1785

De seriebus potestatum reciprocis methodo nova et facillima summandis

Leonhard Euler

§. 29. Ipsae quidem hae summae sine dubio parum attentionis merentur, nisi forte ad quantitates cognitas reduci potuerint. Verum quia in his seriebus neque ipsi termini secundum certam legem progrediuntur, neque etiam in signis plus vel minus certus ordo obseruatur; ista disquisitio primo intuitu plane impossibilis videri potuisset, quamobrem ipsa methodus, qua ad earum summas pertigimus, vtique omni attentione digna est censenda, idque eo magis, quod satis abstrusis serierum potestatum proprietatibus innititur. Nisi enim summae serierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} \text{ etc.}$$

pro casibus quibus n est numerus impar, fuissent cognitae, ota haec inuestigatio frustra fuisset suscepta.

DE

: dubio parum cognitas reieque ipsi terneque etiam ur; ista dis- :ri potuisset, ımmas perti- ısenda, idque testatum pro- um

Tent cognitae,

DE

DE
SERIEBVS POTESTATVM
RECIPROCIS
METHODO NOVA ET FACILLIMA SVMMANDIS.

§. 1.

Cum primum summas harum serierum docuissem, eas ex hoc principio deduxi, quod cuique sinui et cosinui innumerabiles arcus circulares respondent, qui omnes sint radices aequationum infinitarum, quibus arcus per sinum vel cosinum exprimi solent. Hinc enim ex coefficientibus istarum aequationum non solum summas ipsarum radicum, sed etiam earum potestatum quarumcunque assignaui. Postea vero easdem summas etiam ex aliis principiis deriuaui, quae autem omnia memorata circuli proprietate innitebantur. Nunc vero obseruaui, istas summas ex alio principio multo simpliciori, et solis operationibus analyticis innixo, deduci posse, quam methodum hic accuratius exposuisse iuuabit.

§. 2. Hoc autem principium mihi suppeditauit integratio huius formulae: $\int\left(\frac{z^{m-1} + z^{n-m-1}}{1 + z^n}\right) dz$, pro casu quo post integrationem statuitur $z = 1$. Ostendi enim in

Tomo XIX Nov. Comment. per ſolitas integrationum operationes haec integralia ſequenti modo exprimi:

$$\int\left(\frac{z^{m-1}+z^{n-m-1}}{1+z^n}\right)dz=\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}} \text{ et}$$

$$\int\left(\frac{z^{m-1}-z^{n-m-1}}{1-z^n}\right)dz=\frac{\pi}{n\,\text{tang.}\,\frac{m\pi}{n}}.$$

Quod ſi vero eaedem formulae per ſeries infinitas euoluantur, poſito $z=1$ erit

$$\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}}=\frac{1}{m}-\frac{1}{n+m}+\frac{1}{2n+m}-\frac{1}{3n+m}+\frac{1}{4n+m}-\text{etc.}$$

$$+\frac{1}{n-m}-\frac{1}{2n-m}+\frac{1}{3n-m}-\frac{1}{4n-m}+\text{etc. et}$$

$$\frac{\pi}{n\,\text{tang.}\,\frac{m\pi}{n}}=\frac{1}{m}+\frac{1}{n+m}+\frac{1}{2n+m}+\frac{1}{3n+m}+\text{etc.}$$

$$-\frac{1}{n-m}-\frac{1}{2n-m}-\frac{1}{3n-m}-\frac{1}{4n-m}-\text{etc.}$$

quae duae ſeries eo maiori attentione ſunt dignae, quod in iis omnia plane continentur, quae non ſolum circa ſummationes poteſtatum, ſed etiam circa ſummationes ſimiles ſunt prolata.

Euolutio prioris ſeriei generalis.

§. 3. Conſideremus primo formam priorem $\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}}$ ac binis terminis analogis contractis habebimus

$$\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}}=\frac{1}{m}+\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}+\frac{2m}{9nn-mm}-\frac{2m}{16nn-mm}+\text{etc.}$$

Suma-

um ope- euoluan- — etc. + etc. et — etc. quod in ſumma- ſimiles m $\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}}$ + etc. Suma-

Sumatur nunc, quo formulae fiant ſimpliciores, $m=1$ eritque

$$\frac{\pi}{n\,\text{ſin.}\,\frac{\pi}{n}}=1+\frac{2}{nn-1}-\frac{2}{4nn-1}+\frac{2}{9nn-1}-\frac{2}{16nn-1}+\text{etc.}$$

ſiue

$$\frac{\pi}{2n\,\text{ſin.}\,\frac{\pi}{n}}-\frac{1}{2}=\frac{1}{nn-1}-\frac{1}{4nn-1}+\frac{1}{9nn-1}-\frac{1}{16nn-1}+\text{etc.}$$

§. 4. Nunc ſingulas has fractiones more ſolito in ſeries infinitas geometricas reſoluamus eritque

$$\frac{1}{nn-1}=\frac{1}{nn}+\frac{1}{n^4}+\frac{1}{n^6}+\frac{1}{n^8}+\frac{1}{n^{10}}+\text{etc.}$$

$$-\frac{1}{4nn-1}=-\frac{1}{4nn}-\frac{1}{4^2n^4}-\frac{1}{4^3n^6}-\frac{1}{4^4n^8}-\frac{1}{4^5n^{10}}-\text{etc.}$$

$$+\frac{1}{9nn-1}=\frac{1}{9nn}+\frac{1}{9^2n^4}+\frac{1}{9^3n^6}+\frac{1}{9^4n^8}+\frac{1}{9^5n^{10}}+\text{etc.}$$

$$-\frac{1}{16nn-1}=-\frac{1}{16nn}-\frac{1}{16^2n^4}-\frac{1}{16^3n^6}-\frac{1}{16^4n^8}-\frac{1}{16^5n^{10}}-\text{etc.}$$

etc. etc.

Harum igitur ſerierum infinitarum omnium ſumma erit

$$=\frac{\pi}{2n\,\text{ſin.}\,\frac{\pi}{n}}-\frac{1}{2}.$$

§. 5. Nunc igitur has ſeries ſecundum lineas verticales colligamus, quem in finem ſtatuamus breuitatis gratia

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\text{etc.}=A\pi\pi$$

$$1-\frac{1}{4^2}+\frac{1}{9^2}-\frac{1}{16^2}+\frac{1}{25^2}-\text{etc.}=B\pi^4$$

$$1-\frac{1}{4^3}+\frac{1}{9^3}-\frac{1}{16^3}+\frac{1}{25^3}-\text{etc.}=C\pi^6$$

$$1-\frac{1}{4^4}+\frac{1}{9^4}-\frac{1}{16^4}+\frac{1}{25^4}-\text{etc.}=D\pi^8$$

etc. etc.

