1785

De transformatione serierum in fractiones continuas, ubi simul haec theoria non mediocriter amplificatur

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E593

DE TRANSFORMATIONE SERIERVM IN FRACTIONES CONTINVAS; VBI SIMVL HAEC THEORIA NON MEDIOCRITER AMPLIFICATVR.

§. 1.

Consideremus fractionem continuam quamcunque, quae sit

$$s = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

ac primo quaeramus fractiones simplices, quae continuo propius ad valorem ipsius s accedant, quas ita formemus vt sit

$$\frac{A}{\mathfrak{A}} = a;\ \frac{B}{\mathfrak{B}} = a + \frac{1}{b};\ \frac{C}{\mathfrak{C}} = a + \cfrac{1}{b + \cfrac{1}{c}};$$

$$\frac{D}{\mathfrak{D}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

Harum igitur fractionum vltima verum valorem fractionis con-

E
M
NVAS;
R AMPLIFI-
nque, quae
continuo pro-
memus vt sit
;
rem fractionis
con-

continuae propositae exprimet. Hinc igitur statim patet fore

$$\frac{A}{\mathfrak{A}} = \frac{a}{1},\ \frac{B}{\mathfrak{B}} = \frac{ab+1}{b},\ \frac{C}{\mathfrak{C}} = \frac{abc+a+c}{bc+1},$$

Quemadmodum autem hae fractiones vlterius progrediantur sequenti modo inquiramus.

§. 2. Euidens hic est, ex fractione prima secundam oriri, si loco a scribatur $a + \frac{1}{b}$; similique modo ex secunda oriri tertiam, si loco b scribatur $b + \frac{1}{c}$; ex tertia vero quartam, si loco c scribatur $c + \frac{1}{d}$, et ita porro. Hinc ergo, si indefinite fractio $\frac{P}{\mathfrak{P}}$ formata sit ex indicibus $a, b, c, d \ldots p$ binaeque sequentes ponantur $\frac{Q}{\mathfrak{Q}}$ et $\frac{R}{\mathfrak{R}}$, quae respondeant indicibus $a, b, c, d, \ldots q$ et $a, b, c, d, e, \ldots r$, manifestum est, ex fractione $\frac{P}{\mathfrak{P}}$ reperiri sequentem $\frac{Q}{\mathfrak{Q}}$, si loco p scribatur $p + \frac{1}{q}$; ex hac vero $\frac{Q}{\mathfrak{Q}}$ oriri sequentem $\frac{R}{\mathfrak{R}}$, si loco q scribatur $q + \frac{1}{r}$. Nunc vero facile patet, in fractione $\frac{P}{\mathfrak{P}}$ tam numeratorem P quam denominatorem $\mathfrak{P}$ omnes litteras $a, b, c, d, \ldots p$ ita inuoluere, vt nulla earum vltra primam dimensionem exsurgat. Si enim omnes indices a, b, c, d, e, vt inaequales spectentur, nullius quadratum vel altior potestas vsquam occurrere poterit.

§. 3. Quamobrem tam in P quam in $\mathfrak{P}$ duplicis generis occurrent termini, dum alii indicem p plane non continent, alii vero eum tanquam factorem inuoluunt; vnde nume-

numerator P huiusmodi habebit formam: $M + Np$, similique modo denominator $\mathfrak{P}$ hanc: $\mathfrak{M} + \mathfrak{N}p$, ita vt fit $\frac{P}{\mathfrak{P}} = \frac{M+Np}{\mathfrak{M}+\mathfrak{N}p}$. In hac igitur forma loco p fcribamus $p + \frac{1}{q}$ vt obtineamus fractionem $\frac{Q}{\mathfrak{Q}}$, quae ergo, poftquam fupra et infra per q multiplicauerimus, erit

$$\frac{Q}{\mathfrak{Q}} = \frac{Mq + Npq + N}{\mathfrak{M}q + \mathfrak{N}pq + \mathfrak{N}} = \frac{N + (M + Np)q}{\mathfrak{N} + (\mathfrak{M} + \mathfrak{N}p)q}.$$

Nunc vt hinc fequentem fractionem $\frac{R}{\mathfrak{R}}$ obtineamus, loco q fcribamus $q + \frac{1}{r}$, et poftquam fupra et infra per r multiplicauerimus orietur

$$\frac{R}{\mathfrak{R}} = \frac{Nr + (M + Np)qr + M + Np}{\mathfrak{N}r + (\mathfrak{M} + \mathfrak{N}p)qr + \mathfrak{M} + \mathfrak{N}p}, \text{ fiue}$$

$$\frac{R}{\mathfrak{R}} = \frac{M + Np + (N + Mq + Npq)r}{\mathfrak{M} + \mathfrak{N}p + (\mathfrak{N} + \mathfrak{M}q + \mathfrak{N}pq)r}.$$

Cum igitur fit $P = M + Np$, $Q = N + (M + Np)q$, erit $R = P + Qr$. Simili modo cum fit $\mathfrak{P} = \mathfrak{M} + \mathfrak{N}p$ et $\mathfrak{Q} = \mathfrak{N} = (\mathfrak{M} + \mathfrak{N}p)q$ erit $\mathfrak{R} = \mathfrak{P} + \mathfrak{Q}r$. Sicque patet, quomodo quaelibet noftrarum fimplicium fractionum ex binis praecedentibus facile formari poffit.

§. 4. Ecce igitur demonftrationem fatis planam et dilucidam regulae notiffimae pro conuerfione fractionis continuae in fractiones fimplices, vbi tam numeratores quam denominatores fecundum eandem legem ex binis praecedentibus formantur. Cum igitur pro ambabus primis fractionibus fit $A = a$, $\mathfrak{A} = 1$, tum vero $B = ab + 1$ et $\mathfrak{B} = b$, ex his duabus fractionibus fequentes omnes facili negotio formari poterunt. Quod quo clarius appareat fingulis indicibus a, b, c, d, e etc. fractiones refpondentes ordine fubfcribamus

a

- N p, fimi-
, ita vt fit
nus $p + \frac{1}{q}$
quam fupra
ms, loco q
per r multi-
$+ Np)q$,
$= \mathfrak{M} + \mathfrak{N}p$
. r. Sicque
fractionum
planam et
actionis con-
tores quam
praeceden-
nis fractio-
et $\mathfrak{B} = b$,
cili negotio
fingulis in-
ordine fub-
a

a	b	c	d	e	f	g
$\frac{A}{\mathfrak{A}}$	$\frac{B}{\mathfrak{B}}$	$\frac{C}{\mathfrak{C}}$	$\frac{D}{\mathfrak{D}}$	$\frac{E}{\mathfrak{E}}$	$\frac{F}{\mathfrak{F}}$	$\frac{G}{\mathfrak{G}}$

ac tam numeratores quam denominatores fecundum eandem legem ex binis praecedentibus fequenti modo determinabuntur

Pro numeratoribus	Pro denominatoribus
$A = b$	$\mathfrak{A} = 1$
$B = Ab + 1$	$\mathfrak{B} = b$
$C = Bc + A$	$\mathfrak{C} = \mathfrak{B}c + \mathfrak{A}$
$D = Cd + B$	$\mathfrak{D} = \mathfrak{C}d + \mathfrak{B}$
$E = De + C$	$\mathfrak{E} = \mathfrak{D}e + \mathfrak{C}$
$F = Ef + D$	$\mathfrak{F} = \mathfrak{E}f + \mathfrak{D}$
etc.	etc.

Vnde perfpicuum eft, in ferie numeratorum terminum primo anteriorem ex lege progreffionis effe debere $= 1$, in ferie autem denominatorum terminum primo anteriorem effe debere $= 0$, ita vt fractio primam praecedens fit $\frac{1}{0}$.

§. 5. Quoniam per fe fatis eft perfpicuum, has fractiones $\frac{A}{\mathfrak{A}}$, $\frac{B}{\mathfrak{B}}$, $\frac{C}{\mathfrak{C}}$, $\frac{D}{\mathfrak{D}}$, etc. continuo propius ad veritatem accedere, ac tandem verum valorem fractionis continuae exhaurire, neceffe eft vt differentiae inter harum fractionum binas proximas continuo fiant minores, quamobrem has differentias ordine euoluamus. Primo igitur habebimus

$$\text{II} - \text{I} = \frac{B\mathfrak{A} - A\mathfrak{B}}{\mathfrak{A}\mathfrak{B}}.$$

Iam hic loco B et $\mathfrak{B}$ valores ex tabula fubftituantur ac prodibit numerator $A\mathfrak{A}b + \mathfrak{A} - Ab$, quae forma ob $\mathfrak{A} = 1$ abit in 1, ita vt fit $\frac{B}{\mathfrak{B}} - \frac{A}{\mathfrak{A}} = \frac{1}{\mathfrak{A}\mathfrak{B}}$. Porro erit

III

$$\mathrm{III} - \mathrm{II} = \frac{C\mathfrak{B} - B\mathfrak{C}}{\mathfrak{B}\mathfrak{C}}$$

cuius numerator, si loco C et $\mathfrak{C}$ valores assignati scribantur, praebet

$$\mathfrak{B}(Bc + A) - B(\mathfrak{B}c + \mathfrak{A}) = A\mathfrak{B} - B\mathfrak{A}.$$

