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De resolutione fractionum transcendentium in infinitas fractiones simplices

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E592

DE RELATIONE
FRACTIONVM
TRANSCENDENTIVM
IN INFINITAS FRACTIONES SIMPLICES.

§. 1.

Propofita fractione quacunque algebraica $\frac{P}{Q}$, cuius tam numerator P quam denominator Q fint functiones rationales integrae quantitatis z, iam pridem oftendi quomodo ea in fractiones fimplices refolui poffit, quarum denominatores aequentur factoribus fimplicibus denominatoris Q, numeratores vero fint conftantes, fiquidem variabilis z in denominatore Q plures habeat dimenfiones quam in numeratore P. Quin etiam oftendi, quemadmodum pro quolibet factore fimplici denominatoris fractio fimplex refpondens reperiri queat, fine vllo refpectu ad reliquos factores habito. Ita fi conftet, denominatorem Q factorem complecti fimplicem $z-a$, fractio fimplex inde nata, quae erit huius formae: $\frac{\alpha}{z-a}$, facillime hoc modo definitur. Statuatur $\frac{P}{Q} = \frac{\alpha}{z-a} + R$, vbi R complectatur omnes fractiones fimplices ex reliquis oriendas. Multiplicetur vtrinque per $z-a$, vt fiat

$$\frac{P(z-a)}{Q} = \alpha + R(z-a),$$

et

et quia α eft quantitas conftans, ea femper eundem retinebit valorem, quicunque valor variabili z tribuatur; quamobrem fiat vbique $z=a$, vt reliquarum fractionum fimplicium ratio ex calculo excedat, et habebitur $\alpha = \frac{P(z-a)}{Q}$, fiquidem in hac formula fiat $z=a$, tum autem numerator $P(z-a)$ in nihilum abit; verum, quia $z-a$ eft factor denominatoris Q, etiam denominator Q in nihilum abibit. Hinc igitur per regulam confuetam loco numeratoris ac denominatoris fua differentialia fubftituantur, quandoquidem etiamnunc erit $\frac{P\,dz + (z-a)\,dP}{dQ} = \alpha$, fiquidem hic vbique loco z fcribatur a. Ponamus igitur hoc cafu $z=a$ fieri $P=A$ et $\frac{dQ}{dz}=C$, quae ergo quantitates A et C facillime inueniuntur, tum igitur prodibit numerator quaefitus $\alpha=\frac{A}{C}$, ita vt fractio fimplex ex denominatoris factore $z-a$ orienda fit $=\frac{A}{C}(z-a)$, ita vt non opus fit reliquos factores denominatoris noffe. Simili autem modo pro fingulis reliquis factoribus fractiones fimplices refpondentes determinabuntur, quarum omnium fumma aequabitur fractioni propofitae $\frac{P}{Q}$, dummodo variabilis z pauciores habeat dimenfiones in numeratore P quam in denominatore Q.

§. 2. Haec igitur principia fequentes pro denominatore Q eiusmodi affumamus functiones tranfcendentes, quas in infinitos factores fimplices refoluere liceat, id quod euenit, fi eae infinitis cafibus nihilo aequales euadant. Praeterea vero neceffe eft vt omnes ifti factores inter fe fint inaequales, quandoquidem factores aequales peculiarem refolutionem poftulant. Imprimis autem requiritur, vt productum omnium talium factorum ipfam functionem Q penitus exhauriat, quoniam quandoque factores imaginarii fe

inter-

intermiſcere poſſent. Veluti ſi ſumatur $Q = \tang. \phi$, ... vtique omnibus iisdem caſibus euaneſcit, quibus haec functio: $\sin. \phi$, hincque ambae iſtae functiones eosdem factores ſimplices inuoluunt, etiamſi inter ſe neutiquam ſint aequales. Deinde vero numeratorem P ita comparatum eſſe oportet vt cum denominatore Q nullos habeat factores communes. Imprimis autem cauendum eſt, ne quantitas variabilis in numeratore ad totidem vel plures dimenſiones aſſurgat quam in denominatore. Cum autem ea in denominatore ad infinitas dimenſiones aſſurgere ſit cenſenda, iſtud incommodum non erit pertimeſcendum, quamdiu variabilis in numeratore tantum finito dimenſionum numero continetur. Sin autem eius poteſtates etiam in infinitum aſcendant, ſaepenumero difficile erit iudicare, num dimenſionum numerus maior ſit vel minor quam in denominatore. Interim tamen etiam his caſibus fractio propoſita $\frac{P}{Q}$ omnes continebit fractiones ſimplices, ad quas methodus noſtra perducit. Verum euenire poteſt vt praeter eas etiam quasdam partes quaſi integras inuoluat. His igitur praenotatis ſequentes caſus euoluamus.

I. Sumatur $Q = \sin. \phi$, vt fractio reſoluenda ſit $\frac{P}{\sin. \phi}$.

§. 3. Quoniam formula $\sin. \phi$, denotante π ſemiperipheriam circuli cuius radius $= 1$, ſeu angulum duobus rectis aequalem, omnibus his caſibus euaneſcit:

$$\phi = 0,\ \phi = \pm \pi,\ \phi = \pm 2\pi,\ \phi = \pm 3\pi, \text{ etc.}$$

et in genere $\phi = \pm i\pi$, eius factores numero infiniti erunt

$$\phi, (\phi \pm \pi), (\phi \pm 2\pi), (\phi \pm 3\pi) \text{ et in genere } (\phi + i\pi).$$

Aliunde autem certum eſt, hanc formulam $\sin. \phi$ praeter iſtos factores nullos alios ſiue reales ſiue imaginarios inuoluere; cum

cum enim ſit

$$\sin. \phi = \phi - \frac{\phi^3}{1.2.3} + \frac{\phi^5}{1.2.3.4.5} - \text{ etc.}$$

conſtat hanc ſeriem aequari huic producto infinito:

$$\phi(1 - \tfrac{\phi}{\pi})(1 + \tfrac{\phi}{\pi})(1 - \tfrac{\phi}{2\pi})(1 + \tfrac{\phi}{2\pi})(1 - \tfrac{\phi}{3\pi})(1 + \tfrac{\phi}{3\pi}) \text{ etc.}$$

§. 4. Conſideremus igitur noſtri denominatoris $Q = \sin. \phi$ factorem quemcunque $\phi + i\pi$, vbi i denotet omnes plane numeros integros, tam poſitiuos quam negatiuos, ciphra non excepta; ſitque fractio partialis hinc oriunda $\frac{\alpha}{\phi + i\pi}$. Ad eius numeratorem α inueniendum ſtatuatur primo in numeratore P vbique $\phi = i\pi$, ſitque quantitas inde reſultans $= A$; deinde cum ſit $Q = \sin. \phi$, erit $dQ = d\phi \cos. \phi$, ſiue $\frac{dQ}{d\phi} = \cos. \phi$, vbi loco ϕ itidem ſcribi oportet $i\pi$, vt obtineamus C, vnde patet fore $C = \cos. i\pi$, ita vt ſit $C = \pm 1$, vbi ſignum $+$ valebit pro numeris paribus, ſignum vero $-$ pro imparibus numeris loco i aſſumtis. Hoc igitur modo numerator fractionis noſtrae erit $\alpha = \pm A$ ipſaque fractio quaeſita $\pm \frac{A}{\phi - i\pi}$. Hinc autem vlterius progredi non licet, quamdiu numeratorem in genere ſpectamus, vnde eius loco plures valores determinatos accipiamus, ſingulosque in ſequentibus exemplis euoluamus.

1°. Sit $P = 1$ et fractio propoſita $\frac{1}{\sin. \phi}$.

§. 5. Hic igitur ſemper erit $A = 1$ et fractio ſimplex quaecunque $= \frac{\pm 1}{\phi - i\pi}$, vbi ſignum ſuperius valet ſi i numerus par, inferius vero ſi impar. Subſtituamus igitur ſucceſſiue pro i omnes eius valores ordine

$$0, +1, -1, +2, -2, +3, -3, +4, -4 \text{ etc.}$$

 et

et resolutio nostrae fractionis $\frac{1}{\sin.\phi}$ in fractiones simplices ita se habebit:

$$\frac{1}{\sin.\phi} = +\frac{1}{\phi} - \frac{1}{\phi-\pi} - \frac{1}{\phi+\pi} + \frac{1}{\phi-2\pi} + \frac{1}{\phi+2\pi} - \frac{1}{\phi-3\pi} - \frac{1}{\phi+3\pi} + \text{etc.}$$

quae in hanc formam reducatur:

$$\frac{1}{\sin.\phi} = \frac{1}{\phi} + \frac{1}{\pi-\phi} - \frac{1}{\pi+\phi} - \frac{1}{2\pi-\phi} + \frac{1}{2\pi+\phi} + \frac{1}{3\pi-\phi} - \frac{1}{3\pi+\phi} - \text{etc.}$$

Contrahantur post primum terminum bini sequentium, vt nanciscamur hanc seriem:

$$\frac{1}{\sin.\phi} = \frac{1}{\phi} + \frac{2\phi}{\pi\pi-\phi\phi} - \frac{2\phi}{4\pi\pi-\phi\phi} + \frac{2\phi}{9\pi\pi-\phi\phi} - \frac{2\phi}{16\pi\pi-\phi\phi} + \text{etc.}$$

vnde deducitur sequens series memoratu digna

$$\frac{1}{2\phi\sin.\phi} - \frac{1}{2\phi\phi} = \frac{1}{\pi\pi-\phi\phi} - \frac{1}{4\pi\pi-\phi\phi} + \frac{1}{9\pi\pi-\phi\phi} - \frac{1}{16\pi\pi-\phi\phi} \text{ etc.}$$

§. 6. Has quidem series iam olim fusius sum prosecutus: interim tamen pro sequentibus casibus haud inutile erit sequentes transformationes hic repetere. Ponamus igitur primo $\phi = \lambda\pi$, vt littera π ex seriebus elidatur, atque hinc nanciscemur

$$\frac{\pi}{\sin.\lambda\pi} = \frac{1}{\lambda} - \frac{1}{\lambda-1} - \frac{1}{\lambda+1} + \frac{1}{\lambda-2} + \frac{1}{\lambda+2} - \frac{1}{\lambda-3} - \frac{1}{\lambda+3} + \text{etc. et}$$

$$\frac{\pi}{2\lambda\sin.\lambda\pi} - \frac{1}{2\lambda\lambda} = \frac{1}{1-\lambda\lambda} - \frac{1}{4-\lambda\lambda} + \frac{1}{9-\lambda\lambda} - \frac{1}{16-\lambda\lambda} + \text{etc.}$$

atque hinc per differentiationem, spectando λ tanquam quantitatem variabilem, infinitas alias series notatu dignissimas elicere poterimus. Ex priore scilicet nanciscemur

$$\frac{\pi\pi\cos.\lambda\pi}{\sin.\lambda\pi^2} = \frac{1}{\lambda\lambda} - \frac{1}{(\lambda-1)^2} - \frac{1}{(\lambda+1)^2} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} - \frac{1}{(\lambda-3)^2} - \text{etc.}$$

Hinc igitur sequitur, si $\lambda = \frac{1}{2}$ fore

$$0 = 1 - 1 - \tfrac{1}{9} + \tfrac{1}{9} + \tfrac{1}{25} - \tfrac{1}{25} - \text{etc.}$$

quod quidem est manifestum. At si $\lambda = \frac{1}{3}$ erit

$$\frac{2\pi\pi}{27} = 1 - \tfrac{1}{4} - \tfrac{1}{16} + \tfrac{1}{25} + \tfrac{1}{49} - \tfrac{1}{64} - \tfrac{1}{100} + \tfrac{1}{121} + \tfrac{1}{169} - \text{etc.}$$

Si $\lambda = \frac{2}{3}$, oritur series praecedens.

Si $\lambda = \frac{1}{4}$, prodit haec summatio:

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \tfrac{1}{9} - \tfrac{1}{25} + \tfrac{1}{49} + \tfrac{1}{81} - \tfrac{1}{121} - \tfrac{1}{169} + \text{etc.}$$

Quod si denuo differentiemus obtinebitur sequens summatio:

$$\frac{\pi^3}{\sin.\lambda\pi^3} - \frac{\pi^3}{2\sin.\lambda\pi} = \frac{1}{\lambda^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda+1)^3} + \frac{1}{(\lambda-2)^3} + \frac{1}{(\lambda+2)^3} - \text{etc.}$$

sicque continuo vlterius progredi licet.

