1785

De summa seriei ex numeris primis formatae 1/3 – 1/5 + 1/7 + 1/11 – 1/13 – 1/17 + 1/19 + 1/23 – 1/29 + 1/31 etc. ubi numeri primi formae 4*n*–1 habent signum positivum, formae autem 4*n*+1 signum negativum

Leonhard Euler

DE SVMMA SERIEI EX NVMERIS PRIMIS FORMATAE

$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31}$ etc.

VBI NVMERI PRIMI FORMAE $4n-1$ HABENT SIGNVM POSITIVVM, FORMAE AVTEM $4n+1$ SIGNVM NEGATIVVM.

§. 1.

Cum iam *Euclides* demonstrasset, multitudinem numerorum primorum reuera esse infinitam, ego iam pridem ostendi etiam summam seriei reciprocae numerorum primorum: scilicet

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\text{ etc.}$$

esse infinite magnam, atque adeo referre logarithmum summae seriei harmonicae

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{ etc.}$$

id quod non parum mirum videbatur, cum vulgo summa seriei harmonicae ad genus quasi infimum infinitorum referri soleat. Cum autem non solum logarithmus numeri infiniti sit etiam infinitus, sed etiam logarithmi horum ipsorum logarithmorum etiamnunc sint infiniti; manifestum est dari infuper infinitos gradus inferiores infinitorum. Ita si A denotet summam seriei reciprocae numerorum primorum, etiam

lA adhuc erit infinite magnus, sed ad ordinem infinitorum infinities inferiorem pertinere censendus est; tum vero etiamnunc hae formulae: llA, $lllA$, $llllA$, etc. erunt infinitae, quanquam quaelibet earum infinities sit minor quam praecedens.

§. 2. Quoniam porro numeri primi praeter binarium quasi a natura in duas classes distinguuntur, prouti fuerint vel formae $4n+1$, vel formae $4n-1$, dum priores omnes sunt summae duorum quadratorum, posteriores vero ab hac proprietate penitus excluduntur: series reciprocae ex vtraque classe formatae, scilicet:

$$\frac{1}{5}+\frac{1}{13}+\frac{1}{17}+\frac{1}{29}+\text{ etc. et}$$

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{19}+\frac{1}{23}+\text{ etc.}$$

ambae erunt pariter infinitae, id quod etiam de omnibus speciebus numerorum primorum est tenendum. Ita si ex numeris primis ii tantum excerpantur, qui sunt formae $100n+1$, cuiusmodi sunt 101, 401, 601, 701 etc., non solum multitudo eorum est infinita, sed etiam summa huius seriei ex illis formatae, scilicet:

$$\frac{1}{101}+\frac{1}{401}+\frac{1}{601}+\frac{1}{701}+\frac{1}{1201}+\frac{1}{1301}+\frac{1}{1601}+\frac{1}{1801}+\frac{1}{1901}+\text{ etc.}$$

etiam est infinita.

§. 3. Consideremus hic autem imprimis discrimen inter numeros primos formae $4n+1$ et $4n-1$, et quia ambae series ex vtroque ordine formatae sunt infinitae et quasi eiusdem ordinis; nullum est dubium, quin earum differentia habeat valorem determinatum. Hanc ob rem terminis ex forma $4n-1$ formatis tribuamus signum $+$, reliquis vero signum $-$, vt orietur ista series:

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31}\ \text{etc},$$

in qua nullus plane ordo in ſignis cernitur. Hoc tamen non obſtante videamus, quomodo eius ſumma vero ſaltem proxime aſſignari queat. Non parum enim probabile videtur, iſtam ſummam, ſi non fuerit rationalis vel irrationalis, ſaltem ad quodpiam genus notabile quantitatum tranſcendentium pertinere debere. Interim tamen, quoniam non ſolum non in ſignis, verum etiam multo minus in ipſis fractionibus nullus plane ordo perſpicitur, primo intuitu nulla patere videtur via, qua ad eius ſummam peruenire liceat.

§. 4. Quando autem contemplamur ſeriem notiſſimam Leibnitzianam pro quadratura circuli datam

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}\ \text{etc.}$$

videmus, in ea omnes numeros impares formae $4n+1$ habere ſignum $+$, reliquos vero formae $4n-1$ ſignum $-$, vnde mutatis ſignis omnes termini noſtrae ſeriei cum ſuis ſignis in hac ſerie Leibnitziana occurrunt. Quod ſi ergo ex hac ſerie omnes numeros compoſitos expungamus, tandem praeter vnitatem ipſa noſtra ſeries propoſita, contrariis ſignis, remanebit. Quamobrem ſummam noſtrae ſeriei obtinebimus, ſi ex ſerie Leibnitziana ſucceſſiue omnes numeros compoſitos excludamus, quandoquidem omnium terminorum, qui per quamuis operationem excluduntur, ſumma facile aſſignare licebit.