Modo autem vidimus esse $B\mathfrak{A} - A\mathfrak{B} = 1$, vnde iste numerator erit $= -1$ ideoque $\frac{C}{\mathfrak{C}} - \frac{B}{\mathfrak{B}} = -\frac{1}{\mathfrak{B}\mathfrak{C}}$ Porro est

$$\mathrm{IV} - \mathrm{III} = \frac{D\mathfrak{C} - C\mathfrak{D}}{\mathfrak{C}\mathfrak{D}}$$

vbi, si loco D et $\mathfrak{D}$ valores assignati scribantur, erit

$$\mathfrak{C}D - C\mathfrak{D} = \mathfrak{C}(Cd + B) - C(\mathfrak{C}d + \mathfrak{B}) = B\mathfrak{C} - C\mathfrak{B}.$$

Modo autem vidimus esse $C\mathfrak{B} - B\mathfrak{C} = -1$, vnde concluditur $\frac{D}{\mathfrak{D}} - \frac{C}{\mathfrak{C}} = +\frac{1}{\mathfrak{C}\mathfrak{D}}$. Simili modo reperietur pro sequentibus

$$\frac{E}{\mathfrak{E}} - \frac{D}{\mathfrak{D}} = -\frac{1}{\mathfrak{D}\mathfrak{E}};\ \frac{F}{\mathfrak{F}} - \frac{E}{\mathfrak{E}} = +\frac{1}{\mathfrak{E}\mathfrak{F}};\ \text{etc.}$$

§. 6. Hinc igitur singulas nostras fractiones ex sola prima $\frac{A}{\mathfrak{A}} = a$ et fractionibus solas litteras germanicas inuoluentibus definire poterimus, quandoquidem habebimus

$$\frac{B}{\mathfrak{B}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}}$$
$$\frac{C}{\mathfrak{C}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}}$$
$$\frac{D}{\mathfrak{D}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}}$$
$$\frac{E}{\mathfrak{E}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}} - \frac{1}{\mathfrak{D}\mathfrak{E}}$$
$$\frac{F}{\mathfrak{F}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}} - \frac{1}{\mathfrak{D}\mathfrak{E}} + \frac{1}{\mathfrak{E}\mathfrak{F}}$$

etc. etc.

§. 7.

ignati scriban
- B $\mathfrak{A}$.
vnde iste nu-
Porro est
tur, erit
) = B$\mathfrak{C}$ - C$\mathfrak{B}$.
, vnde conclu-
ir pro sequen-
actiones ex sola
germanicas in-
a habebimus
§. 7.

§. 7. Cum igitur harum fractionum vltima; seu infinitesima, verum valorem fractionis continuae propositae, quem designamus littera s, exhibeat, erit

$$s = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}} - \frac{1}{\mathfrak{D}\mathfrak{E}} + \frac{1}{\mathfrak{E}\mathfrak{F}} - \frac{1}{\mathfrak{F}\mathfrak{G}} + \text{etc.}$$

sicque fractionem continuam reduximus ad seriem infinitam fractionum, quarum omnes numeratores sunt alternatim $+1$ et -1, denominatores vero per solas litteras germanicas determinantur, ita vt non opus sit valores litterarum A, B, C, euoluere, sed sufficiat sequentes formulas expediuisse:

$$\mathfrak{A} = 1;\ \mathfrak{B} = b;\ \mathfrak{C} = \mathfrak{B}c + \mathfrak{A};\ \mathfrak{D} = \mathfrak{C}d + \mathfrak{B};\ \mathfrak{E} = \mathfrak{D}e + \mathfrak{C};\ \text{etc.}$$

§. 8. Cum igitur vtraque expressio incipiat a quantitate a, ea prorsus ex calculo egredietur, quoniam litterae germanicae ab ea prorsus non pendent; vnde quae hactenus inuenimus huc redeunt, vt proposita fractione continua

$$s = \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

si ex eius indicibus b, c, d, e, etc. definiantur litterae germanicae, vbi quidem continuo est $\mathfrak{A} = 1$, semper futurum sit

$$s = \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}} - \frac{1}{\mathfrak{D}\mathfrak{E}} + \text{etc.}$$

quae progressio in infinitum progreditur, si fractio continua in infinitum extendatur, contra vero finito terminorum numero constabit.

§. 9. Cum igitur hoc modo fractionem continuam in seriem ordinariam transformauerimus, haud difficile erit, seriem quamcunque propositam in fractionem continuam conuertere. Proposita igitur sit ista series infinita:

$s =$

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \text{etc.}$$

cuius quidem numeratores omnes sint vnitates signo + et − alternatim affectae, denominatores vero progressionem quamcunque constituant, quod tamen non obstat, quo minus omnes plane series in hac forma contineantur, siquidem termini seriei $\alpha, \beta, \gamma, \delta$ non solum numeri fracti, sed etiam negatiui euadere possunt.

§. 10. Quo igitur fractionem continuam isti seriei aequalem eruamus, primo faciamus $\mathfrak{AB} = \alpha$, $\mathfrak{BC} = \beta$, $\mathfrak{CD} = \gamma$, et ita porro, vnde ob $\mathfrak{A} = 1$ sequentes nanciscemur valores:

$$\mathfrak{B} = \alpha; \qquad \mathfrak{C} = \frac{\beta}{\alpha};$$
$$\mathfrak{D} = \frac{\alpha\gamma}{\beta}; \qquad \mathfrak{E} = \frac{\beta\delta}{\alpha\gamma};$$
$$\mathfrak{F} = \frac{\alpha\gamma\varepsilon}{\beta\delta}; \qquad \mathfrak{G} = \frac{\beta\delta\zeta}{\alpha\gamma\varepsilon},$$
$$\mathfrak{H} = \frac{\alpha\gamma\varepsilon\eta}{\beta\delta\zeta}; \qquad \mathfrak{I} = \frac{\beta\delta\zeta\theta}{\alpha\gamma\varepsilon\eta};$$
etc. etc.

Nunc igitur tantum superest, vt ex his valoribus litterarum germanicarum ipsos indices b, c, d, e fractionis continuae eliciamus.

§. 11. Ex formulis autem, quibus supra litterae germanicae per indices fractionis continuae sunt determinatae vicissim ex his litteris ipsos indices b, c, d, e, f etc. definiamus, ac reperiemus

$$b = \mathfrak{B},\ c = \frac{\mathfrak{C} - \mathfrak{A}}{\mathfrak{B}},\ d = \frac{\mathfrak{D} - \mathfrak{B}}{\mathfrak{C}},\ e = \frac{\mathfrak{E} - \mathfrak{C}}{\mathfrak{D}},\ f = \frac{\mathfrak{F} - \mathfrak{D}}{\mathfrak{E}}, \text{ etc.}$$

Hos igitur valores ordine euoluamus, dum loco litterarum $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. formulas ante inuentas substituemus.

§. 12. Primo autem erat $\mathfrak{B} = \alpha$, vnde fit $b = \alpha$; deinde est

𝔈 −

signo + et
progressionem
tat, quo mi-
tur, siquidem
cti, sed etiam
m isti seriei
$= \beta$, $\mathfrak{CD} = \gamma$.
mur valores:
is litterarum
s continuae
pra litterae
leterminatae
f etc. defi-
$\frac{\mathfrak{F} - \mathfrak{D}}{\mathfrak{E}}$, etc.
o litterarum
nus.
fit $b = \alpha$;
𝔈 −

$$\mathfrak{C} - \mathfrak{A} = \frac{\beta - \alpha}{\alpha}, \text{ vnde fit } c = \frac{\beta - \alpha}{\alpha\alpha}.$$

Porro erit

$$\mathfrak{D} - \mathfrak{B} = \frac{\alpha(\gamma - \beta)}{\beta}, \text{ vnde fit } d = \frac{\alpha\alpha(\gamma - \beta)}{\beta\beta}.$$

Deinde habebimus

$$\mathfrak{E} - \mathfrak{C} = \frac{\beta(\delta - \gamma)}{\alpha\gamma} \text{ hincque } e = \frac{\beta\beta(\delta - \gamma)}{\alpha\alpha\gamma\gamma}.$$

Simili modo ob

$$\mathfrak{F} - \mathfrak{D} = [illegible]\ \beta\delta\ [illegible]\ f = \alpha\alpha\gamma\gamma(\varepsilon - \delta) [illegible].$$

Eodem modo ob

$$\mathfrak{G} - \mathfrak{E} = \frac{\beta\delta(\zeta - \varepsilon)}{\alpha\gamma\varepsilon} \text{ erit } g = \frac{\beta\beta\delta\delta(\zeta - \varepsilon)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon}$$

etc.

Hac igitur ratione indices fractionis continuae, quam quaerimus, sequenti modo erunt expressi:

$$b = \alpha \qquad c = \frac{\beta - \alpha}{\alpha\alpha}$$
$$d = \frac{\alpha\alpha(\gamma - \beta)}{\beta\beta} \qquad e = \frac{\beta\beta(\delta - \gamma)}{\alpha\alpha\gamma\gamma}$$
$$f = \frac{\alpha\alpha\gamma\gamma(\varepsilon - \delta)}{\beta\beta\delta\delta} \qquad g = \frac{\beta\beta\delta\delta(\zeta - \varepsilon)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon}$$
$$h = \frac{\alpha\alpha\gamma\gamma\varepsilon\varepsilon(\eta - \zeta)}{\beta\beta\delta\delta\zeta\zeta} \qquad i = \frac{\beta\beta\delta\delta\zeta\zeta(\theta - \eta)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon\eta\eta}$$
etc. etc.

