



1785

# De resolutione fractionum transcendentium in infinitas fractiones simplices

Leonhard Euler

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DE RELATIONE

FRACTIONVM

TRANSCENDENTIVM

IN INFINITAS FRACTIONES

SIMPLICES.

§. 1.

Propofita fractione quacunque algebraica  $\frac{P}{Q}$ , cuius tam numerator  $P$  quam denominator  $Q$  sint functiones rationales integre quantitatis  $x$ , iam pridem ostendi quomodo ea in fractiones simplices refolvi possit, quarum denominatores aequentur factoribus simplicibus denominatoris  $Q$ , numeratores vero sint constantes, siquidem variabilis  $x$  in denominatoris  $Q$  pluris habeat dimensiones quam in numeratore  $P$ . Quin etiam ostendi, quemadmodum pro quolibet factore simplici denominatoris fractio simplex respondens reperiri queat, sine vilo respectu ad reliquos factores habito. Ita si constet, denominatorem  $Q$  factorem completi simplicem  $x - a$ , fractio simplex inde nata, quae erit huius formae:  $\frac{x-a}{x-a} + R$ , facillime hoc modo definitur. Statuatur  $\frac{x-a}{x-a} + R$ , ubi  $R$  complectatur omnes fractiones simplices ex reliquis oriendas. Multiplicetur vtrinque per  $x - a$ , vt fiat

$$\frac{x-a}{x-a} + R(x-a),$$

et

et quia  $a$  est quantitas constans, ea semper eundem retribuebit valorem, quicunque valor variabilis  $x$  tribuatur; quare obrem fiat vbiq;  $x = a$ , vt reliquarum fractionum simplicium ratio ex calculo excedat, et habeatur  $a = \frac{P(a)-aQ}{Q(a)}$ , siquidem in hac formula fiat  $x = a$ , cum autem numerator  $P(x-a)$  in nihilum abiat; verum, quia  $x - a$  est factor denominatoris  $Q$ , etiam denominator  $Q$  in nihilum abibit. Hinc igitur per regulam conseruat loco numeratoris ac denominatoris sua differentialia substituantur, quandoquidem etiamnum erit  $\frac{P(x-a) - aQ}{Q} = a$ , siquidem hic vbiq; loco  $x$  scribatur  $a$ . Ponamus igitur hoc casu  $x = a$  fieri  $P = A$  et  $\frac{dQ}{dx} = C$ , quae ergo quantitates  $A$  et  $C$  facillime inueniuntur, tum igitur prodit numerator quactus  $a = \frac{A}{C}$ , ita vt fractio simplex ex denominatoris factore  $x - a$  orienda sit  $= \frac{A}{C}(x - a)$ , ita vt non opus sit reliquos factores denominatoris nosse. Simili autem modo pro singulis reliquis factoribus fractiones simplices respondentes determinabuntur, quarum omnium summa aequabitur fractioni propofitae  $\frac{P}{Q}$ , dummodo variabilis  $x$  pauciores habeat dimensiones in numeratore  $P$  quam in denominatore  $Q$ .

§. 2. Haec igitur principia sequentes pro denominatoris  $Q$  eiusmodi assumamus transcendentes, quos in infinitos factores simplices resolueri liceat, id quod erit, si eae infinitis casibus nihil aequales euadant. Praeterea vero necesse est vt omnes isti factores inter se sint inaequales, quandoquidem factores aequales peculiarem resolutionem possulant. Imprimis autem requiritur, vt productum omnium salium factorum ipsam functionem  $Q$  penitus exhauriant, quoniam quandoque factores imaginarii se

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internoscere possent. Veluti si sumatur  $Q = \tan \phi$ ,  $\phi$  verique omnibus hinc casibus emanabit, quibus hinc factio:  $\sin \phi$ , hincque ambae hinc fractiones eosdem factores simpliciter involvant, etiam si inter se neutriquam sine aequales. Deinde vero numeratorem P ita comparatum esse oportet ut cum denominatore Q nullos habeat factores communes. Imprimis autem cavendum est, ne quantitas variabilis in numeratore ad eandem vel plures dimensiones affurgat quam in denominatore. Cum autem ea in denominatore ad infinitas dimensiones affurgere sit censenda, istud incommensuratum non erit pertinendum, quando variabilis in numeratore tantum finito dimensionum numero continetur. Sin autem eius potestates etiam in infinitum ascendunt, saepe numero difficile erit iudicare, num dimensionum numerus maior sit vel minor quam in denominatore. Interim tamen etiam his casibus fractio proposita  $\frac{P}{Q}$  omnes continebit fractiones simplices, ad quas methodus nostra perducit. Verum euenire potest ut praeter eas etiam quasdam partes quasi integras involuat. His igitur praenotatis sequentes casus cuolvantur.