§. 7. Simili modo etiam alteram formam differentiemus, quae reducta praebet

$$\frac{1}{2\lambda^4} - \frac{\pi}{4\lambda^3\sin.\lambda\pi} - \frac{\pi\pi\cos.\lambda\pi}{4\lambda\lambda\sin.\lambda\pi^2} = \frac{1}{(1-\lambda\lambda)^2} - \frac{1}{(4-\lambda\lambda)^2} + \frac{1}{(9-\lambda\lambda)^2} - \frac{1}{(16-\lambda\lambda)^2} \text{ etc.}$$

Quod si nunc sumamus $\lambda = \frac{1}{2}$, prodibit ista summatio:

$$\tfrac{1}{2} - \tfrac{\pi}{8} = \tfrac{1}{3^2} - \tfrac{1}{15^2} + \tfrac{1}{35^2} - \tfrac{1}{63^2} + \tfrac{1}{99^2} - \text{etc.},$$

quae series prorsus noua omnem attentionem meretur; neque autem opus est hinc nouam differentiationem instituere.

§. 8. Posteriorem autem summationem

$$\frac{\pi}{2\lambda\sin.\lambda\pi} - \frac{1}{2\lambda\lambda} = \frac{1}{1-\lambda\lambda} - \frac{1}{4-\lambda\lambda} + \frac{1}{9-\lambda\lambda} - \frac{1}{16-\lambda\lambda} + \text{etc.}$$

accuratius perpendamus, ac primo quidem cum ea semper debeat esse vera, quicquid pro λ assumatur, sumamus $\lambda = 0$. Quia autem hoc casu membrum sinistrum abit in $\infty - \infty$, tractetur λ vt quantitas quam minima, et cum sit $\lambda\pi = \lambda\pi - \frac{1}{6}\lambda^3\pi^3$, istud membrum euadet

$$\frac{\pi}{2\lambda(\lambda\pi - \frac{1}{6}\lambda^3\pi^3)} - \frac{1}{2\lambda\lambda},$$

quae fractiones ad communem denominatorem perductae dant

$$\frac{1 - 1 + \frac{1}{6}\lambda\lambda\pi\pi}{2\lambda\lambda(1 - \frac{1}{6}\lambda\lambda\pi\pi)} = \frac{\pi\pi}{12 - 2\lambda\lambda\pi\pi}.$$

Nunc igitur facto $\lambda = 0$, eius factor erit $= \frac{\pi\pi}{12}$, series autem ipsa hoc casu euadet

$$1 + \tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} - \tfrac{1}{25} + \text{etc.}$$

cuius summam constat esse $\frac{\pi\pi}{12}$.

§. 9. Manifestum porro est, quoties pro λ accipiatur numerus integer, vnum terminum seriei, ideoque etiam ipsam seriem fieri infinitam, quod egregie conuenit cum summa inuenta, quandoquidem hoc casu fit $\sin. \lambda\pi = 0$. Atque hinc nata est ista quaestio: si ille terminus seriei in infinitum abiens ad sinistram partem transferatur, quanta futura sit reliquorum terminorum summa. Ponamus scilicet esse $\lambda = 1$, et primus seriei terminus euadet infinitus, qui ergo ad sinistram partem translatus dabit

$$\frac{\pi}{2\lambda \sin. \lambda\pi} - \frac{1}{2\lambda\lambda} - \frac{1}{1-\lambda\lambda} = -\tfrac{1}{3} + \tfrac{1}{8} - \tfrac{1}{15} + \tfrac{1}{24} - \tfrac{1}{35} + \tfrac{1}{48} - \text{etc.}$$

Nunc ad valorem huius seriei inuestigandum statuatur λ vnitati tantum proxime aequale, ponendo $\lambda = 1 - \omega$, eritque

$$\sin. \lambda\pi = \sin. (\pi - \pi\omega) = \sin. \pi\omega; \text{ est vero}$$

$$\sin. \pi\omega = \pi\omega - \tfrac{1}{6}\pi^3\omega^3,$$

quo valore substituto prodibit

$$\frac{1}{2(1-\omega)\omega(1 - \frac{1}{6}\pi^2\omega^2)} - \frac{1}{2(1-\omega)^2} - \frac{1}{2\omega - \omega\omega}.$$

Primum autem membrum

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)}, \text{ ob}$$

$$\frac{1}{1-\omega} = 1 + \omega + \omega^2 \text{ et}$$

$$\frac{1}{1 - \frac{1}{6}\pi^2\omega^2} = 1 + \tfrac{1}{6}\pi^2\omega^2,$$

negli-

negligendo potestates ipsius ω quadrato altiores, transmutatur in hanc formam:

$$\tfrac{1}{2\omega}(1 + \omega + \omega^2 + \tfrac{1}{6}\pi\pi\omega\omega);$$

tertium autem membrum

$$-\frac{1}{2\omega(1 - \frac{1}{2}\omega)}, \text{ ob } \frac{1}{1 - \frac{1}{2}\omega} = 1 + \tfrac{1}{2}\omega + \tfrac{1}{4}\omega\omega$$

abit in

$$-\tfrac{1}{2\omega}(1 + \tfrac{1}{2}\omega + \tfrac{1}{4}\omega\omega),$$

vnde primum et tertium membrum simul faciunt

$$\tfrac{1}{2\omega}(\tfrac{1}{2}\omega + \tfrac{3}{4}\omega\omega + \tfrac{1}{6}\pi\pi\omega\omega) = \tfrac{1}{4} + \tfrac{3}{8}\omega + \tfrac{1}{12}\pi\pi\omega$$

qui valor posito $\omega = 0$ fit $= \frac{1}{4}$, vnde secundum membrum, quod erit $-\frac{1}{2}$, iunctum dabit totam summam quaesitam $-\frac{1}{4}$, ita vt sit mutatis signis

$$\tfrac{1}{4} = \tfrac{1}{3} - \tfrac{1}{8} + \tfrac{1}{15} - \tfrac{1}{24} + \tfrac{1}{35} - \tfrac{1}{48} + \tfrac{1}{63} \text{ etc.}$$

cuius ratio est manifesta, cum sit

$$\tfrac{1}{3} = \tfrac{1}{2}(1 - \tfrac{1}{3});\ \tfrac{1}{8} = \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{4});\ \tfrac{1}{15} = \tfrac{1}{2}(\tfrac{1}{3} - \tfrac{1}{5});\ \tfrac{1}{24} = \tfrac{1}{2}(\tfrac{1}{4} - \tfrac{1}{6});\ \text{etc.}$$

his enim valoribus substitutis et sublatis terminis se destruentibus fiet $\frac{1}{4} = \frac{1}{2} - \frac{1}{2 \cdot 2} = \frac{1}{4}$.

§. 10. Circa eandem autem seriem quaestio magis ardua occurrit, qua quaeritur summa seriei, si $\lambda\lambda$ fuerit numerus negatiuus, ideoque λ quantitas imaginaria. Ponatur igitur $\lambda\lambda = -\mu\mu$, siue $\lambda = \mu\sqrt{-1}$, ac series nihilo minus erit realis, scilicet

$$\frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \frac{1}{25+\mu\mu} - \text{etc}$$

cuius ergo summa erit

$$\frac{\pi}{2\mu\sqrt{-1}\sin. \pi\mu\sqrt{-1}} + \frac{1}{2\mu\mu},$$

cuius

cuius ergo valor realis quaeritur, siquidem nullum est dubium quin seriei valor fiat realis.

§. 11. In doctrina angulorum ostendi solet esse

$$\text{sin.}\,\varphi = \frac{e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}}}{2\sqrt{-1}}.$$

Fiat igitur $\varphi = \mu\pi\sqrt{-1}$, eritque

$$\varphi\sqrt{-1} = -\mu\pi \text{ et } -\varphi\pi\sqrt{-1} + = \mu\pi,$$

vnde concluditur

$$\text{sin.}\,\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}$$

vnde summa quaesita erit

$$\frac{\pi}{\mu(e^{-\mu\pi} - e^{+\mu\pi})} + \frac{1}{2\mu\mu}.$$

Erit igitur

$$\frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \frac{1}{25+\mu\mu} - \text{etc.}$$

$$= \frac{1}{2\mu^2} - \frac{\pi}{\mu(e^{\mu\pi} - e^{-\mu\pi})}.$$

2°. Sit $P = \varphi$ et fractio proposita $\frac{\varphi}{\text{sin.}\,\varphi}$.

§. 12. Hic ergo ob numeratorem $P = \varphi$ factor denominatoris primus φ tollitur, quemadmodum etiam nostra resolutio numeratorem ipsi respondentem nihilo praebet aequalem. Hic igitur pro denominatore $\varphi - i\pi$ fit numerator $\frac{i\pi}{\text{cos.}\,i\pi} = \pm i\pi$, vbi signum superius valet si i numerus par, inferius si impar. Quod si ergo fuerit $i = 2n$, fractio inde nata erit $\frac{2n\pi}{\varphi - 2n\pi}$; at si $i = -2n$, fractio erit

$-\frac{2n\pi}{\varphi + 2n\pi}$, at si fuerit $i = 2n - 1$, fractio erit $-\frac{(2n-1)\pi}{\varphi - (2n-1)\pi}$; denique ex $i = -2n - 1$ oritur $\frac{(2n-1)\pi}{\varphi + (2n-1)\pi}$, quocirca series inventa erit

$$\frac{\varphi}{\text{sin.}\,\varphi} = -\frac{\pi}{\varphi - \pi} + \frac{\pi}{\varphi + \pi} + \frac{2\pi}{\varphi - 2\pi} - \frac{2\pi}{\varphi + 2\pi} - \frac{3\pi}{\varphi - 3\pi} + \frac{3\pi}{\varphi + 3\pi} - \frac{4\pi}{\varphi + 4\pi} + \text{etc.}$$

siue

$$\frac{\varphi}{\text{sin.}\,\varphi} = \frac{\pi}{\pi - \varphi} + \frac{\pi}{\pi + \varphi} - \frac{2\pi}{2\pi - \varphi} - \frac{2\pi}{2\pi + \varphi} + \frac{3\pi}{3\pi - \varphi} + \frac{3\pi}{3\pi + \varphi} - \frac{4\pi}{4\pi - \varphi} - \text{etc.}$$

vnde si bini termini in vnum contrahantur, erit

$$\frac{\varphi}{\text{sin.}\,\varphi} = \frac{2\pi\pi}{\pi\pi - \varphi\varphi} - \frac{8\pi\pi}{4\pi\pi - \varphi\varphi} + \frac{18\pi\pi}{9\pi\pi - \varphi\varphi} - \frac{32\pi\pi}{16\pi\pi - \varphi\varphi} + \text{etc.}$$

quae per $2\pi\pi$ diuisa producit hanc summationem:

$$\frac{\varphi}{2\pi\pi\,\text{sin.}\,\varphi} = \frac{1}{\pi\pi - \varphi\varphi} - \frac{4}{4\pi\pi - \varphi\varphi} + \frac{9}{9\pi\pi - \varphi\varphi} - \frac{16}{16\pi\pi - \varphi\varphi} + \text{etc.}$$

Ac si ponatur $\varphi = \lambda\pi$, prodibit

$$\frac{\lambda\pi}{2\,\text{sin.}\,\lambda\pi} = \frac{1}{1 - \lambda\lambda} - \frac{4}{4 - \lambda\lambda} + \frac{9}{9 - \lambda\lambda} - \frac{16}{16 - \lambda\lambda} + \text{etc.}$$

vnde si fuerit $\lambda\lambda = -\mu\mu$, seu $\lambda = \mu\sqrt{-1}$, ob

$$\text{sin.}\,\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}, \text{ erit}$$

$$\frac{\mu\pi}{e^{\mu\pi} - e^{-\mu\pi}} = \frac{1}{1 + \mu\mu} - \frac{1}{4 + \mu\mu} + \frac{1}{9 + \mu\mu} - \frac{1}{16 + \mu\mu} + \text{etc.}$$

atque hinc per differentiationem infinitas alias summationes deducere licebit.

3°. Sit numerator $P = \varphi^2$ et fractio $\frac{\varphi\varphi}{\text{sin.}\,\varphi}$.