§. 5. Incipiamus igitur ab ipſa ſerie Leibnitziana, ſtatuendo

$$A=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+\ \text{etc.}$$

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ita vt ſit $A=\frac{\pi}{4}$, ex eaque primo omnes numeros compoſitos per 3 diuiſibiles excludamus. Quem in finem hinc formemus iſtam ſeriem:

$$(A-1)\tfrac{1}{3}=-\frac{1}{9}+\frac{1}{15}-\frac{1}{21}+\frac{1}{27}-\frac{1}{33}+\frac{1}{39}-\ \text{etc.}$$

quae vtique continet omnes numeros compoſitos per 3 diuiſibiles, vnde ſi haec ſeries ad illam addatur, omnes iſti numeri compoſiti excludentur fietque

$$\tfrac{4}{3}A-\tfrac{1}{3}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}+\frac{1}{25}\ \text{etc.}=B,$$

ita vt ſit $B=\frac{4}{3}A-\frac{1}{3}$. In hac igitur ſerie, cuius ſumma nouimus, nulli amplius numeri compoſiti per 3 diuiſibiles occurrunt, ſed primus terminus compoſitus hic occurrens eſt $+\frac{1}{25}$.

§. 6. Nunc igitur ex poſtrema hac ſerie omnes terminos per 5 diuiſibiles excludamus, quem in finem hinc formemus iſtam ſeriem:

$$(B-1+\tfrac{1}{3})\tfrac{1}{5}=\frac{1}{25}-\frac{1}{35}-\frac{1}{55}+\frac{1}{65}+\frac{1}{85}-\frac{1}{95}-\ \text{etc.},$$

quae omnes complectitur terminos per quinque diuiſibiles, qui adhuc in ſeriem B ingrediebantur, et quidem iisdem ſignis affectos, quare hac ſerie ab illa ablata remanebit haec:

$$\tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3})=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}+\ \text{etc.}$$

vbi iam ab initio omnes termini ſunt numeri primi, et non primus qui hic occurret erit $\frac{1}{49}$, ſequentes vero $\frac{1}{77}, -\frac{1}{91}$. Ponamus autem ſummam huius ſeriei $=C$, vt ſit

$$C=\tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3}).$$

§. 7. Nunc igitur hinc excludamus omnes terminos qui adhuc per ſeptem ſunt diuiſibiles, quos complectetur ſequens forma:

(C –

$$(C-1+\tfrac{1}{3}-\tfrac{1}{5})\tfrac{1}{7}=-\tfrac{1}{49}-\tfrac{1}{77}+\tfrac{1}{91}+\text{ etc.}$$

quorum terminorum signa sunt contraria, quare haec series ad seriem C addita dabit

$$\tfrac{6}{7}C-\tfrac{1}{7}(1-\tfrac{1}{3}+\tfrac{1}{5})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}+\text{ etc.}$$

in qua primus terminus compositus erit $+\frac{1}{11\cdot 11}$; sequentes vero $-\frac{1}{11\cdot 13}$, $-\frac{1}{11\cdot 17}$ etc. Hanc autem seriem vocemus D vt sit

$$D=\tfrac{6}{7}C-\tfrac{1}{7}(1-\tfrac{1}{3}+\tfrac{1}{5}).$$

§. 8. Iam ex serie modo inuenta D expungamus terminos, qui adhuc sunt per 11 diuisibiles, quos complectetur ista forma:

$$D-(1+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7})\tfrac{1}{11}=-\tfrac{1}{121}+\tfrac{1}{143}+\tfrac{1}{187}\text{ etc.}$$

qui termini in serie D contraria habent signa; quamobrem si haec series ad illam addatur, isti termini excludentur, prodibitque

$$\tfrac{10}{11}D-\tfrac{1}{11}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\text{ etc.}$$

in qua primus terminus non primus est $\frac{1}{169}$; istam autem seriem designemus littera E, ita vt sit

$$E=\tfrac{10}{11}D-\tfrac{1}{11}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}).$$

