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Methodus inveniendi formulas integrales, quae certis casibus datam inter se teneant rationem, ubi sumul methodus traditur fractiones continuas summandi

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E594

METHODVS INVENIENDI FORMVLAS INTEGRALES,

QVAE

CERTIS CASIBVS DATAM INTER SE TENEANT RATIONEM,

VBI SIMVL METHODVS TRADITVR FRACTIONES CONTINVAS SVMMANDI.

§. 1.

Quemadmodum in seriebus recurrentibus quilibet terminus ex vno pluribusue praecedentibus secundum legem quandam constantem determinatur, ita hic eiusmodi series sum consideraturus, in quibus quilibet terminus ex vno pluribusue praecedentibus secundum quampiam legem variabilem determinatur. Quoniam autem in talibus seriebus formula generalis singulos terminos exprimens plerumque non est algebraica, sed transcendens, singulos terminos per formulas integrales exhiberi conueniet, quae vt valores determinatos praebeant, post integrationem quantitati variabili valorem determinatum tribui assumo, ita vt singuli termini prodeant quantitates determinatae; atque nunc quaestio principalis huc redit, quemadmodum istae formulae integrales debeant esse comparatae, vt quilibet terminus secundum datam legem ex vno pluribusue praecedentibus determinetur.

ENDI 'RALES, SE TENEANT IONES CONTINVAS

us quilibet termibus secundum leita hic eiusmodi uilibet terminus ex quampiam legem in talibus seriebus nens plerumque non s terminos per forae vt valores de- quantitati variabili a vt singuli termini nunc quaestio prinformulae integrales minus secundum daibus determinetur.

§. 2. Quod quo clarius perspiciatur, contemplemur seriem notissimam harum formularum integralium:

$$\int\frac{dx}{\sqrt{(1-xx)}},\ \int\frac{xx\,dx}{\sqrt{(1-xx)}},\ \int\frac{x^4\,dx}{\sqrt{(1-xx)}},\ \int\frac{x^6\,dx}{\sqrt{(1-xx)}},\ \text{etc.}$$

quae si singulae ita integrentur, vt euanescant posito $x=0$, tum vero variabili x tribuatur valor $=1$, quilibet terminus a praecedente ita pendet, vt sit

$$\int\frac{xx\,dx}{\sqrt{(1-xx)}}=\tfrac{1}{2}\int\frac{dx}{\sqrt{(1-xx)}},$$
$$\int\frac{x^4\,dx}{\sqrt{(1-xx)}}=\tfrac{3}{4}\int\frac{xx\,dx}{\sqrt{(1-xx)}},$$
$$\int\frac{x^6\,dx}{\sqrt{(1-xx)}}=\tfrac{5}{6}\int\frac{x^4\,dx}{\sqrt{(1-xx)}},$$

atque in genere

$$\int\frac{x^n\,dx}{\sqrt{(1-xx)}}=\frac{n-1}{n}\int\frac{x^{n-2}\,dx}{\sqrt{(1-xx)}}.$$

Vnde patet, hanc formulam generalem spectari posse tanquam terminum generalem illius seriei, atque quemlibet terminum ex praecedente oriri, si iste multiplicetur per $\frac{n-1}{n}$.

§. 3. Ad similitudinem igitur huius casus seriem formularum integralium ita in genere constituamus,

$$\int dv,\ \int x\,dv.\ \int xx\,dv,\ \int x^3\,dv,\ \int x^4\,dv,\ \text{etc.}$$

ita vt terminus indici n respondens sit $\int x^{n-1}\,dv$, quae singula integralia ita accipi sumamus, vt euanescant posito $x=0$; post integrationem autem quantitati variabili x tribuamus quempiam valorem constantem, veluti $x=1$, vel alio cuipiam numero. Quibus positis quaestio huc redit, qualis pro v assumi debeat functio ipsius x, vt quilibet terminus per vnum, vel duos pluresue praecedentes, secundum legem quandam datam vtcunque variabilem, siue ab indice n pendentem, determinetur; vbi quidem imprimis eo erit respiciendum, ad quot dimensiones index n in scala relationis proposita ascendat: plerumque autem non vltra pri-

 mam

mam dimensionem assurgere erit opus. Hinc igitur sequentia Problemata pertractemus.

Problema I.

Inuenire functionem v, *vt ista relatio inter binos terminos sibi succedentes locum habeat:*

$$\int x^n\, dv = \frac{\alpha n + a}{\beta n + b} \int x^{n-1}\, dv.$$

Solutio.

§. 4. Requiritur igitur hic vt sit

$$(\alpha n + a) \int x^{n-1}\, dv = (\beta n + b) \int x^n\, dv$$

si scilicet post integrationem variabili x certus valor tribuatur. Quoniam igitur ista conditio tum demum locum habere debet, postquam variabili x iste valor constans fuerit datus, ponamus in genere, dum x est variabilis, hanc aequationem locum habere:

$$(\alpha n + a) \int x^{n-1}\, dv = (\beta n + b) \int x^n\, dv + V,$$

quantitatem autem V ita esse comparatam, vt euanescat postquam variabili ille valor determinatus fuerit assignatus. Praeterea vero, quia ambo integralia ita capi assumimus, vt euanescant posito $x = 0$, necesse est vt etiam ista quantitas V eodem quoque casu euanescat.

§. 5. Quoniam haec aequalitas subsistere debet pro omnibus indicibus n, quos quidem semper vt positiuos spectamus, facile intelligitur, quantitatem istam V factorem habere debere x^n; quo pacto iam isti conditioni satisfit, vt posito $x = 0$ etiam fiat $v = 0$. Quamobrem statuamus $V = x^n Q$, vbi Q denotet functionem ipsius x proposito accom-

ac igitur sequen-

) inter binos ter-

$\int x^n\, dv$
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conditioni satisfit,
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psius x proposito
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accommodatam, et quam simul ita comparatam esse desideramus, vt euanescat si ipsi x certus quidam valor tribuatur.

§. 6. Cum igitur esse debeat

$$(\alpha n + a) \int x^{n-1}\, dv = (\beta n + b) \int x^n\, dv + x^n Q,$$

differentietur ista aequatio, ac differentiali per x^{n-1} diuiso peruenietur ad hanc aequationem differentialem:

$$(\alpha n + a)\, dv = (\beta n + b)\, x\, dv + n Q\, dx + x\, dQ,$$

quae cum subsistere debeat pro omnibus valoribus ipsius n, termini ista littera affecti seorsim se tollere debent, vnde nanciscimur has duas aequalitates:

$$\text{I. } (\alpha - \beta x)\, dv = Q\, dx \quad \text{et} \quad \text{II. } (a - bx)\, dv = x\, dQ.$$

Ex priore fit $dv = \frac{Q\, dx}{\alpha - \beta x}$, ex altera vero $dv = \frac{x\, dQ}{a - bx}$; qui duo valores inter se aequati suppeditant hanc aequationem: $\frac{dQ}{Q} = \frac{dx}{x} \cdot \frac{a - bx}{\alpha - \beta x}$, quae aequatio resoluitur in has partes

$$\frac{dQ}{Q} = \frac{a}{\alpha} \cdot \frac{dx}{x} + \frac{a\beta - b\alpha}{\alpha} \cdot \frac{dx}{\alpha - \beta x},$$

cuius ergo integrale erit

$$lQ = \frac{a}{\alpha}\, lx - \frac{a\beta - b\alpha}{\alpha\beta}\, l(\alpha - \beta x)$$

vnde deducitur

$$Q = C x^{\frac{a}{\alpha}} \cdot (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

§. 7. Ex hoc valore pro Q inuento statim patet eum euanescere casu $x = \frac{\alpha}{\beta}$, si modo fuerit $\frac{b\alpha - a\beta}{\alpha\beta} > 0$; sin autem secus eueniat, non patet quomodo haec quantitas vllo casu euanescere queat. Inuento autem hoc valore Q, inde reperietur

$$dv = C x^{\frac{a}{\alpha}}\, dx\, (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1}$$

hinc.

hincque noſtrae ſeriei terminus indici n reſpondens erit

$$\int x^{n-1}\,dv = C\int x^{n+\frac{a}{\alpha}-1}\,dx(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}-1},$$

tum vero erit

$$V = Cx^{n+\frac{a}{\alpha}}(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}}.$$

Vbi res imprimis eo redit, vt iſta quantitas praeter caſum $x=0$ inſuper alio caſu euaneſcat.

Corollarium 1.

§. 8. Hic duo caſus occurrunt, qui peculiarem euolutionem poſtulant; prior eſt, quo $\alpha=0$; tum autem inchoandum erit ab aequatione $\frac{dQ}{Q}=-\frac{(a-bx)dx}{\beta xx}$, vnde integrando elicitur $lQ=\frac{a}{\beta x}+\frac{b}{\beta}lx$, hincque ſumendo e pro numero cuius logarithmus hyperbolicus $=1$, colligitur

$$Q=e^{\frac{a}{\beta x}}.\,x^{\frac{b}{\beta}}$$

quae formula in nihilum abire nequit, niſi fiat $\frac{a}{\beta x}=-\infty$, ideoque $x=0$, ſicque non duo haberentur caſus, quibus fieret $V=0$, cum tamen duo deſiderentur. Interim autem hinc fiet

$$dv=\frac{e^{\frac{a}{\beta x}}x^{\frac{b}{\beta}}\,dx}{\alpha-\beta x}.$$

Corollarium 2.

