac{q\,dr - r\,dq}{\sqrt{(qq+rr)}}$. Hoc igitur modo in triangulo characteriſtico ad s rectangulo $z s z'$, ob cathetos $s z$ et $s z'$ datos colligetur

$$(z z')^2 = \frac{qq\,dr - 2qr\,dq\,dr + rr\,dq^2 + dp^2}{qq+rr}.$$

Quia autem eſt

$$p\,dp + q\,dq + r\,dr = 0, \text{ erit}$$

$$q\,dq + r\,dr = -p\,dp \text{ ideoque}$$

$$qq\,dq^2 + 2qr\,dq\,dr + rr\,dr^2 = pp\,dp^2,$$

vnde fit

$$2qr\,dq\,dr = pp\,dp^2 - qq\,dq^2 - rr\,dr^2,$$

qui valor ſupra ſubſtitutus dabit

$$(z z')^2 = \frac{(qq+rr)\,dr^2 + (qq+rr)\,dq^2 + dp^2\,(1-pp)}{qq+rr}.$$

Quia igitur eſt $1 - pp = qq + rr$, fiet

$$(z z')^2 = dp^2 + dq^2 + dr^2 \text{ ideoque}$$

$$z z' = \sqrt{(dp^2 + dq^2 + dr^2)}.$$

§. 9. Hoc igitur elemento $z z'$ inuento reperitur radius oſculi curuae $R = \frac{ds}{\sqrt{(dp^2 + dq^2 + dr^2)}}$; quare cum poſuerimus $p = \frac{dx}{ds}$, $q = \frac{dy}{ds}$, $r = \frac{dz}{ds}$, ſumto elemento ds conſtante fiet $dp = \frac{ddx}{ds}$, $dq = \frac{ddy}{ds}$, $dr = \frac{ddz}{ds}$, ſicque habebimus:

$$R = \frac{ds^2}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}.$$

Verum ſi nullum elementum pro conſtante habeatur, vt fiat

$$dp = \frac{ddx}{ds} - \frac{dx\,dds}{ds^2},$$

$$dq = \frac{ddy}{ds} - \frac{dy\,dds}{ds^2},$$

$$dr = \frac{ddz}{ds} - \frac{dz\,dds}{ds^2},$$

 erit

erit

$$dp^2 + dq^2 + dr^2 = \frac{ddx^2 + ddy^2 + ddz^2}{ds^2}$$
$$- \frac{2dds\,(dx\,ddx + dy\,ddy + dz\,ddz)}{ds^3} + \frac{dds^2(dx^2 + dy^2 + dz^2)}{ds^4},$$

vbi ob $dx^2 + dy^2 + dz^2 = ds^2$ et

$$dx\,ddx + dy\,ddy + dz\,ddz = ds\,dds,$$

haec formula tranſibit in hanc:

$$\frac{ddx^2 + ddy^2 + ddz^2 - dds^2}{ds^2},$$

atque hinc expreſſio generalis, nullo elemento pro conſtante aſſumto, pro radio oſculi erit

$$R = \frac{ds^2}{\sqrt{(ddx^2 + ddy^2 + ddz^2 - dds^2)}},$$

quae formula per Analyſin communem demum poſt calculos maxime perplexos eſt eruta.

§. 10. Quo nunc etiam poſitionem plani, in quo bina curuae elementa proxima ſunt ſita, inueſtigemus: in triangulo characteriſtico $z\,z'\,s$ quaeratur angulus $z\,z'\,s$, atque ex omnibus eius lateribus cognitis concludimus ſequentes formulas:

$$\text{ſin. } z\,z'\,s = \frac{q\,dr - r\,dq}{\sqrt{(qq + rr)(dp^2 + dq^2 + dr^2)}} \text{ et}$$
$$\text{coſ. } z\,z'\,s = \frac{-dp}{\sqrt{(qq + rr)(dp^2 + dq^2 + dr^2)}}, \text{ hincque}$$
$$\text{tang. } z\,z'\,s = \frac{r\,dq - q\,dr}{dp},$$

atque hae eaedem formulae valebunt pro angulo quem particula $z\,z'$ retrocontinuata cum arcu $a\,z$ conſtituet, quandoquidem z et z' infinite parum diſcrepant.

Tab. I. §. 11. Producatur nunc vtcunque elementum $z\,z'$
Fig. 8. inuentum, donec latera noſtri trianguli $a\,b\,c$ ſecet in punctis

ctis q, r et p, et modo vidimus, si angulum $a z r$ vocemus $= \Phi$, fore

$$\text{sin.}\, \Phi = \frac{q\,d\,r - r\,d\,q}{\sqrt{(q\,q + r\,r)(d\,p^2 + d\,q^2 + d\,r^2)}};$$
$$\text{cos.}\, \Phi = \frac{-d\,p}{\sqrt{(q\,q + r\,r)(d\,p^2 + d\,q^2 + d\,r^2)}} \text{ et}$$
$$\text{tang.}\, \Phi = \frac{r\,d\,q - q\,d\,r}{d\,p},$$

atque hinc definiri oportet primo ipsa puncta p, q, r, vbi iste arcus terna latera nostri trianguli secat, deinde etiam angulos, quos ille in his punctis cum lateribus constituit. Quo hoc facilius expediri possit, ex puncto a in arcum $q\,r\,p$ ducamus arcum $a\,m$, qui productus lateri $b\,c$ occurrat in n, et in triangulo rectangulo $a\,z\,m$ cum latere $a\,z$ datur angulus $a\,z\,m$, vnde inuenitur

$$\text{sin.}\, a\,m = \text{sin.}\, a\,z\, \text{sin.}\, a\,z\,m = \frac{q\,d\,r - r\,d\,q}{\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}} \text{ et}$$
$$\text{tang.}\, z\,m = \text{tang.}\, a\,z\, \text{cos.}\, a\,z\,m = \frac{d\,p}{p\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}},$$

denique erit

$$\text{tang.}\, z\,a\,m = \frac{1}{\text{cos.}\, a\,z\, \text{tang.}\, a\,z\,m} = \frac{d\,p}{p(q\,d\,r - r\,d\,q)}.$$

Supra autem vidimus esse $\text{tang.}\, z\,a\,b = \frac{r}{q}$, vnde fit tangens summae horum angulorum, seu

$$\text{tang.}\, b\,a\,m = \frac{p\,r\,(q\,d\,r - r\,d\,q) + q\,d\,p}{p\,q\,(q\,d\,r - r\,d\,q) - r\,d\,p},$$

cuius numerator ob $r\,d\,r = -p\,d\,p - q\,d\,q$, reducitur ad hanc formam: $(q\,q + r\,r)(q\,d\,p - p\,d\,q)$, denominator vero ad hanc $(q\,q + r\,r)p\,d\,r - r\,d\,p)$, vnde fit

$$\text{tang.}\, b\,a\,m = \frac{q\,d\,p - p\,d\,q}{p\,d\,r - r\,d\,p},$$

et quia angulus $b\,a\,c$ est rectus, erit

$$\text{tang.}\, m\,a\,c = \frac{p\,d\,r - r\,d\,p}{q\,d\,p - p\,d\,q}.$$

§. 12. Quia nunc arcus $a n$ eſt quadrans circuli et arcui $b c$ normaliter inſiſtit, erit

$$\text{coſ.}\; m n = \frac{q\,d\,r - r\,d\,q}{\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}},$$

vnde quia ambo arcus $m p$ et $n p$ eidem arcui $m n$ normaliter inſiſtunt, ambo erunt quadrantes et arcus $m n$ erit menſura anguli $m p n$, ſicque erit

$$\text{coſ.}\; m p n = \text{coſ.}\; q p c = \frac{q\,d\,r - r\,d\,q}{\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}},$$

atque ſub hoc angulo planum quaeſitum inclinatur ad planum $b c$, ſiue in quinta figura ad planum B O C. At vero arcus $a m$ metietur angulum, ſub quo iſtud planum ad axem O A inclinatur, cuius ergo ſinus erit

$$\text{ſin.}\; a m = \frac{q\,d\,r - r\,d\,q}{\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}}.$$

§. 13. Inuento autem angulo $z p b$, quo planum quaeſitum ad planum $b c$ inclinatur, per analogiam concludemus angulos, ſub quibus ad reliqua bina latera inclinatur, ſcilicet coſinus inclinationis ad planum $b c$, hoc eſt ad planum C A, erit

$$= \frac{r\,d\,p - p\,d\,r}{\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}},$$

et coſinus inclinationis ad planum A B

$$= \frac{p\,d\,q - q\,d\,p}{\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}}.$$

At analogiam ſequentes, quia prima formula erat coſ. $c p z$ ſecundum ordinem litterarum a, b, c et p, q, r progrediendo, quoniam in poſtremis formulis dubium eſſe poteſt, an pertineant ad angulos $a q z$ et $a r z$, an potius ad angulos $p q z$ et $b r z$, ambiguitas tolletur hoc modo

$$\text{coſ.}\; c p z = \frac{q\,d\,r - r\,d\,q}{\sqrt{(d\,p^2 + d\,q + d\,r^2)}};$$

coſ.

$$\operatorname{cos}. aqz = \frac{rdp - pdr}{\sqrt{(dp^2 + dq^2 + dr^2)}},$$
$$\operatorname{cos}. brz = \frac{pdq - qdp}{\sqrt{(dp^2 + dq^2 + dr^2)}}.$$

Interim tamen analogiae confidere non possumus, quia punctum p extra nostrum triangulum cadit, vnde inuestigationem sequenti modo instituamus.