**I. Sumatur  $Q = \sin \phi$ , ut fractio resolvenda sit  $\frac{1}{\sin \phi}$ .**

§. 3. Quoniam formula  $\sin \phi$ , denotante  $\pi$  semiperipheriam circuli cuius radius = 1, seu angulum duobus rectis aequalem, omnibus his casibus emanabit:

$\phi = 0, \phi = \pi, \phi = \pi + 2\pi, \phi = \pi + 3\pi, \text{ etc.}$

et in genere  $\phi = \pi + i\pi$ , eius factores numero infiniti erunt  $\phi, (\phi + \pi), (\phi + 2\pi), (\phi + 3\pi)$  et in genere  $(\phi + i\pi)$ . Aliunde autem certum est, hanc formulam  $\sin \phi$  praeter istos factores nullos alios sine reales sine imaginarios involvere; cum

cum enim sit  $\sin \phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots$  etc. constat hanc seriem aequari huic producto infinito:  $\phi(1 - \frac{\phi^2}{\pi^2})(1 + \frac{\phi^2}{\pi^2})(1 - \frac{\phi^2}{4\pi^2})(1 + \frac{\phi^2}{4\pi^2}) \text{ etc.}$

§. 4. Consideremus igitur nostri denominatoris  $Q = \sin \phi$  factorem, quemcumque  $\phi + i\pi$ , ubi  $i$  denotet omnes plane numeros integros, cum positius quam negativos, cifra non excepta; sique fractio parialis hinc oriunda  $\frac{1}{\phi + i\pi}$ . Ad eius numeratorem  $e$  inveniendum statimur primo in numeratore P vbi que  $\phi = i\pi$ , sique quantitas inde resultans = A; deinde cum sit  $Q = \sin \phi$ , erit  $dQ = d\phi \cos \phi$ , siue  $dQ = \cos \phi$ , ubi loco  $\phi$  iidem seribi oportet  $i\pi$ , ut obtineamus Q, unde patet fore  $C = \cos i\pi$ , ita ut sit  $C = \pm 1$ , ubi signum + valebit pro numeris paribus, signum vero - pro imparibus numeris loco  $i$  assumis. Hoc igitur modo numerator fractionis nostrae erit  $e = \pm A$  ipsaque fractio quacumque  $\frac{1}{\phi + i\pi}$ . Hinc autem vicibus progredi non licet, quando numeratorem in genere spectamus, unde eius loco plures valores determinatos accipiamus, singulosque in sequentibus exemplis cuolvantur.

**1. Sit  $P = 1$  et fractio proposita  $\frac{1}{\sin \phi}$ .**

§. 5. Hic igitur semper erit  $A = 1$  et fractio simplici quaecumque  $= \frac{1}{\phi + i\pi}$ , ubi signum superius valet si  $i$  numerus par, inferius vero si impar. Substituamus igitur successive pro  $i$  omnes eius valores ordine

$0, +1, -1, +2, -2, +3, -3, +4, -4 \text{ etc.}$   
Euleri Op. Anal. Tom. II. O

me  $\pi$  semiperipheriam duobus etc.  $\sin \phi$  infiniti erunt  $(\phi + i\pi)$ , praeter istos  $i$ , involvere; cum

et resolutio nostrae fractionis  $\frac{1}{\sin \phi}$  in fractiones simplices ita se habebit:

$$\frac{1}{\sin \phi} = \frac{1}{\phi} + \frac{1}{\pi - \phi} + \frac{1}{2\pi - \phi} + \frac{1}{3\pi - \phi} + \dots + \text{etc.}$$

quae in hanc formam reducitur:

$$\frac{1}{\sin \phi} = \frac{1}{\phi} + \frac{1}{\pi - \phi} + \frac{1}{2\pi - \phi} + \frac{1}{3\pi - \phi} + \dots + \text{etc.}$$