§. 9. Ex hac igitur serie excludamus omnes terminos, qui adhuc insunt per 13 diuisibiles, quos ergo complectetur haec forma:

$$(E-1+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11})\tfrac{1}{13}=\tfrac{1}{169}+\tfrac{1}{221}\text{ etc.}$$

hique termini eadem habent signa ac in ipsa serie E. Haec igitur series ab illa debet subtrahi, vnde prodit

$$\tfrac{12}{13}E+\tfrac{1}{13}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\text{ etc.}$$

etc.
re haec series
$+\frac{1}{17}+$ etc.
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$\frac{1}{11}+$ etc.
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etc.
serie E. Haec
dit
$-\frac{1}{7}-\frac{1}{11}+$ etc.
vbi

vbi primus terminus non primus est $\frac{1}{289}$. Totam autem hanc seriem designemus littera F, vt sit

$$F=\tfrac{12}{13}E+\tfrac{1}{13}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}).$$

§. 10. Quod si nunc istas operationes vlterius continuemus, dum successiue hinc excludimus terminos adhuc per 17 diuisibiles, tum vero per 19, per 23, etc. tandem relinquetur tantum series numerorum primorum post vnitatem sequentium, quae si designetur littera Z, quam vt infinitesimam spectari oportet, erit vtique

$$Z=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}\text{ etc.}$$

consequenter summa seriei in titulo propositae summa erit $1-Z$. Ac manifestum est, ad hunc valorem continuo propius accedere istas formulas:

$$1-A,\ 1-B,\ 1-C,\ 1-D,\ 1-E,\ 1-F,\text{ etc.}$$

§. 11. Quemadmodum autem valores omnium harum litterarum successiue ex antecedentibus colligi debeant, ex sequentibus formulis fiet manifestum:

$$B=\tfrac{2}{3}A-\tfrac{1}{3}.1$$
$$C=\tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3})$$
$$D=\tfrac{6}{7}C-\tfrac{1}{7}(1-\tfrac{1}{3}+\tfrac{1}{5})$$
$$E=\tfrac{10}{11}D-\tfrac{1}{11}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7})$$
$$F=\tfrac{12}{13}E+\tfrac{1}{13}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11})$$
$$G=\tfrac{16}{17}F+\tfrac{1}{17}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13})$$
$$H=\tfrac{18}{19}G-\tfrac{1}{19}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17})$$
$$I=\tfrac{22}{23}H-\tfrac{1}{23}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19})$$

etc. etc.

Vbi notandum, si denominator primus fuerit formae $4n+1$, tum numeratorem primae partis fore vnitate minorem, siue $4n$, alteram vero partem addi debere. Sin autem denominator primus fuerit $4n-1$, tum numeratorem primae partis fore vnitate maiorem, siue $4n$, alteram partem vero hoc casu subtrahi debere.

§. 12. Quo nunc omnes hos valores in numeris per fractiones decimales exprimamus, ante omnia notetur esse

$$A = \tfrac{\pi}{4} = 0,\ 7853981634.$$

Pro reliquis autem litteris computentur sequentes valores:

$1-\frac{1}{3}=$ - - - - - - - - - $b=0,6666666666$

$1-\frac{1}{3}+\frac{1}{5}=$ - - - - - - - - $c=0,8666666666$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}=$ - - - - - - - $d=0,7238095238$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}=$ - - - - - - $e=0,6329004329$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}=$ - - - - - $f=0,7098235098$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}=$ - - - - $g=0,7686470392$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}=$ - - $h=0,7160154603$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}=$ - $i=0,6725371994$

in quo ordine primus terminus a vnitate aequatur.

§. 13. Pro computo autem ipsarum litterarum A, B, C, D, E, etc. praestabit sequentibus vti formulis, quibus simul valores numericos harum litterarum adscribamus

$$B=A+\tfrac{1}{3}(A-a)=0,713864$$
$$C=B-\tfrac{1}{5}(B-b)=0,704424$$
$$D=C+\tfrac{1}{7}(C-c)=0,681247$$

E=

formae $4n+1$, minorem, siue em denomina- primae partis n vero hoc ca-
s in numeris nia notetur esse
ntes valores:
, 6666666666
, 8666666666
, 7238095238
, 6329004329
, 7098235098
, 7686470392
, 7160154603
, 6725371994
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litterarum A, ormulis, quibus lscribamus
E=

$$E=D+\tfrac{1}{11}(D-d)=0,677377$$
$$F=E-\tfrac{1}{13}(E-e)=0,673956$$
$$G=F-\tfrac{1}{17}(F-f)=0,676066$$
$$H=G+\tfrac{1}{19}(G-g)=0,671193$$
$$I=H+\tfrac{1}{23}(H-h)=0,669245$$
$$K=I-\tfrac{1}{29}(I-i)=0,669358.$$