Hinc

hinc igitur adipiſcemur ſequentem aequationem:

$$\frac{\pi}{2n\,\text{ſin.}\frac{\pi}{n}}-\frac{1}{2}=\frac{A\pi\pi}{nn}+\frac{B\pi^4}{n^4}+\frac{C\pi^6}{n^6}+\frac{D\pi^8}{n^8}+\text{etc.}$$

§. 6. Ponamus porro breuitatis gratia $\frac{\pi}{n}=x$, vt prodeat ſequens aequatio:

$$\frac{x}{2\,\text{ſin.}\,x}-\frac{1}{2}=Axx+Bx^4+Cx^6+Dx^8+Ex^{10}+\text{etc.}$$

vbi iam intelligitur, per debitam euolutionem omnes coëfficientes aſſumtos A, B, C, etc. definiri poſſe, quibus inventis nanciſcemur ſummas omnium ſerierum in hac forma contentarum:

$$1-\frac{1}{4^i}+\frac{1}{9^i}-\frac{1}{16^i}+\frac{1}{25^i}-\text{etc.}$$

ſiue in hac:

$$1-\frac{1}{2^{2i}}+\frac{1}{3^{2i}}-\frac{1}{4^{2i}}+\frac{1}{5^{2i}}-\text{etc.}$$

denotante i numerum integrum quemcunque.

§. 7. Cum iam per ſeriem notiſſimam ſit

$$\text{ſin.}\,x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1\ldots5}-\frac{x^7}{1\ldots7}+\text{etc.}$$

pro hac ſerie ſimpliciter ſcribamus

$$\text{ſin.}\,x=\alpha x-\beta x^3+\gamma x^5-\delta x^7+\varepsilon x^9-\text{etc.}$$

ita vt ſit

$$\alpha=1,\ \beta=\frac{\alpha}{2.3},\ \gamma=\frac{\beta}{4.5},\ \delta=\frac{\gamma}{6.7},\ \varepsilon=\frac{\delta}{8.9},\ \text{etc.}$$

quo poſito membrum $-\frac{1}{2}$ ad dextram partem transferamus atque vtrinque multiplicemus per hanc ſeriem ipſi ſin x aequalem, fietque

$\frac{x}{2}=\frac{1}{2}$

+ etc.
$=x$, vt
+ etc.
s coëffi-
ibus in-
; forma
etc.
– etc.
$\frac{\delta}{\ }$, etc.
nsferamus
ſi ſin x
$\frac{x}{2}=\frac{1}{2}$

$$\begin{array}{lllllll}
\frac{x}{2}=\frac{1}{2}\alpha x & +\alpha Ax^3 & +\alpha Bx^5 & +\alpha Cx^7 & +\alpha Dx^9 & +\alpha Ex^{11} & +\alpha Fx^{13}\ \text{etc.}\\
 & -\frac{1}{2}\beta & -\beta A & -\beta B & -\beta C & -\beta D & -\beta E\\
 & & +\frac{1}{2}\gamma & +\gamma A & +\gamma B & +\gamma C & +\gamma D\\
 & & & -\frac{1}{2}\delta & -\delta A & -\delta B & -\delta C\\
 & & & & +\frac{1}{2}\varepsilon & +\varepsilon A & +\varepsilon B\\
 & & & & & -\frac{1}{2}\zeta & -\zeta A\\
 & & & & & & -\frac{1}{2}\eta.
\end{array}$$

§. 8. Quoniam haec aequalitas ſubſiſtere debet, quicunque valor litterae x tribuatur, ſingulae eius poteſtate- ſe mutuo ſeorſim deſtruere debent. Primo quidem termini ipſum x continentes ob $\alpha=1$ ſponte ſe tollunt, reliquae poteſtates ob $\alpha=1$ ſequentes dant determinationes:

$$\begin{aligned}
A&=\tfrac{1}{2}\beta\\
B&=\beta A-\tfrac{1}{2}\gamma\\
C&=\beta B-\gamma A+\tfrac{1}{2}\delta\\
D&=\beta C-\gamma B+\delta A-\tfrac{1}{2}\varepsilon\\
E&=\beta D-\gamma C+\delta B-\varepsilon A+\tfrac{1}{2}\zeta\\
&\text{etc.}\qquad\text{etc.}
\end{aligned}$$

Harum igitur formularum ope ſummae quantumuis altarum poteſtatum parium aſſignari poterunt.

§. 9. Inuenta autem ſumma huius ſeriei:

$$s=1-\frac{1}{2^{2i}}+\frac{1}{3^{2i}}-\frac{1}{4^{2i}}+\frac{1}{5^{2i}}-\text{etc.}$$

ex ea quoque ſerierum agnatarum iſtarum ſummae definiri poterunt:

$$t=1+\frac{1}{3^{2i}}+\frac{1}{5^{2i}}+\frac{1}{7^{2i}}+\frac{1}{9^{2i}}+\text{etc. et}$$

$$u=1+\frac{1}{2^{2i}}+\frac{1}{3^{2i}}+\frac{1}{4^{2i}}+\frac{1}{5^{2i}}+\frac{1}{6^{2i}}+\text{etc.}$$

Cum enim ſit

$t=$

$$t = u\left(1 - \frac{1}{2^{2i}}\right) = \left(\frac{2^{2i}-1}{2^{2i}}\right)u \text{ et}$$

$$s = u\left(1 - \frac{2}{2^{2i}}\right) = \left(\frac{2^{2i}-2}{2^{2i}}\right)u, \text{ erit}$$

$$u = \frac{2^{2i}s}{2^{2i}-2}, \text{ hincque } t = \left(\frac{2^{2i}-1}{2^{2i}-2}\right)s$$

in ſequentibus autem harum ſerierum ſummae etiam immediate ex noſtris formulis generalibus elicientur.

Euolutio ſeriei generalis poſterioris.