§. 14. Omni ambiguitati occurremus, si resoluamus triangulum azr, in quo cognita sunt latus az, cum angulis zar et azr, vnde fiet

$$\operatorname{cos}. arz = \frac{pqrdr - prrdq + qdp}{qq + rr\sqrt{(dp^2 + dq^2 + dr^2)}},$$

quae expressio ob $rdr = -pdp - qdq$ transmutatur in hanc:

$$\operatorname{cos}. arz = \frac{qdp - pdq}{\sqrt{(dp^2 + dq^2 + dr^2)}}.$$

Praeterea vero etiam hinc cognoscemus

$$\operatorname{tang}. ar = \frac{qdr - rdq}{pdr - rdp} \text{ et}$$
$$\operatorname{tang}. zr = \frac{r\sqrt{(dp^2 + dq^2 + dr^2)}}{dr}.$$

§. 15. Simili modo resolui poterit triangulum zaq, in quo praeter latus az pariter dantur ambo anguli zaq et azq: reperitur enim

$$\operatorname{cos}. aqz = \frac{pdr - rdp}{\sqrt{(dp^2 + dq^2 + dr^2)}},$$
$$\operatorname{tang}. aq = \frac{qdr - rdq}{qdp - pdq} \text{ et}$$
$$\operatorname{tang}. qz = \frac{q\sqrt{(dp^2 + dq^2 + dr^2)}}{-dq}.$$

Denique iam vidimus arcum cn esse mensuram anguli cam, et quia arcus cb et np sunt quadrantes, erit arcus $bp = cn$, vnde erit

$$\operatorname{tang}. bp = \operatorname{tang}. mac = \frac{pdr - rdp}{qdr - rdq},$$

vnde porro fit

$$\operatorname{tang}. cp = \frac{rdp - qdr}{pdr - rdp}.$$

§. 16. Quae igitur hactenus inuenimus sequenti modo aspectui exponamus; ac primo quidem pro angulis, quae ex punctis p, q, r formantur, habebimus:

$$\cos. bpz = \cos. cpz = \frac{q\,dr - r\,dq}{\sqrt{(dp^2 + dq^2 + dr^2)}},$$

$$\cos. cqz = -\cos. aqz = \frac{r\,dp - p\,dr}{\sqrt{(dp^2 + dq^2 + dr^2)}},$$

$$\cos. arz = -\cos. brz = \frac{q\,dp - p\,dq}{\sqrt{(dp^2 + dq^2 + dr^2)}}.$$

Deinde pro positione ipsorum punctorum p, q, r habebimus:

$$\text{tang. } bp = \frac{p\,dr - r\,dp}{q\,dr - r\,dq};\quad \text{tang. } cp = \frac{r\,dq - q\,dr}{p\,dr - r\,dp},$$

$$\text{tang. } cq = \frac{q\,dp - p\,dq}{q\,dr - r\,dq};\quad \text{tang. } aq = \frac{q\,dr - r\,dq}{q\,dp - p\,dq},$$

$$\text{tang. } ar = \frac{q\,dr - r\,dq}{p\,dr - a\,dp};\quad \text{tang. } br = \frac{p\,dr - r\,dp}{q\,dr - r\,dq}.$$

Euidens autem est, si ducerentur arcus pa, qb et rc, eos fore quadrantes. Denique arcus pz commode reperitur ex triangulo azp, in quo latus ap est quadrans, ac praeterea datur latus az, vna cum angulis azp et zpa, quippe cuius sinus cosinui zpb aequatur, vnde reperitur:

$$\text{tang. } pz = \frac{p\sqrt{(dp^2 + dq^2 + dr^2)}}{dp},$$

iam vero erat

$$\text{tang. } qz = -\frac{q\sqrt{(dp^2 + dq^2 + dr^2)}}{dq},$$

$$\text{tang. } rz = \frac{r\sqrt{(dp^2 + dq^2 + dr^2)}}{dr}.$$

§. 17. Inuentis nunc omnibus, quae ad positionem plani, in quo bina curuae elementa contigua incurvantur, et quod circulo maximo $qzrp$ continetur, spectant, etiam in positionem radii osculi, cuius quantitatem iam invenimus, inquiramus, qui quoniam ad radium Zz est normalis et in ipso plano circuli $qzrp$ situs, eius positio tangenti

genti huius circuli in z erit parallela et, quia ipſi centro ſphaerae Z applicatus eſt cenſendus, ductus concipiatur ex centro Z radius tangenti circuli in z parallelus atque in eius directione ſitus erit radius oculi.

§. 18. Tranſeat igitur iſte radius Sphaerae, tangenti in z parallelus, per punctum R, quod vtique ſitum erit in circulo $p\,r\,q$ continuato, ac manifeſtum eſt arcum $z\,r$ fore quadrantem. Hinc ductus arcus $a\,r$ metietur inclinationem radii oſculi ad axem $Z\,a$, ſiue ad axem OA, quem arcum ex triangulo $a\,z\,R$ definire licebit, cum ſit $z\,R = 90°$ et angulus $R\,z\,a = 180° - \Phi$, latus autem $a\,z$ detur, vnde colligitur Tab. II. Fig. 1.

$$\text{cof.}\, a\,R = \frac{d\,p}{\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}},$$

tum vero erit

$$\text{tang.}\, z\,a\,R = \frac{r\,d\,q - q\,d\,r}{p\,d\,p}.$$

Hinc auferatur angulus $z\,a\,q$, cuius tang. $= \frac{q}{r}$, ac remanebit tang. $q\,a\,R = -\frac{d\,q}{d\,r}$; vnde ſi ducatur arcus $c\,R$, in triangulo $R\,a\,c$ dantur duo latera $a\,c$ et $a\,R$ cum angulo intercepto, vnde colligitur

$$\text{cof.}\, c\,R = \text{ſin.}\, a\,R\, \text{cof.}\, q\,a\,R = \frac{d\,r}{\sqrt{(d\,p^2 + d\,q^2 + d\,r^2)}}.$$

Eademque modo reperietur

$$\text{cof.}\, b\,R = -\frac{d\,q}{\sqrt{(d\,p^2 + d\,p^2 + d\,r^2)}}.$$

Cognitis autem angulis, ſub quibus radius oſculi ad ternos axes principales OA, OB, OC, inclinatur, eorum complementa ad 90° dabunt eius inclinationes ad plana oppoſita, ſcilicet BOC, AOC, AOB. Sicque poſitio radii

oſculi

osculi, cuius quantitas est $= \frac{ds}{\sqrt{(dp^2 + dq^2 + dr^2)}}$, perfecte determinatur.

§. 19. Hae autem determinationes adhuc faciliores reddi possunt, si, quemadmodum inclinationem circuli qzr ad arcum az dedimus, eiusdem quoque inclinationem ad arcus bz et cz definiamus. Hunc in finem quaeramus angulum azb, et ex triangulo azb, quatenus tria laterae sunt cognita, erit

$$\text{cos.}\, azb = \frac{-pq}{\sqrt{(1-pp)(1-qq)}}.$$

Deinde si fiat in eodem triangulo

$$\text{sin.}\, az : \text{sin.}\, abz = \text{sin.}\, ab : \text{sin.}\, azb,$$

inde colligitur:

$$\text{sin.}\, azb = \frac{r}{\sqrt{(1-pp)(1-pp)}},$$

hincque porro tang. $azb = -\frac{r}{pq}$. Eodem modo erit

$$\text{cos.}\, bzc = \frac{-qr}{\sqrt{(1-qq)(1-rr)}} \text{ et}$$

$$\text{sin.}\, bzc = \frac{p}{\sqrt{(1-qq)(1-rr)}},$$

hincque tang. $bzc = \frac{-p}{qr}$. Ac denique

$$\text{cos.}\, cza = \frac{-pr}{\sqrt{(1-pp)(1-rr)}},$$

$$\text{sin.}\, cza = \frac{q}{\sqrt{(1-pp)(1-rr)}},$$

hincque tang. $cza = \frac{-q}{pr}$.