§. 14. Quanquam autem calculum huc vsque produximus, tamen vltra tertiam figuram decimalem de summa nostrae seriei certi esse non possumus, atque adeo in dubio relinquere cogimur, vtrum ista summa aliquanto maior vel minor sit quam 0, 669. Sin autem hunc valorem pro vero assumamus, ipsa series proposita

$$\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19}\ \text{etc.}$$

summam habebit 0, 331, ideoque hic valor tantillo foret minor quam $\frac{1}{3}$. Quoniam vero ablato $\frac{1}{3}$, iterum addi debeat $\frac{1}{7}+\frac{1}{11}$, quarum fractionum summa maior est quam $\frac{1}{5}$, vtique fieri posset, vt verus valor superaret $\frac{1}{3}$, id quod hoc loco in dubio est relinquendum. Datur vero alia methodus, multo accuratius in summam huius seriei inquirendi, quam hic euoluemus, quandoquidem operae pretium videtur, veram huius seriei summam propius cognouisse.

§. 15. Eadem methodo qua hic ex prima serie Leibnitziana successiue terminos compositos expulimus, si omnes plane terminos praeter vnitatem remoueamus, reperiemus

$$\tfrac{\pi}{4}=\tfrac{3}{4}\cdot\tfrac{5}{4}\cdot\tfrac{7}{8}\cdot\tfrac{11}{12}\cdot\tfrac{13}{12}\cdot\tfrac{17}{16}\cdot\tfrac{19}{20}\ \text{etc.}$$

vbi

vbi in numeratoribus omnes numeri primi occurrunt praeter 2, denominatores vero ſunt numeri pariter pares vnitate vel maiores vel minores. Deinde vero ſi iſta ſeries reciproca quadratorum imparium:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\text{etc.}$$

cuius ſummam oſtendi eſſe $=\frac{\pi\pi}{8}$, ſimili modo tractetur, reperietur

$$\frac{\pi\pi}{8}=\frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{10\cdot12}\cdot\frac{13\cdot13}{12\cdot14}\cdot\text{ etc.}$$

vbi iterum in numeratoribus omnes numeri primi bis occurrunt, in denominatoribus vero iidem tam vnitate aucti quam minuti. Quare ſi hanc expreſſionem per quadratum illius, quod eſt

$$\frac{\pi\pi}{16}=\frac{3\cdot3}{4\cdot4}\cdot\frac{5\cdot5}{4\cdot4}\cdot\frac{7\cdot7}{8\cdot8}\cdot\frac{11\cdot11}{12\cdot12}\cdot\frac{13\cdot13}{12\cdot12}\text{ etc.}$$

diuidamus, quotus erit

$$2=\frac{4}{2}\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\text{ etc.}$$

vbi omnes numeri primi tam vnitate aucti quam minuti occurrunt, et numeri pariter pares in numeratore, impariter pares vero in denominatore conſtituuntur.

§. 16. Poſtrema haec expreſſio igitur hoc modo exhiberi poterit:

$$2=\frac{3+1}{3-1}\cdot\frac{5-1}{5+1}\cdot\frac{7+1}{7-1}\cdot\frac{11+1}{11-1}\cdot\frac{13-1}{13+1}\text{ etc.}$$

hinc ergo logarithmis hyperbolicis ſumendis habebimus

$$l\,2=l\frac{3+1}{3-1}+l\frac{5-1}{5+1}+l\frac{7+1}{7-1}+l\frac{11+1}{11-1}+l\frac{13-1}{13+1}+\text{ etc.}$$

Conſtat autem per ſeries infinitas eſſe in genere

$$\frac{1}{2}l\frac{a+1}{a-1}=\frac{1}{a}+\frac{1}{3a^3}+\frac{1}{5a^5}+\frac{1}{7a^7}+\frac{1}{9a^9}+\text{ etc.}$$

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$\frac{13-1}{13+1}$ + etc.
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hincque

$$\frac{1}{2}l\frac{a-1}{a+1}=-\frac{1}{a}-\frac{1}{3a^3}-\frac{1}{5a^5}-\frac{1}{7a^7}-\frac{1}{9a^9}-\text{ etc.}$$

Quodſi igitur harum formularum ope omnes illos logarithmos in ſeries infinitas conuertamus, nanciſcemur quidem innumeras ſeries infinitas, quas autem ad ſeries facilius tractabiles reducere licebit.