§. 13. Pro denominatore igitur $\varphi - i\pi$ numerator erit $\pm ii\pi\pi$, vbi signum superius valet pro i numero pari, inferius vero pro impari. Hinc si loco i successiue scribantur numeri

$$+1 - 1 + 2 - 2 + 3 - 3 + \text{etc.}$$

series

ſeries reſultans erit

$$\frac{\varphi\varphi}{\text{ſin.}\,\varphi} = -\frac{\pi\pi}{\varphi-\pi} - \frac{\pi\pi}{\varphi+\pi} + \frac{4\pi\pi}{\varphi-2\pi} + \frac{4\pi\pi}{\varphi+2\pi} - \frac{9\pi\pi}{\varphi-3\pi} - \frac{9\pi\pi}{\varphi+3\pi} + \text{etc.}$$

ſiue

$$\frac{\varphi\varphi}{\text{ſin.}\,\varphi} = \frac{\pi\pi}{\pi-\varphi} - \frac{\pi\pi}{\pi+\varphi} - \frac{4\pi\pi}{2\pi-\varphi} + \frac{4\pi\pi}{2\pi+\varphi} + \frac{9\pi\pi}{3\pi-\varphi} - \frac{9\pi\pi}{3\pi+\varphi} - \text{etc.}$$

Contractis igitur binis terminis fiet

$$\frac{\varphi\varphi}{\text{ſin.}\,\varphi} = \frac{2\pi\pi\varphi}{\pi\pi-\varphi\varphi} - \frac{8\pi\pi\varphi}{4\pi\pi-\varphi\varphi} + \frac{18\pi\pi\varphi}{9\pi\pi-\varphi\varphi} - \frac{32\pi\pi\varphi}{16\pi\pi-\varphi\varphi}, \text{ ſiue}$$

$$\frac{\varphi}{2\,\text{ſin.}\,\varphi} = \frac{\pi\pi}{\pi\pi-\varphi\varphi} - \frac{4\pi\pi}{4\pi\pi-\varphi\varphi} + \frac{9\pi\pi}{9\pi\pi-\varphi\varphi} - \frac{16\pi\pi}{16\pi\pi-\varphi\varphi} + \text{etc.}$$

Quod ſi nunc quilibet terminus huius ſeriei in duas partes diſcerpatur, quarum prior ſemper eſt 1, binae ſequentes ſeries naſcentur:

$$\frac{\varphi}{2\,\text{ſin.}\,\varphi} = \begin{cases} +1 \quad -1 \quad +1 \quad -1 \quad +1 \quad - \text{etc.} \\ +\frac{\varphi\varphi}{\pi\pi-\varphi\varphi} - \frac{\varphi\varphi}{4\pi\pi-\varphi\varphi} + \frac{\varphi\varphi}{9\pi\pi-\varphi\varphi} - \frac{\varphi\varphi}{16\pi\pi-\varphi\varphi} + \frac{\varphi\varphi}{25\pi\pi-\varphi\varphi} - \text{etc.} \end{cases}$$

Notum autem eſt, ſeriei

$$+1 - 1 + 1 - 1 + 1 - 1 \text{ etc.}$$

ſummam eſſe $= \frac{1}{2}$, qua ad alteram partem translata et per $\varphi\varphi$ diuiſa prodibit

$$\frac{1}{2\varphi\,\text{ſin.}\,\varphi} - \frac{1}{2\varphi\varphi} = \frac{1}{\pi\pi-\varphi\varphi} - \frac{1}{4\pi\pi-\varphi\varphi} + \frac{1}{9\pi\pi-\varphi\varphi} - \text{etc.}$$

quae prorſus conuenit cum ſerie in §. 5 inuenta.

4°. Sit $P = \varphi^\gamma$, denotante γ numerum imparem quemcunque poſitiuum; vt fractio propoſita ſit

$$\frac{\varphi^\gamma}{\text{ſin.}\,\varphi}.$$

§. 14. Cum igitur pro denominatore $\varphi - i\pi$ fiat $A = i^\gamma \pi^\gamma$ et $C = \pm 1$, erit numerator $\pm i^\gamma \pi^\gamma$, vnde cum γ ſit numerus impar, ſigna noſtrorum terminorum eadem

dem lege procedent atque in caſu $P = \varphi$, vbi $\gamma = 1$; vnde ſeries hinc nata erit

$$\frac{\varphi^\gamma}{\text{ſin.}\,\varphi} = \frac{\pi^\gamma}{\pi-\varphi} + \frac{\pi^\gamma}{\pi+\varphi} - \frac{2^\gamma\pi^\gamma}{2\pi-\varphi} - \frac{2^\gamma\pi^\gamma}{2\pi+\varphi} + \frac{3^\gamma\pi^\gamma}{3\pi-\varphi} + \frac{3^\gamma\pi^\gamma}{3\pi+\varphi} - \text{etc.}$$

quae per π^γ diuiſa praebet

$$\frac{\varphi^\gamma}{\pi^\gamma\,\text{ſin.}\,\varphi} = \frac{1}{\pi-\varphi} + \frac{1}{\pi+\varphi} - \frac{2^\gamma}{2\pi-\varphi} - \frac{2^\gamma}{2\pi+\varphi} + \frac{3^\gamma}{3\pi-\varphi} + \frac{3^\gamma}{3\pi+\varphi} - \text{etc.}$$

et binis terminis contractis erit

$$\frac{\varphi^\gamma}{2\pi^{\gamma+1}\,\text{ſin.}\,\varphi} = \frac{1}{\pi\pi-\varphi\varphi} - \frac{2^\gamma}{4\pi\pi-\varphi\varphi} + \frac{3^\gamma}{9\pi\pi-\varphi\varphi} - \frac{4^\gamma}{16\pi\pi-\varphi\varphi} + \text{etc.}$$

Statuamus nunc $\varphi = \lambda\pi$ eritque

$$\frac{\lambda^\gamma \pi}{2\,\text{ſin.}\,\varphi\pi} = \frac{1}{1-\lambda\lambda} - \frac{2^\gamma}{4-\lambda\lambda} + \frac{3^\gamma}{9-\lambda\lambda} - \frac{4^\gamma}{16-\lambda\lambda} + \text{etc.}$$

§. 15. Hinc ſi fuerit $\lambda = \mu\sqrt{-1}$, erit vt ante

$$\text{ſin.}\,\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}.$$

Pro valore autem poteſtatis λ^γ duos caſus euolui oportet, prouti fuerit $\gamma = 4n+1$ vel $\gamma = 4n-1$. Priore caſu erit

$$\lambda^\gamma = (\mu\sqrt{-1})^{4n+1} = (\mu\sqrt{-1})^{4n} \times \mu\sqrt{-1}.$$

 Eſt

Est vero $(\mu\sqrt{-1})^{4n} = \mu^{4n}$, vnde erit $\lambda^{\gamma} = \mu^{4n+1}\sqrt{-1}$, hincque prodit sequens summatio realis:

$$\frac{\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1^{4n+1}}{1+\mu\mu} - \frac{2^{4n+1}}{4+\mu\mu} + \frac{3^{4n+1}}{9+\mu\mu} - \frac{4^{4n+1}}{16+\mu\mu} + \text{etc.}$$

Altero autem casu, quo $\gamma = 4n - 1$, prius membrum capi debet negatiue, eritque

$$-\frac{\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1}{1+\mu\mu} - \frac{2^{4n-1}}{4+\mu\mu} + \frac{4^{4n-1}}{9+\mu\mu} - \text{etc.}$$

Has autem summationes facile patet veras esse non posse, nisi γ sit numerus integer impar et quidem positiuus.

5. Sit numerator $P = \varphi^{\delta}$, denotante δ numerum parem positiuum quemcunque, et fractio $\frac{\varphi^{\delta}}{\text{fin.}\,\varphi}$.

§. 15. Pro denominatore ergo $\varphi - i\pi$ numerator erit $\pm i^{\delta}\pi^{\delta}$, ambiguitate signorum eandem legem tenente. Hoc igitur casu ratio signorum perinde se habebit ac casu $P = \varphi\,\varphi$, eritque idcirco.

$$\frac{\varphi^{\delta}}{\text{fin.}\,\varphi} = \frac{\pi^{\delta}}{\pi-\varphi} - \frac{\pi^{\delta}}{\pi+\varphi} - \frac{2^{\delta}\pi^{\delta}}{2\pi-\varphi} + \frac{2^{\delta}\pi^{\delta}}{2\pi+\varphi} + \frac{3^{\delta}\pi^{\delta}}{3\pi-\varphi} - \frac{3^{\delta}\pi^{\delta}}{3\pi+\varphi} + \text{etc.}$$

Quare si ponamus $\varphi = \lambda\pi$ erit haec series

$$\frac{\lambda^{\delta}\pi}{\text{fin.}\,\lambda\pi} = \frac{1}{1-\lambda} - \frac{1}{1+\lambda} - \frac{2^{\delta}}{2-\lambda} + \frac{2^{\delta}}{2+\lambda} + \frac{3^{\delta}}{3-\lambda} - \frac{3^{\delta}}{3+\lambda} - \text{etc.}$$

hinc binis terminis in vnum contrahendis fiet

$$\frac{\lambda^{\delta-1}\pi}{2\,\text{fin.}\,\lambda\pi} = \frac{1^{\delta}}{1-\lambda\lambda} - \frac{2^{\delta}}{4-\lambda\lambda} + \frac{3^{\delta}}{3-\lambda\lambda} - \frac{4^{\delta}}{4-\lambda\lambda} + \text{etc.}$$

§. 16.

§. 16. Statuamus nunc etiam $\lambda = \mu\sqrt{-1}$, vt sit

$$\text{fin.}\,\lambda\pi = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}.$$

Pro valore autem ipsius $\lambda^{\delta-1}$ duos iterum casus euolui oportet, prout fuerit vel $\delta = 4n$, vel $\delta = 4n + 2$. Priore casu, quo $\delta = 4n$, erit $\lambda^{4n} = \mu^{4n}$, ideoque $\lambda^{4n-1} = \frac{\mu^{4n-1}}{\sqrt{-1}}$; atque hinc orietur ista summatio:

$$\frac{-\mu^{4n-1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1^{4n}}{1+\mu\mu} - \frac{2^{4n}}{4+\mu\mu} + \frac{3^{4n}}{9+\mu\mu} - \frac{4^{4n}}{16+\mu\mu} + \text{etc.}$$

Pro altero autem casu $\delta = 4n + 2$ summatio ita se habebit:

$$\frac{+\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1^{4n+2}}{1+\mu\mu} - \frac{2^{4n+2}}{4+\mu\mu} + \frac{3^{4n+2}}{9+\mu\mu} - \text{etc.}$$

§. 17. Hae autem summationes eatenus tantum veritati erunt consentaneae, quatenus pro exponentibus γ et δ numeri integri, prouti sunt definiti, accipiantur, nihilque impedit quo minus quantumuis magni assumantur. Cum enim denominator

$$\text{fin.}\,\varphi = \varphi - \tfrac{1}{6}\varphi^{3} + \tfrac{1}{120}\varphi^{5} - \text{etc.}$$

ad dimensiones infinitas ipsius φ assurgat, dummodo maxima potestas in numeratore non fiat infinita, resolutio in fractiones semper ad veritatem perducit. Sin autem exponentes illi non essent integri positiui, sed fracti, vel adeo negatiui, resolutio in fractiones partiales locum plane habere nequit. Quamobrem si loco numeratoris P eiusmodi functiones ipsius φ statuamus, quae etiam ad infinitum dimensionum numerum adsurgant, tum de summa inuenta non amplius eri-

 mus

mus certi. Verum fieri poteſt, vt ad fractiones partiales inuentas inſuper quaedam partes integrae adiici debeant. Huiusmodi igitur caſus aliquos euoluamus.

6°. Sit numerator $P = \cos.\varphi$ et fractio $= \frac{\cos.\varphi}{\sin.\varphi}$.