§. 13. Tantum igitur opus est vt isti valores loco indicum b, c, d, e, f etc. in fractione continua

$$s = \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

substituamus; quoniam vero isti valores sunt fracti, quo facilius formam a fractionibus partialibus liberemus, primum ex valoribus inuentis denominatores tollamus eritque

$b = \alpha$

$$b = \alpha \qquad \alpha\alpha c = \beta - \alpha$$
$$\beta\beta d = \alpha\alpha(\gamma - \beta), \qquad \alpha\alpha\gamma\gamma e = \beta\beta(\delta - \gamma)$$
$$\beta\beta\delta\delta f = \alpha\alpha\gamma\gamma(\varepsilon - \delta), \quad \alpha\alpha\gamma\gamma\varepsilon\varepsilon g = \beta\beta\delta\delta(\zeta - \varepsilon)$$
$$\beta\beta\delta\delta\zeta\zeta h = \alpha\alpha\gamma\gamma\varepsilon\varepsilon(\eta - \zeta), \quad \alpha\alpha\gamma\gamma\varepsilon\varepsilon\eta\eta i = \beta\beta\delta\delta\zeta\zeta(\theta - \eta)$$
etc. etc.

§. 14. Nunc ipsam fractionem continuam ... eaedem formulae occurrant, quarum valores hic assignauimus. Secundam scilicet fractionem multiplicemus supra et infra per $\alpha\alpha$, tertiam per $\beta\beta$, quartam per $\alpha\alpha\gamma\gamma$, quintam per $\beta\beta\delta\delta$, sextam per $\alpha\alpha\gamma\gamma\varepsilon\varepsilon$, etc. vt prodeat ista forma:

$$s = \cfrac{1}{b + \cfrac{\alpha\alpha}{\alpha\alpha c + \cfrac{\alpha\alpha\beta\beta}{\beta\beta d + \cfrac{\alpha\alpha\beta\beta\gamma\gamma}{\alpha\alpha\gamma\gamma e + \cfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{\beta\beta\delta\delta f + \text{etc.}}}}}}$$

§. 15. Quod si iam loco horum nouorum indicum $\alpha\alpha c$, $\beta\beta d$, $\alpha\alpha\gamma\gamma e$ etc. valores supra inuentos substituamus, sequens orietur fractio continua:

$$s = \cfrac{1}{\alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\alpha\alpha\beta\beta}{\alpha\alpha(\gamma - \beta) + \cfrac{\alpha\alpha\beta\beta\gamma\gamma}{\beta\beta(\delta - \gamma) + \cfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{\alpha\alpha\gamma\gamma(\varepsilon - \delta) + \text{etc}}}}}}$$

Quod

Qu mu αα per : β(δ−γ) = ββδδ(ζ−ε) = ββδδζζ(θ−η)
e occurrant, qua- scilicet fractionem tiam per ββ, δδ, sextam per
Hi ex nouorum indicum uentos substitua-
ges qui rie Ve dia illi γγ)+ααββγγδδ ααγγ(ε−δ)+ etc
Quod

Quod si hanc formam attentius consideremus, deprehendimus, tertiam fractionem supra et infra deprimi posse per $\alpha\alpha$, tum vero quartam per $\beta\beta$, quintam per $\gamma\gamma$, sextam per $\delta\delta$ etc. quo facto orietur haec fractio continua:

$$s = \cfrac{1}{\alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\beta\beta}{\gamma - \beta + \cfrac{\gamma\gamma}{\delta - \gamma + \cfrac{\delta\delta}{\varepsilon - \delta + \text{etc.}}}}}}$$

Hinc igitur stabiliamus sequens

Theorema I.

§. 16. *Si proposita fuerit talis series infinita:*

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \text{etc.}$$

ex ea semper formari poterit talis fractio continua:

$$\frac{1}{s} = \alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\beta\beta}{\gamma - \beta + \cfrac{\gamma\gamma}{\delta - \gamma + \text{etc.}}}}$$

§. 17. Hanc igitur reductionem per plures ambages ex consideratione fractionis continuae elicuimus, quo quidem proposito nostro satisfecimus, quandoquidem seriem quamcunque in fractionem continuam transformauimus. Verum hic merito desideratur methodus directa, qua immediate ex serie proposita sine illis ambagibus fractio continua illi aequalis deriuari possit. Talem igitur methodum, quippe pe qua Theoria fractionum continuarum non mediocriter illustrabitur, hic sum expositurus.

Problema.

§. 18. *Propositam seriem infinitam*

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \text{etc.}$$

in fractionem continuam transformare.

Solutio.

Cum sit

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \text{etc., statuamus}$$

$$t = + \frac{1}{\beta} - \frac{1}{\gamma} + \frac{1}{\delta} - \frac{1}{\varepsilon} + \text{etc. et}$$

$$u = \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \frac{1}{\zeta} + \text{etc.}$$

Hinc ergo erit $s = \frac{1}{\alpha} - t = \frac{1-\alpha t}{\alpha}$, vnde fit $\frac{1}{s} = \frac{\alpha}{1-\alpha t} = \alpha + \frac{\alpha\alpha t}{1-\alpha t}$. Est autem $\frac{\alpha\alpha t}{1-\alpha t} = \cfrac{\alpha\alpha}{-\alpha + \cfrac{1}{t}}$, vnde fit $\frac{1}{s} = \alpha + \cfrac{\alpha\alpha}{-\alpha + \cfrac{1}{t}}$. Simili ergo modo erit etiam $\frac{1}{t} = \beta + \cfrac{\beta\beta}{-\beta + \cfrac{1}{u}}$ et $\frac{1}{u} = \gamma + \cfrac{\gamma\gamma}{-\gamma + \cfrac{1}{v}}$, etc. quibus valoribus substitutis obtinebitur sequens fractio continua:

$$\frac{1}{s} = \alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\beta\beta}{\gamma - \beta + \cfrac{\gamma\gamma}{\delta - \gamma + \text{etc.}}}}$$

quae est ipsa forma in theoremate exhibita.

§. 19.

mediocriter
is
$\frac{\alpha}{-\alpha t} = \alpha + \frac{\alpha\alpha t}{1-\alpha t}$.
αα. Simili ergo
$+\frac{1}{t}$
etc. quibus
continua:
§. 19.

§. 19. Quod si ergo series proposita sit $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.} = l\,2$, ob $\alpha = 1,\ \beta = 2,\ \gamma = 3,\ \delta = 4$, etc. erit

$$\frac{1}{l\,2} = 1 + \cfrac{1.\,1}{1 + \cfrac{2.\,2}{1 + \cfrac{3.\,3}{1 + \text{etc.}}}}$$

Sin autem assumamus hanc seriem:

$$s = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \frac{\pi}{4}, \text{ ob}$$

$\alpha = 1,\ \beta = 3,\ \gamma = 5,\ \delta = 7$, etc. erit

$$\frac{4}{\pi} = 1 + \cfrac{1.\,1}{2 + \cfrac{3.\,3}{2 + \cfrac{5.\,5}{2 + \text{etc.}}}}$$

quae est ipsa fractio continua olim a *Brounchero* prolata.

§. 20. Sumamus $s = \int \frac{x^{m-1}\,dx}{1+x^n}$, et post integrationem statuamus $x = 1$, quo facto valor ipsius s per sequentem seriem exprimetur:

$$s = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \text{etc.}$$

ita vt sit

$$\alpha = m,\ \beta = m + n,\ \gamma = m + 2n,\ \delta = m + 3n, \text{ etc.}$$

hinc ergo sequens fractio continua emerget:

$\frac{1}{s} = m$

$$\frac{1}{s}=m+\cfrac{mm}{n+\cfrac{(m+n)^2}{n+\cfrac{(m+2n)^2}{n+\text{etc.}}}}$$

quem valorem iam XI Tom. Comentar. Vet. nostrae Academiae dedi.

§. 21. Sin autem proposita sit ista series:

$$s=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\varepsilon}+\text{etc.}$$

cuius omnes termini sint positiui, tantum opus est vt in superiore fractione continua loco litterarum β, δ, ζ, θ, scribatur $-\beta$, $-\delta$, $-\zeta$, etc. tum igitur fiet

$$\frac{1}{s}=\alpha+\cfrac{\alpha\alpha}{-\beta-\alpha+\cfrac{\beta\beta}{\gamma+\beta+\cfrac{\gamma\gamma}{-\delta-\gamma+\cfrac{\delta\delta}{\varepsilon+\delta+\text{etc.}}}}}$$

quae fractio facile transmutatur in hanc formam:

$$\frac{1}{s}=\alpha-\cfrac{\alpha\alpha}{\alpha+\beta-\cfrac{\beta\beta}{\beta+\gamma-\cfrac{\gamma\gamma}{\gamma+\delta-\text{etc.}}}}$$

§. 22. Pluribus autem modis ipsa series proposita transformari potest, vnde continuo aliae atque aliae fractiones continuae eliciuntur. Nonnullas igitur huiusmodi formas hic perpendamus. Sit ergo

$$\alpha=ab,\ \beta=bc,\ \gamma=cd,\ \delta=de \text{ etc.}$$

vt

Vet. nostrae Aca-
ta series:
opus est vt in
n β, δ, ζ, θ, scri-
etc.
formam:
sa series proposita
atque aliae fractio-
ur huiusmodi for-
de etc.
vt

vt habeatur ista series:

$$s=\frac{1}{ab}-\frac{1}{bc}+\frac{1}{ca}-\frac{1}{de}+\text{etc.}$$

hincque formabitur ista fractio continua:

$$\frac{1}{s}=ab+\cfrac{aabb}{b(c-a)+\cfrac{bbcc}{c(d-b)+\cfrac{ccdd}{d(e-c)+\text{etc.}}}}$$

quae facile reducitur ad formam sequentem:

$$\frac{1}{s}=ab+\cfrac{aab}{c-a+\cfrac{bc}{d-b+\cfrac{cd}{e-c+\text{etc.}}}}$$

siue

$$\frac{1}{as}=b+\cfrac{ab}{c-a+\cfrac{bc}{d-b+\text{etc.}}}$$

quae forma nobis suppeditat sequens theorema:

Theorema II.