§. 9. Alter caſus peculiarem integrationem poſtulans erit $\beta=0$; tum autem erit $\frac{dQ}{Q}=\frac{dx(a-bx)}{\alpha x}$, vnde fit $lQ=\frac{a}{\alpha}lx-\frac{bx}{\alpha}$, ideoque $Q=x^{\frac{a}{\alpha}}.\,e^{\frac{-bx}{\alpha}}$, quae formula caſu

pondens erit

is praeter caſum

qui peculiarem ; tum autem in- vnde integran- nendo e pro nu- colligitur

i fiat $\frac{a}{\beta x}=-\infty$, ur caſus, quibus ur. Interim autem

tegrationem poſtu- vnde , quae formula caſu

caſu $x=\infty$ euaneſcit, ſi modo fuerit $\frac{b}{\alpha}$ numerus poſitiuus, ſin autem $\frac{b}{\alpha}$ fuerit numerus negatiuus, tum Q euaneſcit caſu $x=-\infty$. Porro vero hoc caſu fiet

$$dv=\frac{x^{\frac{a}{\alpha}}.\,e^{\frac{-bx}{\alpha}}\,dx}{\alpha-\beta x}$$

Scholion.

§. 8. His in genere obſeruatis aliquot caſus ſpeciales euoluamus, quibus litteris α, β et a, b certos valores tribuemus, qui ad caſus iam ſatis cognitos perducant.

Exemplum 1.

§. 9. *Quaerantur formulae integrales, vt fiat* $\int x^n\,dv=\frac{(2n-1)}{2n}\int x^{n-1}\,dv$.

Cum igitur hic eſſe debeat

$$(2n-1)\int x^{n-1}\,dv=2n\int x^n\,dv,$$

erit hoc caſu $\alpha=2$ et $a=-1$, tum vero $\beta=2$ et $b=0$; hinc fit

$$\frac{dQ}{Q}=-\frac{dx}{2x(1-x)}=-\frac{dx}{2x}-\frac{dx}{2(1-x)},$$ inde integrando

$$lQ=-\tfrac{1}{2}lx+\tfrac{1}{2}l(1-x),$$ ideoque

$$Q=C\sqrt{\tfrac{1-x}{x}},\ \text{ergo}\ V=x^n\sqrt{\tfrac{1-x}{x}}.$$

Porro cum hic ſit $dv=\frac{Q\,dx}{2(1-x)}$, erit

$$dv=\frac{dx\sqrt{\frac{1-x}{x}}}{2(1-x)}=\frac{C\,dx}{2\sqrt{(x-xx)}},$$

ſumto ergo $C=2$ erit $dv=\frac{dx}{\sqrt{(x-xx)}}$, et formula noſtra generalis:

$\int x^{n-}$

$$\int x^{n-1}\,dv = \int \frac{x^{n-1}\,dx}{\sqrt{(x-xx)}},$$

vnde cum fit $V = x^n \sqrt{\frac{1-x}{x}}$, haec quantitas manifesto euanefcit fumto $x = 1$, ita vt noftra formula, fi poft integrationem ftatuatur $x = 1$, quaefito fatisfaciat. Quod fi iam ponamus $x = yy$, ifta formula induet hanc formam: $2\int \frac{y^{2n-2}\,dy}{\sqrt{(1-yy)}}$, quae, pofito poft integrationem $y = 1$, praebet hanc relationem:

$$\int \frac{y^{2n}\,dy}{\sqrt{(1-yy)}} = \frac{2n-1}{2n} \int \frac{y^{2n-2}\,dy}{\sqrt{(1-yy)}},$$

quae continet relationes fupra § 2 commemoratas; hinc enim fiet

$$\int \frac{yy\,dy}{\sqrt{(1-yy)}} = \tfrac{1}{2}\int \frac{dy}{\sqrt{(1-yy)}},$$
$$\int \frac{y^4\,dy}{\sqrt{(1-yy)}} = \tfrac{3}{4}\int \frac{yy\,dy}{\sqrt{(1-yy)}} \text{ et}$$
$$\int \frac{y^6\,dy}{\sqrt{(1-yy)}} = \tfrac{5}{6}\int \frac{y^4\,dy}{\sqrt{(1-yy)}}.$$

Exemplum 2.

§. 10. *Quaerantur formulae integrales, vt fiat*

$$\int x^n\,dv = \frac{\alpha n - 1}{\alpha n} \int x^{n-1}\,dv.$$

Cum igitur hic effe debeat

$$(\alpha n - 1)\int x^{n-1}\,dv = \alpha n \int x^n\,dv,$$

erit hoc cafu $a = -1$, $\beta = \alpha$ et $b = 0$, vnde per formulas fupra datas colligitur

$$Q = C\,x^{\frac{-1}{\alpha}}\,(\alpha - \alpha x)^{\frac{-1}{\alpha}} = C\,x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}$$

quae quantitas manifefto euanefcit pofito $x = 1$. Tum autem erit

is manifefto euanefcit
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Quod fi iam pona-
rmam: $2\int \frac{y^{2n-2}\,dy}{\sqrt{(1-yy)}}$,
praebet hanc re-
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o, vnde per formu-
$(1-x)^{\frac{+1}{\alpha}}$
ofito $x = 1$. Tum

$$dv = \frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}\,dx}{(1-x)},$$

vnde formula noftra generalis erit

$$\int x^{n-1}dv = \int x^{n-\frac{1}{\alpha}-1}(1-x)^{+\frac{1}{\alpha}-1}\,dx = \int \frac{x^{n-\frac{1}{\alpha}-1}\,dx}{(1-x)^{1-\frac{1}{\alpha}}};$$

quae concinnior redditur, faciendo $x = y^\alpha$, tum enim ea induet hanc formam: $\int \frac{y^{\alpha n-2}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}$, vbi iterum poft integrationem ftatui debet $y = 1$. Erit hinc

$$\int \frac{y^{\alpha n+\alpha-2}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha n - 1}{\alpha n} \int \frac{y^{\alpha n-2}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}},$$

atque hinc oriuntur fequentes cafus fpeciales:

$$\int \frac{y^{2\alpha-2}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha-1}{\alpha} \int \frac{y^{\alpha-2}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} \text{ et}$$
$$\int \frac{y^{3\alpha-2}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{2\alpha-1}{2\alpha} \int \frac{y^{2\alpha-2}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}.$$

§. 11. Hinc igitur fi fumatur $\alpha = 1$, vt fieri debeat

$$\int x^n\,dv = \frac{n-1}{n}\int x^{n-1}\,dv,$$

formula noftra generalis iam in y expreffa erit $\int y^{n-2}\,dy$, cuius ergo valor eft $\frac{1}{n-1}y^{n-1} = \frac{1}{n-1}$, vnde tota feries noftrarum formularum integralium abibit in hanc:

$$\tfrac{1}{0},\ \tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \tfrac{1}{7},\ \text{etc.}$$

§. 12.

§. 12. Sumamus etiam $\alpha = \frac{1}{2}$, et iam non amplius opus erit ad y procedere. Hoc igitur casu erit

$$Q = \frac{(1-x)^2}{xx} \text{ et } dv = \frac{(1-x)dx}{xx}$$

vnde formula nostra generalis fit

$$\int x^{n-1} dv = \int x^{n-3}(1-x)dx,$$

cuius ergo valor algebraice expressus erit

$$\frac{1}{n-2}x^{n-2} - \frac{1}{n-1}x^{n-1} = \frac{1}{(n-1)(n-2)}$$

vnde series nostrarum formularum euadet

$$\frac{1}{0.-1}, \frac{1}{0.1}, \frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}, \text{ etc.}$$

Exemplum 3.

§. 13. *Quaerantur formulae integrales, vt sit*

$$\int x^n dv = n\int x^{n-1} dv.$$

Cum igitur esse debeat

$$n\int x^{n-1} dv = 1. \int x^n dv, \text{ erit}$$

$$\alpha = 1,\ a = 0,\ b = 1,\ \beta = 0.$$

Cum igitur sit $\beta = 0$, casus Coroll. 2. hic locum habet, indeque erit $Q = e^{-x}$ ideoque $V = e^{-x}.x^n$, quae quantitas his duobus casibus euanescit: $x = 0$ et $x = \infty$. Porro vero erit $dv = e^{-x}dx$, hincque formula nostra generalis fiet $\int x^{n-1}dx.e^{-x}$, vnde ipsi seriei termini ab initio sequenti modo se habebunt:

$$\int e^{-x}dx,\ \int e^{-x}xdx,\ \int e^{-x}xxdx,\ \int e^{-x}x^3dx \text{ etc.}$$

quibus integratis ita vt euanescant posito $x = 0$, tum vero posito $x = \infty$, orietur sequens series satis simplex:

$$1,\ 1,\ 1.2,\ 1.2.3,\ 1.2.3.4,\ 1.2.3.4.5, \text{ etc.}$$

quae

non amplius
vt fit
um habet, inquantitas his Porro vero generalis fiet sequenti mo-
x^3dx etc.
o, tum vero plex :
4.5, etc.
quae