§. 18. Cum ſit

$$\cos.\varphi = 1 - \tfrac{1}{2}\varphi\varphi + \tfrac{1}{24}\varphi^4 - \tfrac{1}{720}\varphi^6 + \text{etc.}$$

poteſtates ipſius φ in numeratore aeque in infinitum exſurgunt atque in denominatore; vnde fieri poſſet, vt haec fractio partem integram inuolueret, quae cum reperiatur ſi ſumatur $\varphi = \infty$, foret iſta pars integra $= \frac{\cos.\infty}{\sin.\infty} = \cot.\infty$, quae autem in ſe prorſus eſt indeterminata. Interim tamen, quia totidem caſibus euadere poteſt negatiua atque poſitiua, medium ſumendo valor recte videri poteſt $= 0$; ceterum dubium per ſequentem euolutionem tolletur. Cum pro denominatore $\varphi - i\pi$ fiat $A = \cos.i\pi$ et $C = \cos.i\pi$; erit numerator huius fractionis $= 1$; hinc ergo naſcetur ſequens ſeries:

$$\frac{\cos.\varphi}{\sin.\varphi} = \frac{1}{\varphi} + \frac{1}{\varphi-\pi} + \frac{1}{\varphi+\pi} + \frac{1}{\varphi-2\pi} + \frac{1}{\varphi+2\pi} + \text{etc. ſiue.}$$

$$\text{cotag.}\,\varphi = \frac{1}{\varphi} - \frac{1}{\pi-\varphi} + \frac{1}{\pi+\varphi} - \frac{1}{2\pi-\varphi} + \frac{1}{2\pi+\varphi} - \text{etc.}$$

Poſito igitur $\varphi = \lambda\pi$, haec ſeries induet hanc formam:

$$\pi\cot.\lambda\pi = \frac{1}{\lambda} - \frac{1}{1-\lambda} + \frac{1}{1+\lambda} - \frac{1}{2-\lambda} + \frac{1}{2+\lambda} - \frac{1}{3-\lambda} + \text{etc.}$$

quae ſummatio an vera ſit per caſus inueſtigemus. Ac primo quidem ſi λ denotet numerum integrum, veritas confirmatur; ſemper enim aliquis ſeriei terminus ſit infinitus; ſumma vero quoque ſit infinita. Sumamus autem $\lambda = \frac{1}{2}$, erit $\pi\cot.\frac{\pi}{2} = 0$, ipſa autem ſeries prodit

$$\tfrac{2}{1} - \tfrac{2}{1} + \tfrac{2}{3} - \tfrac{2}{3} + \tfrac{2}{5} - \tfrac{2}{5} + \text{etc.}$$

vbi

ones partiales
.diici debeant.
$\varphi = \frac{\cos.\varphi}{\sin.\varphi}$.
etc.
.cum exſurgunt
fractio partem
ſumatur $\varphi = \infty$,
e autem in ſe
ia totidem ca-
medium ſumen-
dubium per
denominato-
erit numera-
ſequens ſeries:
$\frac{1}{+2\pi}$ + etc. ſiue.
$\frac{2}{2\pi+\varphi}$ − etc.
ic formam:
− $\frac{1}{3-\lambda}$ + etc.
mus. Ac primo
ritas confirma-
ſit infinitus;
tem $\lambda = \frac{1}{2}$, erit
vbi

vbi omnes termini ſe manifeſto deſtruunt. Sumamus autem inſuper $\lambda = \frac{1}{4}$, prodibitque

$$\pi = \tfrac{4}{1} - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \tfrac{4}{9} - \tfrac{4}{11} + \text{etc.}$$

quae eſt ſeries notiſſima Leibniziana. Sicque omne dubium circa veritatem huius ſummationis euaneſcit.

§. 19. Contrahamus binos terminos, primo excepto, in ſingulos, et obtinebimus

$$\pi\cot.\lambda\pi = \frac{1}{\lambda} - \frac{2\lambda}{1-\lambda\lambda} - \frac{2\lambda}{4-\lambda\lambda} - \frac{2\lambda}{9-\lambda\lambda} - \frac{2\lambda}{16-\lambda\lambda} - \text{etc.}$$

quae ſeries reducitur ad hanc formam:

$$\frac{1}{2\lambda\lambda} - \frac{\pi\cot.\lambda\pi}{2\lambda} = \frac{1}{1-\lambda\lambda} + \frac{1}{4-\lambda\lambda} + \frac{1}{9-\lambda\lambda} + \frac{1}{16-\lambda\lambda} + \text{etc.}$$

Quod ſi hic iterum ſtatuamus $\lambda = \mu\sqrt{-1}$, ob

$$\cos.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} + e^{\mu\pi}}{2} \text{ et}$$

$$\sin.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}},$$

haec obtinebitur ſummatio:

$$-\frac{1}{2\mu\mu} + \frac{\pi(e^{+\mu\pi} + e^{-\mu\pi})}{(e^{\mu\pi} - e^{-\mu\pi})2\mu} = \frac{1}{1+\mu\mu} + \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} + \frac{1}{16+\mu\mu} + \text{etc.}$$

Nunc autem per ſe eſt manifeſtum, per differentiationem ſimili modo vt ſupra infinitas alias ſummationes obtineri poſſe.

 II.

II. Sumatur $Q = \cos.\zeta - \cos.\varphi$, vt fractio resoluenda fit $\frac{P}{\cos.\zeta - \cos.\varphi}$.

§. 20. Cum fit denominator $Q = \cos.\zeta - \cos.\varphi$, vbi angulus ζ vt datus et constans spectatur, is sequentibus casibus euanescit,

$$\varphi = \pm \zeta;\ \varphi = \pm 2\pi \pm \zeta;\ \varphi = \pm 4\pi \pm \zeta;$$
$$\varphi = \pm 6\pi \pm \zeta;\ \varphi = \pm 8\pi \pm \zeta;\ \text{etc.}$$

ideoque in genere $\varphi = \pm i\pi \pm \zeta$, vbi i denotat omnes numeros pares tam negatiuos quam positiuos; vnde denominatores fractionum simplicium quas quaerimus erunt

$$\varphi - \zeta,\ \varphi + \zeta,\ \varphi - 2\pi - \zeta,\ \varphi - 2\pi + \zeta,\ \varphi + 2\pi - \zeta,\ \varphi + 2\pi + \zeta,\ \text{etc.}$$

hocque modo omnes fractiones simplices reperiemus, quarum omnium summa aequalis esse debet fractioni propositae $\frac{P}{\cos.\zeta - \cos.\varphi}$.

§. 21. Consideremus nunc primo denominatorem simplicem in genere $\varphi - i\pi - \zeta$, ac posito $\varphi = i\pi + \zeta$ abeat numerator P in A. Deinde cum ex denominatore fiat $\frac{dQ}{d\varphi} = \sin.\varphi$, erit $C = \sin.(i\pi + \zeta) = \sin.\zeta$, vnde numerator huius fractionis erit $\frac{A}{C} = \frac{A}{\sin.\zeta}$, ideoque fractio hinc nata $\frac{A}{\sin.\zeta(\varphi - i\pi - \zeta)}$. At vero pro denominatore $\varphi - i\pi + \zeta$, si in numeratore P ponatur $\varphi = i\pi - \zeta$, prodit quantitas B; ex denominatore autem fiet

$$C = \sin.(i\pi - \zeta) = -\sin.\zeta,$$

vnde orietur ista fractio: $-\frac{B}{\sin.\zeta(\varphi - i\pi + \zeta)}$. Nunc igitur tantum opus est vt loco i successiue omnes numeri pares tam positiui quam negatiui substituantur.

oluenda
− cos. φ,
quentibus
± ζ;
omnes nu-
lenomina-
π+ζ, etc.
ius, qua-
propositae
minatorem
π + ζ ab-
natore fiat
numerator
hinc nata
− iπ + ζ,
t quantitas
Nunc igi-
numeri pa-
1°.

1°. Sit numerator $P = 1$ et fractio proposita $\frac{1}{\cos.\zeta - \cos.\varphi}$.

§. 22. Pro binis igitur formulis generalibus erit tam $A = 1$ quam $B = 1$, vnde istae fractiones generales erunt

$$\frac{1}{\sin.\zeta(\varphi - i\pi - \zeta)} - \frac{1}{\sin.\zeta(\varphi - i\pi + \zeta)} = \frac{2\zeta}{\sin.\zeta((\varphi - i\pi)^2 - \zeta\zeta)},$$

consequenter hinc deducemus sequentem summationem:

$$\frac{1}{\cos.\zeta - \cos.\varphi} = \frac{2\zeta}{\sin.\zeta(\varphi\varphi - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi - 2\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi + 2\pi)^2 - \zeta\zeta)}$$
$$+ \frac{2\zeta}{\sin.\zeta((\varphi - 4\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi + 4\pi)^2 - \zeta\zeta)} + \text{etc.}$$

siue habebimus

$$\frac{\sin.\zeta}{2\zeta(\cos.\zeta - \cos.\varphi)} = \frac{1}{\varphi\varphi - \zeta\zeta} + \frac{1}{(\varphi - 2\pi)^2 - \zeta\zeta} + \frac{1}{(\varphi + 2\pi)^2 - \zeta\zeta}$$
$$+ \frac{1}{(\varphi - 4\pi)^2 - \zeta\zeta} + \frac{1}{(\varphi + 4\pi)^2 - \zeta\zeta} + \text{etc.}$$

§. 23. Quod si ergo fuerit $\zeta = 0$, erit

$$\frac{1}{2 - 2\cos.\varphi} = \frac{1}{\varphi^2} + \frac{1}{(\varphi - 2\pi)^2} + \frac{1}{(\varphi + 2\pi)^2} + \frac{1}{(\varphi - 4\pi)^2} + \frac{1}{(\varphi + 4\pi)^2} + \text{etc.}$$

Sit nunc porro $\varphi = \frac{\pi}{2}$, erit haec summatio:

$$\frac{\pi\pi}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \text{etc.}$$

vti quidem satis constat. Ponamus porro $\varphi = \pi$, prodibitque haec summatio:

$$\frac{\pi\pi}{4} = 1 + 1 + \frac{1}{9} + \frac{1}{9} + \frac{1}{25} + \frac{1}{25} + \frac{1}{49} + \frac{1}{49} + \text{etc.}$$

quae series cum praecedente congruit. Sin autem ponamus $\varphi = \lambda\pi$, erit

$$\frac{\pi\pi}{2(1 - \cos.\lambda\pi)} = \frac{1}{\lambda\lambda} + \frac{1}{(\lambda - 2)^2} + \frac{1}{(\lambda + 2)^2} + \frac{1}{(\lambda - 4)^2} + \frac{1}{(\lambda + 4)^2} + \text{etc.}$$

quae summa etiam est $\frac{\pi\pi}{4(\sin.\frac{1}{2}\lambda\pi)^2}$.

§. 24. Ponamus autem in genere $\zeta = \alpha\pi$ et $\Phi = \lambda\pi$, vt obtineatur ista summatio:

$$\frac{\pi \sin. \alpha\pi}{2\alpha(\cos. \alpha\pi - \cos. \lambda\pi)} = \frac{1}{\lambda\lambda - \alpha\alpha} + \frac{1}{(\lambda-2)^2 - \alpha\alpha} + \frac{1}{(\lambda+2)^2 - \alpha\alpha} + \frac{1}{(\lambda-4)^2 - \alpha\alpha} + \frac{1}{(\lambda+4)^2 - \alpha\alpha} + \text{etc.}$$

Quod si iam α fuerit quantitas imaginaria, siue $\alpha = \beta\surd-1$, summatio haec erit

$$\frac{\pi(e^{\beta\pi} - e^{-\beta\pi})}{2\alpha((e^{\beta\pi} + e^{-\beta\pi}) - 2\cos. \lambda\pi)} = \frac{1}{\lambda\lambda + \beta\beta} + \frac{1}{(\lambda-2)^2 + \beta\beta} + \frac{1}{(\lambda+2)^2 + \beta\beta} + \frac{1}{(\lambda-4)^2 + \beta\beta} + \frac{1}{(\lambda+4)^2 + \beta\beta} + \text{etc.}$$

§. 25. Hinc si proponatur haec fractio in seriem resoluenda: $\frac{1}{a - \cos. \Phi}$, siue $\frac{1}{a - \cos. \lambda\pi}$, duos casus considerari oportet, prouti a fuerit vel vnitate minor vel maior. Sit $a < 1$, vt fieri queat $a = \cos. \alpha\pi$, vnde fit $\alpha = \frac{A \cos. a}{\pi}$, atque inuento α reperietur

$$\frac{1}{a - \cos. \lambda\pi} = \frac{2\alpha}{\pi\surd(1-aa)} \left(\frac{1}{\lambda\lambda - \alpha\alpha} + \frac{1}{(\lambda-2)^2 - \alpha\alpha} + \frac{1}{(\lambda+2)^2 - \alpha\alpha} + \frac{1}{(\lambda-4)^2 - \alpha\alpha}\right) + \text{etc.})$$

Sin autem fuerit $a > 1$, quaeri debet β, vt fiat

$$\frac{e^{\beta\pi} + e^{-\beta\pi}}{2} = a.$$

Hinc ergo fiet $e^{+2\beta\pi} + 1 = 2ae^{\beta\pi}$, vnde radice extracta reperitur $e^{\beta\pi} = a + \surd(aa - 1)$ hincque $e^{-\beta\pi} = a - \surd(aa - 1)$, vnde porro fiet

$$\beta\pi = l(a + \surd(aa - 1)), \text{ ergo}$$

$$\beta = \tfrac{1}{\pi} l(a + \surd(aa - 1)).$$

Inuento igitur hoc numero β postrema formula praebet hanc seriem:

$$\frac{\pi\surd(aa-1)}{2\beta(a - \cos. \lambda\pi)} = \frac{1}{\lambda\lambda + \beta\beta} + \frac{1}{(\lambda-2)^2 + \beta\beta} + \frac{1}{(\lambda+2)^2 + \beta\beta} + \frac{1}{(\lambda-4)^2 + \beta\beta} + \text{etc.}$$

conse-

et $\Phi = \lambda\pi$, etc. $= \beta\surd-1$, in seriem rari opor- Sit $a < 1$, atque in-) + etc.) xtracta re- $\surd(aa - 1)$, a praebet + etc. conse-

consequenter habebimus

$$\frac{1}{a - \cos. \lambda\pi} = \frac{2\beta}{\pi\surd(aa-1)} \left(\frac{1}{\lambda\lambda + \beta\beta} + \frac{1}{(\lambda-2)^2 + \beta\beta} + \frac{1}{(\lambda+2)^2 + \beta\beta} + \frac{1}{(\lambda-4)^2 + \beta\beta} + \text{etc.}\right)$$

casu autem medio, quo $a = 1$, fit $\alpha = 0$; tum vero ponatur $a = 1 - \omega$, eritque

$$A \cos. (1 - \omega) = A \sin. \surd(2\omega - \omega\omega) = \surd(2\omega - \omega\omega).$$

Est vero etiam

$$\surd(1 - aa) = \surd(2\omega - \omega\omega),$$

vnde pro hoc casu seriei summatio erit

$$\frac{1}{1 - \cos. \lambda\pi} = \frac{2}{\pi\pi} \left(\frac{1}{\lambda\lambda} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} + \frac{1}{(\lambda-4)^2} + \frac{1}{(\lambda+4)^2} + \text{etc.}\right)$$

Cum igitur sit

$$1 - \cos. \lambda\pi = 2 \sin. \tfrac{1}{2}\lambda\pi^2,$$

habebimus hanc summationem:

$$\frac{\pi\pi}{4 \sin. \frac{1}{2}\lambda\pi^2} = \frac{1}{\lambda\lambda} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} + \frac{1}{(\lambda-4)^2} + \text{etc.}$$

quae series iam supra § 23 est inuenta.