§. 10. Quod ſi hic etiam bini termini analogi contrahantur, orietur iſta ſeries:

$$\frac{\pi}{n\,\text{tang.}\frac{m\pi}{n}} = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} - \frac{2m}{16nn-mm} + \text{etc.}$$

Ponamus hic iterum $m = 1$, et facta diuiſione per 2 habebimus

$$\frac{1}{nn-1} + \frac{1}{4nn-1} + \frac{1}{9nn-1} + \frac{1}{16nn-1} + \frac{1}{25nn-1} + \text{etc.}$$
$$= \frac{1}{2} - \frac{\pi}{2n\,\text{tang.}\frac{\pi}{n}}.$$

Nunc ſingulae iſtae fractiones in ſeries reſoluantur vt ſupra, eritque

$$\frac{1}{nn-1} = \frac{1}{nn} + \frac{1}{n^4} + \frac{1}{n^6} + \frac{1}{n^8} + \frac{1}{n^{10}} + \frac{1}{n^{12}} + \text{etc.}$$
$$\frac{1}{4nn-1} = \frac{1}{4nn} + \frac{1}{4^2n^4} + \frac{1}{4^3n^6} + \frac{1}{4^4n^8} + \frac{1}{4^5n^{10}} + \frac{1}{4^6n^{12}} + \text{etc.}$$
$$\frac{1}{9nn-1} = \frac{1}{9nn} + \frac{1}{9^2n^4} + \frac{1}{9^3n^6} + \frac{1}{9^4n^8} + \frac{1}{9^5n^{10}} + \frac{1}{9^6n^{12}} + \text{etc.}$$
$$\frac{1}{16nn-1} = \frac{1}{16nn} + \frac{1}{16^2n^4} + \frac{1}{16^3n^6} + \frac{1}{16^4n^8} + \frac{1}{16^5n^{10}} + \frac{1}{16^6n^{12}} + \text{etc.}$$
etc. etc.

cuncta-

m imme-
alogi con-
$\frac{2m}{9nn-mm}$
r 2 habe-
-1 + etc.
ır vt ſupra,
$\frac{1}{n^{12}}$ + etc.
$\frac{1}{4^6n^{12}}$ + etc.
$\frac{1}{9^6n^{12}}$ + etc.
$\frac{1}{16^6n^{12}}$ + etc.
cuncta-

cunctarum igitur harum ſerierum iunctim ſumtarum ſumma erit $= \frac{1}{2} - \frac{\pi}{2n\,\text{tang.}\frac{\pi}{n}}$.

§. 11. Nunc igitur, vt ſupra fecimus, per columnas verticales ſummam colligamus, quem in finem ſtatuamus

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \mathfrak{A}\pi^2$$
$$1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \text{etc.} = \mathfrak{B}\pi^4$$
$$1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \text{etc.} = \mathfrak{C}\pi^6$$
$$1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \text{etc.} = \mathfrak{D}\pi^8$$
etc. etc.

Quibus poſitis aequatio noſtra erit

$$\frac{1}{2} - \frac{\pi}{2n\,\text{tang.}\frac{\pi}{n}} = \frac{\mathfrak{A}\pi^2}{n^2} + \frac{\mathfrak{B}\pi^4}{n^4} + \frac{\mathfrak{C}\pi^6}{n^6} + \frac{\mathfrak{D}\pi^8}{n^8} + \text{etc.}$$

§. 12. Faciamus nunc $\frac{\pi}{n} = x$, quo pacto ambae litterae π et n ſimul ex calculo elidentur, eritque

$$\frac{1}{2} - \frac{x}{2\,\text{tang.}\,x} = \mathfrak{A}xx + \mathfrak{B}x^4 + \mathfrak{C}x^6 + \mathfrak{D}x^8 + \mathfrak{E}x^{10} + \text{etc.}$$

vbi loco huius ſeriei breuitatis gratia ſcribamus litteram s, vt ſit

$$s = \frac{1}{2} - \frac{x}{2\,\text{tang.}\,x} = \frac{\text{ſin.}\,x - x\,\text{coſ.}\,x}{2\,\text{ſin.}\,x}$$

quae aequatio per ſin. x multiplicata praebet

$$s\,\text{ſin.}\,x = \tfrac{1}{2}\,\text{ſin.}\,x - \tfrac{1}{2}x\,\text{coſ.}\,x.$$

§. 13. Statuamus nunc, vt in praecedente euolutione,

$$\text{ſin.}\,x = \alpha x - \beta x^3 + \gamma x^5 - \delta x^7 + \varepsilon x^9 - \text{etc.}$$

exiſten-

existente

$$\alpha = 1,\ \beta = \frac{1}{1.2.3},\ \gamma = \frac{1}{1.2.3.4.5},\ \delta = \frac{1}{1\ldots7}, \text{ etc.}$$

Quia nunc est

$$\cos. x = 1 - \frac{xx}{1.2} + \frac{x^4}{1\ldots4} - \frac{x^6}{1\ldots6} + \frac{x^8}{1\ldots8} - \text{etc.}$$

erit

$$\cos. x = \alpha - 3\beta xx + 5\gamma x^4 - 7\delta x^6 + 9\varepsilon x^8 - \text{etc.}$$

Tum autem erit

$$\tfrac{1}{2}\sin. x - \tfrac{1}{2}x\cos. x = \beta x^3 - 2\gamma x^5 + 3\delta x^7 - 4\varepsilon x^9 + 5\zeta x^{11} - \text{etc.}$$

cui ergo expressioni formula $s \sin. x$ debet esse aequalis.

§. 14. Binas igitur series per s et $\sin. x$ indicatas inuicem multiplicemus, et productum reperietur

$$\begin{array}{l} s\sin. x = \alpha\mathfrak{A}x^3 + \alpha\mathfrak{B}x^5 + \alpha\mathfrak{C}x^7 + \alpha\mathfrak{D}x^9 + \alpha\mathfrak{E}x^{11} + \alpha\mathfrak{F}x^{13} + \text{etc.} \\ \qquad\qquad -\beta\mathfrak{A} \quad -\beta\mathfrak{B} \quad -\beta\mathfrak{C} \quad -\beta\mathfrak{D} \quad -\beta\mathfrak{E} - \text{etc.} \\ \qquad\qquad\qquad\quad +\gamma\mathfrak{A} \quad +\gamma\mathfrak{B} \quad +\gamma\mathfrak{C} \quad +\gamma\mathfrak{D} + \text{etc.} \\ \qquad\qquad\qquad\qquad\quad -\delta\mathfrak{A} \quad -\delta\mathfrak{B} \quad -\delta\mathfrak{C} - \text{etc.} \\ \qquad\qquad\qquad\qquad\qquad\quad +\varepsilon\mathfrak{A} \quad +\varepsilon\mathfrak{B} + \text{etc.} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad -\zeta\mathfrak{A} - \text{etc.} \end{array}$$

quae expressio praecedenti debet esse aequalis.