§. 23. *Si proposita fuerit series huius formae:*

$$s=\frac{1}{ab}-\frac{1}{bc}+\frac{1}{cd}-\frac{1}{de}+\frac{1}{ef}-\text{etc.}$$

ex ea sequens oritur fractio continua:

$$\frac{1}{as}=b+\cfrac{ab}{e-a+\cfrac{bc}{d-b+\cfrac{cd}{e-c+\cfrac{de}{f-d+\text{etc.}}}}}$$

§. 24.

§. 24. Haec forma, etsi facile ex praecedente deriuatur, ideo est notatu digna, quod fractionem continuam formae maxime diuersae praebeat, vnde operae pretium erit exempla supra allata etiam ad hanc formam accommodare. Cum igitur fuerit

$$l\,2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.}, \text{ erit}$$

$$l\,2 - 1 = -\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.}$$

et his seriebus addendis oritur

$$2\,l\,2 - 1 = + \tfrac{1}{1.2} - \tfrac{1}{2.3} + \tfrac{1}{3.4} - \tfrac{1}{4.5} + \tfrac{1}{5.6} - \text{etc.}$$

Hic ergo est

$$s = 2\,l\,2 - 1 \text{ et } a = 1,\ b = 2,\ c = 3,\ d = 4, \text{ etc.}$$

hinc igitur formabitur ista fractio continua:

$$\frac{1}{2\,l\,2-1} = 2 + \cfrac{1.2}{2 + \cfrac{2.3}{2 + \cfrac{3.4}{2 + \text{etc.}}}}$$

§. 25. Simili modo quia est

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc. erit}$$

$$\tfrac{\pi}{4} - 1 = -\tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} - \text{etc.}$$

quarum serierum summa dat

$$\tfrac{\pi}{2} - 1 = \tfrac{2}{1.3} - \tfrac{2}{3.5} + \tfrac{2}{5.7} - \tfrac{2}{7.9} + \text{etc. siue}$$

$$\tfrac{\pi}{4} - \tfrac{1}{2} = \tfrac{1}{1.3} - \tfrac{1}{3.5} + \tfrac{1}{5.7} - \tfrac{1}{7.9} + \text{etc.}$$

Hic igitur erit $s = \frac{\pi}{4} - \frac{1}{2}$; tum vero

$$a = 1,\ b = 3,\ c = 5,\ d = 7 \text{ etc.}$$

quare fractio continua hinc nata erit

$\frac{4}{\pi-2}$

praecedente de-
nem continuam
operae pretium
nam accommo-
, erit
$+ \tfrac{1}{5.6} - \text{etc.}$
3, $d = 4$, etc.
: siue.
$\frac{4}{\pi-2}$

$$\frac{4}{\pi-2} = 3 + \cfrac{1.3}{4 + \cfrac{3.5}{4 + \cfrac{5.7}{4 + \cfrac{7.9}{4 + \text{etc.}}}}}$$

§. 26. Generalius nunc etiam hanc transformationem contemplemur. Denoret igitur Δ valorem formulae integralis $\int \frac{x^{m-1}\,dx}{1+x^n}$, posito post integrationem $x = 1$, et cum sit vt supra § 20. vidimus

$$\Delta = \tfrac{1}{m} - \tfrac{1}{m+n} + \tfrac{1}{m+2n} - \text{etc. erit}$$

$$\Delta - \tfrac{1}{m} = - \tfrac{1}{m+n} + \tfrac{1}{m+2n} - \tfrac{1}{m+3n} + \text{etc.}$$

quibus seriebus additis fit

$$2\,\Delta - \tfrac{1}{m} = \tfrac{n}{m(m+n)} - \tfrac{n}{(m+n)(m+2n)} + \tfrac{n}{(m+2n)(m+3n)} - \text{etc.}$$

hinc diuidendo per n erit

$$\tfrac{2m\Delta-1}{mn} = \tfrac{1}{m(m+n)} - \tfrac{1}{(m+n)(m+2n)} + \tfrac{1}{(m+2n)(m+3n)} - \text{etc.}$$

Hic igitur habemus $s = \frac{2m\Delta-1}{mn}$, tum vero

$$a = m,\ b = m+n,\ c = m+2n,\ d = m+3n \text{ etc.}$$

quocirca fractio continua hinc formata erit

$$\frac{n}{2m\Delta-1} = m+n+\cfrac{m(m+n)}{2n + \cfrac{(m+n)(m+2n)}{2n + \cfrac{(m+2n)(m+3n)}{2n + \cfrac{(m+3n)(m+4n)}{2n + \text{etc.}}}}}$$

quae forma praecedenti simplicitate nihil cedit.

§. 27.

§. 27. Tribuamus nunc etiam feriei initio affumtae

$\frac{1}{a} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} +$ etc.

numeratores quoscunque, fitque

$$s = \frac{a}{\alpha} - \frac{b}{\beta} + \frac{c}{\gamma} - \frac{d}{\delta} + \text{etc.}$$

atque in Theoremate primo loco litterarum $\alpha, \beta, \gamma, \delta$, etc. fcribi oportet $\frac{\alpha}{a}, \frac{\beta}{b}, \frac{\gamma}{c}, \frac{\delta}{d}$, etc. quo facto fractio continua ita fe habebit:

$$\frac{1}{s} = \frac{\alpha}{a} + \cfrac{\frac{\alpha\alpha}{aa}}{\frac{\beta}{b} - \frac{\alpha}{a} + \cfrac{\frac{\beta\beta}{bb}}{\frac{\gamma}{c} - \frac{\beta}{b} + \cfrac{\frac{\gamma\gamma}{cc}}{\frac{\delta}{d} - \frac{\gamma}{c} + \text{etc.}}}}$$

Iam ad fractiones tollendas prima fractio fupra et infra multiplicetur per ab, fecunda per bc, tertia per cd, et ita porro; tum vero vtrinque per a multiplicando obtinebitur

$$\frac{a}{s} = \alpha + \cfrac{\alpha\alpha b}{a\beta - b\alpha + \cfrac{ac\beta\beta}{b\gamma - c\beta + \cfrac{bd\gamma\gamma}{c\delta - d\gamma + \text{etc.}}}}$$

Hinc igitur formetur fequens

Theorema III.

§. 28. *Si propofita fuerit feries infinita huius formae*

$$s = \frac{a}{\alpha} - \frac{b}{\beta} + \frac{c}{\gamma} - \frac{d}{\delta} + \text{etc.}$$

ex ea formabitur fequens fractio continua:

$\frac{a}{s} =$

ei initio affumtae

n $\alpha, \beta, \gamma, \delta$, etc. fractio continua

fupra et infra tia per cd, et ita ando obtinebitur

$\gamma\gamma$ $-d\gamma +$ etc.

nfinita huius formae

$\frac{a}{s} =$

$$\frac{a}{s} = \alpha + \cfrac{\alpha\alpha b}{a\beta - b\alpha + \cfrac{ac\beta\beta}{b\gamma - c\beta + \cfrac{bd\gamma\gamma}{c\delta - d\gamma + \text{etc.}}}}$$

§. 29. Ad hoc illuftrandum propofita fit haec feries:

$$\frac{1}{1} - \frac{2}{2} + \frac{3}{3} - \frac{4}{4} + \frac{5}{5} - \text{etc.} = \frac{1}{2},$$

ita vt fit $s = \frac{1}{2}$; fractio ergo continua hinc orta erit

$$2 = 1 + \cfrac{2}{0 + \cfrac{3.4}{0 + \cfrac{8.9}{0 + \cfrac{15.16}{0 + \text{etc.}}}}}$$

quae forma reducitur ad iftud productum infinitum:

$$2 = 1 + \frac{2.1^2.2}{1.3.2^2}.\frac{4.3^2.4}{3.5.4^2}.\frac{6.5^2}{5.7.6^2}.\frac{6.8.7^2.\text{etc.}}{7.9.8^2\text{ etc.}}$$

cuius veritas non facile perfpicitur, quoniam numeri factorum in numeratore et denominatore non aequales ftatui p flunt, etiamfi ambo fint infiniti. Nullum vero dubium effe poteft, quin valor iftius producti fit $= 1$.

§. 30. Confideremus nunc iftam feriem:

$$s = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \text{etc.}$$

cuius fumma eft $s = l\,2 - \frac{1}{2}$. Quia igitur eft

$a = 1$, $b = 2$, $c = 3$, $d = 4$, etc.