2°. Sit nunc $P = \sin. \Phi$ et fractio proposita $\frac{\sin. \Phi}{\cos. \zeta - \cos. \Phi}$.

§. 26. Cum igitur sit $P = \sin. \Phi$ sumto, $\Phi = i\pi + \zeta$ erit $A = \sin. (i\pi + \zeta) = \sin. \zeta$; at posito $\Phi = i\pi - \zeta$ prodit $B = -\sin. \zeta$; hinc binae fractiones inde resultantes erunt

$$\frac{1}{\Phi - i\pi - \zeta} + \frac{1}{\Phi - i\pi + \zeta} = \frac{2\Phi - 2i\pi}{(\Phi - i\pi)^2 - \zeta\zeta}.$$

Quare si loco i successiue omnes eius valores scribamus, nanciscemur sequentem seriem:

$$\frac{\sin. \Phi}{\cos. \zeta - \cos. \Phi} = \frac{2\Phi}{\Phi\Phi - \zeta\zeta} + \frac{2(\Phi - 2\pi)}{(\Phi - 2\pi)^2 - \zeta\zeta} + \frac{2(\Phi + 2\pi)}{(\Phi + 2\pi)^2 - \zeta\zeta} + \frac{2(\Phi - 4\pi)}{(\Phi - 4\pi)^2 - \zeta\zeta} + \text{etc.}$$

 siue

siue

$$\frac{\sin.\phi}{2(\cos.\zeta-\cos.\phi)}=\frac{\phi}{\phi\phi-\zeta\zeta}+\frac{\phi-2\pi}{(\phi-2\pi)^2-\zeta\zeta}+\frac{\phi+2\pi}{(\phi+2\pi)^2-\zeta\zeta}+\frac{\phi-4\pi}{(\phi-4\pi)^2-\zeta\zeta}+\text{etc.}$$

§. 27. Hinc si fuerit $\zeta=0$, erit

$$\frac{\sin.\phi}{2(1-\cos.\phi)}=\frac{1}{\phi}+\frac{1}{\phi-2\pi}+\frac{1}{\phi+2\pi}+\frac{1}{\phi-4\pi}+\frac{1}{\phi+4\pi}+\text{etc.}$$

cuius igitur seriei summa est $\frac{1}{2}\cot.\frac{1}{2}\phi$. Hinc si ponamus $\phi=\lambda\pi$, erit

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi=\frac{1}{\lambda}+\frac{1}{\lambda-2}+\frac{1}{\lambda+2}+\frac{1}{\lambda-4}+\frac{1}{\lambda+4}+\text{etc.}$$

et contrahendis binis terminis

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi=\frac{1}{\lambda}+\frac{2\lambda}{\lambda\lambda-4}+\frac{2\lambda}{\lambda\lambda-16}+\frac{2\lambda}{\lambda\lambda-36}+\text{etc.}$$

hincque

$$\frac{1}{2\lambda\lambda}-\frac{\pi\cot.\frac{1}{2}\lambda\pi}{4\lambda}=\frac{1}{4-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\frac{1}{36-\lambda\lambda}+\text{etc.}$$

Quod si hic loco λ scribamus 2λ, habebimus

$$\frac{1}{8\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{8\lambda}=\frac{1}{4-4\lambda\lambda}+\frac{1}{16-4\lambda\lambda}+\frac{1}{36-4\lambda\lambda}+\text{etc.}$$

siue

$$\frac{1}{2\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{2\lambda}=\frac{1}{1-\lambda\lambda}+\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\text{etc.}$$

quae series est plane eadem, quam supra § 19 inuenimus.

§. 28. Ponamus nunc vt supra $\zeta=\alpha\pi$ et $\phi=\lambda\pi$, vt obtineatur sequens series:

$$\frac{\pi\sin.\lambda\pi}{2(\cos.\alpha\pi-\cos.\lambda\pi)}=\frac{\lambda}{(\lambda\lambda-\alpha\alpha)}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\frac{\lambda-4}{(\lambda-4)^2-\alpha\alpha}+\text{etc.}$$

Sin autem hic ponatur $\alpha=\beta\sqrt{-1}$, ista series sequentem induet formam:

$$\frac{\pi\sin.\lambda\pi}{e^{\beta\pi}+e^{-\beta\pi}-2\cos.\lambda\pi}=\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}$$
$$+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\frac{\lambda-4}{(\lambda-4)^2+\beta\beta}+\text{etc.}$$

§. 29.

§. 29. Quod si igitur proposita fuerit haec fractio: $\frac{\sin.\phi}{a-\cos.\phi}$ siue $\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}$, iterum duos casus euolui conuenit, alterum quo $a<1$, alterum quo $a>1$. Priore quidem casu, quo $a<1$, statuatur $\cos.\alpha\pi=a$, vnde fit $\alpha=\frac{\text{Arc.cos.}a}{\pi}$, quo inuento erit

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}=\frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda-\alpha\alpha}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\text{etc.}\right)$$

Sin autem $a>1$, quaeri debet β, ita vt sit vt ante

$$\beta=\tfrac{1}{\pi}l(a+\sqrt{(aa-1)})$$

quo valore inuento erit

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}=\frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\text{etc.}\right)$$

Sin autem fuerit $a=1$, tum fit tam $\alpha=0$ quam $\beta=0$, eademque series resultat, quam supra ex casu $\zeta=0$ elicuimus. Hinc ergo si sumatur $\lambda=\frac{1}{2}$ prodibit series

$$\frac{\pi}{2\cos.\alpha\pi}=\frac{2}{1-4\alpha\alpha}-\frac{6}{9-4\alpha\alpha}+\frac{10}{25-4\alpha\alpha}-\frac{14}{49-4\alpha\alpha}+\text{etc.}$$

vel etiam haec:

$$\frac{\pi}{e^{\beta\pi}-e^{-\beta\pi}}=\frac{2}{1+4\beta\beta}-\frac{6}{9+4\beta\beta}+\frac{10}{25+4\beta\beta}-\frac{14}{49+4\beta\beta}+\text{etc.}$$

Ceterum per se intelligitur, per differentiationem plurimas alias series formari posse.

III. Sit fractio resoluenda $\frac{1}{\cos.\phi-\cos.2\Phi}$.

§. 30. Ante omnia igitur hic quaeri debet, quibusnam casibus iste denominator euanescat. Cum igitur in genere sit $\cos.\phi=\cos.(i\pi\pm\Phi)$, denotante i numerum parem, similique modo $\cos.2\phi=\cos.(i'\pi+2\Phi)$, habebimus $i\pi\pm\phi=i'\pi\pm2\Phi$, vnde ob ambiguitatem signorum sequentes casus eruuntur: $\phi=i\pi$, $\phi=\frac{i\pi}{3}$. Hic autem probe be eſt obſeruandum, caſus priores bis occurrere, ſeu factores hinc natos $\phi - i\pi$ bis eſſe collocandos, ita vt factor denominatoris ſit $(\phi - i\pi)^2$. Quod cum minus clare appareat, ita oſtendamus: quoniam in genere eſt

$$\operatorname{cof.} a - \operatorname{cof.} b = 2 \operatorname{fin.} \tfrac{a+b}{2}, \operatorname{fin.} \tfrac{b-a}{2}$$

erit noſter denominator $2 \operatorname{fin.} \frac{1}{2}\phi \operatorname{fin.} \frac{3}{2}\phi$, qui igitur euaneſcit tam quando $\operatorname{fin.} \frac{1}{2}\phi = 0$ quam quando $\operatorname{fin.} \frac{3}{2}\phi = 0$. Fit autem $\operatorname{fin.} \frac{\phi}{2} = 0$ quoties $\frac{\phi}{2} = i\pi$, denotante i omnes numeros integros, ideoque $\phi = 2i\pi$. Similique modo $\operatorname{fin.} \frac{3\phi}{2}$ euaneſcit, ſi $\frac{3\phi}{2} = i\pi$ ideoque $\phi = \frac{2i\pi}{3}$, quae poſterior formula, quoties $\frac{i}{3}$ eſt numerus integer, priores caſus ſuppeditat; ſicque manifeſtum eſt, in factoribus occurrere omnia quadrata $(\phi - i\pi)^2$. Reliqui vero factores $\phi - \frac{2i\pi}{3}$, quando i per 3 non eſt diuiſibile, erunt ſimplices.

§. 31. Cum igitur formula $(\phi - 2i\pi)^2$ ſit factor noſtri denominatoris $\operatorname{cof.} \phi - \operatorname{cof.} 2\phi$, ſecundum regulam pro huiusmodi caſibus ſtatuamus

$$\frac{1}{\operatorname{cof.}\phi - \operatorname{cof.} 2\phi} = \frac{\alpha}{(\phi - 2i\pi)^2} + \frac{\beta}{\phi - 2i\pi} + R,$$

vbi R complectitur omnes reliquas fractiones. Nunc vtrinque multiplicemus per $(\phi - 2i\pi)^2$ et habebimus

$$\frac{(\phi - 2i\pi)^2}{\operatorname{cof.}\phi - \operatorname{cof.} 2\phi} = \alpha + \beta(\phi - 2i\pi) + R(\phi - 2i\pi)^2$$

Faciamus $\phi = 2i\pi$ fietque $\alpha = \frac{(\phi - 2i\pi)^2}{\operatorname{cof.}\phi - \operatorname{cof.} 2\phi}$, cuius fractionis numerator et denominator euaneſcent, hinc differentialibus ſubſtitutis fiet $\alpha = \frac{2(\phi - 2i\pi)}{-\operatorname{fin.}\phi + 2\operatorname{fin.} 2\phi}$, vbi cum numerator et denominator iterum euaneſcant, denuo eorum loco differentialia ſcribantur eritque $\alpha = \frac{2}{-\operatorname{cof.}\phi + 4\operatorname{cof.} 2\phi}$. Nunc igitur poſito $\phi = 2i\pi$ reperietur $\alpha = \frac{2}{3}$.

§. 32.

occurrere, ſeu facto-
andos, ita vt factor
m minus clare ap-
nere eſt
, qui igitur euaneſcit
o fin. $\frac{3}{2}\phi = 0$. Fit
tante i omnes nume-
milique modo fin. $\frac{3\phi}{2}$
, quae poſterior for-
priores caſus ſuppe-
us occurrere omnia
ctores $\phi - \frac{2i\pi}{3}$, quan-
implices.
$\phi - 2i\pi)^2$ ſit factor
, ſecundum regulam
$+ R$,
ctiones. Nunc vtrin-
t habebimus
$) + R(\phi - 2i\pi)^2$
cuius fractionis
hinc differentialibus ſub
numerator et denomina-
loco differentialia ſcri-
Nunc igitur poſito
§. 32.