§. 15. Singulae igitur potestates ipsius x seorsim inter se aequentur, indeque formentur sequentes determinationes:

$\mathfrak{A} = \beta$

$\mathfrak{B} = \beta\mathfrak{A} - 2\gamma$

$\mathfrak{C} = \beta\mathfrak{B} \quad \gamma\mathfrak{A} + 3\delta$

$\mathfrak{D} = \beta\mathfrak{C} - \gamma\mathfrak{B} + \delta\mathfrak{A} - 4\varepsilon$

$\mathfrak{E} =$

$\frac{1}{1\ldots7}$, etc.
$+ \frac{x^8}{1\ldots8} -$ etc.
$-9\varepsilon x^8 -$ etc.
$x^9 + 5\zeta x^{11} -$ etc.
esse aequalis.
t sin. x indicatas
rietur
$x^{11} + \alpha\mathfrak{F}x^{13} +$ etc.
$-\beta\mathfrak{E} -$ etc.
$+\gamma\mathfrak{D} +$ etc.
$-\delta\mathfrak{C} -$ etc.
$+\varepsilon\mathfrak{B} +$ etc.
$-\zeta\mathfrak{A} -$ etc.
alis.
ipsius x seorsim
sequentes deter-
$\mathfrak{E} =$

$\mathfrak{E} = \beta\mathfrak{D} - \gamma\mathfrak{C} + \delta\mathfrak{B} - \varepsilon\mathfrak{A} + 5\zeta$

$\mathfrak{F} = \beta\mathfrak{E} - \gamma\mathfrak{D} + \delta\mathfrak{C} - \varepsilon\mathfrak{B} + \vartheta\mathfrak{A} - 6\eta$

etc. etc.

§. 16. Quanquam ope harum formularum determinatio coëfficientium $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc., quousque libuerit continuari potest, tamen ex iisdem principiis aliae relationes inter hos coëfficientes deriuari possunt, quibus calculus haud mediocriter subleuabitur. Resumamus scilicet aequationem $\frac{1}{2} - \frac{x}{2\,tang.\,x} = s$, vnde fit $\frac{x}{2\,tang.\,x} = \frac{1}{2} - s$, hincque porro $\frac{1-2s}{x} = \cot. x$, quae cotangens statuatur $= t$, vt sit $t = \frac{1-2s}{x}$, ergo loco s serie substituta fiet

$$t = \frac{1}{x} - 2\mathfrak{A}x - 2\mathfrak{B}x^3 - 2\mathfrak{C}x^5 - 2\mathfrak{D}x^7 - \text{etc.}$$

§. 17. Cum igitur posuerimus $\cot. x = t$, ideoque $x = A\cot. t$, erit differentiando $dx = -\frac{dt}{1+tt}$ hincque

$$dt + dx(1+tt) = 0, \text{ siue}$$

$$\frac{dt}{dx} + 1 + tt = 0. \text{ Est vero}$$

$$\frac{dt}{dx} = -\frac{1}{xx} - 2\mathfrak{A} - 6\mathfrak{B}xx - 10\mathfrak{C}x^4 - 14\mathfrak{D}x^6 - 18\mathfrak{E}x^8 - \text{etc.}$$

praeterea vero reperitur

$$\begin{array}{l} 1 + tt = \frac{1}{xx} - 4\mathfrak{A} - 4\mathfrak{B}xx - 4\mathfrak{C}x^4 - 4\mathfrak{D}x^6 - 4\mathfrak{E}x^8 - 4\mathfrak{F}x^{10} - \text{etc.} \\ \qquad\qquad\quad +1 \quad +4\mathfrak{A}\mathfrak{A} \quad +8\mathfrak{A}\mathfrak{B} + 8\mathfrak{A}\mathfrak{C} + 8\mathfrak{A}\mathfrak{D} + 8\mathfrak{A}\mathfrak{E} + \text{etc.} \\ \qquad\qquad\qquad\qquad\qquad\qquad +4\mathfrak{B}\mathfrak{B} + 8\mathfrak{B}\mathfrak{C} + 8\mathfrak{B}\mathfrak{D} + \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +4\mathfrak{C}\mathfrak{C}. \end{array}$$

§. 18. In aequalitate igitur $\frac{dt}{dx} + 1 + tt = 0$, prima membra sponte se tollunt; ex sequentibus autem colliguntur sequentes determinationes:

$\mathfrak{A} =$

$\mathfrak{A} = \frac{1}{6};$

$\mathfrak{B} = \frac{2}{5}\mathfrak{A}\mathfrak{A};$

$\mathfrak{C} = \frac{2}{7}.2\mathfrak{A}\mathfrak{B};$

$\mathfrak{D} = \frac{2}{9}(2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B})$

$\mathfrak{E} = \frac{2}{11}(2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C});$

$\mathfrak{F} = \frac{2}{13}(2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C})$

$\mathfrak{G} = \frac{2}{15}(2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{E} + 2\mathfrak{C}\mathfrak{D});$

$\mathfrak{H} = \frac{2}{17}(2\mathfrak{A}\mathfrak{G} + 2\mathfrak{B}\mathfrak{F} + 2\mathfrak{C}\mathfrak{E} + \mathfrak{D}\mathfrak{D}).$

§. 19. Ex his formulis iam olim in introductione mea in Analysin infinitorum valores istarum litterarum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. satis longe computaui, deinceps vero ad aliquot terminos longius continuaui, quos valores igitur hic apponam:

$\mathfrak{A} = \frac{1}{1.2.3}\cdot\frac{1}{1} = \frac{1}{6},$ - - - - - pro potestatibus secundis

$\mathfrak{B} = \frac{2^2}{1 \ldots 5}\cdot\frac{1}{3} = \frac{1}{90},$ - - - - pro potestatibus quartis

$\mathfrak{C} = \frac{2^4}{1 \ldots 7}\cdot\frac{1}{3} = \frac{1}{945}$ - - - - - - - - - - sextis

$\mathfrak{D} = \frac{2^6}{1 \ldots 9}\cdot\frac{3}{5} = \frac{1}{9450}$ - - - - - - - - - octauis

$\mathfrak{E} = \frac{2^8}{1 \ldots 11}\cdot\frac{5}{3} = \frac{1}{93555}$ - - - - - - - - - decimis

$\mathfrak{F} = \frac{2^{10}}{1 \ldots 13}\cdot\frac{691}{105}$ - - - - - - - - duodecimis

$\mathfrak{G} = \frac{2^{12}}{1 \ldots 15}\cdot\frac{35}{1}$ - - - - - - - decimis quartis

$\mathfrak{H} = \frac{2^{14}}{1 \ldots 17}\cdot\frac{3617}{15}$ - - - - - - - decimis sextis

$\mathfrak{I} = \frac{2^{16}}{1 \ldots 19}\cdot\frac{43867}{21}$ - - - - - - - decimis octauis

$\mathfrak{K} = \frac{2^{18}}{1 \ldots 21}\cdot\frac{1222277}{55}$ - - - - - - - - - vigesimis

$\mathfrak{L} = \frac{2^{20}}{1 \ldots 23}\cdot\frac{854513}{3}$ - - - - - - - - - viges. sec.