quae est series hypergeometrica *Wallisii*, cuius ergo terminus generalis est

$$\int x^{n-1}e^{-x}dx = 1.2.3.4 \ldots (n-1).$$

§. 14. Ope ergo huius termini generalis hanc seriem interpolare licebit. Ita si quaeratur terminus medius inter duos primos, poni debet $n = \frac{1}{2}$, ac valor huius termini erit $\int e^{-x}dx \sqrt{x}$, cuius autem valor nullo modo algebraice exprimi potest. Inueni autem singulari modo hunc ipsum terminum aequari $\frac{1}{2}\sqrt{\pi}$, denotante π peripheriam circuli cuius diameter $= 1$, vnde hic vicissim cognoscimus esse $\int e^{-x}dx\sqrt{x} = \frac{\sqrt{\pi}}{2}$, posito scilicet post integrationem $x = \infty$. Terminus autem hunc praecedens, indici $\frac{1}{2}$ respondens, erit $= \sqrt{\pi}$, cui ergo aequatur formula $\int \frac{e^{-x}dx}{\sqrt{x}}$. Quod si hic ponamus $e^x = y$, ita vt posito $x = 0$ fit $y = 1$, at posito $x = \infty$ fiat $y = \infty$, tum ergo ista formula $\int \frac{e^{-x}dx}{\sqrt{x}}$ abit in hanc $\int \frac{dy}{yy\sqrt{ly}}$, quae formula si ita integretur vt euanescat posito $y = 1$, tum vero fiat $y = \infty$, praebet valorem ipsius $\sqrt{\pi}$. Si porro fiat $y = \frac{1}{z}$, erunt termini integrationis $z = 1$, et $z = 0$, et formula integralis erit

$$-\int \frac{dz}{\sqrt{-lz}} \begin{bmatrix} \text{a} & z = 1 \\ \text{ad} & z = 0 \end{bmatrix} = \sqrt{\pi},$$

siue permutatis terminis integrationis erit

$$\int \frac{dz}{\sqrt{-lz}} \begin{bmatrix} \text{a} & z = 0 \\ \text{ad} & z = 1 \end{bmatrix} = \sqrt{\pi},$$

quemadmodum iam olim obseruaui

Exem-

Exemplum 4.

§. 15. *Quaerantur formulae integrales, vt sit*

$$\int x^n\, dv = \tfrac{1}{n}\int x^{n-1}\, dv, \text{ siue}$$
$$\int x^{n-1}\, dv = n\int x^n\, dv.$$

Hic est $\alpha = 0$ et $a = 1$, $\beta = 1$ et $b = 0$; qui ergo est casus in Coroll. 1. tractatus, vnde colligitur fore $Q = e^{\frac{1}{x}}$, ideoque $V = x^n e^{\frac{1}{x}}$, quae formula nequidem euanescit sumto $x = 0$, quandoquidem formula $e^{\frac{1}{0}}$ aequiualet infinito infinitesimae potestatis. Hic autem miro modo euenit, vt casus $x = -0$ reddat formulam $e^{\frac{-1}{0}}$ subito euanescentem. Scilicet, si ω denotet quantitatem infinite paruam, erit $e^{\frac{1}{\omega}} = \infty^{\infty}$, tum vero repente fiet $e^{\frac{-1}{\omega}} = \frac{1}{\infty^{\infty}} = 0$, quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem $dv = -e^{\frac{1}{x}}\frac{dx}{x}$, ita vt formula nostra generalis futura sit $-\int x^{n-2}\, dx\, e^{\frac{1}{x}}$, quae autem nobis nullum vsum praestare potest.

§. 16. Quod si hic ponamus $\frac{1}{x} = y$, formula ista generalis transit in hanc: $+\int\frac{e^y\, dy}{y^n}$. At vero nunc erit $V = \frac{e^y}{y^n}$, quae formula euanescit posito $y = -\infty$. Quomodocunque autem hanc expressionem transformemus, semper idem incommodum occurret. Interim tamen etiam hunc casum sequenti modo resoluere licebit. Sit enim seriei, quam quaerimus

, vt sit
qui ergo est
fore $Q = e^{\frac{1}{x}}$,
euanescit sumto
infinito infini-
ienit, vt casus
entem. Scilicet,
$e^{\frac{1}{\omega}} = \infty^{\infty}$, tum
causam for-
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hunc casum se-
i, quam quae-
rimus

rimus, primus terminus $= \omega$, ex quo per regulam praescriptam sequentes ordine ita procedent

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & & n \\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1.2}, & \frac{\omega}{1.2.3}, & \frac{\omega}{1.2.3.4}, & \ldots & \frac{\omega}{1.2.3\ldots(n-1)} \end{array}$$

Supra autem vidimus huius formulae $1.2.3.4.5\ldots(n-1)$ valorem exprimi per hoc integrale: $\int x^{n-1} e^{-x}\, dx$, integratione ab $x = 0$ ad $x = \infty$ extensa; tantum igitur opus est vt hanc formulam integralem in denominatorem transferamus, et seriei quam quaerimus terminus generalis erit

$$\frac{1}{\int x^{n-1} e^{-x}\, dx},$$

vnde satis intelligitur, negotium non per simplicem formulam integralem expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantitas V non duobus casibus euanescere potest; tum enim tantum opus est fractionem $\frac{\alpha n + a}{\beta n + b}$ inuertere, atque formulam integralem in denominatorem transferre.

Scholion.

§. 17. Nisi sit vel $\alpha = 0$ vel $\beta = 0$, quos casus iam expediuimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae α et β sunt aequales vnitati. Cum enim esse debeat

$$\int x^n\, dv = \frac{\alpha n + a}{\beta n + b}\int x^{n-1}\, dv,$$

ponatur $x = \frac{\alpha y}{\beta}$, fietque

$$\frac{\alpha}{\beta}\int y^n\, dv = \frac{\alpha n + a}{\beta n + b}\int y^{n-1}\, dv,$$

quae aequatio reducitur ad hanc formam:

$\int y^n$

$$\int y^n\,dv = \frac{n+a\alpha}{n+b\beta}\int y^{n-1}\,dv.$$

Quod si iam nunc loco $\frac{a}{\alpha}$ scribamus a, et b loco $\frac{b}{\beta}$, resoluenda erit haec formula:

$$\int y^n\,dv = \frac{n+a}{n+b}\int y^{n-1}\,dv,$$

cuius resolutio, si loco x scribamus y et loco litterarum α et β vnitatem, ex superiori solutione praebet primo

$$Q = C\,y^a(1-y)^{b-a},$$

quod ergo euanescit posito $y=1$, si modo fuerit $b>a$, tum autem erit ipsa formula

$$\int y^{n-1}\,dv = C\int y^{n+a-1}\,dy\,(1-y)^{b-a-1};$$

sin autem fuerit $b<a$, haec solutio, vti vidimus, locum habere nequit; verum hoc casu pro termino nostrae seriei assumi debet haec forma: $\frac{1}{\int y^{n-1}\,dv}$, ita vt tum esse debeat

$$\frac{1}{\int y^n\,dv} = \frac{n+a}{n+b}\,\frac{1}{\int y^{n-1}\,dv}, \text{ siue}$$

$$\int y^n\,dv = \frac{n+b}{n+a}\int y^{n-1}\,dv,$$

cuius resolutio permutatis litteris a et b praebet

$$Q = C\,y^b(1-y)^{a-b}$$

quae iam casu $y=1$ euanescit, si fuerit $a>b$, atque tum erit formula generalis

$$\int y^{n-1}\,dv = C\int y^{n+b-1}\,dy\,(1-y)^{a-b-1}.$$

Siue igitur sit $b>a$ siue $a>b$, solutio nulla amplius laborat difficultate.

§. 18. Sin autem fuerit vel $\alpha=0$ vel $\beta=0$, loco alterius etiam scribi poterit vnitas; vnde si esse debeat

$\int x^n$

loco $\frac{b}{\beta}$, resol-
co litterarum
bet primo
fuerit $b>a$,
$-a-1$;
us, locum ha-
nostrae seriei
m esse debeat
bet
$>b$, atque tum
$-b-1$.
amplius labo-
vel $\beta=0$, lo-
si esse debeat
$\int x^n$

$$\int x^n\,dv = \frac{n+a}{b}\int x^{n-1}\,dv,$$

ob $\alpha=1$ et $\beta=0$ solutio nostra generalis dat

$$\frac{dQ}{Q} = \frac{dx}{x}(a-bx)$$

vnde colligitur $Q = C\,x^a.e^{-bx}$, quae formula euanescit posito $x=\infty$, si modo b fuerit numerus positiuus; tum autem fit terminus generalis

$$\int x^{n-1}\,dv = C\int x^{n+a-1}\,dx.e^{-x}.$$

At vero numerus b negatiuus esse nequit, quia alioquin conditio praescripta esset incongrua.