§. 32. Iam in aequatione

$$\frac{(\phi - 2i\pi)^2}{\operatorname{cof.}\phi - \operatorname{cof.} 2\phi} = \alpha + \beta(\phi - 2i\pi) + R(\phi - 2i\pi)^2$$

terminus $\alpha = \frac{2}{3}$ ad alteram partem transferatur et ad eandem denominationem reducatur et reſultabit haec aequatio:

$$\frac{(\phi - 2i\pi)^2 - \frac{2}{3}(\operatorname{cof.}\phi - \operatorname{cof.} 2\phi)}{\operatorname{cof.}\phi - \operatorname{cof.} 2\phi} = \beta(\phi - 2i\pi) + R(\phi - 2i\pi)^2$$

vnde per $\phi - 2i\pi$ diuidendo fiet

$$\frac{(\phi - 2i\pi)^2 - \frac{2}{3}(\operatorname{cof.}\phi - \operatorname{cof.} 2\phi)}{(\phi - 2i\pi)(\operatorname{cof.}\phi - \operatorname{cof.} 2\phi)} = \beta + R(\phi - 2i\pi)$$

Quod ſi iam ſtatuatur $\phi = 2i\pi$, β aequabitur fractioni, cuius tam numerator quam denominator ter euaneſcit, ita vt triplici differentiatione ſit opus.

Prima autem differentiatio dabit:

$$\beta = \frac{2(\phi - 2i\pi) + \frac{2}{3}(\operatorname{fin.}\phi - 2\operatorname{fin.} 2\phi)}{\operatorname{cof.}\phi - \operatorname{cof.} 2\phi - (\phi - 2i\pi)(\operatorname{fin.}\phi - 2\operatorname{fin.} 2\phi)}.$$

Secunda differentiatio dabit:

$$\beta = \frac{2 + \frac{2}{3}(\operatorname{cof.}\phi - 4\operatorname{cof.} 2\phi)}{-2\operatorname{fin.}\phi + 4\operatorname{fin.} 2\phi - (\phi - 2i\pi)(\operatorname{cof.}\phi - 4\operatorname{cof.} 2\phi)}.$$

Tertia denique differentiatio dat:

$$\beta = \frac{-\frac{2}{3}(\operatorname{fin.}\phi - 8\operatorname{fin.} 2\phi)}{-3\operatorname{cof.}\phi + 12\operatorname{cof.} 2\phi + (\phi - 2i\pi)(\operatorname{fin.}\phi - 8\operatorname{fin.} 2\phi)}.$$

Nunc autem facto $\phi = 2i\pi$ numerator quidem iterum euaneſcit, denominator vero euadit 9, ita vt ſit $\beta = 0$.

§. 33. At vero iſte valor pro β ſine differentiatione facilius erui poteſt ponendo $\phi = 2i\pi + \omega$, exiſtente ω infinite paruo; tum autem erit

cof.

$$\cos\Phi = \cos\omega \text{ et } \cos 2\Phi = \cos 2\omega;$$

aequatio autem fiet

$$\frac{\omega\omega}{\cos\omega - \cos 2\omega} = \tfrac{2}{3} + \beta\omega + R\omega\omega.$$

Nunc ambos cosinus proxime exhibeamus vsque ad quartam potestatem ipsius ω procedendo, et cum sit

$$\cos\omega = 1 - \tfrac{1}{2}\omega\omega + \tfrac{1}{24}\omega^4 \text{ et}$$
$$\cos 2\omega = 1 - 2\omega\omega + \tfrac{16}{24}\omega^4, \text{ erit}$$
$$\cos\omega - \cos 2\omega = \tfrac{3}{2}\omega\omega - \tfrac{5}{8}\omega^4 = \tfrac{3}{2}\omega\omega(1 - \tfrac{5}{12}\omega\omega),$$

quo valore substituto habebimus

$$\frac{2}{3(1 - \frac{5}{12}\omega\omega)} = \tfrac{2}{3}(1 + \tfrac{5}{12}\omega\omega) = \tfrac{2}{3} + \beta\omega + R\omega\omega,$$

hincque fit $\beta = \frac{5}{18}\omega$; sicque facto $\omega = 0$ erit etiam $\beta = 0$.

§. 34. Hanc obrem pro denominatoris factore quadrato $(\Phi - 2i\pi)^2$ ob $\alpha = \frac{2}{3}$ fractio inde nata erit $\frac{2}{3(\Phi - 2i\pi)^2}$. Pro reliquis autem factoribus simplicibus $\Phi - \frac{2}{3}i\pi$ statuamus

$$\frac{1}{\cos\Phi - \cos 2\Phi} = \frac{\alpha}{\Phi - \frac{2}{3}i\pi} + R,$$

quae aequatio multiplicetur per $\Phi - \frac{2}{3}i\pi = \omega$, vt prodeat

$$\frac{\omega}{\cos\Phi - \cos 2\Phi} = \alpha + R\omega.$$

Vbi notetur numerum i non esse per 3 diuisibilem, vnde $\frac{2i\pi}{3}$ sequentes angulos exprimet:

$$\tfrac{2}{3}\pi,\ \tfrac{4}{3}\pi,\ \tfrac{8}{3}\pi,\ \tfrac{10}{3}\pi,\ \tfrac{14}{3}\pi,$$

at anguli $\frac{4i\pi}{3}$ valores sunt $\frac{4\pi}{3}, \frac{8\pi}{3}, \frac{16\pi}{3}$ quorum angulorum cosinus est idem $-\frac{1}{2}$, sinus autem horum angulorum sunt $\sin.\frac{2i\pi}{3} = \pm\frac{\sqrt{3}}{2}$, vbi signum superius valet, si i sit $3n + 1$, inferius vero si fuerit $i = 3n + 2$. At vero $\sin.\frac{4i\pi}{3}$ semper est $\mp\frac{\sqrt{3}}{2}$, vbi iterum signum superius valet si

si i gul gau ω; sque ad quarn sit vnc 1 − 5/12 ωω), + Rωω, vbi feri nor rit etiam β = 0. oris factore quaata erit −2/3 iπ statuamus vnd α = : ω, vt prodeat iuisibilem, vnde feri nur mei feq quorum angun horum anguiperius valet, si 3n + 2. At veum superius valet si

si $i = 3n + 1$, inferius vero si $i = 3n + 2$. Haecque regula semper valet, siue n sit numerus positiuus siue negatiuus.

§. 35. His praenotatis erit

$$\cos\varphi = -\tfrac{1}{2}\cos\omega \mp \tfrac{\sqrt{3}}{2}\sin\omega \text{ et}$$
$$\cos 2\varphi = -\tfrac{1}{2}\cos 2\omega \pm \tfrac{\sqrt{3}}{2}\sin 2\omega,$$

vnde vero proxime habebimus

$$\cos\Phi = -\tfrac{1}{2}(1 - \tfrac{1}{2}\omega\omega) \mp \tfrac{\sqrt{3}}{2}\omega \text{ et}$$
$$\cos 2\varphi = -\tfrac{1}{2}(1 - 2\omega\omega) \pm \tfrac{\sqrt{3}}{2}.2\omega,$$

vbi perpetuo signa superiora valent si $i = 3n + 1$, inferiora autem si $i = 3n + 2$. Hinc igitur erit noster denominator

$$\cos\varphi - \cos 2\varphi = -\tfrac{3}{4}\omega\omega \mp \tfrac{3\sqrt{3}}{2}.\omega$$

vnde fit $\frac{1}{-\frac{3}{4}\omega \mp \frac{3\sqrt{3}}{2}} = \alpha$. Posito igitur $\omega = 0$ erit $\alpha = \mp\frac{2}{3\sqrt{3}}$, ita vt ex factore $\varphi - \frac{2i\pi}{3}$ nascatur ista fractio:

$$\mp\frac{2}{3\sqrt{3}}\left(\frac{1}{\varphi - \frac{2i\pi}{3}}\right) = \mp\frac{2}{(3\varphi - 2i\pi)\sqrt{3}}.$$

§. 36. Euoluamus igitur primo omnes terminos seriei ex factoribus geminatis $(\varphi - 2i\pi)^2$ natos, et cum numerator fuisset $\frac{2}{3}$, si loco i successiue omnes scribamus numeros integros tam positiuos quam negatiuos, series orietur sequens:

$$\frac{2}{3\Phi\Phi} + \frac{2}{3(\Phi - 2\pi)^2} + \frac{2}{3(\Phi + 2\pi)^2} + \frac{2}{3(\Phi - 4\pi)^2} + \frac{2}{3(\Phi + 4\pi)^2} + \frac{2}{3(\Phi - 6\pi)^2} + \text{etc.}$$

Pro

Pro altera ferie fit 1°) $i = 3n + 1$, hincque fractio fiet $-\frac{2}{(3\phi - 2(3n+1)\pi)\sqrt{3}}$; fin autem fit $i = -3n - 1$ fignum inferius valebit et fractio erit $+\frac{2}{(3\phi + 2(3n+1)\pi)\sqrt{3}}$, qui duo termini contracti praebent $-\frac{8(3n+1)\pi}{(9\phi\phi - 4(3n+1)^2\pi\pi)\sqrt{3}}$; fin autem fuerit $i = 3n + 2$, tum vero etiam $i = -3n - 2$, binae fractiones in vnam contractae praebent

$$+\frac{8(3n+2)\pi}{(9\phi\phi - 4(3n+2)^2\pi\pi)\sqrt{3}}.$$

Quare cum valores negatiuos ipfius i iam fimus complexi, loco n tantum omnes numeros pofitiuos 0, 1, 2, 3, 4, 5 etc. poni oportet, vnde fequens refultabit feries:

$$-\frac{8\pi}{(9\phi\phi - 4\pi\pi)\sqrt{3}} - \frac{8\cdot 4\pi}{(9\phi\phi - 4\cdot 16\pi\pi)\sqrt{3}} - \frac{8\cdot 7\pi}{(9\phi\phi - 4\cdot 49\pi\pi)\sqrt{3}}$$
$$+\frac{16\,\pi}{(9\phi\phi - 16\pi\pi)\sqrt{3}} + \frac{8\cdot 5\pi}{(9\phi\phi - 4\cdot 25\pi\pi)\sqrt{3}} + \frac{8\cdot 8\pi}{(9\phi\phi - 4\cdot 64\pi\pi)\sqrt{3}}.$$

§. 37. Propofita igitur fractio $\frac{1}{\cos.\phi - \cos.2\phi}$ refoluitur in has duas feries

$$\tfrac{2}{3}\left(\frac{1}{\phi^2} + \frac{1}{(\phi-2\pi)^2} + \frac{1}{(\phi+2\pi)^2} + \frac{1}{(\phi-4\pi)^2} + \frac{1}{(\phi+4\pi)^2} + \text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\phi\phi - 4\cdot 1\pi\pi} + \frac{4}{9\phi\phi - 4\cdot 4^2\pi\pi} + \frac{7}{9\phi\phi - 4\cdot 7^2\pi\pi} + \frac{10}{9\phi\phi - 4\cdot 10^2\pi\pi} + \text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\phi\phi - 4\cdot 2^2\pi\pi} + \frac{5}{9\phi\phi - 4\cdot 5^2\pi\pi} + \frac{8}{9\phi\phi - 4\cdot 8^2\pi\pi} + \frac{11}{9\phi\phi - 4\cdot 11^2\pi\pi} + \text{etc.}\right)$$

Hinc ergo fi faciamus vt fupra $\phi = \lambda\pi$, erit fractio

$$\frac{\pi\pi}{\cos.\lambda\pi - \cos.2\lambda\pi} =$$
$$\tfrac{2}{3}\left(\frac{1}{\lambda^2} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} + \frac{1}{(\lambda-4)^2} + \frac{1}{(\lambda+4)^2} + \frac{1}{(\lambda-6)^2} + \frac{1}{(\lambda+6)^2} + \text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\lambda\lambda - 4\cdot 1^2} + \frac{4}{9\lambda\lambda - 4\cdot 4^2} + \frac{7}{9\lambda\lambda - 4\cdot 7^2} + \frac{10}{9\lambda\lambda - 4\cdot 10^2} + \text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\lambda\lambda - 4\cdot 2^2} + \frac{5}{9\lambda\lambda - 4\cdot 5^2} + \frac{8}{9\lambda\lambda - 4\cdot 8^2} + \frac{11}{9\lambda\lambda - 4\cdot 11^2} + \text{etc.}\right)$$

§. 38. Vt exemplum afferamus fit $\lambda = \frac{1}{3}$, vt fiat $9\lambda\lambda = 1$ ac prodibit ifta fummatio:

$\pi\pi$

$$\pi\pi = \tfrac{2}{3}\left(\frac{9}{1^2} + \frac{9}{5^2} + \frac{9}{7^2} + \frac{9}{11^2} + \frac{9}{13^2} + \frac{9}{17^2} + \text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{1}{4\cdot 1^2 - 1} + \frac{4}{4\cdot 4^2 - 1} + \frac{7}{4\cdot 7^2 - 1} + \frac{10}{4\cdot 10^2 - 1} + \frac{13}{4\cdot 13^2 - 1} + \text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{2}{4\cdot 2^2 - 1} + \frac{5}{4\cdot 5^2 - 1} + \frac{8}{4\cdot 8^2 - 1} + \frac{11}{4\cdot 11^2 - 1} + \frac{14}{4\cdot 14^2 - 1} + \text{etc.}\right)$$

quae fummatio etiam hoc modo referri poteft:

$$\pi\pi = 6\left(\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.}\right)$$
$$+\frac{4\pi}{\sqrt{3}}\left(\frac{2}{2^2 - 1} - \frac{4}{4^2 - 1} + \frac{8}{8^2 - 1} - \frac{10}{10^2 - 1} + \frac{14}{14^2 - 1} - \frac{16}{16^2 - 1} + \text{etc.}\right)$$

elegantior autem forma erit fequens:

$$\pi\pi = 6\left(\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.}\right)$$
$$+\frac{2\pi}{\sqrt{3}}\left(1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \text{etc.}\right)$$

§. 39. Quoniam hoc cafu occurrerunt factores quadrati, etiam eiusmodi fractiones refoluere poterimus, quarum denominatores ipfi funt quadrati, ideoque meros factores fimplices quadratos inuoluunt. Atque adeo hanc refolutionem extendere licebit ad denominatores cubicos altiorumque poteftatum, fi modo in fubfidium vocentur ea praecepta, quae pro huiusmodi refolutionibus olim dedi.