$\mathfrak{M} = \frac{2^{22}}{1 \ldots 25}\cdot\frac{1181820455}{273}$ - - - - - - - viges. quart.

luctione mea
$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc.
ot terminos
ponam:
ous secundis
tibus quartis
- - sextis
- octauis
- decimis
duodecimis
cimis quartis
decimis sextis
cimis octauis
- vigesimis
- viges. sec.
viges. quart.
$\mathfrak{N} =$

$\mathfrak{N} = \frac{2^{24}}{1 \ldots 27}\cdot\frac{76977927}{1}$ - - - - - - - - viges. sext.

$\mathfrak{O} = \frac{2^{26}}{1 \ldots 29}\cdot\frac{23749461029}{15}$ - - - - - - - viges. oct.

$\mathfrak{P} = \frac{2^{28}}{1 \ldots 31}\cdot\frac{8615841276005}{231}$ - - - - - - - trigesimis

$\mathfrak{Q} = \frac{2^{30}}{1 \ldots 33}\cdot\frac{84802531453387}{85}$ - - - - - - - triges. sec.

$\mathfrak{R} = \frac{2^{32}}{1 \ldots 35}\cdot\frac{90219075042845}{3}$ - - - - - - triges. quart.

Praeparatio Formularum generalium ad alios vsus.

§. 20. Hactenus posuimus $m = 1$, nunc autem statuamus $m = \frac{n-1}{2}$, eritque

$\frac{m\pi}{n} = \frac{(n-1)\pi}{2n} = \frac{1}{2}\pi - \frac{\pi}{2n}$, vnde fit

$\sin.\frac{m\pi}{n} = \cos.\frac{\pi}{2n}$ et $\tang.\frac{m\pi}{n} = \cot.\frac{\pi}{2n}$.

Ipsae autem series ita se habebunt:

$$\frac{\pi}{2n\cos.\frac{\pi}{2n}} = \frac{1}{n-1} - \frac{1}{3n-1} + \frac{1}{5n-1} - \frac{1}{7n-1} + \frac{1}{9n-1} - \text{etc.}$$

$$+ \frac{1}{n+1} - \frac{1}{3n+1} + \frac{1}{5n+1} - \frac{1}{7n+1} + \frac{1}{9n+1} - \text{etc.}$$

$$\frac{\pi}{2n\cot.\frac{\pi}{2n}} = \frac{1}{n-1} + \frac{1}{3n-1} + \frac{1}{5n-1} + \frac{1}{7n-1} + \frac{1}{9n-1} + \text{etc.}$$

$$- \frac{n}{n+1} - \frac{1}{3n+1} - \frac{1}{5n+1} - \frac{1}{7n+1} - \frac{1}{9n+1} - \text{etc.}$$

Euolutio seriei prioris § 20.

§. 21. Contrahantur hic etiam bini termini analogi, ac prodibit haec series:

$$\frac{\pi}{2n\cos.\frac{\pi}{2n}} = \frac{2n}{nn-1} - \frac{6n}{9nn-1} + \frac{10n}{25nn-1} - \frac{14n}{49nn-1} + \text{etc.}$$

 fiue

fiue

$$\frac{\pi}{4n\,\mathrm{cof.}\frac{\pi}{2n}} = \frac{n}{nn-1} - \frac{3n}{9nn-1} + \frac{5n}{25nn-1} - \frac{7n}{49nn-1} + \text{etc.}$$

§. 22. Hic igitur omnes iftae fractiones continentur in hac forma generali: $\frac{in}{i^2n^2-1}$, vbi i denotat omnes numeros impares. Haec autem fractio in feriem infinitam conuerfa praebet

$$\frac{1}{in} + \frac{1}{i^3n^3} + \frac{1}{i^5n^5} + \frac{1}{i^7n^7} + \frac{1}{i^9n^9} + \text{etc.}$$

Hinc igitur fingulas fractiones per feries euoluamus:

$$\begin{aligned}
\frac{n}{nn-1} &= \frac{1}{n} + \frac{1}{n^3} + \frac{1}{n^5} + \frac{1}{n^7} + \frac{1}{n^9} + \text{etc.}\\
-\frac{3n}{9nn-1} &= -\frac{1}{3n} - \frac{1}{3^3n^3} - \frac{1}{3^5n^5} - \frac{1}{3^7n^7} - \frac{1}{3^9n^9} - \text{etc.}\\
\frac{5n}{25nn-1} &= \frac{1}{5n} + \frac{1}{5^3n^3} + \frac{1}{5^5n^5} + \frac{1}{5^7n^7} + \frac{1}{5^9n^9} + \text{etc.}\\
-\frac{7n}{49nn-1} &= -\frac{1}{7n} - \frac{1}{7^3n^3} - \frac{1}{7^5n^5} - \frac{1}{7^7n^7} - \frac{1}{7^9n^9} - \text{etc.}\\
&\text{etc.} \qquad \text{etc}
\end{aligned}$$

quarum igitur ferierum omnium fumma eft $\frac{\pi}{4n\,\mathrm{cof.}\frac{\pi}{2n}}$.

§. 23. Nunc etiam has feries per columnas verticales colligamus, ac ftatuamus

$$\begin{aligned}
1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.} &= \mathfrak{a}\,\tfrac{\pi}{2}\\
1 - \tfrac{1}{3^3} + \tfrac{1}{5^3} - \tfrac{1}{7^3} + \tfrac{1}{9^3} - \tfrac{1}{11^3} + \text{etc.} &= \mathfrak{b}\,\tfrac{\pi^3}{2^3}\\
1 - \tfrac{1}{3^5} + \tfrac{1}{5^5} - \tfrac{1}{7^5} + \tfrac{1}{9^5} - \tfrac{1}{11^5} + \text{etc.} &= \mathfrak{c}\,\tfrac{\pi^5}{2^5}\\
1 - \tfrac{1}{3^7} + \tfrac{1}{5^7} - \tfrac{1}{7^7} + \tfrac{1}{9^7} - \tfrac{1}{11^7} + \text{etc.} &= \mathfrak{d}\,\tfrac{\pi^7}{2^7}\\
\text{etc.} \qquad \text{etc.}
\end{aligned}$$

qui-

$\frac{7n}{nn-1}$ + etc.

nes continen-
:rat omnes nu-
1 infinitam con-

luamus:
+ etc.
– etc.
+ etc.
– etc.