§. 17. Primo igitur omnium illorum logarithmorum ſemiſſes accipi oportet, et quia hic de logarithmis hyperbolicis agitur, ob

$$l\,2=0,6931471805,\text{ erit}$$
$$\tfrac{1}{2}l\,2=0,3465735902,$$

ex altera autem parte logarithmi ita ordinentur:

$$\frac{1}{2}l\frac{3+1}{3-1}=\frac{1}{3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\frac{1}{7\cdot3^7}+\frac{1}{9\cdot3^9}+\text{ etc.}$$
$$\frac{1}{2}l\frac{5-1}{5+1}=-\frac{1}{5}-\frac{1}{3\cdot5^3}-\frac{1}{5\cdot5^5}-\frac{1}{7\cdot5^7}-\frac{1}{9\cdot5^9}-\text{ etc.}$$
$$\frac{1}{2}l\frac{7+1}{7-1}=\frac{1}{7}+\frac{1}{3\cdot7^3}+\frac{1}{5\cdot7^5}+\frac{1}{7\cdot7^7}+\frac{1}{9\cdot7^9}+\text{ etc.}$$
$$\frac{1}{2}l\frac{11+1}{11-1}=\frac{1}{11}+\frac{1}{3\cdot11^3}+\frac{1}{5\cdot11^5}+\frac{1}{7\cdot11^7}+\frac{1}{9\cdot11^9}+\text{ etc.}$$
$$\frac{1}{2}l\frac{13-1}{13+1}=-\frac{1}{13}-\frac{1}{3\cdot13^3}-\frac{1}{5\cdot13^5}-\frac{1}{7\cdot13^7}-\frac{1}{9\cdot13^9}-\text{ etc.}$$

etc. etc.

§. 18. Hinc iam verticaliter deſcendendo conſideremus ſequentes ſeries pariter infinitas:

$$O=\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\text{ etc.}$$
$$P=\frac{1}{3^3}-\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{11^3}-\frac{1}{13^3}-\frac{1}{17^3}+\frac{1}{19^3}+\text{ etc.}$$
$$Q=\frac{1}{3^5}-\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{11^5}-\frac{1}{13^5}-\frac{1}{17^5}+\frac{1}{19^5}+\text{ etc.}$$

R =

$$R = \frac{1}{3^7} - \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} - \frac{1}{13^7} - \frac{1}{17^7} + \frac{1}{19^7} + \text{etc.}$$

$$S = \frac{1}{3^9} - \frac{1}{5^9} + \frac{1}{7^9} + \frac{1}{11^9} - \frac{1}{13^9} - \frac{1}{17^9} + \frac{1}{19^9} + \text{etc.}$$

etc. etc.

Quarum ſerierum prima O eſt ea ipſa, cuius ſummam hic inueſtigare nobis eſt propoſitum.

§. 19. His igitur ſeriebus ita per litteras maiuſculas deſignatis habebimus iſtam aequationem:

$$\tfrac{1}{2}\,l\,2 = O + \tfrac{1}{3}P + \tfrac{1}{5}Q + \tfrac{1}{7}R + \tfrac{1}{9}S + \tfrac{1}{11}T + \text{etc.}$$

vnde ſi ſummae ſerierum P, Q, R, S eſſent cognitae, inde impetraremus facile ſummam ſeriei O quaeſitam; foret enim

$$O = \tfrac{1}{2}\,l\,2 - \tfrac{1}{3}P - \tfrac{1}{5}Q - \tfrac{1}{7}R - \tfrac{1}{9}S - \text{etc.}$$

§. 20. At vero ſummas ſerierum P, Q, R etc. ex ſeriebus ordinis, vbi omnes numeri impares occurrunt, concludere poterimus eodem modo quo ſupra ſeriem ipſam O ex ſerie Leibnitziana

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} \text{ etc.}$$

elicuimus. Hunc in finem euolui hac methodo oportebit ſequentes ſeries ordinatas:

$$\mathfrak{P} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + \text{etc.}$$

$$\mathfrak{Q} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \frac{1}{13^5} - \frac{1}{15^5} + \text{etc.}$$

$$\mathfrak{R} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \frac{1}{13^7} - \frac{1}{15^7} + \text{etc.}$$

$$\mathfrak{S} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \frac{1}{11^9} + \frac{1}{13^9} - \frac{1}{15^9} + \text{etc.}$$

$$\mathfrak{T} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \frac{1}{11^{11}} + \frac{1}{13^{11}} - \frac{1}{15^{11}} + \text{etc.}$$

etc. etc.