§. 19. Consideremus etiam alterum casum, quo $\alpha=0$ et $\beta=1$, ideoque conditio praescripta

$$\int x^n\,dv = \frac{a}{n+b}\int x^{n-1}\,dv,$$

vnde fit

$$\frac{dQ}{Q} = -\frac{dx}{xx}(a-bx).$$

Hinc autem pro Q orietur valor, qui praeter casum $x=0$ euanescere non posset; quam ob caussam formula generalis statui debet $\frac{1}{\int x^{n-1}\,dv}$, ita vt esse debeat

$$\int x^n\,dv = \frac{n+b}{a}\int x^{n-1}\,dv$$

vnde prodit

$$\frac{dQ}{Q} = \frac{dx}{x}(b-ax), \text{ ideoque } Q = C\,e^{-ax}.x^b,$$

quae expressio euanescit posito $x=\infty$, quoniam a necessario debet esse numerus positiuus; tum autem erit

$$dv = C\,e^{-ax}.x^b\,dx,$$

vnde

vnde formula generalis feriei erit

$$\frac{1}{C\int x^{a+b-1}\,d\,x.\,e^{-ax}}$$

Problema II.

§. 20. *Denotet* T *terminum indici* n *refpondentem in ferie quam confiderandam fufcepimus, at vero* T′ *terminum fequentem, atque proponatur haec conditio adimplenda:*

$$T' = \frac{(\alpha n + a)(\alpha' n + a')}{(\beta n + b)(\beta' n + b')} T$$

Solutio.

Quoniam hic valores geminati occurrunt, huic conditioni commodiffime fatisfiet, fi terminus generalis T tanquam productum ex duobus factoribus fpectetur. Statuatur igitur $T = RS$, fitque terminus fequens $= R'S'$, et quaerantur formulae R et S, vt fiat

$$R' = \frac{\alpha n + a}{\beta n + b} R \text{ et } S' = \frac{\alpha' n + a'}{\beta' n + b'} S,$$

tum enim fumendo $T = RS$ conditioni praefcriptae manifefto fatisfiet Hoc igitur modo pro R et S vel huiusmodi formulae: $\int x^{n-1}dv$, vel inuerfae $\frac{1}{\int x^{n-1}\,d\,v}$ reperientur, id quod pro folutione generali fufficit, vnde rem exemplo illuftremus.

Exemplum.

§. 21. *Quaeratur formula generalis* T, *vt fiat*

$$T' = \frac{nn - cc}{nn} T.$$

Refoluamus igitur T in duos factores R et S, ac ftatuamus

$$R' = \frac{n-c}{n} R \text{ et } S' = \frac{n+c}{n} S.$$

Pro

n refpondentem
t vero T′ ter-
iditio adimplenda:
rrunt, huic con-
generalis T tan-
ectetur. Statuatur
$= R'S'$, et quae-
raefcriptae mani-
S vel huiusmodi
reperientur, id
rem exemplo il-
is T, vt fiat
et S, ac ftatuamus
Pro

Pro priore forma fi ftatuamus $R = \int x^{n-1}\,d\,v$, ex folutione generali, vbi erit $\alpha = 1$, $a = -c$, $\beta = 1$ et $b = 0$, fiet

$$Q = C\,x^{-c}(1-x)^c;$$

quae forma manifefto euanefcit pofito $x=1$, hincque quia fit

$$V = C\,x^{n-c}(1-x)^c,$$

haec forma etiam cafu $x = 0$ euanefcit, fi modo n fuerit $> c$, id quod tuto affumi poteft, quia exponentem n fucceffiue in infinitum crefcere affumimus, ac plerumque pro c fractiones tantum accipi folent. Hinc ergo erit

$$R = C\int x^{n-c-1}(1-x)^{c-1}\,d\,x.$$

§. 22. Hinc iam alter valor litterae S deduci poffet, fcribendo tantum $-c$ loco c, tum autem non amplius fieret $Q = 0$ pofito $x = 1$, quamobrem pro S formulam inuerfam $\frac{1}{\int x^{n-1}\,d\,v}$ affumi oportet, vt effe debeat

$$\int x^n\,d\,v = \frac{n}{n+c}\int x^{n-1}\,d\,v,$$

vbi cum fit $\alpha = 1$, $a = 0$, $\beta = 1$ et $b = c$, reperitur $Q = C(1-x)^c$, quae forma manifefto fit $= 0$ pofito $x = 1$, hinc autem prodit

$$d\,v = C(1-x)^{c-1}\,d\,x,$$

ergo habebimus

$$S = \frac{1}{C\int x^{n-1}(1-x)^{c-1}\,d\,x},$$

confequenter formula noftra generalis quaefita erit

$$T = \frac{\int x^{n-c-1}(1-x)^{c-1}\,d\,x}{\int x^{n-1}(1-x)^{c-1}\,d\,x}.$$

 §. 23.

§. 23. Quod si ergo nostrae seriei per factores procedentis primum terminum ponamus $= A$, ipsa series erit

I. II. III. IV.

$A, \frac{1-cc}{1}. A, \frac{1-cc}{1}. \frac{4-cc}{4}. A, \frac{1-cc}{1}. \frac{4-cc}{4}. \frac{9-cc}{9} A$, etc.

vnde si sumamus $c = \frac{1}{2}$, erit haec series

$A, \frac{1.3}{2.2}. A, \frac{1.3}{2.2}. \frac{3.5}{4.4}. A, \frac{1.3}{2.2}. \frac{3.5}{4.4}. \frac{5.7}{6.6}. A$, etc.

cuius ergo terminus indici n respondens est

$$\frac{\int x^{n-\frac{3}{2}}(1-x)^{-\frac{1}{2}}\,dx}{\int x^{n-1}(1-x)^{-\frac{1}{2}}\,dx},$$

qui posito $x = yy$ transit in hanc formam:

$$\frac{\int y^{2n-2}(1-yy)^{-\frac{1}{2}}\,dy}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}\,dy},$$

vnde patet, terminum primum fore

$$A = \int \frac{dy}{\sqrt{(1-yy)}} : \int \frac{y\,dy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$$

posito scilicet post integrationem $y = 1$.

Problema.

Denotet T *terminum seriei indici* n *respondentem, sintque* T′ *et* T″ *termini sequentes pro indicibus* $n+1$ *et* $n+2$, *si proponatur inter ternos terminos se insequentes talis relatio, vt sit*

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'';$$

inuestigare formulam pro T, *qua terminus generalis huius seriei exprimatur.*

Solutio.

seriei per factores
$= A$, ipsa series erit
IV.
$\frac{cc}{1}. \frac{4-cc}{4}. \frac{9-cc}{9} A$, etc.
$\frac{5.7}{6.6}. A$, etc.
est
nam:
1.
ci n respondentem, sint-
indicibus $n+1$ et
rinos se insequentes talis
$-(\gamma n+c)T''$;
rminus generalis huius
Solutio.

Solutio.

§. 24. Assumatur pro T formula integralis $\int x^{n-1} dv$, huiusque integrale ita capiatur, vt euanescat posito $x = 0$, eruntque termini sequentes

$$T' = \int x^n\,dv \text{ et } T'' = \int x^{n+1}\,dv,$$

siquidem post integrationem variabili x certus valor determinatus tribuatur. Quamdiu autem haec quantitas x vt variabilis spectatur, ponamus esse

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'' + x^n Q,$$

ac perspicuum est Q eiusmodi functionem esse debere ipsius x, quae euanescat, si loco x valor ille determinatus substituatur, quem autem a ciphra diuersum esse oportet, quoniam iam assumsimus, omnes istas formulas in nihilum abire posito $x = 0$. Quodsi vero, absoluto calculo, huic conditioni nullo modo satisfieri poterit, id erit indicio, problema nostrum hac ratione resolui non posse, vt scilicet eius terminus generalis T per talem formulam differentialem simplicem $\int x^{n-1} dv$ exhibeatur.

§. 25. Differentiemus nunc aequationem modo stabilitam, ac diuisione facta per x^{n-1} sequens prodibit aequatio:

$$(\alpha n + a)dv = (\beta n + b)x\,dv + (\gamma n + c)xx\,dv + nQ\,dx + x\,dQ,$$

quae, quia termini littera n affecti seorsim se destruere debent, discerpetur in binas sequentes aequationes:

$$1^\circ\ \alpha\,dv = \beta\,x\,dv + \gamma\,xx\,dv + Q\,dx,$$

$$2^\circ\ a\,dv = b\,x\,dv + c\,xx\,dv + x\,dQ,$$

 ex

ex quarum priore fit

$$d v = \frac{Q\, d x}{a - \beta x - \gamma x x},$$

ex altera vero fit

$$d v = \frac{x\, d Q}{a - b x - c x x},$$

quorum valorum posterior per priorem diuisus praebet

$$\frac{dQ}{Q} = \frac{d x (a - b x - c x x)}{x (\alpha - \beta x - \gamma x x)},$$

ex cuius ergo integratione valor ipsius Q elici debet, quo facto facile patebit, vtrum is certo quodam casu praeter $x = 0$ euanescere possit. Imprimis autem hic notari conuenit, si hoc integrale inuoluat huiusmodi factorem $e^{\frac{\lambda}{x}}$, tum solutionem quoque successu esse carituram, quandoquidem posito $x = 0$ iste factor tantam inuoluet infiniti potestatem, vt, etiamsi per x^n multiplicetur, productum etiamnum infinitum maneat.