§. 20. Quoniam igitur supra inuenimus esse

$$\text{cos.}\, azr = \frac{-dp}{\sqrt{(qq+rr)(dp^2+dq^2+dr^2)}},$$

$$\text{sin.}\, azr = \frac{qdr - rdq}{\sqrt{(qr+rr)(dp^2+dq^2+dr^2)}},$$

$$\text{tang.}\, azr = \frac{rdq - qdr}{dp},$$

subtra-

ſubtrahamus hunc angulum azr ab angulo azb, ac reperiemus

$$\text{tang. } bzr = \frac{-r\,dp - pqr\,dq + pqq\,dr}{pq\,dp - rr\,dq + qr\,dr} = \frac{r\,dp - p\,dr}{dq}.$$

Eodem modo cum ſit

$$\text{tang. } azq = \frac{q\,dr - r\,dq}{dp},$$

ſubtrahatur iſte angulus ab angulo azc, cuius tangens eſt $\frac{-q}{pr}$, ac reperietur

$$\text{tang. } czq = \frac{-q\,dp - pqr\,dr + prr\,dq}{pr\,dp - qq\,dr + qr\,dq} = \frac{q\,dp - p\,dq}{dr}.$$

§. 21. Quoniam igitur omnia determinauimus, quae ad poſitionem tam plani, in quo duo elementa proxima curuae inclinantur, quam radii oſculi ſpectant, atque adeo multo vberius quam per methodum comunem fieri ſolet, coronidis loco ſubiungamus quaedam theoremata ad doctrinam ſphaericam pertinentia, ad quae iſta tractatio perduxit et quae omni attentione digna videntur.

Theorema I.

§. 22. *Propoſito triangulo ſphaerico* abc, *cuius omnia latera ſunt quadrantes, ſi vel intra id vel extra capiatur punctum quodcunque* z, *ex eoque ad angulos trianguli educantur arcus* za, zb, zc, *ſemper erit* Tab II. Fig. 2

$$\text{coſ. } za^2 + \text{coſ. } zb^2 + \text{coſ. } zc^2 = 1.$$

Demonſtratio.

Ex triangulo azb, ſi eius latera vt cognita ſpectentur, reperitur

$$\text{cos}. \, b\,a\,z = \frac{\text{cos}. \, b\,z}{\text{sin}. \, a\,z};$$

eodem modo ex triangulo $a\,z\,c$ erit

$$\text{cos}. \, c\,a\,z = \frac{\text{cos}. \, c\,z}{\text{sin}. \, a\,z},$$

vnde quia angulus $b\,a\,c$ eſt rectus, ſi quadrata addantur prodit

$$\text{cos}. \, b\,a\,z^2 + \text{cos}. \, c\,a\,z^2 = 1 = \frac{\text{cos}. \, b\,z^2 + \text{cos}. \, c\,z^2}{\text{sin}. \, a\,z^2},$$

ideoque

$$\text{sin}. \, a\,z^2 = \text{cos}. \, b\,z^2 + \text{cos}. \, c\,z^2 = 1 - \text{cos}. \, a\,z^2,$$

vnde manifeſto fit

$$\text{cos}. \, a\,z^2 + \text{cos}. \, b\,z^2 + \text{cos}. \, c\,z^2 = 1. \quad \text{Q. E. D.}$$

Theorema II.

Tab. II. Fig. 2. §. 23. *Propoſito triangulo ſphaerico* a b c, *cuius omnia latera ſint quadrantes, ſi intra vel extra id capiatur punctum quodcunque* z, *ex eoque in ſingula latera ducantur arcus perpendiculares* z p, z q, z r, *ſemper erit*

$$\text{sin}. \, z\,p^2 + \text{sin}. \, z\,q^2 + \text{sin}. \, z\,r^2 = 1.$$

Demonſtratio.

Euidens eſt haec perpendicula oriri, ſi praecedentes arcus $a\,z$, $b\,z$, $c\,z$ continuentur, vnde quia arcus $a\,p$, $b\,q$, $c\,r$ ſunt quadrantes erit

$$\text{sin}. \, z\,p^2 = \text{cos}. \, z\,a^2,$$

$$\text{sin}. \, z\,q^2 = \text{cos}. \, z\,b^2 \text{ et}$$

$$\text{sin}. \, z\,r^2 = \text{cos}. \, z\,c^2;$$

quare cum modo inuenerimus

$$\text{cos}. \, z\,a^2 + \text{cos}. \, z\,b^2 + \text{cos}. \, z\,c^2 = 1, \text{ erit vtique}$$

$$\text{sin}. \, z\,p^2 + \text{sin}. \, z\,q^2 + \text{sin}. \, z\,r^2 = 1. \quad \text{Q. E. D.}$$

Theo-

Theorema III.

§. 24. *Proposito triangulo sphaerico* a b c, *cuius omnia latera sint quadrantes, si ducatur circulus maximus quicunque* $\beta\gamma\alpha$, *qui secet latera trianguli, si opus est producta, in punctis* α, β, γ, *is ad haec latera ita inclinabitur, vt sit* Tab. II. Fig. 3.

$$\text{cof. } b\alpha\gamma^2 + \text{cof. } a\beta\gamma^2 + \text{cof. } a\gamma\beta^2 = 1.$$

Demonstratio.

In triangulo $\beta a \gamma$, ad a rectangulo, erit

$$\text{cof. } a\beta\gamma = \text{fin. } a\beta\gamma \text{ cof. } a\gamma;$$

similique modo in triangulo $b\alpha\gamma$, ad b rectangulo, erit

$$\text{cof. } b\alpha\gamma = \text{fin. } b\gamma\alpha \text{ cof. } b\gamma.$$

Addantur nunc quadrata harum duarum formularum eritque

$$\text{cof. } b\alpha\gamma^2 + \text{cof. } a\beta\gamma^2 = \text{fin. } a\gamma\beta^2 \text{ cof. } a\gamma^2 + \text{fin. } b\gamma\alpha^2 \text{ cof. } b\gamma^2,$$

et quia

$$b\gamma\alpha = a\gamma\beta \text{ et cof. } a\gamma^2 + \text{cof. } b\gamma^2 = 1,$$

erit

$$\text{cof. } b\alpha\gamma^2 + \text{cof. } a\beta\gamma^2 = \text{fin. } a\gamma\beta^2 = 1 - \text{cof. } a\gamma\beta^2,$$

vnde sequitur, quod demonstrari debet, fore

$$\text{cof. } b\alpha\gamma^2 + \text{cof. } a\beta\gamma^2 + \text{cof. } a\gamma\beta^2 = 1.$$

Theorema IV.

§. 25. *Proposito triangulo sphaerico* a b c, *cuius singula latera sint quadrantes, si pro lubitu ducatur arcus circuli maximi quicunque* $\alpha\beta\gamma$, *in eumque ex angulis* a b c *ducantur arcus perpendiculares* a p, b q, c r *semper erit*

$$\text{fin. } ap^2 + \text{fin. } bq^2 + \text{fin. } cr^2 = 1.$$

Demonſtratio.

Ducantur arcus αa, βb et γc, qui erunt quadrantes, quoniam omnes arcus ex a in latus $b\,c$ ducti ſunt quadrantes, ac praeterea etiam normaliter inſiſtunt. Hinc quia anguli ad p, q et r ſunt etiam recti, erunt quoque arcus αp, βq et γr quadrantes, vnde ſequitur fore arcum $a\,p =$ angulo $a\,\alpha\,p$, ideoque

$$\text{ſin.}\ a\,p = \text{ſin.}\ a\,\alpha\,p = \text{coſ.}\ b\,\alpha\,\gamma.$$

Eodem modo erit $b\,q = b\,\beta\,q$, ideoque

$$\text{ſin.}\ b\,q = \text{ſin.}\ b\,\beta\,q = \text{coſ.}\ a\,\beta\,\gamma;$$

denique erit $c\,r = c'\,\gamma\,r$ hincque

$$\text{ſin.}\ c\,r = \text{ſin.}\ c\,\gamma\,r = \text{coſ.}\ a\,\gamma\,\beta.$$

Quare cum ſumma quadratorum horum coſinuum ſit $= 1$, erit quoque

$$\text{ſin.}\,a\,p^2 + \text{ſin.}\,b\,q^2 + \text{ſin.}\,c\,r^2 = 1.\ \text{Q. E. D.}$$

ME-

METHODVS FACILIS
OMNIA SYMPTOMATA
LINEARVM CVRVARVM
NON IN EODEM PLANO SITARVM
INVESTIGANDI.