harum

etc.
etc.
is ſummam hic
itteras maiuſcu-
T + etc.
at cognitae, in-
quaeſitam; foret
etc.
P, Q, R etc.
impares occur-
uo ſupra ſeriem
ethodo oportebit
$\frac{1}{13^3} - \frac{1}{15^3} + \text{etc.}$
$\frac{1}{13^5} - \frac{1}{15^5} + \text{etc.}$
$\frac{1}{13^7} - \frac{1}{15^7} + \text{etc.}$
$\frac{1}{13^9} - \frac{1}{15^9} + \text{etc.}$
$\frac{1}{13^{11}} - \frac{1}{15^{11}} + \text{etc.}$
harum

Harum autem omnium ſerierum ſummas iam pridem per quadraturam circuli, ſcilicet per ſimiles poteſtates ipſius π expreſſas dedi, ſequenti modo:

$$\mathfrak{P} = \frac{1}{1.\,2}\cdot\frac{\pi^3}{2^4} \qquad \mathfrak{T} = \frac{50521}{1\,.\,.\,.\,.\,.\,.\,10}\cdot\frac{\pi^{11}}{2^{12}}$$

$$\mathfrak{Q} = \frac{5}{1.\,2.\,3.\,4}\cdot\frac{\pi^5}{2^6} \qquad \mathfrak{U} = \frac{2702765}{1\,.\,.\,.\,.\,.\,.\,12}\cdot\frac{\pi^{13}}{2^{14}}$$

$$\mathfrak{R} = \frac{61}{1\,.\,.\,.\,.\,.\,.\,6}\cdot\frac{\pi^7}{2^8} \qquad \mathfrak{V} = \frac{199360981}{1\,.\,.\,.\,.\,.\,.\,14}\cdot\frac{\pi^{15}}{2^{16}}$$

$$\mathfrak{S} = \frac{1385}{1\,.\,.\,.\,.\,.\,.\,8}\cdot\frac{\pi^9}{2^{10}} \qquad \mathfrak{W} = \frac{19391512145}{1\,.\,.\,.\,.\,.\,.\,16}\cdot\frac{\pi^{17}}{2^{18}}$$

etc. etc.

§. 21. Hos igitur valores in fractionibus decimalibus vsque ad ſextam figuram euoluamus, eritque

	Differentiae
$\mathfrak{P} = 0,9689462$	0,0272095
$\mathfrak{Q} = 0,9961557$	0,0033990
$\mathfrak{R} = 0,9995547$	0,0003952
$\mathfrak{S} = 0,9999499$	0,0000448
$\mathfrak{T} = 0,9999947$	0,0000050
$\mathfrak{U} = 0,9999997$	0,0000005
etc.	etc.

§. 22. Vt nunc hinc valores litterarum P, Q, R, etc. eruamus, eadem methodo vtamur, qua ſupra ex ſerie

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}$$

omnes terminos compoſitos exterminauimus, quandoquidem loco horum numerorum ſimplicium eorum poteſtates ſcribi conuenit. Hanc igitur operationem in genere pro his litteris doceamus. Conſideremus igitur hanc ſeriem:

$\mathfrak{Z} =$

$$\mathfrak{Z} = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \text{etc.}$$

cuius summam, vt supra factum est, littera A designemus, vt sit $A = \mathfrak{Z}$, hincque sequentes litteras B, C, D etc. eliciamus per sequentes formulas:

$$B = A + \frac{1}{3^n}(A - a) \text{ existente } a = 1$$

$$C = B - \frac{1}{5^n}(B - b) \;\text{- - - -}\; b = 1 - \frac{1}{3^n}$$

$$D = C + \frac{1}{7^n}(C - c) \;\text{- - - -}\; c = 1 - \frac{1}{3^n} + \frac{1}{5^n}$$

$$E = D + \frac{1}{11^n}(D - d) \;\text{- - - -}\; d = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n}$$

$$F = E - \frac{1}{13^n}(E - e) \;\text{- - - -}\; e = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n}$$

etc. etc.

Quibus valoribus inuentis eorum complementa ad vnitatem, scilicet: $1 - A$, $1 - B$, $1 - C$, $1 - D$, etc. promptissime ad valorem quaesitum

$$Z = \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} - \frac{1}{13^n} - \text{etc.}$$

appropinquabunt.