§. 26. Quod si igitur his conditionibus praescriptis satisfacere licuerit, tum inuento valore litterae Q, quem ponamus fieri $= 0$ posito $x = f$, habebitur

$$d v = \frac{Q\, d x}{\alpha - \beta x - \gamma x x},$$

et formula generalis naturam seriei complectens erit

$$T = \int x^{n-1}\, d v = \int \frac{x^{n-1}\, Q\, d x}{\alpha - \beta x - \gamma x x},$$

quippe cuius integrale, a termino $x = 0$ vsque ad terminum $x = f$ extensum, praebebit valorem termini T, indici cuicunque n respondentis.

Scholion.

te fo re ne tu m de Ex ded Sim vnde tes conti nume conti
praebet
i debet, quo casu praeter c notari con- torem $e^{\frac{\lambda}{x}}$, tum quandoquidem i potestatem, etiamnum in-
us praescriptis Q, quem po-
ns erit
que ad termi- nini T, indici
Scholion.

Scholion.

§. 27. Inuenta autem tali relatione inter ternos terminos cuiuspiam seriei sibi inuicem succedentes, inde more solito formari poterit fractio continua, cuius valorem assignare licebit. Si enim characteres T', T'', T''', T^{IV}, etc. denotent ordine omnes terminos post T sequentes in infinitum, ex relationibus, quas inter se tenent, sequentes formulae deducentur. Ex relatione

$$(\alpha n + a) T = (\beta n + b) T' + (\gamma n + c) T''$$

deducitur

$$(\alpha n + a) \frac{T}{T'} = \beta n + b + \frac{(\gamma n + c)(\alpha n + \alpha + a)}{(\alpha n + \alpha + a) T' : T''}.$$

Ex relatione sequente

$$(\alpha n + \alpha + a) T' = (\beta n + \beta + b) T'' + (\gamma n + \gamma + c) T'''$$

deducitur

$$(\alpha n + \alpha + a) \frac{T'}{T''} = \beta n + \beta + b + \frac{(\gamma n + \gamma + c)(\alpha n + 2\alpha + a)}{(\alpha n + 2\alpha + a) T'' : T'''}.$$

Simili modo sequentes relationes suppeditabunt:

$$(\alpha n + 2\alpha + a) \frac{T''}{T'''} = \beta n + 2\beta + b + \frac{(\gamma n + 2\gamma + c)(\alpha n + 3\alpha + a)}{(\alpha n + 3\alpha + a) T''' : T''''};$$

$$(\alpha n + 3\alpha + a) \frac{T'''}{T''''} = (\beta n + 3\beta + b) + \frac{(\gamma n + 3\gamma + c)(\alpha n + 4\alpha + a)}{(\alpha n + 4\alpha + a) T'''' : T'''''};$$

vnde manifestum est, si in prima formula continuo sequentes valores ordine substituantur, prodituram esse fractionem continuam, cuius valor aequalis erit formulae $(\alpha n + a) \frac{T}{T'}$.

§. 28. Quod si ergo loco n successiue scribamus numeros 1, 2, 3, 4, etc., sequens problema circa fractiones continuas resoluere poterimus.

 Pro-

Problema.

Propofita fractione continua huius formae:

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{ etc.}}}}}}$$

eius valorem inueftigare.

Solutio.

Confideretur in genere ifta relatio inter ternas quantitates fibi fuccedentes T, T′, T″, quae fit

$$(\alpha n+a)\mathrm{T}=(\beta n+b)\mathrm{T}'+(\gamma n+c)\mathrm{T}'',$$

atque ex praecedente Problemate quaeratur valor ipfius T, fiquidem fieri poteft, hoc modo expreffus:

$$\mathrm{T}=\int x^{n-1}\,dv=\int\frac{x^{n-1}\,\mathrm{Q}\,dx}{\alpha-\beta x-\gamma xx},$$

cuius integrale ab $x=0$ vsque ad $x=f$ extendatur, quae formula inuenta ponatur

$$\int\frac{\mathrm{Q}\,dx}{\alpha-\beta x-\gamma xx}=\mathrm{A} \text{ et } \int\frac{n\,\mathrm{Q}\,dx}{\alpha-\beta x-\gamma xx}=\mathrm{B},$$

ita vt A et B fint valores ipfius T, pro cafibus $n=1$ et $n=2$, quibus definitis fractionis continuae propofitae valor per praecedentia erit $=\frac{(\alpha+a)\mathrm{A}}{\mathrm{B}}$. Hanc igitur inueftigationem ad fequentia exempla accommodemus.

Exem-

m q Q ft er fra fac ernas quan- ″, vn r ipfius T, vn har idatur, quae qua is $n=1$ et ɔpofitae valor vnd inueftigatio-

Exem-

Exemplum 1.

§. 29. *Inueftigare valorem fractionis continuae notiffimae, quam olim* Brouncherus *pro quadratura circuli protulit, quae eft*

$$2+\cfrac{1.1}{2+\cfrac{3.3}{2+\cfrac{5.5}{2+\text{ etc.}}}}$$

Quia omnes partes integrae laeuam refpicientes funt conftantes $=2$, pro noftra forma generali fiet

$$\beta+b=2,\ 2\beta+b=2,\ 3\beta+b=2 \text{ etc.}$$

erit ergo $\beta=0$ et $b=2$; at pro numeratoribus fequentium fractionum, quandoquidem conftant binis factoribus, erit pro factoribus prioribus

$$\gamma+c=1,\ 2\gamma+c=3,\ 3\gamma+c=5,\ 4\gamma+c=7,$$

vnde concluditur $\gamma=2$ et $c=-1$, pro alteris vero erit

$$2\alpha+a=1,\ 3\alpha+a=3,\ 4\alpha+a=5 \text{ etc.}$$

vnde $\alpha=2$ et $a=-3$. Ex his autem valoribus colligimus hanc aequationem

$$\frac{d\mathrm{Q}}{\mathrm{Q}}=-\frac{dx(3+2x-xx)}{2x(1-xx)},$$

quae per $1+x$ depreffa praebet

$$\frac{d\mathrm{Q}}{\mathrm{Q}}=-\frac{dx(3-x)}{2x(1-x)},$$

vnde integrando fit

$$l\mathrm{Q}=-\tfrac{3}{2}\,lx+(1-x) \text{ et hinc } \mathrm{Q}=\frac{1-x}{x^{\frac{3}{2}}},$$

ex

ex quo valore porro ſequitur,

$$A = \int \frac{(1-x)\,dx}{2x^{\frac{3}{2}}(1-xx)} = \int \frac{dx}{2x(1+x)\sqrt{x}}$$

$$B = \int \frac{(1-x)\,dx}{2x^{\frac{1}{2}}(1-xx)} = \int \frac{dx}{2(1+x)\sqrt{x}}.$$

§. 30. In his autem valoribus iſtud incommodum deprehenditur, quod prius integrale euaneſcens reddi nequit poſito $x = 0$. Hoc autem incommodum facile remoueri poteſt, ſi fractionem continuam ſupremo membro truncemus et quaeramus valorem iſtius fractionis:

$$2 + \cfrac{3.\,3}{2 + \cfrac{5.\,5}{2 + \text{etc.}}}$$

qui ſi repertus fuerit $= s$, erit ipſius propoſitae valor $= b + \frac{1}{s}$. Nunc vero, comparatione inſtituta, fit quidem vt ante $\beta = 0$ et $b = 2$, tum vero $\gamma = 2$ et $c = +1$, $\alpha = 2$ et $a = -1$, vnde ſequitur

$$\frac{dQ}{Q} = -\frac{dx(1+2x+xx)}{2x(1-xx)} = -\frac{dx(1+x)}{2x(1-x)},$$

vnde integrando fit

$$lQ = -\tfrac{1}{2}lx + l(1-x) \text{ ideoque } Q = \frac{1-x}{\sqrt{x}},$$

ex quo valore iam habebimus

$$A = \int \frac{(1-x)\,dx}{2(1-xx)\sqrt{x}} = \tfrac{1}{2}\int \frac{dx}{(1+x)\sqrt{x}} \text{ et}$$

$$B = \tfrac{1}{2}\int \frac{dx\sqrt{x}}{1+x}$$

vbi cum ſit $Q = \frac{1-x}{\sqrt{x}}$, eius valor manifeſto euaneſcit poſito $x = 1$, quamobrem illa integralia a termino $x = 0$, vsque ad $x = 1$ ſunt extendenda.

§. 31.

mmodum
ddi nequit
noueri po-
ncemus et
lor $= b + \frac{1}{s}$
ante $\beta = 0$
et $a = -1$,
$\frac{-x}{\sqrt{x}}$,
eſcit poſito
$= 0$, vsque
§. 31.

§. 31. Quo nunc haec integralia facilius eruamus, ſtatuamus $x = zz$, ita vt termini integrationis etiamnunc ſint $z = 0$ et $z = 1$, eritque

$$A = \int \frac{dz}{1+zz} = \text{A tang. } z = \frac{\pi}{4} \text{ et}$$

$$B = \int \frac{zz\,dz}{1+zz} = 1 - \frac{\pi}{4},$$

ſicque habebimus $s = \frac{\pi}{4-\pi}$, quocirca ipſius fractionis *Broungcherianae* valor eſt $1 + \frac{4}{\pi}$, omnino vti olim *Brounckerus* iam inuenerat.

Exemplum 2.