Differtatio altera.

§. 1.

Quod fi forte methodus ante tradita et doctrinae fphaericae innixa minus placeat, vtpote ex principiis alienis deducta; aliam hic fum propofiturus methodum, ex principio magis directo petitam, qua omnia fymptomata linearum curuarum non in eodem plano fitarum etiam multo fuccinctius et clarius demonftrari poffunt quam methodo hactenus vfitata. Hanc autem methodum mihi fuppeditauit confideratio illius plani, in quo bina curuae elementa contigua funt fita, et in quo proinde tam tangens curuae quam eius radius curuaturae exiftere debent; vnde totam inueftigationem ex confideratione huius plani fum petiturus.

Tab. II. Fig. 4. §. 2. Constitutis igitur vt ante ternis axibus principalibus O A, O B, O C, quibus coordinatae, quoduis punctum curuae z determinantes, sint parallelae, scilicet

$$O\,x = x,\; x\,y = y,\; y\,z = z,$$

statim in calculum introducam illud planum, quod per bina curuae elementa contigua determinatur, siue in quo curua hoc loco incuruatur. Secet igitur istud planum ternos nostros axes in punctis u, v et w, ac ponamus distantias $O\,u = u$, $O\,v = v$, $O\,w = w$, quandoquidem his tribus quantitatibus positio plani penitus determinatur. Quatenus igitur curuae punctum z in hoc ipso plano reperitur, aequatio inter ternas coordinatus x, y, z erit vnius dimensionis, huius formae, $A\,x + B\,y + C\,z = D$, ad quam penitus determinandam transferamus primo punctum z in ipsum punctum u fietque $x = u$, $y = 0$ et $z = 0$, vnde erit $A\,u = D$. Simili modo, translato puncto z in ipsum punctum v, fiet $x = 0$, $y = v$, $z = 0$, vnde erit $B\,v = D$. Denique translato puncto z in ipsum punctum w, erit $z = w$, $x = 0$, $y = 0$, vnde erit $C\,w = D$. Cum igitur hinc fit

$$A = \frac{D}{u},\; B = \frac{D}{v},\; C = \frac{D}{w},$$

erit aequatio localis pro plano $u\,v\,w$ haec:

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 1.$$

§. 3. Quod si nunc vt ante elementum curuae statuatur $= d\,s$, ac per hoc elementum reliqua ita definiantur, vt sit

$$d\,x = p\,d\,s,\; d\,y = q\,d\,s,\; d\,z = r\,d\,s,$$

erit vtique $p\,p + q\,q + r\,r = 1$, ac differentiando

$p\,d\,p$

$$p\,dp + q\,dq + r\,dr = 0.$$

Transferatur nunc punctum curuae z in proximum, cui respondeant coordinatae $x + dx$, $y + dy$, $z + dz$, quoniam istud punctum proximum denuo in plano $u\,v\,w$ est situm, erit etiam

$$\frac{x + dx}{u} + \frac{y + dy}{v} + \frac{z + dz}{w} = 1,$$

a qua aequatione si prior subtrahatur, habebimus:

$$\frac{dx}{u} + \frac{dy}{v} + \frac{dz}{w} = 0, \text{ ideoque } \frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0.$$

Denique quia etiam sequens elementum in idem planum incidere debet, erit etiam denuo differentiando

$$\frac{dp}{u} + \frac{dq}{v} + \frac{dr}{w} = 0.$$

Vlterius vero differentiando progredi non licet, quia sequentia elementa vtique in aliud planum incidere possunt.

§. 4. Consideratio igitur istius plani $u\,v\,w$ cum indole curuae coniuncta nobis suppeditauit has tres aequationes:

$$\text{I. } \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 1.$$

$$\text{II. } \frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0.$$

$$\text{III. } \frac{dp}{u} + \frac{dq}{v} + \frac{dr}{w} = 0.$$

ex quibus aequationibus vicissim ternas quantitates u, v et w per coordinatas curuae earumque differentialia determinare poterimus. Hunc in finem adhibeamus hanc combinationem: II. dr — III. r, quae nobis praebebit

$$\frac{p\,dr - r\,dp}{u} + \frac{q\,dr - r\,dq}{v} = 0, \text{ siue}$$

$$\frac{p\,dr - r\,dp}{u} = \frac{r\,dq - q\,dr}{v},$$

vnde

vnde deducitur haec proportio:

$$\frac{1}{u} : \frac{1}{v} = (r\,d\,q - q\,d\,r) : (p\,d\,r - r\,d\,p).$$

Statuamus igitur

$$\frac{1}{u} = \frac{r\,d\,q - q\,d\,r}{t}, \text{ erit } \frac{1}{v} = \frac{p\,d\,r - r\,d\,p}{t},$$

qui valores in ſecunda aequatione ſubſtituti dant

$$r\,\frac{(p\,d\,q - q\,d\,p)}{t} + \frac{r}{w} = 0, \text{ vnde colligimus}$$

$$\frac{1}{w} = \frac{q\,d\,p - p\,d\,q}{t}.$$

§. 5. Ex his autem valoribus pro $\frac{1}{u}$, $\frac{1}{v}$, $\frac{1}{w}$ inuentis, ſi ii in prima aequatione ſubſtituantur, perueniemus ad hanc aequationem:

$$x\,(r\,d\,q - q\,d\,r) + y\,(p\,d\,r - r\,d\,p) + z\,(q\,d\,p - p\,d\,q) = t,$$

quae nobis igitur valorem quantitatis t exhibet, ita vt deinceps hac littera t tanquam cognita vti queamus, vbi quidem neceſſe erit valores formularum $\frac{1}{u}$, $\frac{1}{v}$, $\frac{1}{w}$ in calculum introducere.

Tab. II. Fig. 5. §. 6. Nunc igitur determinata poſitione plani $u\,v\,w$ inueſtigemus eius inclinationem ad terna noſtra plana principalia A O B, B O C, C O A. Ac primo quidem quia iſtud planum ſecat planum A O B per lineam $u\,v$, quae eſt $= \sqrt{(u\,u + v\,v)}$, ex O ducatur in hanc $u\,v$ perpendiculum O r, eritque $u\,r$: O u = Ov : O r, vnde fit O $r = \frac{u\,v}{\sqrt{(u\,u + v\,v)}}$. Iungatur nunc recta $r\,w$, et quia triangulum O $w\,r$ eſt verticale, recta $w\,r$ quoque normalis erit ad rectam $u\,v$, vnde angulus O $r\,w$ dabit inclinationem plani $u\,v\,w$ ad planum A O B; quare cum angulus wOr ſit rectus et O $w = w$, erit hypothenuſa

$r\,w$

$$r\,w = \sqrt{\left(\frac{u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w}{u\,u + v\,v}\right)},$$

ex qua colligimus

$$\text{cof. } O\,r\,w = \frac{O\,r}{r\,w} = \frac{u\,v}{\sqrt{(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)}}, \text{ fiue}$$

$$\text{cof. } O\,r\,w = \frac{1}{\sqrt{\left(1 + \frac{w\,w}{u\,u} + \frac{w\,w}{v\,v}\right)}} = \frac{1}{w\sqrt{\left(\frac{1}{u\,u} + \frac{1}{v\,v} + \frac{1}{w\,w}\right)}}.$$

Sicque innotefcit inclinatio plani $u\,v\,w$ ad planum AOB, quippe cuius cofinus eft $\frac{1}{w\sqrt{\left(\frac{1}{u\,u} + \frac{1}{v\,v} + \frac{1}{w\,w}\right)}}$. Simili modo per analogiam concluditur inclinatio plani $u\,v\,w$ ad planum BOC, cuius cofinus eft $\frac{1}{u\sqrt{\left(\frac{1}{u\,u} + \frac{1}{v\,v} + \frac{1}{w\,w}\right)}}$; ac denique inclinatio plani $u\,v\,w$ ad planum COA, cuius cofinus eft $\frac{1}{v\sqrt{\left(\frac{1}{u\,u} + \frac{1}{v\,v} + \frac{1}{w\,w}\right)}}$.