§. 23. Haec igitur praecepta generalia applicemus primo ad valorem litterae P, vnde incipiendum erit a valore

$$\mathfrak{P} = 0,9689462 = A,$$

et quia hic est $n = 3$, habebimus

$a = 1$, $b = 0,9629630$, $c = 0,9709630$, $d = 0,9680476$; pluri-

– etc.
A designemus, C, D etc. eli-
$-\frac{1}{7^n}$
$-\frac{1}{7^n} - \frac{1}{11^n}$
e.
ita ad vnitatem, promptissime ad
etc.
ralia applicemus dum erit a valore
$d = 0,9680476$; pluri-

pluribus valoribus non erit opus. Hinc igitur colligemus sequentes valores:

$$B = A - \tfrac{1}{3^3} \cdot 0,0310538 = 0,9677961$$

$$C = B - \tfrac{1}{5^3} \cdot 0,0048331 = 0,9677574$$

$$D = C - \tfrac{1}{7^3} \cdot 0,0032056 = 0,9677481$$

$$E = D - \tfrac{1}{11^3} \cdot 0,0002995 = 0,9677479$$

Vlterius procedi non est opus; quamobrem hinc habebimus

$$P = 1 - E = 0,0322521,$$

vnde iam colligimus

$$\tfrac{1}{4} l 2 - \tfrac{1}{4} \cdot 0,0322521 = 0,3358229.$$

§. 24. Sumamus nunc $n = 5$ et habebimus

$$A = \mathfrak{Q} = 0,9961557,$$

tum vero erit

$a = 1$; $b = 0,9958847$; $c = 0,9962048$; $d = 0,9961753$, hinc igitur reperiemus

$$B = A - \tfrac{1}{3^5} \cdot 0,0038443 = 0,9961899$$

$$C = B - \tfrac{1}{5^5} \cdot 0,0002551 = 0,9961398.$$

Erit igitur

$$Q = 1 - C = 0,0038602, \text{ ideoque}$$

$$\tfrac{1}{4} l 2 - \tfrac{1}{4} P - \tfrac{1}{4} Q = 0,3350509.$$

§. 25. Sit nunc $n = 7$ et $A = \mathfrak{R} = 0,9995547$, tum vero $a = 1$; $b = 0,9995428$, hinc igitur fiet

$$B = A - \tfrac{1}{3^7} \cdot 0,0004453 = 0,9995545,$$

 vnde

vnde iam habemus

$$R = 1 - B = 0,0004455 \text{ ideoque}$$

$$\tfrac{1}{2}l2 - \tfrac{1}{3}P - \tfrac{1}{5}Q - \tfrac{1}{7}R = 0,3349873.$$

§. 26. Cum in hoc calculo tantum non fuerit $B = A$, in sequentibus nequidem littera B erit opus, quamobrem habebimus

$$S = 1 - \mathfrak{S} = 0,0000501 \text{ hincque}$$

$$\tfrac{1}{9}S = 0,0000056.$$

Deinde vero erit

$$T = 1 - \mathfrak{T} = 0,0000053 \text{ hincque}$$

$$\tfrac{1}{11}T = 0,0000005, \text{ denique}$$

$$U = 1 - \mathfrak{U} = 0,0000003 \text{ et } \tfrac{1}{13}U = 0,000000.$$

Particulis igitur his a praecedente valore ablatis prodit

$$O = 0,3349812.$$

Vnde patet, hunc valorem adhuc aliquanto maiorem esse quam $\frac{1}{3}$.

§. 27. Nunc igitur certi esse possumus summam seriei infinitae huius

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} + \text{etc.}$$

esse satis exacte $= 0,3349812$. Inuestigandum iam foret, num iste valor non quampiam teneat rationem notabilem, siue ad peripheriam circuli π, siue ad eius logarithmum hyperbolicum, quandoquidem supra obseruauimus, seriem reciprocam numerorum primorum

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \text{etc.}$$

expri-

3.
tum non fuerit
erit opus, quam-
= 0, 000000.
ablatis prodit
nto maiorem esse
ossumus summam
tc.
gandum iam fo-
at rationem nota-
ue ad eius loga-
pra obseruauimus,
expri-

exprimere logarithmum hyperbolicum seriei harmonicae completae

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \text{etc.}$$

vnde videri potest, istam seriem numerorum primorum

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \text{etc.}$$

etiam continere logarithmum eiusdem seriei completae:

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}$$

cuius summa est $\frac{\pi}{4}$. Hunc in finem subiungam logarithmum hyperbolicum ipsius π, quem olim reperi

1, 14472, 98858, 49400, 17414, 34273, 51353, 05865.