§. 31. *Inueſtigare valorem huius fractionis continuae* Brounckeriana *latius patentis:*

$$b + \cfrac{1.\,1}{b + \cfrac{3.\,3}{b + \cfrac{5.\,5}{b + \text{etc.}}}}$$

Vt hic incommodum ſuperius euitemus, omittamus membrum ſupremum et quaeramus

$$s = b + \cfrac{3.\,3}{b + \cfrac{5.\,5}{b + \text{etc.}}}$$

quandoquidem tum erit valor quaeſitus $= b + \frac{1}{s}$. Nunc igitur erit $\beta = 0$ et $b = b$, $\gamma = 2$, $c = 1$, $\alpha = 2$ et $a = -1$, vnde fit

$$\frac{dQ}{Q} = -\frac{dx(1+bx+xx)}{2x(1-xx)}, \text{ ac proinde}$$

$$lQ = -\tfrac{1}{2}lx - \tfrac{b-2}{4}l(1+x) + \tfrac{b+2}{4}l(1-x),$$

Q =

hincque

$$Q = \frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}} \sqrt{x}};$$

quae formula manifesto fit $= 0$ ponendo $x = 1$, siquidem $b + 2$ fuerit numerus positiuus, vnde fit

$$dv = \frac{(1-x)^{\frac{b-2}{4}}}{2(1+x)^{\frac{b+2}{4}} \sqrt{x}}.$$

Hinc autem colligetur

$$A = \tfrac{1}{2}\int \frac{(1-x)^{\frac{b-2}{4}}\, dx}{(1+x)^{\frac{b+2}{4}} \sqrt{x}} \text{ et}$$

$$B = \tfrac{1}{2}\int \frac{(1-x)^{\frac{b-2}{4}}\, dx \sqrt{x}}{(1+x)^{\frac{b+2}{4}}},$$

siue ponendo $x = zz$ habebimus

$$A = \int \frac{(1-zz)^{\frac{b-2}{4}}\, dz}{(1+zz)^{\frac{b+2}{4}}} \text{ et}$$

$$B = \int \frac{(1-zz)^{\frac{b-2}{4}}\, zz\, dz}{(1+zz)^{\frac{b+2}{4}}},$$

quae ambo integralia a $z = 0$ vsque ad $z = 1$ sunt extendenda. Ex his autem valoribus A et B erit $s = \frac{A}{B}$; ipsius igitur fractionis propositae valor erit $= b + \frac{1}{s} = b + \frac{B}{A}$.

§. 42.

$= 1$, siquidem

$= 1$ sunt exten-
$s = \frac{A}{B}$; ipsius
$+ \frac{1}{s} = b + \frac{B}{A}$.
§. 42.

§. 32. Quod si hic ponatur $b = 2$, prodit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes $\frac{b-2}{4}$ et $\frac{b+2}{4}$ non sunt numeri integri, tum litteras A et B neque per arcus circulares, neque per logarithmos exprimere licet. Veluti si fuerit $b = 4$, erit

$$A = \int \frac{dz \sqrt{(1-zz)}}{(1+zz)^{\frac{3}{2}}},$$

cuius valor per arcus ellipticos exhiberi posset. At si b fuerit numerus impar, hi valores multo magis euadunt transcendentes, ita vt his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi fiant numeri integri, totum negotium per arcus circulares expedire licebit.

§. 33. Exponentes autem illi $\frac{b-2}{4}$ et $\frac{b+2}{4}$ erunt numeri integri, quoties fuerit b numerus huius formae:

$$b = 4i + 2$$

tum enim erit

$$A = \int \frac{(1-zz)^{i}\, dz}{(1+zz)^{i+1}} \text{ et}$$

$$B = \int \frac{(1-zz)^{i}\, zz\, dz}{(1+zz)^{i+1}};$$

quos ergo casus quomodo euolui oporteat operae pretium erit docere, quoniam *Wallisius* eos iam est contemplatus.

§. 34. Quoniam hoc negotium totum redit ad reductionem huiusmodi formularum integralium ad formas sim-

simpliciores, consideremus in genere formam $P = \frac{z^m}{(1+zz)^n}$, cuius differentiale sub sequentibus formis exhiberi potest:

$$1^\circ)\ dP = \frac{m z^{m-1}\, dz}{(1+zz)^n} - \frac{2n z^{m+1}\, dz}{(1+zz)^{n+1}}$$

$$2^\circ)\ dP = \frac{m z^{m-1}\, dz}{(1+zz)^{n+1}} - \frac{(2n-m)\, z^{m+1}\, dz}{(1+zz)^{n+1}}$$

$$3^\circ)\ dP = -\frac{(2n-m)\, z^{m-1}\, dz}{(1+zz)^n} - \frac{2n z^{m-1}\, dz}{(1+zz)^{n+1}}$$

vnde hanc triplicem reductionem integralium deducimus:

$$\text{I. } \int \frac{z^{m+1}\, dz}{(1+zz)^{n+1}} = \frac{m}{2n} \int \frac{z^{m-1}\, dz}{(1+zz)^n} - \frac{1}{2n} \cdot \frac{z^m}{(1+zz)^n}$$

$$\text{II. } \int \frac{z^{m+1}\, dz}{(1+zz)^{n+1}} = \frac{m}{2n-m} \int \frac{z^{m-1}\, dz}{(1+zz)^{n+1}} - \frac{1}{2n-m} \cdot \frac{z^m}{(1+xz)^n}$$

$$\text{III. } \int \frac{z^{m-1}\, dz}{(1+zz)^{n+1}} = \frac{2n-m}{2n} \int \frac{z^{m-1}\, dz}{(1+zz)^n} + \frac{1}{2n} \cdot \frac{z^m}{(1+zz)^n}$$

quarum reductionum ope casibus $b = 4i + 2$ totum negotium absolui et ad formulam $\frac{\pi}{4}$ reduci poterit, siquidem post integrationem sumatur $z = 1$.

§. 35. Sit $i = 1$ ideoque $b = 6$ eritque

$$A = \int \frac{(1-zz)\,dz}{(1+zz)^2} \text{ et } B = \int \frac{(1-zz)\,zz\,dz}{(1+zz)^2}.$$

Nunc igitur reperiemus per reductionem tertiam

$$\int \frac{dz}{(1+zz)^2} = \frac{1}{2} \int \frac{dz}{1+zz} + \frac{1}{2} \cdot \frac{z}{1+zz} = \frac{\pi}{8} + \frac{1}{4}$$

et per reductionem primam

$$\int \frac{zz\,dz}{(1+zz)^2} = \frac{1}{2} \int \frac{dz}{1+zz} - \frac{1}{2} \cdot \frac{z}{1+zz} = \frac{\pi}{8} - \frac{1}{4}, \text{ porro}$$

$$\int \frac{z^4\,dz}{(1+zz)^2} = \frac{3}{2} \int \frac{zz\,dz}{1+zz} - \frac{1}{2} \cdot \frac{z^3}{1+zz} = \frac{5}{4} - \frac{3\pi}{8}.$$

Ex

$P = \frac{z^m}{(1+zz)^n}$, xhiberi potest: deducimus: totum negotium, siquidem eritque tertiam $\frac{\pi}{8} - \frac{1}{4}$, porro Ex

Ex his iam valoribus colligitur $A = \frac{1}{2}$ et $B = \frac{\pi}{2} - \frac{3}{2}$, ideoque $\frac{B}{A} = \pi - 3$, quocirca oritur ista summatio:

$$3 + \pi = 6 + \cfrac{1.1}{6 + \cfrac{3.3}{6 + \cfrac{5.5}{6 + \cfrac{7.7}{6 + \text{etc.}}}}}$$

§. 36. Sit nunc $i = 2$ et $b = 10$, eritque

$$A = \int \frac{(1-zz)^2\,dz}{(1+zz)^3} \text{ et } B = \int \frac{zz(1-zz)^2\,dz}{(1+zz)^3}.$$

Quo harum integralium valores inuestigemus, sequentes euoluamus formulas:

$$\int \frac{dz}{(1+zz)^3} = \frac{3}{4} \int \frac{dz}{(1+zz)^2} + \frac{1}{4} \cdot \frac{z}{(1+zz)^2} = \frac{3\pi}{32} + \frac{1}{4}$$

$$\int \frac{zz\,dz}{(1+zz)^3} = \frac{1}{4} \int \frac{dz}{(1+zz)^2} - \frac{1}{4} \cdot \frac{z}{(1+zz)^2} = \frac{\pi}{32}$$

$$\int \frac{z^4\,dz}{(1+zz)^3} = \frac{3}{4} \int \frac{zz\,dz}{(1+zz)^2} - \frac{1}{4} \cdot \frac{z^3}{(1+zz)^2} = \frac{3\pi}{32} - \frac{1}{4}$$

$$\int \frac{z^6\,dz}{(1+zz)^3} = \frac{5}{4} \int \frac{z^4\,dz}{(1+zz)^2} - \frac{1}{4} \cdot \frac{z^5}{(1+zz)^2} = \frac{3}{2} - \frac{15\pi}{32}.$$

Ex quibus iam valoribus deducitur $A = \frac{\pi}{8}$ et $B = 2 - \frac{5\pi}{8}$, ideoque $\frac{B}{A} = \frac{16 - 5\pi}{\pi}$, vnde emergit sequens summatio:

$$\frac{5\pi + 16}{\pi} = 10 + \cfrac{1.1}{10 + \cfrac{3.3}{10 + \cfrac{5.5}{10 + \text{etc.}}}}$$

§. 37. Si b esset numerus negatiuus, inuestigatio nulla prorsus laboraret difficultate. Si enim in genere fuerit

 $s =$

$$s = -a + \cfrac{\alpha}{-b + \cfrac{\beta}{-c + \cfrac{\gamma}{-d + \cfrac{\delta}{-e + \text{etc.}}}}}$$

semper erit

$$s = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{\varepsilon + \text{etc.}}}}}$$

vnde si habeatur valor istius expressionis, idem negatiue sumtus dabit valorem illius.