Constat autem post primum terminum hinc sequentium, ut nascatur hanc seriem:

$$\frac{1}{\sin \phi} = \frac{1}{\phi} + \frac{1}{\pi - \phi} + \frac{1}{2\pi - \phi} + \frac{1}{3\pi - \phi} + \dots + \text{etc.}$$

unde deducitur sequens series memorari digna

$$\frac{1}{\sin \phi} = \frac{1}{\phi} + \frac{1}{\pi - \phi} + \frac{1}{2\pi - \phi} + \frac{1}{3\pi - \phi} + \dots + \text{etc.}$$

§. 6. Has quidem series iam olim finis sum profectus: interim tamen pro sequentibus casibus haud inutile erit sequentes transformationes hic repetere. Ponamus igitur primo  $\phi = \lambda \pi$ , ut littera  $\pi$  ex seriebus elidatur, atque hinc nascemur

$$\frac{1}{\sin \lambda \pi} = \frac{1}{\lambda \pi} + \frac{1}{\pi - \lambda \pi} + \frac{1}{2\pi - \lambda \pi} + \frac{1}{3\pi - \lambda \pi} + \dots + \text{etc. et}$$

atque hinc per differentiationem, spectando  $\lambda$  tanquam quantitatam variabilem, infinitas alias series notari dignissimas elicere poterimus. Ex priore scilicet nascemur

$$\frac{\pi \cos \lambda \pi}{\sin^2 \lambda \pi} = \frac{\pi}{\lambda \pi} - \frac{\pi}{\pi - \lambda \pi} + \frac{\pi}{2\pi - \lambda \pi} - \frac{\pi}{3\pi - \lambda \pi} + \dots - \text{etc.}$$

Hinc igitur sequitur, si  $\lambda = \frac{1}{2}$  fore

$$0 = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots - \text{etc.}$$

quod quidem est manifestum. At si  $\lambda = \frac{1}{3}$  erit

$$\frac{2\pi \pi}{3} = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} + \dots - \text{etc.}$$

Si  $\lambda = \frac{1}{2}$ , oriuntur series praecedens.

Si  $\lambda = \frac{1}{3}$ , prodit haec summatio:

$$\frac{\pi \pi}{3} = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} + \dots - \text{etc.}$$

Quod si denuo differentiemus obtinebitur sequens summatio:

$$\frac{\pi^2}{\sin \lambda \pi} = \frac{\pi^2}{2\lambda \pi} - \frac{\pi^2}{\lambda \pi} + \frac{\pi^2}{2\pi - \lambda \pi} - \frac{\pi^2}{3\pi - \lambda \pi} + \dots - \text{etc.}$$

sique continuo vicinius progredi licet.

§. 7. Simili modo etiam alteram formam differentiemus, quae redunda praebet

$$\frac{1}{\sin \lambda \pi} = \frac{1}{\lambda \pi} + \frac{1}{\pi - \lambda \pi} + \frac{1}{2\pi - \lambda \pi} + \frac{1}{3\pi - \lambda \pi} + \dots + \text{etc.}$$

Quod si nunc summamus  $\lambda = \frac{1}{2}$ , prodibit illa summatio:

$$\frac{1}{\sin \frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} + \frac{1}{\pi - \frac{\pi}{2}} + \frac{1}{2\pi - \frac{\pi}{2}} + \frac{1}{3\pi - \frac{\pi}{2}} + \dots + \text{etc.}$$

quae series prorsus noua omnem attentionem meretur; neque autem opus est hinc nouam differentiationem influere.

§. 8. Posteriores autem summationem

$$\frac{\pi}{\sin \lambda \pi} = \frac{\pi}{\lambda \pi} + \frac{\pi}{\pi - \lambda \pi} + \frac{\pi}{2\pi - \lambda \pi} + \frac{\pi}{3\pi - \lambda \pi} + \dots + \text{etc.}$$

accuratius perpendamus, ac primo quidem cum ea seriem per debeat esse vera, quicquid pro  $\lambda$  assumatur, summamus  $\lambda = 0$ . Quia autem hoc casu membrum finitimum abire in  $\infty - \infty$ , tractetur  $\lambda$  ut quantitas quam minimam, et cum sit  $\lambda \pi = \lambda \pi - \frac{1}{2} \lambda^2 \pi^2$ , istud membrum evadet

$$\frac{2\lambda(\lambda \pi - \frac{1}{2} \lambda^2 \pi^2)}{\pi} = \frac{2\lambda^2 \pi}{\pi}$$

quae fractionem ad communem denominatorem perductae dant

$$\frac{2\lambda \lambda (1 - \frac{1}{2} \lambda \lambda \pi \pi)}{2} = \frac{2\lambda \lambda \pi \pi}{2} = 2\lambda \lambda \pi \pi$$

Nunc

Nunc igitur facto  $\lambda = 0$ , eius factor erit  $= \pi$ ; series autem ipsa hoc casu eadem

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots \text{ etc.}$$

cuius summam constat esse  $\frac{\pi}{2}$ .