§. 7. Cum igitur fit

$$\frac{1}{u} = \frac{r\,d\,q - q\,d\,r}{t},$$

$$\frac{1}{v} = \frac{p\,d\,r - r\,d\,p}{t} \text{ et}$$

$$\frac{1}{w} = \frac{q\,d\,p - p\,d\,q}{t}, \text{ erit}$$

$$\frac{1}{u\,u} + \frac{1}{v\,v} + \frac{1}{w\,w} = \frac{d\,p^2(q\,q + r\,r) + d\,q^2(p\,p + r\,r) + d\,r^2(p\,p + q\,q)}{t\,t}$$
$$- \frac{2\,p\,q\,d\,p\,d\,q - 2\,p\,r\,d\,p\,d\,r - 2\,q\,r\,d\,q\,d\,r}{t\,t},$$

quia vero eft

$$q\,q + r\,r = 1 - p\,p;\; p\,p + r\,r = 1 - q\,q \text{ et}$$
$$p\,p + q\,q = 1 - r\,r,$$

his valoribus substitutis formula ista in duas partes distribuetur, alteram positiuam, alteram negatiuam; ac positiua quidem erit $\frac{dp^2 + dq^2 + dr^2}{tt}$, termini vero negatiui erunt

$$\frac{-ppdp^2 - qqdq^2 - rrdr^2 - 2pqdpdq - 2prdpdr - 2qrdqdr}{tt},$$

quae forma cum manifesto sit quadratum huius formulae: $\frac{pdp + qdq + rdr}{t}$, ea ob

$$pdp + qdq + rdr = 0,$$

sponte euanescit, ita vt habeamus:

$$\sqrt{\left(\frac{1}{uu} + \frac{1}{vv} + \frac{1}{ww}\right)} = \sqrt{\left(\frac{dp^2 + dq^2 + dr^2}{t}\right)},$$

quamobrem inclinationes supra inuentae ita succincte exprimentur:

I. cos. incl. ad planum A O B $= \frac{qdp - pdq}{\sqrt{(dp^2 + dq^2 + dr^2)}}$.

II. cos. incl. ad planum B O C $= \frac{rdq - qdr}{\sqrt{(dp^2 + dq^2 + dr^2)}}$.

III. cos. incl. ad planam C O A $= \frac{pdr - rdp}{\sqrt{(dp^2 + dq^2 + dr^2)}}$.

Sicque iam praecipua symptomata sumus nacti, quae praecedente methodo demum post plures ambages sunt eruta.

§. 8. Quo nunc etiam reliqua symptomata facilius eliciamus, ponamus breuitatis gratia angulum $Ouv = \alpha$ atque habebimus

$$\sin.\ \alpha = \frac{v}{\sqrt{(uu + vv)}},\ \cos.\ \alpha = \frac{u}{\sqrt{(uu + vv)}}\ \text{et}\ \tang.\ \alpha = \frac{v}{u}.$$

Deinde ponatur inclinatio seu angulus $Orw = \theta$, eritque

$$\sin.\ \theta = \frac{Ow}{rw} = \frac{w\sqrt{(uu + vv)}}{\sqrt{(uuvv + uuww + vvww)}},$$

$$\cos.\ \theta = \frac{Or}{rw} = \frac{uv}{\sqrt{(uuvv + uuww + vvww)}}\ \text{et}$$

$$\tang.\ \theta = \frac{w\sqrt{(uu + vv)}}{uv},$$

vbi

vbi notandum eſt hos valores manere conſtantes, dum a puncto curuae z per duo elementa ſequentia progredimur.

§. 9. His praemiſſis inueſtigemus locum puncti z in ipſo plano $u\,v\,w$, et cum ſit $Ox = x$, $xy = y$, $yz = z$, ex punctis x et y ad rectam uv ducamus perpendicula xp et yq, et quoniam $xu = u - x$ et angulus $xup = \alpha$, erit $xp = (u - x)\,\text{ſin.}\,\alpha$ et $up = (u - x)\,\text{coſ.}\,\alpha$. Tum vero ex y in xp demiſſo perpendiculo yo, erit $yo = y\,\text{ſin.}\,\alpha$ et $xo = y\,\text{coſ.}\,\alpha$. Quare cum ſit $pq = yo$ et $yq = po$, fiet interuallum Tab II Fig. 4.

$$uq = (u - x)\,\text{coſ.}\,\alpha + y\,\text{ſin.}\,\alpha \text{ et}$$
$$yq = (u - x)\,\text{ſin.}\,\alpha - y\,\text{coſ.}\,\alpha.$$

Nunc igitur quia $yz = z$ eſt verticalis, ducta recta qz triangulum yzq non ſolum plano AOB normaliter inſiſtit, ſed etiam rectae yq et zq ad rectam uv erunt perpendiculares, et angulus yqz ipſi inclinationi θ aequabitur, vnde erit recta

$$qz = \frac{(u-x)\,\text{ſin.}\,\alpha - y\,\text{coſ.}\,\alpha}{\text{coſ.}\,\theta},$$

vel etiam erit

$$qz = \frac{yz}{\text{ſin.}\,\theta} = \frac{z}{\text{ſin.}\,\theta},$$

vnde has duas formulas aequales eſſe neceſſe eſt, ex quo ſequitur fore

$$\frac{z\sqrt{(uuvv + uuww + vvww)}}{w\sqrt{(uu + vv)}} = \frac{(u-x)\,\text{ſin.}\,\alpha - y\,\text{coſ.}\,\alpha)\sqrt{(uuvv + uuww + vvww)}}{uv},$$

ſiue

$$\frac{z}{w\sqrt{(uu + vv)}} = \frac{(u-x)\,v - uy}{uv\sqrt{(uu + vv)}},$$

quae aequatio reducitur ad hanc: $\frac{z}{w} = \frac{(u-x)}{u} - \frac{y}{v}$, quae eſt

ipſa

ipſa aequatio primo conſtituta, ſcilicet

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 1.$$

§. 10. Quoniam igitur recta zq in ipſo plano uvw ad rectam uv eſt normalis, in hoc plano pro puncto z ſpectemus rectam uq tanquam abſciſſam et qz tanquam applicatam, et vocemus $uq = X$ et $qz = Y$, quocirca habebimus

$$X = (u - x) \operatorname{coſ.} \alpha + y \operatorname{ſin.} \alpha \text{ et}$$

$$Y = \frac{(u - x) \operatorname{ſin.} \alpha - y \operatorname{coſ.} \alpha}{\operatorname{coſ.} \theta} = \frac{z}{\operatorname{coſ.} \theta}.$$

Quae quo clarius perſpici queant, hoc planum uvw cum puncto z ſeorſim in tabulam coniiciamus.

Tab. II. Fig. 6. §. 11. Nunc igitur differentiando per elementum curuae zz' progrodiamur, atque ob

$$dx = p\,ds, \; dy = q\,ds \text{ et } dz = r\,ds,$$

quoniam quantitates u, v, w, cum angulis α et θ manent conſtantes, erit

$$dX = -p\,ds \operatorname{coſ.} \alpha + q\,ds \operatorname{ſin.} \alpha \text{ et}$$

$$dY = -\frac{p\,ds \operatorname{ſin.} \alpha - q\,ds \operatorname{coſ.} \alpha}{\operatorname{coſ.} \theta} = \frac{r\,ds}{\operatorname{ſin.} \theta},$$

quos duos valores ipſius dY pariter aequales eſſe neceſſe eſt, ideoque erit

$$-\frac{p \operatorname{ſin.} \alpha - q \operatorname{coſ.} \alpha}{\operatorname{coſ.} \theta} = \frac{r}{\operatorname{ſin.} \theta},$$

quae aequatio, ſubſtitutis valoribus, tranſit in hanc:

$$\frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0,$$

quae eſt ipſa aequatio ſecunda initio conſtituta. Praeterea, cum poſuerimus elementum curuae $= ds$, quod nunc ex

ex coordinatis X et Y fit $ds = \sqrt{(dX^2 + dY^2)}$, necesse est vt euadat $dX^2 + dY^2 = ds^2$. Est vero

$$dX = \frac{ds\,(qv - pu)}{\sqrt{(uu + vv)}} \text{ et}$$

$$dY = -\frac{ds\,(pv + qu)\,\sqrt{(uuvv + uuww + vvww)}}{uv\sqrt{(uu + vv)}},$$

vel etiam

$$dY = \frac{r\,ds\,\sqrt{(uuvv + uuww + vvww)}}{w\,(\sqrt{uu + vv})}.$$

Hinc autem erit

$$\frac{dX^2 + dY^2}{ds^2} = \frac{uuvv\,(qqvv - 2pquv + ppuu) + (ppvv + 2pquv) + qquu\,(qquu + uuww + vvww)}{uuvv\,(uu + vv)},$$

quae expressio reducitur ad hanc:

$$\frac{dX^2 + dY^2}{ds^2} = \frac{pp\,(uu + ww)}{uu} + \frac{qq\,(vv + ww)}{vv} + \frac{2pqww}{uv}$$

$$= (pp + qq) + ww\left(\frac{p}{u} + \frac{q}{v}\right)^2.$$

Cum autem sit $\frac{p}{u} + \frac{q}{v} = -\frac{r}{w}$, resultabit haec forma:

$$\frac{dX^2 + dY^2}{ds^2} = pp + qq + rr = 1.$$

Sicque patet, reuera esse $dX^2 + dY^2 = ds^2$.