Exemplum 3.

§. 38. *Proposita sit fractio continua, cuius valorem inuestigari oporteat, ista:*

$$1 + \cfrac{1.1}{3 + \cfrac{3.3}{5 + \cfrac{5.5}{7 + \cfrac{7.7}{9 + \text{etc.}}}}}$$

Quo fractiones supra allegatas, omisso membro supremo, sint

$$3 + \cfrac{3.3}{5 + \cfrac{5.5}{7 + \cfrac{7.7}{9 + \text{etc.}}}}$$

eritque $\beta + b = 3$, $2\beta + b = 5$, ideoque $\beta = 2$ et $b = 1$; tum vero vt ante $\alpha = 2$, $a = -1$, $\gamma = 2$ et $c = +1$;

Inuento

m negatiue
ius valorem
upremo, sint
2 et $b = 1$;
et $c = +1$;
Inuento

inuento autem s erit valor quaesitus $= 1 + \frac{1}{s}$. Nunc igitur habebimus

$$\frac{dQ}{Q} = -\frac{dx(1+x+xx)}{2x(1-x-xx)}. \quad \text{Est vero}$$

$$\frac{1+x+xx}{x(1-x-xx)} = \frac{1}{x} + \frac{2+2x}{1-x-xx}, \text{ vnde fit}$$

$$lQ = -\tfrac{1}{2} lx - \int \frac{dx(1+x)}{1-x-xx}.$$

Porro vero pro formula $\int \frac{dx(1+x)}{1-x-xx}$ inuenienda, statuamus denominatorem

$$1 - x - xx = (1 - fx)(1 - gx)$$

eritque $f + g = 1$ et $fg = -1$, vnde fit

$$f = \frac{1+\sqrt{5}}{2} \text{ et } g = \frac{1-\sqrt{5}}{2}.$$

Nunc statuatur

$$\frac{1+x}{1-x-xx} = \frac{\mathfrak{A}}{1-fx} + \frac{\mathfrak{B}}{1-gx},$$

vnde reperietur

$$\mathfrak{A} = \frac{1+f}{f-g} \text{ et } \mathfrak{B} = -\frac{(1+g)}{f-g}$$

siue substitutis pro f et g valoribus supra datis erit

$$\mathfrak{A} = \frac{\sqrt{5}+1}{2\sqrt{5}} \text{ et } \mathfrak{B} = \frac{\sqrt{5}-1}{2\sqrt{5}},$$

quibus inuentis erit

$$\int \frac{dx(1+x)}{1-x-xx} = -\frac{\mathfrak{A}}{f} l(1-fx) - \frac{\mathfrak{B}}{g} l(1-gx) = \\ -\frac{(1+\sqrt{5})}{2\sqrt{5}} l(1-fx) - \frac{(\sqrt{5}-1)}{2\sqrt{5}} l(1-gx)$$

quocirca fiet

$$lQ = -\tfrac{1}{2} lx + \frac{(\sqrt{5}+1)}{2\sqrt{5}} l(1-fx) + \frac{(\sqrt{5}-1)}{2\sqrt{5}} l(1-gx)$$

consequenter

$$Q = \frac{(1-fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}} (1-gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

qui

qui valor duobus casibus euanescit: altero quo

$$x=\frac{1}{f}=\frac{2}{1+\sqrt{5}}=\frac{\sqrt{5}-1}{2},$$

altero vero quo $x=\frac{1}{g}=-\frac{1-\sqrt{5}}{2}$; vtrouis autem vtamur, res eodem redibit.

§. 39. Ex hoc autem valore habebimus

$$A=\int\frac{Q\,dx}{1-x-xx}\ \text{et}\ B=\int\frac{Qx\,dx}{1-x-xx};$$

vnde porro deducitur

$$s=(\alpha+a)\frac{A}{B}=\frac{A}{B}$$

hinc propositae fractionis summa erit $1+\frac{B}{A}$. Hinc autem nihil vlterius concludere licet, ob formulas differentiales non solum irrationales, sed etiam vere transscendentes ob exponentes surdos.

Exemplum 4.

§. 40. *Proposita sit haec fractio continua:*

$$b+\cfrac{1.1}{b+\cfrac{2.2}{b+\cfrac{3.3}{b+\cfrac{4.4}{b+\text{etc.}}}}}$$

vbi est $\beta=0$, $\mathfrak{b}=b$.

Nunc consideremus hanc formam:

$$s=b+\cfrac{2.2}{b+\cfrac{3.3}{b+\text{etc.},}}$$

quippe quo valore inuento quaesitus erit $=b+\frac{1}{s}$. Habebimus igitur $\gamma+c=2$, $2\gamma+c=3$, ideoque $\gamma=1$ et $c=1$,

$c=1$, deinde erit $\alpha=\gamma=1$, $a=0$ et $c=1$. Hinc igitur colligimus

$$\frac{dQ}{Q}=-\frac{dx(bx+xx)}{x(1-xx)}=-\frac{dx(b+x)}{1-xx},\ \text{ideoque}$$

$$lQ=-\frac{b}{2}l\frac{1+x}{1-x}+\frac{1}{2}l(1-xx)\ \text{hincque}$$

$$Q=\frac{(1-x)^{\frac{b}{2}}\sqrt{(1-xx)}}{(1+x)^{\frac{b}{2}}}=\frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

quae quantitas manifesto euanescit posito $x=1$. Hinc igitur fiet

$$A=\int\frac{Q\,dx}{1-xx}=\int\frac{(1-x)^{\frac{b+1}{2}}dx}{(1+x)^{\frac{b-1}{2}}(1-xx)}=\int\frac{(1-x)^{\frac{b-1}{2}}dx}{(1+x)^{\frac{b+1}{2}}}\ \text{et}$$

$$B=\int\frac{x(1-x)^{\frac{b-1}{2}}dx}{(1+x)^{\frac{b+1}{2}}},$$

tum autem erit $s=(\alpha+a)\frac{A}{B}=\frac{A}{B}$ ideoque summa quaesita $b+\frac{B}{A}$.

§. 41. Percurramus nunc casus praecipuos: ac primo sit $b=1$ eritque

$$A=\int\frac{dx}{1+x}=l(1+x)=l2\ \text{et}$$

$$B=\int\frac{x\,dx}{1+x}=x-\int\frac{dx}{1+x}=1-l2,$$

ideoque $b+\frac{B}{A}=\frac{1}{l2}$; ergo hinc prodiit ista summatio:

$\frac{1}{l2}=$

$$\frac{1}{l2} = 1 + \cfrac{1.1}{1 + \cfrac{2.2}{1 + \cfrac{3.3}{1 + \text{etc.}}}}$$

§. 42. Sit nunc $b = 2$ eritque

$$A = \int \frac{dx\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}} \text{ et } B = \int \frac{x\,dx\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

Ad has formulas rationales reddendas statuamus

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = z, \text{ eritque } x = \frac{1-zz}{1+zz},$$

vnde terminis integrationis $x = 0$ et $x = 1$ respondebunt $z = 1$ et $z = 0$; tum vero erit

$$1 + x = \frac{2}{1+zz} \text{ et } dx = -\frac{4z\,dz}{(1+zz)^2},$$

hincque colligitur

$$A = -2\int\frac{zz\,dz}{1+zz} = -2z + 2\,\text{A tang.}\,z + 2 - \frac{\pi}{2} = 2 - \frac{\pi}{2},$$

porro fit

$$B = -2\int\frac{zz\,dz}{(1+zz)^2} + 2\int\frac{z^4\,dz}{(1+zz)^2}.$$

Per reductiones igitur supra § 35 monstratas, si hic scilicet terminos integrationis $z = 1$ et $z = 0$ permutemus, vt habeamus

$$B = +2\int\frac{zz\,dz}{(1+zz)^2} - 2\int\frac{z^4\,dz}{(1+zz)^2}, \text{ erit}$$

$$B = 2\left(\tfrac{\pi}{8} - \tfrac{1}{4}\right) - 2\left(\tfrac{3}{4} - \tfrac{\pi}{4}\right) = \pi - 3,$$

vnde sequitur ista summatio:

$\frac{2}{4-\pi}$

unt

$-\frac{\pi}{2}$,

licet
ha-

$\frac{2}{4-\pi}$

$$\frac{2}{4-\pi} = 2 + \cfrac{1.1}{2 + \cfrac{3.3}{2 + \cfrac{4.4}{2 + \text{etc.}}}}$$

quae *Brouncherianae* simplicitate nihil cedit.