$\frac{\pi}{4n\,\mathrm{cof.}\frac{\pi}{2n}}$.

columnas ver-

$\mathfrak{a}\,\frac{\pi}{2}$
$\mathfrak{b}\,\frac{\pi^3}{2^3}$
$\mathfrak{c}\,\frac{\pi^5}{2^5}$
$\mathfrak{d}\,\frac{\pi^7}{2^7}$

qui-

quibus pofitis aequatio noftra erit

$$\frac{\pi}{4n\,\mathrm{cof.}\frac{\pi}{2n}} = \frac{\mathfrak{a}\pi}{2n} + \frac{\mathfrak{b}\pi^3}{2^3n^3} + \frac{\mathfrak{c}\pi^5}{2^5n^5} + \frac{\mathfrak{d}\pi^7}{2^7n^7} + \text{etc.}$$

§. 24. Ponamus nunc $\frac{\pi}{2n} = x$, et aequatio noftra hanc induet formam:

$$\frac{x}{2\,\mathrm{cof.}\,x} = \mathfrak{a}x + \mathfrak{b}x^3 + \mathfrak{c}x^5 + \mathfrak{d}x^7 + \mathfrak{e}x^9 + \text{etc.}$$

Nunc igitur, fi breuitatis gratia ponamus

$$\mathrm{cof.}\,x = \alpha - \beta xx + \gamma x^4 - \delta x^6 + \varepsilon x^8 - \text{etc.}$$

ita vt fit

$$\alpha = 1,\ \beta = \frac{1}{1.2},\ \gamma = \frac{1}{1\ldots 4},\ \delta = \frac{1}{1\ldots 6},\ \text{etc.}$$

fi per hanc feriem vtrinque multiplicemus, orietur ifta aequatio:

$$\left.\begin{array}{lllllll}
\frac{x}{2} = \alpha\mathfrak{a}x & +\alpha\mathfrak{b}x^3 & +\alpha\mathfrak{c}x^5 & +\alpha\mathfrak{d}x^7 & +\alpha\mathfrak{e}x^9 & +\alpha\mathfrak{f}x^{11} & +\alpha\mathfrak{g}x^{13}\\
 & -\beta\mathfrak{a} & -\beta\mathfrak{b} & -\beta\mathfrak{c} & -\beta\mathfrak{d} & -\beta\mathfrak{e} & -\beta\mathfrak{f}\\
 & & +\gamma\mathfrak{a} & +\gamma\mathfrak{b} & +\gamma\mathfrak{c} & +\gamma\mathfrak{d} & +\gamma\mathfrak{e}\\
 & & & -\delta\mathfrak{a} & -\delta\mathfrak{b} & -\delta\mathfrak{c} & -\delta\mathfrak{d}\\
 & & & & +\varepsilon\mathfrak{a} & +\varepsilon\mathfrak{b} & +\varepsilon\mathfrak{c}\\
 & & & & & -\zeta\mathfrak{a} & -\zeta\mathfrak{b}\\
 & & & & & & +\eta\mathfrak{a}
\end{array}\right\}\ \text{etc.}$$

§. 25. Singulis igitur poteftatibus ad nihilum reductis nancifcemur fequentes determinationes:

$$\begin{aligned}
\mathfrak{a} &= \tfrac{1}{2}\\
\mathfrak{b} &= \beta\mathfrak{a}\\
\mathfrak{c} &= \beta\mathfrak{b} - \gamma\mathfrak{a}
\end{aligned}$$

$\mathfrak{d} =$

$$\mathfrak{d} = \beta\mathfrak{c} - \gamma\mathfrak{b} + \delta\mathfrak{a}$$
$$\mathfrak{e} = \beta\mathfrak{d} - \gamma\mathfrak{c} + \delta\mathfrak{b} - \varepsilon\mathfrak{a}$$
$$\mathfrak{f} = \beta\mathfrak{e} - \gamma\mathfrak{d} + \delta\mathfrak{c} - \varepsilon\mathfrak{b} + \zeta\mathfrak{a}$$

etc. etc.

§. 26. Ope harum formularum iam olim ſummas iſtarum ſerierum exhibui, vnde valores pro praeſentibus litteris $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. ita reperientur determinati:

$\mathfrak{a} = \frac{1}{2}$	pro poteſtatibus	I.
$\mathfrak{b} = \frac{1}{1.2}\cdot\frac{1}{2}$	- - - - - - - - - -	III.
$\mathfrak{c} = \frac{5}{1.2.3.4}\cdot\frac{1}{2}$	- - - - - - - - - -	V.
$\mathfrak{d} = \frac{61}{1\ldots\ldots 6}\cdot\frac{1}{2}$	- - - - - - - - -	VII.
$\mathfrak{e} = \frac{1385}{1\ldots\ldots 8}\cdot\frac{1}{2}$	- - - - - - - - -	IX.
$\mathfrak{f} = \frac{50521}{1\ldots\ldots 10}\cdot\frac{1}{2}$	- - - - - - - - -	XI.
$\mathfrak{g} = \frac{2702765}{1\ldots\ldots 12}\cdot\frac{1}{2}$	- - - - - - - - -	XIII.
$\mathfrak{h} = \frac{199360981}{1\ldots\ldots 14}\cdot\frac{1}{2}$	- - - - - - - - -	XV.
$\mathfrak{i} = \frac{19391512145}{1\ldots\ldots 16}\cdot\frac{1}{2}$	- - - - - - - - -	XVII.
$\mathfrak{k} = \frac{2404879661671}{1\ldots\ldots 18}\cdot\frac{1}{2}$	- - - - - - - - -	XIX.