$\alpha = 2$, $\beta = 3$, $\gamma = 4$, $\delta = 5$, etc.

fractio continua hinc nata erit

$\frac{1}{l\,2 - \frac{1}{2}}$

$$\frac{1}{l2-\frac{1}{2}}=2+\cfrac{1.2.2^2}{-1+\cfrac{1.3.3^2}{-1+\cfrac{2.4.4^2}{-1+\cfrac{3.5.5^2}{-1+\text{etc.}}}}}$$

§. 31. Quod si autem hanc accipiamus seriem:

$$s=\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\text{etc.}$$

cuius valor est $\frac{1}{2}+l2$, habebimus

$$a=2,\ b=3,\ c=4,\ d=5,\ \text{etc.}$$
$$\alpha=1,\ \beta=2,\ \gamma=3,\ \delta=4,\ \text{etc.}$$

hinc ergo orietur haec fractio continua:

$$\frac{2}{\frac{1}{2}+l2}=1+\cfrac{1.3.1^2}{1+\cfrac{2.4.2^2}{1+\cfrac{3.5.3^2}{1+\cfrac{4.6.4^2}{1+\text{etc.}}}}}$$

siue

$$\frac{4}{2l2+1}=1+\cfrac{1^2.3}{1+\cfrac{2^2.4}{1+\cfrac{3^2.5}{1+\text{etc.}}}}$$

Problema II.

Propositam seriem infinitam

s =

$$s=\frac{x}{\alpha}-\frac{xx}{\beta}+\frac{x^3}{\gamma}-\frac{x^4}{\delta}+\text{etc.}$$

in fractionem continuam transformare.

Solutio.

§. 32. Considerentur sequentes series ex proposita serie formatae:

$$t=\frac{x}{\beta}-\frac{xx}{\gamma}+\frac{x^3}{\delta}-\frac{x^4}{\varepsilon}+\text{etc., porro}$$
$$u=\frac{x}{\gamma}-\frac{xx}{\delta}+\frac{x^3}{\varepsilon}-\frac{x^4}{\zeta}+\text{etc.}$$
$$v=\frac{x}{\delta}-\frac{xx}{\varepsilon}+\frac{x^3}{\zeta}-\frac{x^4}{\eta}+\text{etc., eritque}$$
$$s=\frac{x}{\alpha}-tx=\frac{x(1-\alpha t)}{\alpha},\ \text{vnde fit}$$
$$\frac{x}{s}=\frac{\alpha}{1-\alpha t}=\alpha+\frac{\alpha\alpha t}{1-\alpha t}=\alpha+\cfrac{\alpha\alpha}{-\alpha+\cfrac{1}{t}}$$

Hinc ergo erit

$$\frac{x}{s}=\alpha+\cfrac{\alpha\alpha x}{-\alpha x+\cfrac{x}{t}}$$

simili autem modo erit

$$\frac{x}{t}=\beta+\cfrac{\beta\beta x}{-\beta x+\cfrac{x}{u}}$$

Hi ergo valores si omnes ordine substituantur, orietur ista fractio continua:

$$\frac{x}{s}=\alpha+\cfrac{\alpha\alpha x}{\beta-\alpha x+\cfrac{\beta\beta x}{\gamma-\beta x+\cfrac{\gamma\gamma x}{\delta-\gamma x+\text{etc.}}}}$$

§. 33.

§. 33. Quod si hic vbique loco x scribamus $\frac{x}{y}$; vt habeamus hanc seriem:

$$s=\frac{x}{\alpha y}-\frac{xx}{\beta yy}+\frac{x^3}{\gamma y^3}-\frac{x^4}{\delta y^4}+\text{etc.}$$

tum fractio continua hinc nata erit

$$\frac{1}{sy}=\alpha+\cfrac{\alpha\alpha x:y}{\beta-\frac{\alpha x}{y}+\cfrac{\beta\beta x:y}{\gamma-\frac{\beta x}{y}+\text{etc.}}}$$

quae a fractionibus partialibus liberata dat

$$\frac{x}{sy}=\alpha+\cfrac{\alpha\alpha x}{\beta y-\alpha x+\cfrac{\beta\beta xy}{\gamma y-\beta x+\cfrac{\gamma\gamma xy}{\delta y-\gamma x+\text{etc.}}}}$$

vnde nascitur sequens

Theorema IV.

§. 34. *Si proposita fuerit huiusmodi series infinita:*

$$s=\frac{x}{\alpha y}-\frac{xx}{\beta yy}+\frac{x^3}{\gamma y^3}-\frac{x^4}{\delta y^4}+\text{etc.}$$

ex ea formari poterit ista fractio continua:

$$\frac{x}{s}=\alpha y+\cfrac{\alpha\alpha xy}{\beta y-\alpha x+\cfrac{\beta\beta xy}{\gamma y-\beta x+\cfrac{\gamma\gamma xy}{\delta y-\gamma x+\cfrac{\delta\delta xy}{\varepsilon y-\delta x+\text{etc}}}}}$$

§. 35. Cum sit

$$l\left(1+\frac{x}{y}\right)=\frac{x}{y}-\frac{xx}{2yy}+\frac{x^3}{3y^3}-\frac{x^4}{4y^4}+\text{etc.}$$

posito

scribamus $\frac{x}{y}$;
$x+$ etc.
: *series infinita:*
$+\delta\delta xy$ / $\varepsilon y-\delta x+$ etc
etc.
posito

posito $s=l\left(1+\frac{x}{y}\right)$ erit

$$\alpha=1,\ \beta=2,\ \gamma=3,\ \delta=4,\ \text{etc.}$$

hincque nascetur ista fractio continua:

$$\frac{x}{l\left(1+\frac{x}{y}\right)}=y+\cfrac{1xy}{2y-x+\cfrac{4xy}{3y-2x+\cfrac{9xy}{4y-3x+\text{etc.}}}}$$

§. 36. Cum arcus cuius tangens t hac serie exprimatur:

$$\text{A tang.}\,t=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\frac{t^9}{9}-\text{etc.}\quad\text{erit}$$

$$t\,\text{A tang.}\,t=\frac{tt}{1}-\frac{t^4}{3}+\frac{t^6}{5}-\frac{t^8}{7}+\text{etc.}$$

Nunc ponatur $tt=\frac{x}{y}$, ita vt sit $t=\sqrt{\frac{x}{y}}$, fietque

$$\sqrt{\frac{x}{y}}.\,\text{A tang.}\sqrt{\frac{x}{y}}=\frac{x}{y}-\frac{xx}{3yy}+\frac{x^3}{5y^3}-\frac{x^4}{7y^4}+\text{etc.}$$

Hinc ergo est $s=\sqrt{\frac{x}{y}}.\,\text{A tang.}\sqrt{\frac{x}{y}}$, tum vero

$$\alpha=1,\ \beta=3,\ \gamma=5,\ \delta=7,\ \text{etc.},$$

quare fractio continua hinc nata erit

$$\frac{\sqrt{xy}}{\text{A tang.}\sqrt{\frac{x}{y}}}=y+\cfrac{xy}{3y-x+\cfrac{9xy}{5y-3x+\cfrac{25xy}{7y-5x+\text{etc.}}}}$$

Veluti si fuerit $x=1$ et $y=3$, ob A tang. $\frac{1}{\sqrt{3}}=\frac{\pi}{6}$, habebitur ista fractio continua:

6 V 3

$$\frac{4\sqrt{s}}{\pi} = 3 + \cfrac{1.3}{8 + \cfrac{3.9}{12 + \cfrac{3.25}{16 + \text{etc.}}}}$$

§. 37. Quod si in casu Theorematis loco litterarum α, β, γ, δ, etc. scribamus fractiones

$$\frac{\alpha}{a},\ \frac{\beta}{b},\ \frac{\gamma}{c},\ \frac{\delta}{d},\ \text{etc.}$$

vt habeamus hanc seriem:

$$s = \frac{ax}{\alpha y} - \frac{bxx}{\beta\gamma y} + \frac{cx^3}{\gamma y^3} - \frac{dx^4}{\delta y^4} + \text{etc.}$$

fractio continua hinc formata ita se habebit:

$$\frac{x}{s} = \frac{\alpha}{a}y + \alpha\alpha xy : aa \over \frac{\beta}{b}y - \frac{\alpha}{a}x + \beta\beta xy : bb \over \frac{\gamma}{c}y - \frac{\beta}{b}x + \gamma\gamma xy : cc \over \frac{\delta}{d}y - \frac{\gamma}{c}x + \text{etc.}$$

Hic iam primo vtrinque multiplicetur per a, deinde primae fractionis numerator et denominator multiplicentur per ab, secundae per bc, tertiae per cd, etc. et fractio continua hanc induet formam:

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha\alpha bxy}{a\beta y - \alpha bx + \cfrac{\beta\beta acxy}{b\gamma y - c\beta x + \cfrac{\gamma\gamma bdxy}{c\delta y - d\gamma x + \text{etc.}}}}$$

vnde operae pretium erit sequens apponere

Theo-

ematis loco littera-
ones
bebit:
$+\gamma\gamma xy : cc$
$\frac{\delta}{d}y - \frac{\gamma}{c}x +$ etc.
per a, deinde primae
ultiplicentur per ab,
, et fractio continua
$\gamma bdxy$
$\delta y - d\gamma x +$ etc.
onere
Theo-

Theorema V.