IV. Sit fractio refoluenda propofita $\frac{1}{\sin.\phi^2}$.

§. 40. Cum igitur hic omnes factores quadrati denominatoris in hac forma contineantur: $\frac{1}{(\phi - i\pi)^2}$, denotante i omnes numeros integros tam pofitiuos quam negatiuos, ponamus pro refolutione generali

$$\frac{1}{\sin.\phi^2} = \frac{\alpha}{(\phi - i\pi)^2} + \frac{\beta}{\phi - i\pi} + R$$

vbi R complectitur omnes fractiones reliquas. Hinc per $(\phi - i\pi)^2$ multiplicando erit

$$\frac{(\phi - i\pi)^2}{\sin.\phi^2} = \alpha + \beta(\phi - i\pi) + R(\phi - i\pi)^2.$$

 Iam

Iam fiat $\phi = i\pi$, et quoniam hoc casu numerator ac denominator nostrae fractionis euanescunt, statuamus $\phi - i\pi = \alpha$, eritque

$\sin.\phi = \sin.(i\pi+\omega) = \sin.i\pi\cos.\omega + \sin.\omega\cos.i\pi = \pm\sin.\omega$

ob $\sin.i\pi = 0$ et $\cos.i\pi = \pm 1$; vbi signum superius valet si i sit numerus par, inferius vero si impar, quod tamen discrimen hic non in censum venit, cum sit $\sin.\phi^2 = \sin.\omega$. Hinc igitur erit

$$\frac{\omega\omega}{\sin.\omega} = \alpha + \beta\omega + R\omega\omega.$$

Cum igitur sit

$$\sin.\omega = \omega - \tfrac{1}{6}\omega^3 = \omega(1 - \tfrac{1}{6}\omega\omega), \text{ erit}$$

$$\frac{1}{(1-\frac{1}{6}\omega\omega)^2} = 1 + \tfrac{1}{3}\omega\omega = \alpha + \beta\omega + R\omega\omega$$

vnde fit statim $\alpha = 1$. Tum vero aequatio erit $\frac{1}{3}\omega = \beta + R\omega$, sicque facto $\omega = 0$ fit $\beta = 0$, consequenter ex denominatoris factore $(\phi - i\pi)^2$ oritur haec fractio $\frac{1}{(\phi-i\pi)^2}$.

§. 41. Tribuantur nunc ipsi i omnes valores debiti ac reperietur haec series:

$$\frac{1}{\sin.\phi^2} = \frac{1}{\phi\phi} + \frac{1}{(\phi-\pi)^2} + \frac{1}{(\phi+\pi)^2} + \frac{1}{(\phi-2\pi)^2} + \frac{1}{(\phi+2\pi)^2} + \frac{1}{(\phi-3\pi)^2} + \text{etc.}$$

quae quidem series deduci potuisset ex § 18, vbi inuenimus

$$\frac{\cos.\phi}{\sin.\phi} = \frac{1}{\phi} + \frac{1}{\phi-\pi} + \frac{1}{\phi+\pi} + \frac{1}{\phi-2\pi} + \frac{1}{\phi+2\pi} + \text{etc.},$$

vnde per differentiationem signis mutatis ea ipsa oritur series quam hic inuenimus.

§. 42. Quod si fractio fuisset proposita $\frac{\cos.\phi^2}{\sin.\phi^2}$ et eodem modo resolutio instituatur, ob

cos.

merator ac denoamus $\phi - i\pi = \alpha$,
$i\pi = \pm\sin.\omega$
um superius valet
par, quod tamen
it $\sin.\phi^2 = \sin.\omega$.
, erit
$+R\omega\omega$
erit $\frac{1}{3}\omega = \beta + R\omega$,
er ex denomina-
o $\frac{1}{(\phi-i\pi)^2}$.
nnes valores de-
$+ \frac{1}{(\phi-3\pi)^2} +$ etc.
18, vbi inuenimus
$+$ etc.,
ea ipsa oritur se-
proposita $\frac{\cos.\phi^2}{\sin.\phi^2}$ et
cos.

$$\cos.(i\pi + \omega) = \pm\cos.\omega \text{ ideoque}$$
$$\cos.\phi^2 = \cos.\omega^2 = 1 - \omega\omega$$

quoniam secundae potestates ipsius ω non in computum veniunt, numerator foret vt in casu praecedente $= 1$, ideoque eadem plane series prodiisset, id quod vtique foret absurdum. Supra autem iam animaduertimus, huiusmodi resolutiones veritati non esse consentaneas, nisi quantitas variabilis ϕ in numeratore pauciores habeat dimensiones quam in denominatore, quia alioquin praeter seriem fractionum partes integrae essent accessurae, id quod hoc casu manifesto euenit, cum sit $\frac{\cos.\phi^2}{\sin.\phi^2} = \frac{1}{\sin.\phi^2} - 1$, ita vt pars integra hoc casu sit $= -1$.

V. Sit fractio resoluenda $= \frac{1}{\sin.\phi^3}$.

§. 43. Pro hoc ergo casu poni oportebit

$$\frac{1}{\sin.\phi^3} = \frac{\alpha}{(\phi-i\pi)^3} + \frac{\beta}{(\phi-i\pi)^2} + \frac{\gamma}{\phi-i\pi} + R.$$

Ponamus nunc iterum $\phi = i\pi + \omega$, et cum sit

$$\frac{1}{\sin.\phi^3} = \pm\frac{1}{\omega^3}(1 + \tfrac{1}{2}\omega\omega)$$

(vbi ratio signorum legem supra datam seruet) haec resultat aequatio, postquam per ω^3 fuerit multiplicata:

$$\pm\frac{\omega^3(1+\frac{1}{2}\omega\omega)}{\omega^3} = \alpha + \beta\omega + \gamma\omega\omega + R\omega^3 = 1 + \tfrac{1}{2}\omega\omega,$$

vnde manifesto fit $\alpha = \pm 1$, tum vero $\beta + \gamma\omega + R\omega\omega = \frac{1}{2}\omega$ sicque erit $\beta = 0$ et $\gamma = \pm\frac{1}{2}$. Hoc igitur modo ex denominatoris factore cubico $(\phi - i\pi)^3$ nascentur hae duae fractiones: $\frac{1}{(\phi-i\pi)^3} \pm \frac{1}{2(\phi-i\pi)}$.

§. 44.

§. 44. Tribuamus igitur litterae i ſucceſſiue omnes valores tam poſitiuos quam negatiuos atque obtinebimus ſequentem reſolutionem:

$$\frac{1}{\sin.\Phi^3} = \frac{1}{\Phi^3} - \frac{1}{(\Phi-\pi)^3} - \frac{1}{(\Phi+\pi)^3} + \frac{1}{(\Phi-2\pi)^3} + \frac{1}{(\Phi+2\pi)^3} - \text{etc.}$$
$$+ \frac{1}{2\Phi} - \frac{1}{2(\Phi-\pi)} - \frac{1}{2(\Phi+\pi)} + \frac{1}{2(\Phi-2\pi)} + \frac{1}{2(\Phi+2\pi)} - \text{etc.}$$

Hic obſeruaſſe iuuabit, inferiorem ſeriem iam ſupra in primo exemplo eſſe inuentam; vnde intelligimus fore ſummam huius ſeriei $= \frac{1}{2\sin.\Phi}$: quamobrem ſeries ſuperior cuborum ſola aequabitur huic formulae: $\frac{1}{\sin.\Phi^3} - \frac{1}{2\sin.\Phi}$.

§. 45. Egregie hoc quoque conuenit cum principiis ſupra ſtabilitis, ex quibus per differentiationem continuo alias nouas ſeries eruere docuimus. Cum enim ſit

$$\frac{1}{\sin.\Phi} = \frac{1}{\Phi} - \frac{1}{\Phi-\pi} - \frac{1}{\Phi+\pi} + \frac{1}{\Phi-2\pi} + \frac{1}{\Phi+2\pi} + \text{etc.}$$

hinc deducitur differentiando

$$-\frac{\cos.\Phi}{\sin.\Phi^2} = -\frac{1}{\Phi^2} + \frac{1}{(\Phi-\pi)^2} + \frac{1}{(\Phi+\pi)^2} - \frac{1}{(\Phi-2\pi)^2} - \frac{1}{(\Phi+2\pi)^2} + \text{etc.}$$

atque hinc denuo differentiando

$$\frac{1}{\sin.\Phi} + \frac{2\cos.\Phi^2}{\sin.\Phi^3} = \frac{2}{\Phi^3} - \frac{2}{(\Phi-\pi)^3} - \frac{2}{(\Phi+\pi)^3} + \frac{2}{(\Phi-2\pi)^3} + \frac{2}{(\Phi+2\pi)^3} + \text{etc.}$$

quae reducitur ad hanc formam: $\frac{2}{\sin.\Phi^3} - \frac{1}{\sin.\Phi}$, id quod egregie conuenit cum valore praecedente.

VI. Sit fractio propoſita reſoluenda $= \frac{1}{\text{tang.}\Phi - \sin.\Phi}$.

§. 46. Denominator iſte tang. Φ − ſin. Φ manifeſto euaneſcit caſibus quibus $\Phi = i\pi$, denotante i omnes numeros integros tam poſitiuos quam negatiuos, vnde fractiones ſimplices, quarum denominatores continent iſtum factorem $\Phi - i\pi$, euadunt infiniti caſu $\Phi = i\pi$, dum reliquae fractiones retinent valorem finitum. Haecque conſideratio nobis aperit nouam methodum, omnes fractiones ſimplices inueſtigandi. Pro quouis enim tali factore euaneſcente quaeratur valor ipſius fractionis propoſitae, qui cum ſiat infinitus, ei aequales eſſe debebunt ii ſeriei termini, qui eodem caſu euadunt infiniti. Hanc ob rem ſtatui oportebit $\Phi - i\pi = \omega$; denotante ω angulum infinite paruum. Quo facto fractio propoſita fiet certa quaedam functio ipſius ω, quam ſecundum eius dimenſiones euolui conueniet.

§. 47. Hanc igitur ideam ſequentes, duos caſus diſtinguere debemus, prouti i fuerit numerus vel par vel impar, quoniam priore caſu fit ſin. Φ = ſin. ω, poſteriore vero caſu fit ſin. Φ = − ſin. ω, dum vtroque caſu manet

tang. Φ = tang. ω.