§. 12. Quod si iam statuamus $dY = P\,dX$, notum est radium osculi curuae, quatenus cadet in rectam $zN\Omega$, esse $z\Omega = -\frac{dX\,(1 + PP)^{\frac{3}{2}}}{dP}$, ita vt Ω sit centrum circuli nostram curuam in z osculantis. Modo autem vidimus reuera fieri $dX\sqrt{(1 + PP)} = ds$, vnde cum sit $dX = \frac{ds\,(qv - pu)}{\sqrt{uu + vv}}$, erit

$$\sqrt{(1 + PP)} = \frac{ds}{dX} = \frac{\sqrt{(uu + vv)}}{qv - pu}, \text{ siue}$$

$$\frac{1}{\sqrt{(1 + PP)}} = \frac{qv - pu}{\sqrt{(uu + vv)}},$$

 vnde

vnde differentiando colligimus:

$$-\frac{\mathrm{P}\,d\,\mathrm{P}}{(1+\mathrm{P\,P})^{\frac{3}{2}}}=\frac{v\,d\,q-u\,d\,p}{\sqrt{(u\,u+v\,v)}},$$

vnde deducitur

$$(1+\mathrm{P\,P})^{\frac{3}{2}}=\frac{\mathrm{P}\,d\,\mathrm{P}\sqrt{(u\,u+v\,v)}}{u\,d\,p-v\,d\,q},$$

atque hinc expreſſio pro radio oſculi erit

$$z\,\Omega=\frac{\mathrm{P}\,d\,\mathrm{X}\sqrt{(u\,u+v\,v)}}{v\,d\,q-u\,d\,p}-\frac{d\,\mathrm{Y}\sqrt{(u\,u+v\,v)}}{v\,d\,q-u\,d\,p},$$

ac loco $d\,\mathrm{Y}$ ſubſtituto valore obtinebitur tandem radius oſculi

$$z\,\Omega=\frac{d\,s\,(p\,v+q\,u)\sqrt{(u\,u\,v\,v+u\,u\,w\,w+v\,v\,w\,w)}}{u\,v\,(u\,d\,p-v\,d\,q)}.$$

§. 13. Quo nunc hanc expreſſionem radii oſculi ab elementis u, v et w liberemus, diuidamus numeratorem et denominatorem ſeorſim per $u\,v\,w$, et obtinebimus numeratorem

$$=d\,s\,(p\,v+q\,u)\sqrt{(\tfrac{1}{u\,u}+\tfrac{1}{v\,v}+\tfrac{1}{w\,w})},$$

denominator vero erit $\frac{u\,d\,p-v\,d\,q}{w}$. Supra autem iam oſtendimus eſſe

$$\sqrt{(\tfrac{1}{u\,u}+\tfrac{1}{v\,v}+\tfrac{1}{w\,w})}=\frac{\sqrt{(d\,p^2+d\,q^2+d\,r^2)}}{t},$$

vnde fit numerator

$$=d\,s\,(\tfrac{p\,v}{t}+\tfrac{q\,u}{t})\sqrt{(d\,p^2+d\,q^2+d\,r^2)}.$$

Cum nunc ſit

$$\frac{v}{t}=\frac{1}{-p\,d\,r-r\,d\,p}\ \text{et}\ \frac{u}{t}=\frac{1}{r\,d\,q-q\,d\,r},$$

erit noſter numerator

$d\,s$

$$ds\left(\frac{p}{p\,dr - r\,dp} + \frac{q}{r\,dq - q\,dr}\right)\sqrt{(dp^2 + dq^2 + dr^2)}$$
$$= \frac{r\,ds\,(p\,dq - q\,dp)}{(p\,dr - r\,dp)(r\,dq - q\,dr)}\sqrt{(dp^2 + dq^2 + dr^2)};$$

denominator vero, ob

$$\frac{u}{w} = \frac{q\,dp - p\,dq}{r\,dq - q\,dr} \text{ et } \frac{v}{w} = \frac{q\,dp - p\,dq}{p\,dr - r\,dp},$$

induet hanc formam:

$$(q\,dp - p\,dq)\left(\frac{dp}{r\,dq - q\,dr} - \frac{dq}{p\,dr - r\,dp}\right) =$$
$$(q\,dp - p\,dq)\left((p\,dp + q\,dq)\,dr - (dp^2 + dq^2)\,r\right).$$

Cum autem ſit

$$p\,dp + q\,dq = -r\,dr,$$

erit iſte denominator:

$$-\frac{r\,(q\,dp - p\,dq)(dp^2 + dq^2 + dr^2)}{(r\,dq - q\,dr)(p\,dr - r\,dp)}.$$

§. 14. Cum igitur euoluerimus tam numeratorem quam denominatorem, inde colligetur radius oſculi

$$z\,\Omega = \frac{ds}{\sqrt{(dp^2 + dq^2 + dr^2)}},$$

qui valor perfecte congruit cum eo qui ſuperiore methodo erat inuentus. Supereſt igitur vt eius quoque poſitionem reſpectu ternorum axium principalium definiamus. Hic autem vidimus eum cadere in normalem $z\,N$ productam, ſiquidem eius expreſſio fuerit poſitiua. Eſt vero ſubnormalis $q\,N = \frac{Y\,dY}{dX} = P\,Y$, atque ſi inſuper ducamus tangentem $z\,T$, erit ſubtangens $q\,T = \frac{Y\,dX}{dY} = \frac{Y}{P}$. At vero poſitionem huius tangentis $z\,T$ in praecedente diſſertatione tam ſimpliciter determinauimus, vt angulorum, ſub quibus ea ad axes principales $O\,A$, $O\,B$ et $O\,C$ inclinatur, coſinus ſint p, q et r.

§. 15.

§. 15. Consideremus primo tangentem zT, pro cuius positione respectu rectae uv habemus tangentem anguli $qTz = \frac{qz}{qT} = P = \frac{dY}{dX}$; tum vero quia elementum curuae est ds, si ponamus breuitatis ergo angulum $qTz = \pi$, erit

$$\sin.\tau = \frac{dY}{ds} = -\frac{(pv+qu)\sqrt{(uuvv+uuww+vvww)}}{uv\sqrt{(uu+vv)}} \text{ et}$$
$$\cos.\tau = \frac{dX}{ds} = \frac{qv-pu}{\sqrt{(uu+vv)}},$$

vnde fit

$$\text{tang.}\tau = -\frac{(pv+qu)\sqrt{(uuvv+uuww+vvww)}}{uv(qv-pu)}.$$

Porro vero erit subtangens $qT = \frac{Y}{tang.\tau}$, est vero

$$Y = \frac{(v(u-x)-yu)\sqrt{(uuvv+uuww+vvww)}}{uv\sqrt{(uu+vv)}},$$

vnde fit subtangens

$$qT = \frac{(v(u-x)-uy)(pu-qv)}{(pv+qu)\sqrt{(uu+vv)}},$$

quae subtracta ab asciffa X relinquet spatium

$$uT = \frac{py+q(u-x)\sqrt{(uu+vv)}}{pv+qu}.$$

Denique fiat $dX : ds = qT$ ad zT,

hincque prodibit ipsa

$$\text{tang.}\,zT = -\frac{(v(u-x)-uy)}{pv+qu} = \frac{uy-v(u-x)}{pv+qu}.$$

§. 16. Eodem modo definiamus positionem normalis zN, ac primo quidem erit angulus $qNz = 90° - \tau$; tum vero subnormalis

$$qN = YP = Y\,\text{tang.}\tau = \frac{(v(u-x)-uy)(pv+qu)(uuvv+uuww+vvww)}{uuvv(pu-qv)\sqrt{(uu+vv)}},$$

ipsa autem normalis erit

zN

$$zN = \frac{Y}{cos. \tau} = \frac{(v(u-x) - uy)\sqrt{(uuvv + uuww + vvww)}}{uv(qv - pu)}.$$

Tandem si subnormalem abscissae X addamus, prodibit interuallum

$$UN = \frac{(u-x)(pvv(uu+ww) + quvww)\sqrt{(uu+vv)}}{uuvv(pu-qv)}$$
$$- \frac{y(quu(vv+ww) + puvww)\sqrt{(uu+vv)}}{uuvv(pu-qv)}.$$

§. 17. Transferamus nunc iterum hanc figuram in planum inclinatum uvw, et cum sit recta zy ad planum AOB perpendicularis, ducta recta yT erit triangulum zyT ad y rectangulum, vnde anguli yzT cosinus erit $\frac{zy}{zT}$. Hic autem angulus aequalis est illi, sub quo tang. zT ad axem OC inclinatur, quia zy ipsi OC est parallela. Cum igitur sit $zy = z$ et $zT = \frac{uy - v(u-x)}{pv + qu}$, at vero ex prima aequatione: Tab. II. Fig. 7.