§. 43. Si ponamus $b = 0$, fractio continua abit in sequens continuum productum:

$$\frac{1.1}{2.2} \cdot \frac{3.3}{4.4} \cdot \frac{5.5}{6.6} \cdot \frac{7.7}{8.8} \cdot \text{etc.}$$

hoc autem casu fit

$$A = \int\frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ et } B = \int\frac{x\,dx}{\sqrt{(1-xx)}} = 1$$

vnde istius producti valor colligitur $\frac{2}{\pi}$, id quod egregie convenit cum iam dudum cognitis, quandoquidem hoc productum est ipsa progressio *Wallisiana*.

Exemplum 5.

§. 44. *Proposita sit haec fractio continua, vbi* $\beta = 0$, $b = b$ *et numeratores numeri trigonales:*

$$b + \cfrac{1}{b + \cfrac{3}{b + \cfrac{6}{b + \cfrac{10}{b + \text{etc.}}}}}$$

Omisso supremo membro statuamus

$$s = b + \cfrac{3}{b + \cfrac{6}{b + \text{etc.}}}$$

et

et primo numeratores per producta repraesentemus, hoc modo:

$$3 = 2 \cdot \tfrac{3}{2}, \quad 6 = 3 \cdot \tfrac{4}{2}, \quad 10 = 4 \cdot \tfrac{5}{2}$$

quorum priores comparentur cum formulis

$$\gamma + c, \quad 2\gamma + c, \quad 3\gamma + c,$$

posteriores vero cum formulis $2\alpha + a$, $3\alpha + a$, $4\alpha + a$, eritque $\gamma = 1$, $c = 1$, $\alpha = \frac{1}{2}$, $a = \frac{1}{2}$, vnde erit

$$\frac{dQ}{Q} = \frac{dx(\frac{1}{2} - bx - xx)}{x(\frac{1}{2} - xx)} = \frac{dx(1 - 2bx - 2xx)}{x(1 - 2xx)}$$

siue

$$\frac{dQ}{Q} = \frac{dx}{x} - \frac{2b\,dx}{1 - 2xx}$$

cuius integrale est

$$lQ = lx - \tfrac{b}{\sqrt{2}}\, l\tfrac{1 + x\sqrt{2}}{1 - x\sqrt{2}}, \text{ ergo}$$

$$Q = \frac{x(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}}{(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

quae formula euanescit casu $x = \frac{1}{\sqrt{2}}$. Hinc igitur erit

$$dv = \frac{2x(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}\,dx}{(1 - 2xx)(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

Sit $\frac{b}{\sqrt{2}} = \lambda$ eritque

$$A = 2\int \frac{x(1 - x\sqrt{2})^{\lambda}\,dx}{(1 - 2xx)(1 + x\sqrt{2})^{\lambda}} = 2\int \frac{x(1 - x\sqrt{2})^{\lambda - 1}\,dx}{(1 + x\sqrt{2})^{\lambda + 1}}$$

et

$$B = 2\int \frac{xx(1 - x\sqrt{2})^{\lambda - 1}\,dx}{(1 + x\sqrt{2})^{\lambda + 1}},$$

vbi post integrationem statuitur $x = \frac{1}{\sqrt{2}}$; cum autem fit $s = \frac{A}{B}$, hincque valor fractionis propositae $= b + \frac{B}{A}$.

§. 45.

hoc modo:
+a, erit-
$1 - 2xx)$
$.xx)$
r erit
$\sqrt{2})^{\lambda-1}dx$
$x\sqrt{2})^{\lambda+1}$
m fit $s = \frac{A}{B}$,
§. 45.

§. 45. Nisi igitur fuerit $\lambda = \frac{b}{\sqrt{2}}$ numerus rationalis, hos valores commode assignare non licet. Sit igitur $b = \sqrt{2}$, siue $\lambda = 1$, eritque

$$A = 2\int \frac{x\,dx}{(1 + x\sqrt{2})^2} \text{ et } B = 2\int \frac{xx\,dx}{(1 + x\sqrt{2})^2}.$$

Hinc integrando colligitur

$$A = l(1 + x\sqrt{2}) - \frac{x\sqrt{2}}{1 + x\sqrt{2}}$$

ideoque posito $x\sqrt{2} = 1$ fiet $A = l2 - \frac{1}{2}$; tum vero reperitur

$$B = \tfrac{3}{2\sqrt{2}} - \sqrt{2}.\,l2,$$

quare ob $b = \sqrt{2}$ erit $b + \frac{B}{A} = \frac{1}{\sqrt{2}(2l2 - 1)}$, vnde sequitur haec summatio:

$$\frac{1}{\sqrt{2}(2l2 - 1)} = \sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{3}{\sqrt{2} + \cfrac{6}{\sqrt{2} + \text{etc.}}}}$$

Scholion.

§. 46. Fractiones autem continuae, ad quas plerumque calculo numerico deducimur, huiusmodi formam habere solent:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc.}}}}}$$

vbi omnes numeratores sunt vnitates, denominatores vero a, b, c, d, etc. numeri integri. Verum ope nostrae methodi difficulter talium formarum valores eruere licet, etiamsi numeri

meri a, b, c, d, e progressionem arithmeticam constituant, id quod sequenti exemplo ostendamus.

Exemplum.

§. 47. *Proposita sit ista fractio continua:*

$$\beta+b+\cfrac{1}{2\beta+b+\cfrac{1}{3\beta+b+\cfrac{1}{4\beta+b+\cfrac{1}{5\beta+b+\text{etc.}}}}}$$

vbi $\alpha=0$, $\gamma=0$, $a=1$, $c=1$.
Hinc fit

$$\frac{dQ}{Q}=-\frac{dx(1+bx-xx)}{\beta xx}, \text{ vnde}$$

$$lQ=\frac{1}{\beta x}+\frac{b}{\beta}lx+\frac{x}{\beta} \text{ et}$$

$$Q=e^{\frac{1+xx}{\beta x}}.x^{\frac{b}{\beta}},$$

quae autem expressio nullo casu euanescere potest, etiamsi per x^n multiplicetur, siquidem β fuerit numerus positiuus. Verum si pro β sumamus numeros negatiuos puta, $\beta=-m$, tum valor $Q=x^{\frac{-b}{m}}\times e^{\frac{-(1+xx)}{mx}}$, manifesto euanescit, tam si $x=0$, quam si $x=\infty$. Hinc autem erit

$$dv=\frac{x^{\frac{-b}{m}}.e^{\frac{-1-xx}{mx}}dx}{mxx},$$

quamobrem habebimus

$$A=\frac{1}{m}\int\frac{dx}{x^{2+\frac{b}{m}}.e^{\frac{1+xx}{mx}}} \text{ et}$$

B =

constituant,

$b+$ etc.

, etiamsi per
uus. Verum
$=-m$, tum
am si $x=0$;

B =

$$B=\frac{1}{m}\int\frac{dx}{x^{1+\frac{b}{m}}.e^{\frac{1+xx}{mx}}}.$$

His valoribus inuentis formula $\frac{A}{B}$ exprimet summam huius fractionis continuae:

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

quamobrem formula illa negatiue sumta $-\frac{A}{B}$ exprimet valorem huius fractionis continuae:

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

quem igitur assignare liceret, si modo formulae integrales A et B expediri et a termino $x=0$ ad $x=\infty$ extendi possent. Verum istae formulae ita sunt comparatae, vt earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impedit, quo minus fractio $\frac{A}{B}$ valores satis cognitos inuoluere queat, etiamsi eos nullo adhuc modo assignare valeamus.

§. 49. Talium autem fractionum continuarum mihi quidem binae sequentes innotuere, quarum valores commode exhibere licet:

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}} = \frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1} \text{ et}$$

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}} = \cot.\tfrac{1}{n}$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m-b=n$, $2m-b=3n$, ideoque $m=2n$ et $b=n$, vnde fit

$$A=\tfrac{1}{2n}\int\frac{dx}{x^{\frac{1}{2}}\,e^{\frac{1+xx}{2nx}}} \text{ et}$$

$$B=\tfrac{1}{2n}\int\frac{dx}{x^{\frac{1}{2}}\,e^{\frac{1+xx}{2n}}};$$

vnde iam discimus si hae duae formulae integrentur à termino $x=0$ vsque ad terminum $x=\infty$, tum fore

$$\frac{A}{B}=\frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet, hanc conuenientiam demonstrandi.

SVM-

exempli
$m=2n$
ur à ter-
re
conueni-
SVM-

SVMMATIO FRACTIONIS CONTINVAE, CVIVS INDICES PROGRESSIONEM ARITHMETICAM CONSTITVVNT, DVM NVMERATORES OMNES SVNT VNITATES; VBI SIMVL RESOLVTIO AEQVATIONIS RICCATIANAE PER HVIVSMODI FRACTIONES DOCETVR.

§. 1.

Cum in praecedente dissertatione methodum exposuissem, fractiones continuas ad duas formulas integrales reducendi, ea quidem infinitis casibus feliciter successit: at vero casus, qui simplicissimus videtur, vbi omnes numeratores inter se ponuntur aequales, ad eiusmodi formulas integrales perduxit, quas nullo adhuc modo euoluere et inter se comparare licuit, cum tamen ex hoc genere binae fractiones continuae habeantur, quarum valores satis commode exhiberi possunt:

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\text{etc.}}}} = \frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1} \text{ et}$$

$n-1$