§. 38. *Si proposita fuerit series infinita huius formae:*

$$s = \frac{ax}{\alpha y} - \frac{bxx}{\beta\gamma y} + \frac{cx^3}{\gamma y^3} - \frac{dx^4}{\delta y^4} + \text{etc.}$$

inde formabitur sequens fractio continua:

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha\alpha bxy}{a\beta y - \alpha bx + \cfrac{\beta\beta acxy}{b\gamma y - c\beta x + \cfrac{\gamma\gamma bdxy}{c\delta y - d\gamma x + \text{etc.}}}}$$

Problema III.

Propositam hanc seriem infinitam:

$$s = \frac{1}{\alpha} - \frac{1}{\alpha\beta} + \frac{1}{\alpha\beta\gamma} - \frac{1}{\alpha\beta\gamma\delta} + \text{etc.}$$

in fractionem continuam conuertere.

Solutio.

§. 39. Ex serie proposita formemus sequentes series:

$$t = \frac{1}{\beta} - \frac{1}{\beta\gamma} + \frac{1}{\beta\gamma\delta} - \frac{1}{\beta\gamma\delta\varepsilon} + \text{etc.},$$

$$u = \frac{1}{\gamma} - \frac{1}{\gamma\delta} + \frac{1}{\gamma\delta\varepsilon} - \frac{1}{\gamma\delta\varepsilon\zeta} + \text{etc.}$$

atque habebimus

$$s = \frac{1-t}{\alpha},\ t = \frac{1-u}{\beta},\ u = \frac{1-v}{\gamma}\ \text{etc.}$$

hinc igitur deducimus

$$\frac{1}{s} = \frac{\alpha}{1-t} = \alpha + \frac{\alpha t}{1-t} = \alpha + \cfrac{\alpha}{-1 + \cfrac{1}{t}}.$$

Simili autem modo erit

$\frac{1}{t} =$

$$\frac{1}{t}=\beta+\cfrac{\beta}{-1+\cfrac{1}{u}},\quad \frac{1}{u}=\gamma+\cfrac{\gamma}{-1+\cfrac{1}{v}},\ \text{etc.}$$

quare posterioribus valoribus in prioribus substitutis obtinebitur ista fractio continua:

$$\frac{1}{s}=\alpha+\cfrac{\alpha}{\beta-1+\cfrac{\beta}{\gamma-1+\cfrac{\gamma}{\delta-1+\text{etc.}}}}$$

vnde deducimus sequens Theorema.

Theorema VI.

§. 40. *Si proposita fuerit huiusmodi series infinita*

$$s=\frac{1}{\alpha}-\frac{1}{\alpha\beta}+\frac{1}{\alpha\beta\gamma}-\frac{1}{\alpha\beta\gamma\delta}+\text{etc.}$$

exinde formari poterit haec fractio continua:

$$\frac{1}{s}=\alpha+\cfrac{\alpha}{\beta-1+\cfrac{\beta}{\gamma-1+\cfrac{\gamma}{\delta-1+\text{etc.}}}}$$

§. 41. Si e denotet numerum cuius logarithmus hyperbolicus est vnitas, notum est esse

$$\frac{1}{e}=1-\frac{1}{1}+\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}-\text{etc. siue}$$

$$\frac{e-1}{e}=\frac{1}{1}-\frac{1}{1.2}+\frac{1}{1.2.3}-\frac{1}{1.2.3.4}+\text{etc.}$$

Hic igitur fit $s=\frac{e-1}{e}$, tum vero

$$\alpha=1,\ \beta=2,\ \gamma=3,\ \delta=4,\ \text{etc.}$$

quare fractio continua hinc oriunda est

$\frac{e}{e-1}$

, etc. substitutis obtine- etc. di series infinita : etc. cuius logarithmus - etc. siue - etc. etc. $\frac{e}{e-1}$

$$\frac{e}{e-1}=1+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\text{etc.}}}}$$

§. 42. Cum igitur sit

$$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\text{etc.}}}}}=\frac{1}{e-1}$$

haud difficulter autem demonstrari queat, si fuerit

$$\cfrac{a}{a+\cfrac{b}{b+\cfrac{c}{c+\text{etc.}}}}=s$$

tum fore

$$a+\cfrac{a}{b+\cfrac{b}{c+\cfrac{c}{d+\text{etc.}}}}=\frac{1}{1-s}$$

pro nostro casu erit

$$s=\frac{1}{e-1},\ a=1,\ b=2,\ c=3,\ \text{etc.}$$

quibus valoribus substitutis fiet

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}}=\frac{1}{e-2}$$

X 2 §. 43.

§. 43. Quodsi in serie Theorematis VI loco literarum $\alpha, \beta, \gamma, \delta$, etc. scribantur fractiones
$\frac{\alpha}{a}, \frac{\beta}{b}, \frac{\gamma}{c}, \frac{\delta}{d}$ etc., vt sit

$$s = \frac{a}{\alpha} - \frac{ab}{\alpha\beta} + \frac{abc}{\alpha\beta\gamma} - \frac{abcd}{\alpha\beta\gamma\delta} + \text{etc.}$$

fractio continua hinc nata erit

$$\frac{1}{s} = \frac{\alpha}{a} + \alpha : a \over \frac{\beta}{b} - 1 + \beta : b \over \frac{\gamma}{c} - 1 + \gamma : c \over \frac{\delta}{d} - 1 + \text{etc.}$$

Quodsi iam primo multiplicetur vtrinque per a, tum vero prima fractio supra et infra per b, secunda per c, tertia per d, etc., orietur ista forma:

$$\frac{a}{s} = \alpha + \alpha b \over \beta - b + \beta c \over \gamma - c + \gamma d \over \delta - d + \text{etc.}$$

quod sequenti Theoremati includatur

Theorema VII.

§. 44. *Si proposita fuerit huiusmodi series infinita:*

$$s = \frac{a}{\alpha} - \frac{ab}{\alpha\beta} + \frac{abc}{\alpha\beta\gamma} - \frac{abcd}{\alpha\beta\gamma\delta} + \text{etc.}$$

inde deducitur haec fractio continua:

$$\frac{a}{s} = \alpha + \alpha b \over \beta - b + \beta c \over \gamma - c + \gamma d \over \delta - d + \text{etc.}$$

§. 45.

matis VI loco li-
actiones
+ etc.
= per a, tum vero
unda per c, tertia
modi series infinita:
tc.
tc.
§. 45.

§. 45. Applicemus hoc ad sequentem seriem infinitam:

$$s = \frac{1}{2} - \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} - \frac{1.3.5.7}{2.4.6.8} + \text{etc.}$$

cuius summam constat esse $s = \frac{\sqrt{2} - 1}{\sqrt{2}}$; tum igitur erit

$$a = 1,\ b = 3,\ c = 5,\ d = 7,\ \text{etc.}$$
$$\alpha = 2,\ \beta = 4,\ \gamma = 6,\ \delta = 8,\ \text{etc.}$$

fractio ergo continua hinc nata erit

$$\frac{\sqrt{2}}{\sqrt{2}-1} = 2 + 2.3 \over 1 + 4.5 \over 1 + 6.7 \over 1 + \text{etc.}$$

Si vtrinque vnitas auferatur erit

$$\frac{1}{\sqrt{2}-1} = 1 + 2.3 \over 1 + 4.5 \over 1 + 6.7 \over 1 + \text{etc.}$$

vnde deducitur

$$\sqrt{2} = 1 + 1.1 \over 1 + 2.3 \over 1 + 4.5 \over 1 + 6.7 \over 1 + \text{etc.}$$

Problema IV.

Propositam seriem infinitam huius formae:

$$s = \frac{x}{\alpha} - \frac{xx}{\alpha\beta} + \frac{x^3}{\alpha\beta\gamma} - \frac{x^4}{\alpha\beta\gamma\delta} + \text{etc.}$$

in fractionem continuam conuertere

Solutio.

§. 46. Statuamus vt hactenus

$$t = \frac{x}{\beta} - \frac{xx}{\beta\gamma} + \frac{x^3}{\beta\gamma\delta} - \frac{x^4}{\beta\gamma\delta\varepsilon} + \text{etc. et}$$

$$u = \frac{x}{\gamma} - \frac{xx}{\gamma\delta} + \frac{x^3}{\gamma\delta\varepsilon} - \frac{x^4}{\gamma\delta\varepsilon\zeta} + \text{etc.}$$

ita vt sit $s = \frac{x - tx}{\alpha}$, vnde sit $\frac{x}{s} = \frac{\alpha}{1-t} = \alpha + \frac{\alpha t}{1-t}$. Est vero

$$\frac{\alpha t}{1-t} = \frac{\alpha}{-1 + \frac{1}{t}} = \frac{\alpha x}{-x + \frac{x}{t}},$$

sicque erit

$$\frac{x}{s} = \alpha + \frac{\alpha x}{-x + \frac{x}{t}}.$$

Simili igitur modo reperietur

$$\frac{x}{t} = \beta + \frac{\beta x}{-x + \frac{x}{u}}, \quad \frac{x}{u} = \gamma + \frac{\gamma x}{-x + \frac{x}{v}}, \text{ etc.}$$

Quodsi ergo hi valores continuo in praecedentibus substituantur, obtinebitur sequens fractio continua:

$$\frac{x}{s} = \alpha + \cfrac{\alpha x}{\beta - x + \cfrac{\beta x}{\gamma - x + \cfrac{\gamma x}{\delta - x + \text{etc.}}}}$$

hincque nascitur

Theorema VIII.