§. 27. Hinc igitur ſummas iſtarum ſerierum vsque ad poteſtatem vigeſimam apponamus:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \frac{1}{1}\cdot\frac{\pi}{2^2}$$
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{1}{1.2}\cdot\frac{\pi^3}{2^4}$$
$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = \frac{5}{1.2.3.4}\cdot\frac{\pi^5}{2^6}$$
$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \text{etc.} = \frac{61}{1\ldots\ldots 6}\cdot\frac{\pi^7}{2^8}$$
$$1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \text{etc.} = \frac{1385}{1\ldots\ldots 8}\cdot\frac{\pi^9}{2^{10}}$$

1 —

n olim ſummas
praeſentibus lit-
rminati :
I.
- - III.
- - V.
- - VII.
- - IX.
- - XI.
- - XIII.
- - XV.
- - XVII.
- - XIX.
m ſerierum vsque
1 —

$$1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \text{etc.} = \frac{50521}{1\ldots\ldots 10}\cdot\frac{\pi^{11}}{2^{12}}$$
$$1 - \frac{1}{3^{13}} + \frac{1}{5^{13}} - \frac{1}{7^{13}} + \frac{1}{9^{13}} - \text{etc.} = \frac{2702765}{1\ldots\ldots 12}\cdot\frac{\pi^{13}}{2^{14}}$$
$$1 - \frac{1}{3^{15}} + \frac{1}{5^{15}} - \frac{1}{7^{15}} + \frac{1}{9^{15}} - \text{etc.} = \frac{199360981}{1\ldots\ldots 14}\cdot\frac{\pi^{15}}{2^{16}}$$
$$1 - \frac{1}{3^{17}} + \frac{1}{5^{17}} - \frac{1}{7^{17}} + \frac{1}{9^{17}} - \text{etc.} = \frac{19391512145}{1\ldots\ldots 16}\cdot\frac{\pi^{17}}{2^{18}}$$
$$1 - \frac{1}{3^{19}} + \frac{1}{5^{19}} - \frac{1}{7^{19}} + \frac{1}{9^{19}} - \text{etc.} = \frac{2404879661671}{1\ldots\ldots 18}\cdot\frac{\pi^{19}}{2^{20}}$$

etc. etc.

Euolutio ſeriei poſterioris § 20.

§. 28. Binis terminis analogis contractis haec ſeries hanc induet formam:

$$\frac{\pi}{4n\cot.\frac{\pi}{2n}} = \frac{1}{nn-1} + \frac{1}{9nn-1} + \frac{1}{25nn-1} + \frac{1}{49nn-1} + \text{etc.}$$

quae fractiones in ſeries euolutae dabunt:

$$\frac{1}{nn-1} = \frac{1}{nn} + \frac{1}{n^4} + \frac{1}{n^6} + \frac{1}{n^8} + \frac{1}{n^{10}} + \text{etc.}$$
$$\frac{1}{9nn-1} = \frac{1}{9nn} + \frac{1}{9^2 n^4} + \frac{1}{9^3 n^6} + \frac{1}{9^4 n^8} + \frac{1}{9^5 n^{10}} + \text{etc.}$$
$$\frac{1}{25nn-1} = \frac{1}{25nn} + \frac{1}{25^2 n^4} + \frac{1}{25^3 n^6} + \frac{1}{25^4 n^8} + \frac{1}{25^5 n^{10}} + \text{etc.}$$

etc. etc.

quarum igitur omnium ſumma eſt $\frac{\pi}{4n}\,\text{tang.}\,\frac{\pi}{2n}$.

§. 29. Quo nunc has ſeries verticaliter colligere queamus, ſtatuamus:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} = \mathfrak{A}'\frac{\pi\pi}{2^3}$$
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = \mathfrak{B}'\frac{\pi^4}{2^4}$$
$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = \mathfrak{C}'\frac{\pi^6}{2^6}$$

etc. etc.

Vnde

Vnde aequatio nostra fiet

$$\frac{\pi}{4n}\,\text{tang.}\,\frac{\pi}{2n} = \frac{\mathfrak{A}'\pi^2}{2^2 n^2} + \frac{\mathfrak{B}'\pi^4}{2^4 n^4} + \frac{\mathfrak{C}'\pi^6}{2^6 n^6} + \frac{\mathfrak{D}'\pi^8}{2^8 n^8} + \text{etc.}$$

§. 30. Ponamus nunc $\frac{\pi}{2n} = x$, vt aequatio nostra fiat

$$\frac{x}{2}\,\text{tang.}\,x = \mathfrak{A}'xx + \mathfrak{B}'x^4 + \mathfrak{C}'x^6 + \text{etc.}$$

vnde erit

$$\text{tang.}\,x = 2\mathfrak{A}'x + 2\mathfrak{B}'x^3 + 2\mathfrak{C}'x^5 + 2\mathfrak{D}'x^7 + \text{etc.}$$

cuius seriei loco scribamus litteram t, vt sit $\text{tang.}\,x = t$, hincque differentiando $dx = \frac{dt}{1+tt}$, ideoque habebimus $\frac{dt}{dx} = 1 + tt$. Est vero

$$\frac{dt}{dx} = 2\mathfrak{A}' + 2.3\mathfrak{B}'xx + 2.5\mathfrak{C}'x^4 + 2.7\mathfrak{D}'x^6 + \text{etc.}$$

§. 31. Eodem modo facta euolutione erit

$$1 + tt = 1 + 4\mathfrak{A}'\mathfrak{A}'xx + 8\mathfrak{A}'\mathfrak{B}'x^4 + 8\mathfrak{A}'\mathfrak{C}'x^6 + 8\mathfrak{A}'\mathfrak{D}'x^8 + \text{etc.}$$
$$+ 4\mathfrak{B}'\mathfrak{B}' + 8\mathfrak{B}'\mathfrak{C}' + \text{etc.}$$

Hinc igitur sequentes deducuntur determinationes:

$\mathfrak{A}' = \frac{1}{2}$

$\mathfrak{B}' = \frac{2}{3}.\mathfrak{A}'\mathfrak{A}'$

$\mathfrak{C}' = \frac{2}{5}.2\mathfrak{A}'\mathfrak{B}'$

$\mathfrak{D}' = \frac{2}{7}(2\mathfrak{A}'\mathfrak{C}' + \mathfrak{B}'\mathfrak{B}')$

$\mathfrak{E}' = \frac{2}{9}(2\mathfrak{A}'\mathfrak{D}' + 2\mathfrak{B}'\mathfrak{C}')$

$\mathfrak{F}' = \frac{2}{11}(2\mathfrak{A}'\mathfrak{E}' + 2\mathfrak{B}'\mathfrak{D}' + \mathfrak{C}'\mathfrak{C}')$

$\mathfrak{G}' = \frac{2}{13}(2\mathfrak{A}'\mathfrak{F}' + 2\mathfrak{B}'\mathfrak{E}' + 2\mathfrak{C}'\mathfrak{D}')$

etc. etc.