Videndum igitur erit num forte sit summa inuenta $O = l\pi - lN$, ita vt N sit numerus satis simplex. Verum huiusmodi inuestigationes plerumque sine vllo successu instituuntur.

§. 28. Ope posterioris methodi autem non solum summam seriei propositae elicuimus, sed etiam eius potestatum imparium, quas summas hic conspectui exponamus.

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} + \text{etc.} = 0,3349812$$

$$\tfrac{1}{3^3} - \tfrac{1}{5^3} + \tfrac{1}{7^3} + \tfrac{1}{11^3} - \tfrac{1}{13^3} - \tfrac{1}{17^3} + \tfrac{1}{19^3} + \text{etc.} = 0,0322522$$

$$\tfrac{1}{3^5} - \tfrac{1}{5^5} + \tfrac{1}{7^5} + \tfrac{1}{11^5} - \tfrac{1}{13^5} - \tfrac{1}{17^5} + \tfrac{1}{19^5} + \text{etc.} = 0,0038602$$

$$\tfrac{1}{3^7} - \tfrac{1}{5^7} + \tfrac{1}{7^7} + \tfrac{1}{11^7} - \tfrac{1}{13^7} - \tfrac{1}{17^7} + \tfrac{1}{19^7} + \text{etc.} = 0,0004455$$

$$\tfrac{1}{3^9} - \tfrac{1}{5^9} + \tfrac{1}{7^9} + \tfrac{1}{11^9} - \tfrac{1}{13^9} - \tfrac{1}{17^9} + \tfrac{1}{19^9} + \text{etc.} = 0,0000501$$

$$\tfrac{1}{3^{11}} - \tfrac{1}{5^{11}} + \tfrac{1}{7^{11}} + \tfrac{1}{11^{11}} - \tfrac{1}{13^{11}} - \tfrac{1}{17^{11}} + \tfrac{1}{19^{11}} + \text{etc.} = 0,0000056$$

$$\tfrac{1}{3^{13}} - \tfrac{1}{5^{13}} + \tfrac{1}{7^{13}} + \tfrac{1}{11^{13}} - \tfrac{1}{13^{13}} - \tfrac{1}{17^{13}} + \tfrac{1}{19^{13}} + \text{etc.} = 0,0000005$$

etc. etc.

§. 29.

§. 29. Ipſae quidem hae ſummae ſine dubio parum attentionis merentur, niſi forte ad quantitates cognitas reduci potuerint. Verum quia in his ſeriebus neque ipſi termini ſecundum certam legem progrediuntur, neque etiam in ſignis plus vel minus certus ordo obſeruatur; iſta disquiſitio primo intuitu plane impoſſibilis videri potuiſſet, quamobrem ipſa methodus, qua ad earum ſummas pertigimus, vtique omni attentione digna eſt cenſenda, idque eo magis, quod ſatis abſtruſis ſerierum poteſtatum proprietatibus innititur. Niſi enim ſummae ſerierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} \text{ etc.}$$

pro caſibus quibus n eſt numerus impar, fuiſſent cognitae, ota haec inueſtigatio fruſtra fuiſſet ſuſcepta.

DE

dubio parum
cognitas re-
neque ipſi ter-
neque etiam
ur; iſta dis-
ri potuiſſet,
mmas perti-
ſenda, idque
teſtatum pro-
um

ſent cognitae,

DE

DE SERIEBVS POTESTATVM RECIPROCIS

METHODO NOVA ET FACILLIMA SVMMANDIS.

§. 1.

Cum primum ſummas harum ſerierum docuiſſem, eas ex hoc principio deduxi, quod cuique ſinui et coſinui innumerabiles arcus circulares reſpondent, qui omnes ſint radices aequationum infinitarum, quibus arcus per ſinum vel coſinum exprimi ſolent. Hinc enim ex coefficientibus iſtarum aequationum non ſolum ſummas ipſarum radicum, ſed etiam earum poteſtatum quarumcunque aſſignaui. Poſtea vero easdem ſummas etiam ex aliis principiis deriuaui, quae autem omnia memorata circuli proprietate innitebantur. Nunc vero obſeruaui, iſtas ſummas ex alio principio multo ſimpliciori, et ſolis operationibus analyticis innixo, deduci poſſe, quam methodum hic accuratius expoſuiſſe iuuabit.

§. 2. Hoc autem principium mihi ſuppeditauit integratio huius formulae: $\int\left(\frac{z^{m-1} + z^{n-m-1}}{1 + z^n}\right) dz$, pro caſu quo poſt integrationem ſtatuitur $z = 1$. Oſtendi enim in