§. 47. *Si proposita fuerit huiusmodi series infinita:*
$s =$

$\alpha + \frac{\alpha t}{1-t}$. Est
etc.
entibus substi-
series infinita:
$s =$

$$s = \frac{x}{\alpha} - \frac{xx}{\alpha\beta} + \frac{x^3}{\alpha\beta\gamma} - \frac{x^4}{\alpha\beta\gamma\delta} + \text{etc.}$$

inde formabitur sequens fractio continua:

$$\frac{x}{s} = \alpha + \cfrac{\alpha x}{\beta - x + \cfrac{\beta x}{\gamma - x + \cfrac{\gamma x}{\delta - x + \text{etc.}}}}$$

§. 48. Quod si hic loco x scribamus $\frac{x}{y}$, vt habeamus hanc seriem:

$$s = \frac{x}{\alpha y} - \frac{xx}{\alpha\beta yy} + \frac{x^3}{\alpha\beta\gamma y^3} - \frac{x^4}{\alpha\beta\gamma\delta y^4} + \text{etc.}$$

hinc nascetur sequens fractio continua:

$$\frac{x}{s} = \alpha y + \cfrac{\alpha x y}{\beta y - x + \cfrac{\beta x y}{\gamma y - x + \cfrac{\gamma x y}{\delta y - x + \text{etc.}}}}$$

§. 49. Sumamus $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\delta = 4$ etc. vt sit

$$s = \frac{x}{y} - \frac{xx}{1.2.yy} + \frac{x^3}{1.2.3\,y^3} - \frac{x^4}{1.2.3.4\,y^4} + \text{etc.}$$

vbi igitur est $s = 1 - e^{\frac{-x}{y}}$, hincque formabitur ista fractio continua:

$$\frac{x}{1 - e^{\frac{-x}{y}}} = y + \cfrac{xy}{2y - x + \cfrac{2xy}{3y - x + \cfrac{3xy}{4y - x + \text{etc.}}}}$$

$= \frac{x e^{\frac{x}{y}}}{e^{\frac{x}{y}} - 1}$, vnde obtinebuntur ſequentes formulae ſpeciales: ſumendo $x = 1$ et loco y ſucceſſiue numeros 1, 2, 3, 4, 5 etc.

$$\frac{e}{e-1} = 1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \text{etc.}}}}$$

$$\frac{\sqrt{e}}{\sqrt{e}-1} = 2 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \text{etc.}}}}$$

$$\frac{\sqrt[3]{e}}{\sqrt[3]{e}-1} = 3 + \cfrac{3}{5 + \cfrac{6}{8 + \cfrac{9}{11 + \text{etc.}}}}$$

$$\frac{\sqrt[4]{e}}{\sqrt[4]{e}-1} = 4 + \cfrac{4}{7 + \cfrac{8}{11 + \cfrac{12}{15 + \text{etc.}}}}$$

Problema V.

Si in genere propoſita fuerit ſeries huius formae:

$$s = \frac{ax}{\alpha y} - \frac{abx^2}{\alpha\beta y^2} + \frac{abcx^3}{\alpha\beta\gamma y^3} - \frac{abcdx^4}{\alpha\beta\gamma\delta y^4} + \text{etc.}$$

rmulae ſpeciales:
eros 1, 2, 3, 4, 5
huius formae:
etc.
eam

eam

eam in fractionem continuam conuertere

Solutio.

§. 50. Ex ſerie propoſita formemus ſequentes:

$$t = \frac{bx}{\beta y} - \frac{bcxx}{\beta\gamma yy} + \frac{bcdx^3}{\beta\gamma\delta y^3} - \frac{bcdex^4}{\beta\gamma\delta\varepsilon y^4} + \text{etc. et}$$

$$u = \frac{cx}{\gamma y} - \frac{cdxx}{\gamma\delta yy} + \frac{cdex^3}{\gamma\delta\varepsilon y^3} - \frac{cdefx^4}{\gamma\delta\varepsilon\zeta y^4} + \text{etc.}$$

ita vt ſit $s = \frac{ax}{\alpha y}(1 - t)$ hincque

$$\frac{ax}{s} = \frac{\alpha y}{1-t} = \alpha y + \frac{\alpha y t}{1-t}.$$

Eſt vero

$$\frac{\alpha y t}{1-t} = \frac{\alpha y}{-1 + \frac{1}{t}} = \frac{\alpha b x y}{-bx + \frac{bx}{t}}$$

vnde fit

$$\frac{ax}{s} = \alpha y + \frac{\alpha b x y}{-bx + \frac{bx}{t}};$$

ſimili igitur modo ex relatione $\frac{bx}{\beta y}(1-u)$ fiet

$$\frac{bx}{t} = \beta y + \frac{\beta c x y}{-cx + \frac{cx}{u}}$$

ſicque porro. Quare his valoribus continuo ſubſtitutis orietur iſta fractio continua:

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha b x y}{\beta y - bx + \cfrac{\beta c x y}{\gamma y - cx + \cfrac{\gamma d x y}{\delta y - dx + \text{etc.}}}}$$

hinc ſequens

Theo-

Theorema IX.

§. 51. *Si propofita fuerit ifta feries generalis:*

$$s=\frac{\alpha x}{\alpha y}-\frac{\alpha b x x}{\alpha\beta y y}+\frac{\alpha b c x^3}{\alpha\beta\gamma y^3}-\frac{\alpha b c d x^4}{\alpha\beta\gamma\delta y^4}+\text{ etc.}$$

inde formabitur fractio continua

$$\frac{\alpha x}{s}=\alpha y+\cfrac{\alpha b x y}{\beta y-b x+\cfrac{\beta c x y}{\gamma y-c x+\cfrac{\gamma d x y}{\delta y-d x+\text{ etc.}}}}$$

§. 52. Vt hoc Theorema per exemplum notatu dignum illuftremus, confideremus hanc formulam integralem:

$$Z=\int z^{m-1}\,dz\,(1+z^n)^{\frac{k}{n}-1}$$

quod integrale ita fumatur vt euanefcat pofito $z=0$, ac ftatuamus $Z=v(1+z^n)^{\frac{k}{n}}$, eritque differentiando

$$dZ=z^{m-1}\,dz(1-z^n)^{\frac{k}{n}-1}=dv(1+z^n)^{\frac{k}{n}}$$
$$+kvz^{n-1}\,dz(1+z^n)^{\frac{k}{n}-1},$$

quae aequatio per $(1+z^n)^{\frac{k}{n}-1}$ diuifa praebet

$$z^{m-1}\,dz=dv(1+z^n)+kvz^{n-1}\,dz,\text{ ideoque}$$
$$\frac{dv}{dz}(1+z^n)+kvz^{n-1}-z^{m-1}=0.$$

§. 53. Quoniam fumto z infinite paruo fit

$$Z=\frac{z^m}{m}=v,$$

inde difcimus, quantitatem v per eiusmodi feriem infinitam exprimi, cuius primus terminus fit poteftas z^m; in fequentibus

feries generalis:
+ etc.
-γdxy
δy−dx+ etc.
exemplum notatu
ormulam integralem:
it pofito z = 0, ac
ifferentiando
(1+z^n)^{k/n}
−1,
i praebet
dz, ideoque
0.
ite paruo fit
iodi feriem infinitam
eftas z^n; in fequentibus

tibus autem terminis exponentes ipfius z continuo numero n augeri, quamobrem pro v fingamus fequentem feriem infinitam:

$$v=Az^m-Bz^{m+n}+Cz^{m+2n}-Dz^{m+3n}+\text{ etc.}$$

quem valorem in aequatione differentiali fubftituamus, fimilesque poteftates ipfius z fibi fubfcribamus fequendi modo:

$$\frac{dv}{dz}=mAz^{m-1}-(m+n)Bz^{m+n-1}+(m+2n)Cz^{m+2n-1}-(m+3n)Dz^{m+3n-1}+\text{ etc.}$$

$$\begin{array}{lllll}
\frac{z^n\,dv}{dz}= & +mA & -(m+n)B & +(m+2n)C & -\text{ etc.}\\
+kvz^{n-1}=+kA & & -kB & +kC & \\
-z^{m-1}=-1. & & & &
\end{array}$$

§. 54. Quod fi nunc fingulae ipfius z poteftates feorfim ad nihilum redigantur, obtinebuntur fequentes valores:

$$\begin{array}{ll}
mA-1=0 & \text{ergo } A=\frac{1}{m}\\
-(m+n)B+(m+k)A=0 & \text{ergo } B=\frac{(m+k)A}{m+n}\\
(m+2n)C-(m+n+k)B=0 & \text{ergo } C=\frac{(m+n+k)B}{m+2n}\\
-(m+3n)D+(m+2n+k)C=0 & \text{ergo } D=\frac{(m+2n+k)C}{m+3n}\\
\text{etc} & \text{etc.}
\end{array}$$

§. 55. Subftituamus igitur hos valores inuentos ac pro v reperiemus fequentem feriem infinitam:

$$v=\frac{z^m}{m}-\frac{m+k}{m(m+n)}z^{m+n}+\frac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{m+2n}$$
$$-\frac{(m+k)(m+n+k)(m+2n+k)}{m(m+n)(m+2n)(m+3n)}z^{m+3n}\text{ etc.}$$

 quam

quam feriem, vt ad formam noſtri theorematis reducamus, hoc modo repraeſentemus:

$$v=\frac{z^{m-n}}{m}\left(z^n-\frac{m+k}{m+n}z^{2n}+\frac{(m+k)(m+n+k)}{(m+n)(m+2n)}z^{3n}\right.$$
$$\left.-\frac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n}\text{ etc.}\right)$$