§. 32. Istae determinationes fere prorsus conueniunt cum iis, quas supra pro litteris $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. inuenimus; totum enim discrimen reperitur in coëfficientibus numericis. Neutiquam vero opus est istos valores seorsim com-

+ etc.
tio nostra fiat
)'x^7 + etc.
tang. x = t,
habebimus
)'x^6 + etc.
erit
'D'x^8 + etc.
B'C' + etc.
:s:
sus conueni-
: etc. inueni-
cientibus nu-
lores seorsim
com-

computare, cum ii iam ex superioribus facillime deduci queant. Cum enim sit $\frac{\mathfrak{A}'}{2^2} = \frac{\mathfrak{A}(2^2-1)}{2^2}$, erit $\mathfrak{A}' = (2^2-1)\mathfrak{A}$. Simili modo erit

$$\mathfrak{B}' = (2^4-1)\mathfrak{B},\ \mathfrak{C}' = (2^6-1)\mathfrak{C},\ \mathfrak{D}' = (2^8-1)\mathfrak{D},\ \text{etc.}$$

Conclusio.

§. 33. In gratiam eorum, qui hos valores penitus numerice per fractiones decimales exprimere voluerint, subiungamus sequentem tabulam, in qua omnes potestates ipsius π per fractiones decimales sunt euolutae, vbi loco $\frac{\pi}{2}$ scripsimus q.

$\frac{q}{1}$	= 1, 57079, 63267, 94896, 61923, 13216, 916
$\frac{q^2}{1.2}$	= 1, 23370, 05501, 36169, 82735, 43113, 745
$\frac{q^3}{1.2.3}$	= 0, 64596, 40975, 06246, 25365, 57565, 636
$\frac{q^4}{1\ldots4}$	= 0, 25366, 95079, 01048, 01363, 65633, 659
$\frac{q^5}{1\ldots5}$	= 0, 07969, 26262, 46167, 04512, 05055, 487
$\frac{q^6}{1\ldots6}$	= 0, 02086, 34807, 63352, 96087, 30516, 364
$\frac{q^7}{1\ldots7}$	= 0, 00468, 17541, 35318, 68810, 06854, 633
$\frac{q^8}{1\ldots8}$	= 0, 00091, 92602, 74839, 42658, 02417, 158
$\frac{b^9}{1\ldots9}$	= 0, 00016, 04411, 84787, 35982, 18726, 605
$\frac{q^{10}}{1\ldots10}$	= 0, 00002, 52020, 42373, 06060, 54810, 526
$\frac{q^{11}}{1\ldots11}$	= 0, 00000, 35988, 43235, 21208, 53404, 580
$\frac{q^{12}}{1\ldots12}$	= 0, 00000, 04710, 87477, 88181, 71503, 665
$\frac{q^{13}}{1\ldots13}$	= 0, 00000, 00569, 21729, 21967, 92681, 170
$\frac{q^{14}}{1\ldots14}$	= 0, 00000, 00063, 86603, 08379, 18522, 408

$\frac{q^{15}}{1 \cdot \ldots \cdot 15} = 0,00000,00006,68803,51098,11467,225$

$\frac{q^{16}}{1 \cdot \ldots \cdot 16} = 0,00000,00000,65659,63114,97947,230$

$\frac{q^{17}}{1 \cdot \ldots \cdot 17} = 0,00000,00000,06066,93573,11061,950$

$\frac{q^{18}}{1 \cdot \ldots \cdot 18} = 0,00000,00000,00529,44002,00734,620$

$\frac{q^{19}}{1 \cdot \ldots \cdot 19} = 0,00000,00000,00043,77065,46731,370$

$\frac{q^{20}}{1 \cdot \ldots \cdot 20} = 0,00000,00000,00003,43773,91790,981$

$\frac{q^{21}}{1 \cdot \ldots \cdot 21} = 0,00000,00000,00000,25714,22892,855$

$\frac{q^{22}}{1 \cdot \ldots \cdot 22} = 0,00000,00000,00000,01835,99165,212$

$\frac{q^{23}}{1 \cdot \ldots \cdot 23} = 0,00000,00000,00000,00125,38995,403$

$\frac{q^{24}}{1 \cdot \ldots \cdot 24} = 0,00000,00000,00000,00008,20675,327$

$\frac{q^{25}}{1 \cdot \ldots \cdot 25} = 0,00000,00000,00000,00000,51564,550$

$\frac{q^{26}}{1 \cdot \ldots \cdot 26} = 0,00000,00000,00000,00000,03115,285$

$\frac{q^{27}}{1 \cdot \ldots \cdot 27} = 0,00000,00000,00000,00000,00181,239$

$\frac{q^{28}}{1 \cdot \ldots \cdot 28} = 0,00000,00000,00000,00000,00010,165$

$\frac{q^{29}}{1 \cdot \ldots \cdot 29} = 0,00000,00000,00000,00000,00000,549$

$\frac{q^{30}}{1 \cdot \ldots \cdot 30} = 0,00000,00000,00000,00000,00000,026$

$\frac{q^{31}}{1 \cdot \ldots \cdot 31} = 0,00000,00000,00000,00000,00000,000$

Hae quidem poteſtates diuiſae ſunt per certos numeros, qui autem plerumque ſunt ii ipſi, per quos eaedem poteſtates ipſius π in ſuperioribus formulis diuiſi occurrunt, vnde euolutio in fractiones decimales eo facilior redditur.

098, 11467, 225
114, 97947, 230
573, 11061, 950
002, 00734, 620
065, 46731, 370
773, 91790, 981
714, 22892, 855
835, 99165, 212
125, 38995, 403
008, 20675, 327
000, 51564, 550
000, 03115, 285
000, 00181, 239
000, 00010, 165
000, 00000, 549
000, 00000, 026
000, 00000, 000
rtos numeros, qui
eaedem poteſtates
urrunt, vnde euo-
dditur.
DE

DE INSIGNI PROMOTIONE SCIENTIAE NVMERORVM.

§. 1.

Eximia omnino ſunt, quae celeberr. *La Grange* in Comment. Academiae Regiae Boruſſicae pro Anno 1773 de diuiſoribus formulae generaliſſimae $Btt + Ctu + Duu$ demonſtrauit, et maximam lucem in ſcientia numerorum, quae etiamnunc tantis tenebris eſt inuoluta, accendunt. Ob hoc ipſum autem, quod iſta tractatio maxime eſt generalis, ii qui non ſatis ſunt exercitati in huiusmodi ſpeculationibus, non parum difficultatis offendunt, neque vim talium ſublimium demonſtrationum ſatis perſpicere valent. Quamobrem haud inutile erit omnia momenta, quibus hae demonſtrationes innituntur, diligentius explicare atque ad formulas magis ſpeciales accommodare, quandoquidem hoc modo omnia facilius intelligi poterunt. Deinde imprimis accuratius exponam, quantum firmamentum hinc plurimis theorematibus, quorum veritatem per ſolam inductionem mihi quidem cognoſcere licuit, afferri poſſit, vnde multo clarius patebit, quantum adhuc ad eorum perfectam demonſtrationem deſideretur.