§. 9. Manifestum porro est, quoties pro  $\lambda$  accipitur numerus integer, vna terminum seriei; ideoque etiam ipsam seriem fieri infinitam, quod egregie conuenit cum summa inuenta, quandoquidem hoc casu sit  $\sin. \lambda \pi = 0$ . Arque hinc nata est ista quæstio: si ille terminus seriei in infinitum abiens ad sinistram partem transferatur, quænam futura sit reliquorum terminorum summa. Ponnus seriei esse  $\lambda = 1$ , et primus seriei terminus eadem infinitus, qui ergo ad sinistram partem transferens dabit

$$\frac{\pi}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \text{ etc.}$$

Nunc ad valorem huius seriei investigandum statuitur  $\lambda$  variari tantum proximo æquale, ponendo  $\lambda = 1 - \omega$ , erique

$$\sin. \lambda \pi = \sin. (\pi - \pi \omega) = \sin. \pi \omega; \text{ est vero}$$

$$\sin. \pi \omega = \pi \omega - \frac{1}{6} \pi^3 \omega^3,$$

quo valore substituto prodibit

$$\frac{1}{2(1-\omega)} \omega (1 - \frac{1}{2} \pi^2 \omega^2) = \frac{1}{2(1-\omega)} - \frac{1}{2} \frac{\pi^2 \omega^3}{1-\omega}$$

Primum autem membrum

$$\frac{2(1-\omega)\omega(1 - \frac{1}{2} \pi^2 \omega^2)}{1} \text{, ob}$$

$$\frac{1}{1-\omega} = 1 + \omega + \omega^2 \text{ et}$$

$$\frac{1}{1 - \frac{1}{2} \pi^2 \omega^2} = 1 + \frac{1}{2} \pi^2 \omega^2,$$

negli-

es autem

negligendo potestates ipsius  $\omega$  quadrato altiores, transmutetur in hanc formam:

$$\frac{1}{2\omega} (1 + \omega + \omega^2 + \frac{1}{2} \pi^2 \omega \omega^2);$$

tertium autem membrum

$$-\frac{1}{2\omega} \frac{1}{(1 - \frac{1}{2} \omega)} \text{, ob } \frac{1}{1 - \frac{1}{2} \omega} = 1 + \frac{1}{2} \omega + \frac{1}{4} \omega^2$$

abte in

$$-\frac{1}{2\omega} (1 + \frac{1}{2} \omega + \frac{1}{4} \omega^2 \omega),$$

unde primum et tertium membrum simul faciunt

$$\frac{1}{2\omega} (\frac{1}{2} \omega + \frac{1}{4} \omega^2 + \frac{1}{2} \pi^2 \omega \omega) = \frac{1}{4} + \frac{1}{4} \omega + \frac{1}{2} \pi^2 \omega$$

qui valor posito  $\omega = 0$  fit  $= \frac{1}{4}$ , unde secundum membrum, quod erit  $-\frac{1}{4}$ , iungtum dabit totam summam quæsitam  $-\frac{1}{4}$ , ita vt sit mutatis signis

$$\frac{1}{4} = \frac{1}{4} - \frac{1}{4} + \frac{1}{8} - \frac{1}{8} + \frac{1}{16} - \frac{1}{16} + \frac{1}{32} \text{ etc.}$$

cuius ratio est manifesta, cum sit

$$\frac{1}{4} = \frac{1}{4} (1 - \frac{1}{2}); \frac{1}{8} = \frac{1}{8} (1 - \frac{1}{2}); \frac{1}{16} = \frac{1}{16} (1 - \frac{1}{2}); \text{ etc.}$$

his enim valoribus substitutis et sublati terminis se destructibus fiet  $\frac{1}{4} = \frac{1}{4} - \frac{1}{8} + \frac{1}{8} - \frac{1}{16} + \frac{1}{16} = \frac{1}{4}$ .