§. 56. Cum iam ſit

$$Z=\int z^{m-1}dz(1+z^n)^{\frac{k}{n}}-1,\ \text{ſtatuamus}$$

$$V=\frac{mZ}{z^{m-n}(1+z^n)^{\frac{k}{n}}},\ \text{vt fiat}$$

$$V=z^n-\frac{m+k}{m+n}z^{2n}+\frac{(m+k)(m+n+k)}{(m+n)(m+2n)}z^{3n}-\frac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n}+\text{etc.}$$

erit igitur V functio ipſius z per integrationem formulae differentialis eruenda, quae ergo pro quouis valore ipſius z determinatum adipiſcetur valorem, ſiquidem integrale ita capi ponimus, vt euaneſcat facto $z=0$. Quo igitur facilius iſtos valores ipſius v aſſignare queamus, quando variabili z valores fracti tribuantur, ſtatuamus in genere $z^n=\frac{x}{y}$, ita vt hanc formam nanciſcamur:

$$V=\frac{x}{y}-\frac{(m+k)xx}{(m+n)yy}+\frac{(m+k)(m+n+k)x^3}{(m+n)(m+2n)y^3}-\frac{(m+k)(m+n+k)(m+2n+k)x^4}{(m+n)(m+2n)(m+3n)y^4}+\text{etc.}$$

quae feries cum noſtro Theoremate collata praebet $s=V$; tum vero

$$a=1,\ b=m+k,\ c=m+n+k,\ \text{etc.}$$
$$\alpha=1,\ \beta=m+n,\ \gamma=m+2n,\ \text{etc.}$$

§. 57. His notatis formula integralis aſſumta ſequentem nobis ſuppeditat fractionem continuam:

$\frac{x}{V}=$

oremutis reducamus;

$$)(m+n+k)z^{3n}\ \ n)(m+2n)$$
$$-2n+k)z^{4n}\text{ etc.})\ \ +3n)$$

ſtatuamus

$$+n+k)(m+2n+k)z^{4n}+\text{etc.}$$

grationem formulae
juouis valore ipſius
quidem integrale ita
o. Quo igitur faeamus, quando vamus in genere $z^n=\frac{x}{y}$,

$$+n+k)(m+2n+k)x^4 / (m+2n)(m+3n)y^4+\text{etc.}$$

lata praebet $s=V$;

-k, etc.
. etc.

ıtegralis aſſumta ſe-
ontinuam:

$\frac{x}{V}=$

$$\frac{x}{V}=y+\cfrac{(m+k)xy}{(m+n)y-(m+k)x+\cfrac{(m+n)(m+n+k)xy}{(m+2n)y-(m+n+k)x+\cfrac{(m+2n)(m+2n+k)xy}{(m+3n)y-(m+2n+k)x-\text{etc.}}}}$$

cuius valor erit $\frac{xz^{m-n}(1+z^n)^{\frac{k}{n}}}{mZ}$.

§. 58. Exempli gratia ſumamus hanc formam:

$$Z=\int\frac{dz}{\sqrt{(1+zz)}}=l(z+\sqrt{(1+zz)}),$$

erit igitur $m=1,\ n=2,\ k=1$, fietque $V=\frac{zl(z+\sqrt{(1+zz)})}{\sqrt{(1+zz)}}$ qui valor aequatur huic feriei:

$$zz-\tfrac{2}{3}z^4+\tfrac{2.4}{3.5}z^6-\tfrac{2.4.6}{3.5.7}z^8+\text{etc.}$$

quae, ſi iam loco zz ſcribamus $\frac{x}{y}$, fiet

$$V=\frac{\sqrt{x}}{\sqrt{(x+y)}}l\left(\frac{\sqrt{x}+\sqrt{(x+y)}}{\sqrt{y}}\right)$$
$$=\frac{x}{y}-\frac{2}{3}.\frac{xx}{yy}+\frac{2.4}{3.5}.\frac{x^3}{y^3}-\frac{2.4.6}{3.5.7}.\frac{x^4}{y^4}+\text{etc.}$$

quare fractio continua hinc formata erit

$$\frac{\sqrt{x(x+y)}}{l\frac{(\sqrt{x}+\sqrt{(x+y)})}{\sqrt{y}}}=y+\cfrac{1.2xy}{3y-2x+\cfrac{3.4xy}{5y-4x+\cfrac{5.6xy}{7y-6x+\cfrac{7.8xy}{9y-8x+\text{etc.}}}}}$$

§. 59. Quodſi ergo ſumamus $x=1$ et $y=1$, habebimus iſtam fractionem continuam:

$$\frac{\sqrt{2}}{l(1+\sqrt{2})} = 1 + \cfrac{1.2}{1 + \cfrac{3.4}{1 + \cfrac{5.6}{1 + \text{etc.}}}}$$

ipſa ſerie infinita exiſtente

$$\frac{l(1+\sqrt{2})}{\sqrt{2}} = 1 - \frac{2}{1.3} + \frac{2.4}{3.5} - \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} - \text{etc.}$$

Sin autem manente $x = 1$ ſumamus $y = 2$, ſeries infinita erit

$$\frac{1}{2} - \frac{2}{3}.\frac{1}{4} + \frac{2.4}{3.5}.\frac{1}{8} - \frac{2.4.6}{3.5.7}.\frac{1}{16} + \text{etc.}$$

fractio vero continua haec:

$$\frac{\sqrt{3}}{l\frac{1+\sqrt{3}}{\sqrt{2}}} = 2 + \cfrac{1.2.2}{4 + \cfrac{3.4.2}{6 + \cfrac{5.6.2}{8 + \text{etc.}}}}$$

vnde ſequens forma deducitur:

$$\frac{\sqrt{3}}{2\,l\frac{1+\sqrt{3}}{\sqrt{2}}} = 1 + \cfrac{1}{2 + \cfrac{6}{3 + \cfrac{15}{4 + \cfrac{28}{5 + \text{etc.}}}}}$$

vbi numeratores ſunt numeri trigonales alternatim ſumti.

§. 60. Euoluamus adhuc caſum quo $x = 1$ et $y = 3$, quoniam irrationalitas hoc modo tolletur; erit autem hoc caſu

$\frac{1}{2} l 3$

$\frac{2.4.6.8}{3.5.7.9}$ − etc.
= 2, ſeries infinita
c.
+ etc.
alternatim ſumti.
ſum quo $x = 1$ et
o tolletur; erit autem
$\frac{1}{2} l 3$

$$\frac{1}{2} l 3 = \frac{1}{3} - \frac{2}{3}.\frac{1}{9} + \frac{2.4}{3.5}.\frac{1}{27} - \frac{2.4.6}{3.5.7}.\frac{1}{81} + \text{etc.}$$

fractio autem continua hinc nata erit

$$\frac{1}{l3} = 3 + \cfrac{1.2.3}{7 + \cfrac{3.4.3}{11 + \cfrac{5.6.3}{15 + \cfrac{7.8.3}{19 + \text{etc.}}}}}$$

§. 61. Quoniam igitur hic nouam plane methodum aperui, ſeries quaſcunque infinitas in fractiones continuas transformandi, merito equidem mihi videor doctrinam fractionum continuarum haud mediocriter locupletaſſe. His igitur tantum ſubiungam theorema notatu digniſſimum, quo ſupra § 42. fractionem

$$\cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \text{etc.}}}}} = \frac{1}{e-1}$$

transformauimus in hanc:

$$1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \text{etc.}}}} = \frac{1}{e-2}$$

quod multo latius patens ita ſe habet

Theo-

Theorema

§. 62. *Si fuerit*

$$s=\cfrac{aA}{\alpha A+\cfrac{bB}{\beta B+\cfrac{cC}{\gamma C+\text{etc.}}}}$$

erit

$$\frac{bs}{a-\alpha s}=\beta A+\cfrac{cA}{\gamma B+\cfrac{dB}{\delta C+\cfrac{eC}{\varepsilon D+\text{etc.}}}}$$

Demonſtratio

Cum enim ſit

$$s=\cfrac{aA}{\alpha A+\cfrac{bB}{\beta B+\cfrac{cC}{\gamma C+\text{etc.}}}}$$

ſi primam fractionem per A, ſecundam per B, tertiam per C, etc. deprimamus, prodibit

$$s=\cfrac{a}{\alpha+\cfrac{b:A}{\beta+\cfrac{c:B}{\gamma+\cfrac{d:C}{\delta+\text{etc.}}}}}$$

Nunc huius formae ſecundam fractionem ſupra et infra per A, tertiam per B, quartam per C multiplicemus et ita porro, et nanciſcemur hanc formam:

+ etc.

m per B, tertiam per.

ıem ſupra et infra per
multiplicemus et ita

s =

$$s=\cfrac{a}{\alpha+\cfrac{b}{\beta A+\cfrac{cA}{\gamma B+\cfrac{dB}{\delta C+\text{etc.}}}}}$$

quare ſi ſtatuamus

$$t=\beta A+\cfrac{cA}{\gamma B+\cfrac{dB}{\delta C+\text{etc.}}}$$

erit

$$s=\frac{a}{\alpha+\frac{b}{t}}=\frac{at}{\alpha t+b}$$

vnde reperitur $t=\frac{bs}{a-\alpha s}$ q. e. d.

 ME-