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} =, \text{ fiat}$$
$$\frac{z}{w} = \frac{u-x}{u} - \frac{y}{v} = \frac{v(u-x) - uy}{uv}, \text{ erit}$$
$$zT = \frac{uvz}{w(pv+qu)};$$

tum vero ex secunda aequatione:

$$\frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0, \text{ erit}$$
$$\frac{r}{w} = -\frac{(pv+qu)}{uv},$$

quo valore substituto fit $zT = -\frac{z}{r}$, quocirca habebimus cos. $yzT = -r$. Supra autem vidimus cosinum anguli, quo tangens curuae ad axem OC inclinatur, esse $= +r$; verum notandum est ibi tangentem sursum fuisse productam, cum hic deorsum dirigatur, vnde signum cosinus mutatur.

§. 18 Quoniam autem hic tantum de inclinatione agitur, quam tangens curuae tenet respectu trium axium, loco ipsius tangentis zT aliam quamcunque ipsi parallelam substituere licebit, quippe quae ad axes nostros pariter erit inclinata. Hanc ob rem in plano inclinato
Tab. II. Fig. 8. uvw ex u ducatur recta ut, tangenti curuae parallela, ita vt sit angulus $vut = \tau$. Capiatur autem haec recta ut, calculi gratia, $= 1$, ita vt, si ex t in uv demittatur perpendiculum td, sit $ud = \text{cos.}\,\tau$ et $td = \text{sin.}\,\tau$. Tum vero in ipso plano AOB ex d ad rectam uv ducatur normalis de, in eamque ex t demittatur perpendiculum te, et quia angulus tde est inclinatio plani uvw ad planum AOB, erit iste angulus $tde = \theta$, vnde fit interuallum $de = td\,\text{cos.}\,\theta = \text{sin.}\,\tau\,\text{cos.}\,\theta$ et perpendiculum in normalem $te = \text{sin.}\,\tau\,\text{sin.}\,\theta$; vbi notetur rectam te axi OC esse parallelam. Hinc si ducatur recta ue triangulum teu ad e erit rectangulum, vnde anguli ute cosinus erit

$$\text{cos.}\, ute = \frac{te}{ut} = \text{sin.}\,\theta\,\text{sin.}\,\tau.$$

Supra autem vidimus esse

$$\text{sin.}\,\theta = \frac{w\sqrt{(uu+vv)}}{\sqrt{(uuvv+uuww+vvww)}};$$

tum vero erat

$$\text{sin.}\,\tau = -\frac{(pv+qu)\sqrt{(uuvv+uuww+vvww)}}{uv\sqrt{(uu+vv)}},$$

quae expressio ob

$$pv+qu = -r.\frac{uv}{w},$$

transformatur in hanc:

$$\text{sin.}\,\tau = \frac{r\sqrt{(uuvv+uuww+vvww)}}{w\sqrt{(uu+vv)}},$$

vnde fit

$$\sin.\,\theta \sin.\,\tau = r = \cos.\,u\,t\,e,$$

hoc est cosinus anguli, sub quo tangens sursum ducta ad axem O C inclinatur $= r$, (Cf. §. 19.); recta autem $u\,e$ sinum eiusdem anguli exhibebit, ita vt sit

$$u\,e = \sqrt{(1 - r\,r)} = \sqrt{(p\,p + q\,q)},$$

quae etiam definiri potest ex triangulo $e\,d\,u$, in quo est

$$u\,d = \cos.\,\tau \text{ et } d\,e = \sin.\,\tau \cos.\,\theta,$$

vnde fit

$$u\,e = \sqrt{(1 - \sin.\,\theta^2 \sin.\,\tau^2)}.$$

Hinc autem deducitur tangens anguli $d\,u\,e$, scil.

$$\text{tang.}\, d\,u\,e = \frac{d\,e}{u\,d} = \cos.\,\theta\ \text{tang.}\,\tau,$$

ad quam euoluendam ponamus breuitatis gratia:

$$\sqrt{(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)} = \Theta, \text{ eritque}$$

$$\cos.\,\theta = \frac{u\,v}{\Theta} \text{ et tang.}\,\tau = \frac{\Theta\,r}{w\,(q\,v - p\,u)},$$

vnde fit

$$\text{tang.}\, d\,u\,e = \frac{r\,.\,u\,v}{w\,(q\,v - p\,u)}.$$

Quare cum anguli O $u\,w$ tangens sit $\frac{v}{u}$, hinc reperiemus tangentem anguli O $u\,e$, scil.

$$\text{tang. O}\, u\,e = \frac{v\,w\,(q\,v - p\,u) - r\,u\,u\,v}{u\,w\,(q\,v - p\,u) + r\,u\,v\,v},$$

cuius fractionis numerator ob $\frac{p}{u} + \frac{r}{w} = -\frac{q}{v}$, siue

$$u\,r + w\,p = -\frac{q.\,u\,w}{v},$$

reducitur ad hanc formam: $q\,w\,(u\,u + v\,v)$; denominator vero, ob

$$q\,w + r\,v = -\frac{v\,w\,p}{u},$$

reducitur ad: $-p\,w\,(u\,u + v\,v)$, ex quo colligitur

$$\text{tang. O}\, u\,e = -\frac{q}{p}.$$

§. 20.

§. 20. Iam ex e in Ou demittatur perpendiculum ef; et cum ſit $ue = \sqrt{(pp+qq)}$, tum vero

$$\text{ſin.}\, euf = \frac{-q}{\sqrt{(pp+qq)}} \text{ et cof.}\, euf = \frac{p}{\sqrt{(pp+qq)}},$$

hinc erit

$$ef = -q \text{ et } uf = p.$$

Quod ſi nunc ducta concipiatur recta tf, ea erit ad Ou normalis, hinque triangulum tfu ad f rectangulum, ex quo obtinebitur coſinus anguli Out, ſub quo tangens tu ad axem OA inclinatur $= p$, prorſus vt ſupra. Eodem modo ſi ex w ducatur ug parallela et aequalis ipſi ef, erit quoque triangulum tug ad g rectangulum, vnde coſinus anguli tug, ſub quo noſtra tangens ad axem OB inclinatur $= -q$. Sicque has inclinationes, quae ſuperiore methodo ſponte ſe offerebant, hac methodo per plures ambages tandem ſunt erutae.

§. 21. His expeditis videamus etiam quomodo normalis ad curuam zN (fig. 6.) ad ternos axes inclinetur. Hunc in finem concipiatur quoque ex puncto u in plano uvw educta recta normali illi zN parallela, quae peitidem vnitati aequalis ſtatuatur: neque vero opus erit peculiarem figuram conſtituere. Nam ſi nunc recta ut pro directione huius normalis accipiatur, omnia manere poſſunt vt ante, ſi modo angulus dut capiatur $= 90° + \tau$. Sicque in praecedentibus formulis tantum opus erit ſcribere cof. τ loco ſin. τ et ſin. τ loco cof. τ. Sicque erit

$$td = \text{cof.}\, \tau \text{ et } ud = -\text{ſin.}\, \tau.$$

Hincque porro erit

$$de = \text{cof.}\, \tau\, \text{cof.}\, \theta \text{ et } te = \text{cof.}\, \tau\, \text{ſin.}\, \theta.$$

At

At vero per reductiones ante adhibitas habebimus

$$\text{fin.}\, \tau = \frac{\Theta r}{w \sqrt{(uu + vv)}} \text{ et } \text{cof.}\, \tau = \frac{qv - pu}{\sqrt{(uu + vv)}},$$

porro

$$\text{fin.}\, \theta = \frac{w \sqrt{(uu + vv)}}{\Theta} \text{ et } \text{cof.}\, \theta = \frac{uv}{\Theta},$$

quorum valorum ope erit

$$ud = -\frac{\Theta r}{w \sqrt{(uu + vv)}},$$

$$dt = \frac{uv}{\Theta},$$

$$de = \frac{uv(qv - pu)}{\Theta \sqrt{(uu + vv)}} \text{ et}$$

$$te = \frac{w(qv - pu)}{\Theta},$$

hincque

$$\text{tang.}\, due = -\frac{uvw(qv - pu)}{\Theta \Theta r} = \frac{de}{ud} = -\frac{\text{cof.}\,\theta}{\text{tang.}\,\tau}.$$