Sit igitur primo i numerus impar et erit caſu $\Phi = i\pi$ noſtra fractio $\frac{1}{\text{tang.}\omega + \sin.\omega}$. Eſt vero proxime tang. $\omega = \omega + \frac{1}{3}\omega^3$ et ſin. $\omega = \omega - \frac{1}{6}\omega^3$, vnde iſta fractio fiet

$$\frac{1}{2\omega + \frac{1}{6}\omega^3} = \frac{1}{2\omega(1 + \frac{1}{12}\omega\omega)} = \frac{1}{2\omega}(1 - \tfrac{1}{12}\omega\omega).$$

Haec iam expreſſio ſponte praebet has duas fractiones $\frac{1}{2\omega} - \frac{1}{24}\omega$, vnde ob $\omega = \Phi - i\pi$ pro iſto factore oritur haec fractio ſimplex $\frac{1}{2(\Phi - i\pi)}$, quia altera pars euaneſcit. Quare ſi nunc loco i ordine ſcribamus numeros impares, ſequentem fractionum ſeriem adipiſcemur:

$$\frac{1}{2(\Phi-\pi)} + \frac{1}{2(\Phi+\pi)} + \frac{1}{2(\Phi-3\pi)} + \frac{1}{2(\Phi+3\pi)} + \frac{1}{2(\Phi-5\pi)} + \text{etc.}$$

§. 48. Sit nunc etiam i numerus par, vnde fit tang. Φ = tang. ω et ſin. Φ = ſin. ω, hinc fractio noſtra erit $\frac{1}{\text{tang.}\omega - \sin.\omega}$, vbi facta euolutione primi termini ω ſe se tollunt, ita vt in hoc denominatore infima potestas ipsius ω futura sit ω^3. Atque ob hanc caussam approximationem vlterius continuari oportet quam casu praecedente. Hunc in finem loco tang. ω scribamus $\frac{\sin.\,\omega}{\cos.\,\omega}$, vt fractio nostra sit $\frac{\cos.\,\omega}{\sin.\,\omega - \sin.\,\omega\,\cos.\,\omega}$. Cum iam sit

$$\sin.\,\omega\,\cos.\,\omega = \tfrac{1}{2}\sin.\,2\,\omega,\ \text{erit per series}$$

$$\sin.\,\omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5 \ \text{et}$$

$$\sin.\,2\,\omega = 2\,\omega - \tfrac{4}{3}\omega^3 + \tfrac{32}{120}\omega^5$$

vnde totus denominator erit

$$+\tfrac{1}{2}\omega^3 - \tfrac{1}{8}\omega^5 = \tfrac{1}{2}\omega^3(1 - \tfrac{1}{4}\omega\omega)$$

numerator vero est cos. $\omega = 1 - \frac{1}{2}\omega\omega$, vnde tota fractio nostra erit

$$\frac{1 - \frac{1}{2}\omega\omega}{\frac{1}{2}\omega^3(1 - \frac{1}{4}\omega\omega)} = \frac{1 - \frac{1}{4}\omega\omega}{\frac{1}{2}\omega^3};$$

hincque partes resultantes erunt $\frac{2}{\omega^3} - \frac{1}{2\omega}$, quae ambae casu $\omega = 0$ fiunt infinitae. Facile autem patet, si approximationem vlterius extendissemus, in sequenti termino litteram ω iam in numeratorem transituram fuisse. Scribatur igitur $\Phi - i\pi$ loco ω, et partes ex hoc factore denominatoris oriundae erunt $\frac{2}{(\Phi - i\pi)^3} - \frac{1}{2(\Phi - i\pi)}$, vnde loco i successiue omnes numeros pares scribendo ista prodibit series geminata:

$$\frac{2}{\Phi^3} + \frac{2}{(\Phi-2\pi)^3} + \frac{2}{(\Phi+2\pi)^3} + \frac{2}{(\Phi-4\pi)^3} + \frac{2}{(\Phi+4\pi)^3} + \frac{2}{(\Phi-6\pi)^3} + \text{etc.}$$

$$-\frac{1}{2\Phi} - \frac{1}{2(\Phi-2\pi)} - \frac{1}{2(\Phi+2\pi)} - \frac{1}{2(\Phi-4\pi)} - \frac{1}{2(\Phi+4\pi)} - \text{etc.}$$

§. 49. Iungamus igitur has series ex vtroque casu deductas et fractio proposita $\frac{1}{\text{tang.}\,\Phi - \sin.\,\Phi}$ resolui reperitur in ternas series sequentes:

potestas ipapproximapraecedente. vt fractio notota fractio
ae ambae casu approximationo litteram ω cribatur igitur denominatoris successiue omies geminata:
+ etc.
etc.
vtroque casu resolui reperitur
I 2

$$\frac{1}{2(\Phi-\pi)} + \frac{1}{2(\Phi+\pi)} + \frac{1}{2(\Phi-3\pi)} + \frac{1}{2(\Phi+3\pi)} + \frac{1}{2(\Phi-5\pi)} + \text{etc.}$$

$$-\frac{1}{2\Phi} - \frac{1}{2(\Phi-2\pi)} - \frac{1}{2(\Phi+2\pi)} - \frac{1}{2(\Phi-4\pi)} - \frac{1}{2(\Phi+4\pi)} - \text{etc.}$$

$$+\frac{2}{\Phi^3} + \frac{2}{(\Phi-2\pi)^3} + \frac{2}{(\Phi+2\pi)^3} + \frac{2}{(\Phi-4\pi)^3} + \frac{2}{(\Phi+4\pi)^3} + \text{etc.}$$

§. 50. Qnilibet hic facile sentiet, istam methodum non parum antecellere illi, qua ante vsi sumus, quandoquidem hoc modo statim fractiones ex quolibet denominatoris factore oriundas nacti sumus, neque opus fuerat earum numeratores per litteras indefinitas designare. Praeterea etiam hac ratione non opus erat sollicite inquirere, quoties singuli factores simplices in denominatore contineantur, siquidem nostra methodus hoc sponte declarat.

§. 51. In huiusmodi autem seriebus generalibus, vbi quorundam terminorum denominatores certo casu euanescunt ideoque hi termini in infinitum excrescunt, quaeri solet, his terminis sublatis, quanta futura sit summa reliquorum terminorum. Ita pro casu quo i est numerus impar, terminus $\frac{1}{2(\Phi - i\pi)}$ fit infinitus casu $\Phi = i\pi$. Hoc igitur termino deleto quaeritur, quanta futura sit summa reliquorum terminorum casu $\Phi = i\pi$. Ad hanc quaestionem soluendam ponatur $\Phi - i\pi = \omega$, atque ex §47 patet fore

$$\frac{1}{2\,\omega} - \tfrac{1}{24}\,\omega = \frac{1}{2(\Phi - i\pi)} + R$$

vbi R complectitur omnes reliquos terminos, quorum summa desideratur casu $\Phi = i\pi$. Transferatur igitur terminus $\frac{1}{2(\Phi - i\pi)} = \frac{1}{2\,\omega}$ in alteram partem ac statim elucet fore

$$R = -\tfrac{1}{24}\,\omega = 0 \ \text{ob}\ \omega = 0,$$

ita vt omisso termino illo infinito summa omnium reliquorum casu $\Phi = i\pi$ semper sit 0.

§. 52. Quando autem i eſt numerus par, eadem concluſio locum habebit, ad quod oſtendendum neceſſe eſt approximationem adhibitam vlterius continuare. Tum autem erit numerator

$$\operatorname{cof.} \omega = 1 - \tfrac{1}{2}\omega\omega + \tfrac{1}{24}\omega^4;$$

pro denominatore vero

$$\operatorname{fin.} \omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5 - \tfrac{1}{5040}\omega^7 \text{ et}$$

$$\operatorname{fin.} 2\omega = 2\omega - \tfrac{8}{6}\omega^3 + \tfrac{32}{120}\omega^5 - \tfrac{128}{5040}\omega^7,$$

vnde fit ipſe denominator

$$\tfrac{1}{2}\omega^3 - \tfrac{1}{8}\omega^5 + \tfrac{1}{80}\omega^7 = \tfrac{1}{2}\omega^3(1 - \tfrac{1}{4}\omega\omega + \tfrac{1}{40}\omega^4);$$

hinc factor poſterior in numeratorem translatus praebet

$$1 + \tfrac{1}{4}\omega\omega + \tfrac{3}{80}\omega^4$$

hincque tota fractio iam erit

$$\frac{1 - \tfrac{1}{4}\omega\omega - \tfrac{11}{240}\omega^4}{\tfrac{1}{2}\omega^3}$$

quae aequari debet toti ſeriei poſito $\Phi = i\pi$, hoc eſt terminis inuentis $\frac{2}{(\Phi - i\pi)^3} - \frac{1}{2(\Phi - i\pi)}$ cum omnibus reliquis R, vnde elicitur $R = -\frac{11}{120}\omega = 0$; vnde patet etiam his caſibus ſummam omnium reliquorum eſſe $= 0$.

§. 53. Quod ſi ergo ſumamus $\Phi = 0$ et terminos in infinitum exreſcentes deleamus, termini remanentes erunt

$$-\tfrac{1}{2\pi} + \tfrac{1}{2\pi} - \tfrac{1}{6\pi} + \tfrac{1}{6\pi} - \tfrac{1}{10\pi} + \tfrac{1}{10\pi} - \text{etc.}$$

$$+\tfrac{1}{4\pi} - \tfrac{1}{4\pi} + \tfrac{1}{8\pi} - \tfrac{1}{8\pi} + \tfrac{1}{12\pi} - \tfrac{1}{12\pi} + \text{etc.}$$

$$-\tfrac{2}{8\pi^3} + \tfrac{2}{8\pi^3} - \tfrac{2}{64\pi^3} + \tfrac{2}{64\pi^3} - \tfrac{2}{216\pi^3} + \tfrac{2}{216\pi^3} - \text{etc.}$$

vbi omnes termini manifeſto ſe tollunt, id quod etiam omnibus reliquis caſibus, quibus ponitur $\Phi = i\pi$, contingit.

§. 54.

rus par, eadem
ndum neceſſe eſt
uare. Tum au-
ω^7,
ω^4);
atus praebet
π, hoc eſt ter-
s reliquis R, vnde
iam his caſibus
$= 0$ et termi-
mini remanentes
etc.
etc.
$\frac{2}{216\pi^3}$ — etc.
quod etiam om-
$= i\pi$, contingit.
§. 54.

§. 54. Sin autem binos terminos contiguos contraxiſſemus, hae ſeries prodiiſſent:

$$-\tfrac{1}{2\Phi} + \tfrac{\Phi}{\Phi\Phi - \pi\pi} - \tfrac{\Phi}{\Phi\Phi - 4\pi\pi} + \tfrac{\Phi}{\Phi\Phi - 9\pi\pi} - \tfrac{\Phi}{\Phi\Phi - 16\pi\pi} + \tfrac{\Phi}{\Phi\Phi - 25\pi\pi} - \text{etc.}$$

$$+\tfrac{2}{\Phi^3} + \tfrac{4\Phi(\Phi\Phi + 3.4\pi\pi)}{(\Phi\Phi - 4\pi\pi)^3} + \tfrac{4\Phi(\Phi\Phi + 3.16\pi\pi)}{(\Phi\Phi - 16\pi\pi)^3} + \tfrac{4\Phi(\Phi\Phi + 3.36\pi\pi)}{(\Phi\Phi - 36\pi\pi)^3} + \text{etc.}$$

quarum ſerierum ſumma eſt $\frac{1}{\operatorname{tang.}\Phi - \operatorname{fin.}\Phi}$. Quod ſi nunc hic ponamus $\Phi = 0$, ſiue $\Phi = \omega$, quoniam omnes termini ſunt diuiſibiles per $\Phi = 0$, eorumque autem ſumma inuenta eſt $-\frac{11}{120}\omega$, ſi vtrinque per ω diuidamus, ſumma erit $= -\frac{11}{120}$; ipſae autem ſeries euadent

$$-\tfrac{1}{\pi\pi} + \tfrac{1}{4\pi\pi} - \tfrac{1}{9\pi\pi} + \tfrac{1}{16\pi\pi} - \tfrac{1}{25\pi\pi} + \text{etc.}$$

$$-\tfrac{3.4}{(4\pi\pi)^2} - \tfrac{3.4}{(16\pi\pi)^2} - \tfrac{3.4}{(36\pi\pi)^2} - \tfrac{3.4}{(64\pi\pi)^2} - \text{etc.}$$

§. 55. Mutatis igitur ſignis et reductis terminis ad formam ſimpliciſſimam impetrabimus hanc ſummationem:

$$\tfrac{11}{120} = \tfrac{1}{\pi\pi}(1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \tfrac{1}{36} + \text{etc.})$$

$$+\tfrac{3}{4\pi^4}(1 + \tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \tfrac{1}{25^2} + \tfrac{1}{36^2} + \text{etc.})$$

Notum autem eſt eſſe

$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} \text{ etc.} = \tfrac{\pi\pi}{12} \text{ et}$$

$$1 + \tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \tfrac{1}{25^2} + \tfrac{1}{36^2} = \tfrac{\pi^4}{90}$$

vnde haec aequalitas manifeſto in oculos incurrit.