§. 22. Nunc autem angulus ute exprimit inclinationem radii ofculi ad axem OC, cuius ergo cofinus erit $= \frac{w(qv - pu)}{\Theta}$; vbi cum fit

$$\Theta = \sqrt{(uuvv + uuww + vvww)}, \text{ erit}$$

$$\Theta = uvw \sqrt{\left(\tfrac{1}{uu} + \tfrac{1}{vv} + \tfrac{1}{ww}\right)} = \frac{uvw}{t} \sqrt{(dp^2 + dq^2 + dr^2)},$$

vnde ille cofinus erit

$$= t \frac{(qv - pu)}{uv \sqrt{(dp^2 + dq^2 + dr^2)}}.$$

Diuidatur tam numerator quam denominator per uv, et haec expreffio euadet

$$\text{cof.}\, ute = \frac{\left(\frac{qt}{u} - \frac{pt}{v}\right)}{\sqrt{(dp^2 + dq^2 + dr^2)}}.$$

§. 23. Supra autem vidimus effe

$$\frac{t}{u} = r\,dq - q\,dr \text{ et } \frac{t}{v} = p\,dr - r\,dp,$$

quibus valoribus introductis numerator noftrae formulae

erit

erit dr, vnde manifesto colligitur anguli ſub quo radius oſculi ad axem O C inclinatur coſinus

$$= \frac{-dr}{\surd(dq^2 + dp^2 + dr^2)},$$

vbi diſcrimen ſigni non eſt attendendum, propterea quod ſignum per ſe mutatur, ſiue radium oſculi ſurſum vel deorſum tendere accipiamus.

§. 24. Cum igitur ſit recta

$$te = -\frac{dr}{\surd(dp^2 + dq^2 + dr^2)},$$

erit recta

$$ue = \frac{\surd(dp^2 + dq^2)}{\surd(dp^2 + dq^2 + dr^2)}.$$

at vero tang. $eud = \frac{uvw(qv - pu)}{\Theta\Theta r}$. Ante autem erat

$$\Theta = \frac{uvw}{t}\surd(dp^2 + dq^2 + dr^2),$$

vnde iſta tangens euadet

$$\frac{-(qv - pu)tt}{uvwr(dp^2 + dq^2 + dr^2)} = \frac{-\left(\frac{qt}{u} - \frac{pt}{v}\right)t}{wr(dp^2 + dq^2 + dr^2)},$$

cuius fractionis numerator transformatur in $+ tdr$, vnde iſta tangens fiet

$$\text{tang. } due = \frac{dr(qdp - pdq)}{r(dp^2 + dq^2 + dr^2)}.$$

Hunc igitur angulum ſubtrahamus ab angulo Ouv, cuius tangens eſt $\frac{v}{u} = \frac{rdq - qdr}{pdr - rdp}$, et factis omnibus reductionibus, quibus hactenus ſumus vſi, tandem elicitur

$$\text{tang. } Oue = -\frac{dq}{dp}, \text{ ergo}$$

$$\text{ſin.} = Oue \frac{-dq}{\surd(dp^2 + dq^2)} \text{ et coſ. } Oue \frac{dp}{\surd(dp^2 + dq^2)}.$$

Cum igitur ſit

$$ue = \frac{\surd(dp^2 + dq^2)}{\surd(dp^2 + dq^2 + dr^2)}, \text{ erit}$$

$uf = ue \cos. Oue = \frac{dp}{\sqrt{(dp^2 + dq^2 + dr^2)}}$ et

$ef = \frac{-dq}{\sqrt{(dp^2 + dq^2 + dr^2)}}$.

§. 25. Cum autem ſit $ut = 1$, recta uf exprimit coſinum anguli fut, quem radius oſculi cum axe OA conſtituit, recta vero $fe = ug$ exprimit coſinum anguli gut, quem radius oſculi facit cum axe OB, quocirca habebimus:

$$\text{coſ. incl. rad. oſc. ad axem } OA = \frac{dp}{\sqrt{(dp^2 + dq^2 + dr^2)}},$$

$$\text{coſ. incl. rad. oſc. ad axem } OB = \frac{dq}{\sqrt{(dp^2 + dq^2 + dr^2)}},$$

$$\text{coſ. incl. rad. oſc. ad axem } OC = \frac{dr}{\sqrt{(dp^2 + dq^2 + dr^2)}}.$$

Ceterum iſta methodus ideo notatu maxime digna videtur, quod per reductiones hic expoſitas formulae maxime complexae ad ſimpliciſſimas, quaſi praeter omnem expectationem, reuocantur.

§. 26. Quo omnes iſtae reductiones facilius inſtitui queant, ſequentes proprietates probe notaſſe iuuabit. Cum enim ſit

$$\frac{t}{u} = r\,dq - q\,dr;$$

$$\frac{t}{v} = p\,dr - r\,dp;$$

$$\frac{t}{w} = q\,dp - p\,dq;$$

exiſtente vti aſſumſimus

$$t = x(r\,dq - q\,dr) + y(p\,dr - r\,dp) + z(q\,dp - p\,dq),$$

ita vt t denotet quantitatem differentialem, hinc erit

I. $\frac{pt}{v} - \frac{qt}{u} = p(p\,dr - r\,dp) - q(r\,dq - q\,dr) = dr;$

II.

II. $\frac{qt}{w} - \frac{rt}{v} = q(qdp - pdq) - r(pdr - rdp) = dp;$

III. $\frac{rt}{u} - \frac{pt}{w} = r(rdq - qdr) - p(qdp - pdq) = dq.$

Deinde vero ſequentes formulae etiam commodam reductionem patiuntur.

$$\text{I. } \frac{tdp}{v} - \frac{tdq}{u} = dp(pdr - rdp) - dq(rdq - qdr)$$
$$= -r(dp^2 + dq^2 + dr^2),$$

$$\text{II. } \frac{tdq}{w} - \frac{tdr}{v} = dq(qdp - pdq) - dr(pdr - rdp)$$
$$= -p(dp^2 + dq^2 + dr^2),$$

$$\text{III. } \frac{tdr}{u} - \frac{tdp}{w} = dr(rdq - qdr) - dp(qdp - pdq)$$
$$= -q(dp^2 + dq^2 + dr^2).$$

Praeterea vero hae reductiones notatu dignae ſunt:

$$\frac{tt}{uu} + \frac{tt}{vv} = dr^2 + rr(dp^2 + dq^2 + dr^2),$$
$$\frac{tt}{vv} + \frac{tt}{ww} = dp^2 + pp(dp^2 + dq^2 + dr^2),$$
$$\frac{tt}{ww} + \frac{tt}{uu} = dq^2 + qq(dp^2 + dq^2 + dr^2).$$

Tab. II. Fig. 9.

§. 27. Denique quoniam omnes has determinationes perduximus ad parallelepipeda circa axes principales exſtructa, cuiusmodi in hac figura exhibetur, vbi, prouti ternis lateribus O P, O Q, O R certi valores tribuuntur, diagonales O Ω poſitionem rectarum notabilium ad curvam pertinentium repraeſentant: quae hactenus hic ſunt allata ſequentibus parallelepipedis complecti licet. I.) Si parallelepipedi latera ſtatuantur

$$OP = x;\; OQ = y;\; OR = z;$$

diago-

diagonalis, quae erit $\sqrt{(xx + yy + zz)}$, dat positionem rectae ex puncto O ad curuae punctum z ductae.
II. Si parallelepipedi latera capiantur

$$OP = p; \; OQ = q; \; OR = r;$$

tum diagonalis $O\Omega$ dabit positionem tangentis curuae in puncto z, siue ista tangens parallela erit diagonali $O\Omega$; ipsa autem haec diagonalis erit

$$\sqrt{(pp + qq + rr)} = \mathrm{r}.$$

III.) Si parallelepipedi latera statuantur:

$$OP = \frac{dp}{ds}, \; OQ = \frac{dq}{ds}, \; OR = \frac{dr}{ds},$$

tum diagonalis parallelogrammi dabit directionem radii osculi in puncto z, seu ipsi erit parallela: ipsa vero diagonalis erit $\frac{\sqrt{(dp^2 + dq^2 + dr^2)}}{ds}$, ita vt ipse radius osculi sit $\frac{1}{O\Omega}$. IV.) Si latera parallelepipedi statuantur:

$$OP = \frac{rdq - qdr}{ds}, \; OQ = \frac{pdr - rdp}{ds}, \; OR = \frac{qdp - pdq}{ds},$$

tum diagonalis $O\Omega$ erit normalis in planum in quo bina elementa curuae proxima incuruantur: ipsa autem diagonalis iterum erit $\frac{\sqrt{(dp^2 + dq^2 + dr^2)}}{ds}$. Postremum hoc theorema inde manifesto sequitur, quod normalis ad planum incuruationis perinde inclinatur ad ternos axes, atque ipsum planum inclinatur ad plana principalia opposita.