§. 10. Circa eandem autem seriem quæstio magis ardua occurrit, qua quaeritur summa seriei, si  $\lambda$  fuerit numerus negativus, ideoque  $\lambda$  quantitas imaginaria. Ponatur igitur  $\lambda = -\mu + i$ , sine  $\lambda = \mu + i - 1$ , ac series nihilominus erit realis, scilicet

$$\frac{1}{1 + \mu + i} - \frac{1}{1 + \mu + i} + \frac{1}{1 + \mu + i} - \frac{1}{1 + \mu + i} + \dots \text{ etc}$$

cuius ergo summa erit

$$\frac{\pi}{2\mu + \pi} + \frac{1}{2\mu + \pi} \text{,}$$

0 3

cuius

cuus ergo valor realis quaeritur, siquidem nullum est du-  
bium quin ferrei valor fiat realis.

§. 11. In doctrina angulorum ostendi solet esse

$$\sin \varphi = \frac{e^{\varphi \gamma - 1} - e^{-\varphi \gamma - 1}}{2 \gamma - 1}$$

Fiat igitur  $\varphi = \mu \pi \gamma - 1$ , eritque

$$\varphi \gamma - 1 = -\mu \pi \text{ et } -\varphi \pi \gamma - 1 = 2 \mu \pi,$$

unde concluditur

$$\sin \mu \pi \gamma - 1 = \frac{e^{-\mu \pi} - e^{+\mu \pi}}{2 \gamma - 1}$$

unde summa quaesita erit

$$\frac{\mu (e^{-\mu \pi} - e^{+\mu \pi})}{\pi} + \frac{1}{2 \mu \mu}$$

Erit igitur

$$\frac{1}{4 + \mu \mu} + \frac{1}{4 + \mu \mu} + \frac{1}{16 + \mu \mu} + \frac{1}{36 + \mu \mu} - \text{etc.}$$

$$= \frac{1}{2 \mu^2} - \frac{\mu (e^{2 \mu \pi} - e^{-2 \mu \pi})}{\pi}$$

2°. Sit  $P = \varphi$  et fractio proposita  $\frac{\varphi}{\mu \pi}$ .

§. 12. Hic ergo ob numeratorem  $P = \varphi$  factor  
denominatoris primus  $\varphi$  tollitur, quemadmodum etiam no-  
stra resolutio numeratorem ipsi respondentem nihil praebet  
aequaltem. Hic igitur pro denominatore  $\varphi - i \pi$  sic nume-  
rator  $\frac{\varphi - i \pi}{\varphi - i \pi} = 1 + i \pi$ , ubi signum superius valet si  $i$  nu-  
merus par, inferius si impar. Quod si ergo fuerit  $i = 2 n$ ,  
fractio inde nata erit  $\frac{1 + i \pi}{\varphi - 2 n \pi}$ ; at si  $i = 2 n - 1$ , fractio erit

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=  $\varphi$  factor  
etiam no-  
nullo praebet  
1 sic nume-  
rator  $i = 2 n$ ,  
erit  $i = 2 n$ ,  
erit  $i = 2 n$ ,  
fractio erit

denique ex  $i = 2 n - 1$  oritur  $\frac{1 + i \pi}{\varphi + (2 n - 1) \pi}$ , quocirca series in-  
venia erit

$$\frac{\varphi}{\mu \pi} = \frac{\varphi}{\mu \pi} + \frac{\pi}{\varphi + \pi} + \frac{\pi}{\varphi + 2 \pi} + \frac{\pi}{\varphi + 3 \pi} + \frac{\pi}{\varphi + 4 \pi} + \frac{\pi}{\varphi + 5 \pi} + \frac{\pi}{\varphi + 6 \pi} + \frac{\pi}{\varphi + 7 \pi} + \frac{\pi}{\varphi + 8 \pi} + \frac{\pi}{\varphi + 9 \pi} + \frac{\pi}{\varphi + 10 \pi} + \text{etc.}$$

unde si bini termini in unum contrahantur, erit

$$\frac{\varphi}{\mu \pi} = \frac{2 \pi \pi}{\pi - \varphi} + \frac{4 \pi \pi}{\pi + \varphi} + \frac{6 \pi \pi}{\pi - \varphi} + \frac{8 \pi \pi}{\pi + \varphi} + \frac{10 \pi \pi}{\pi - \varphi} + \frac{12 \pi \pi}{\pi + \varphi} + \text{etc.}$$

quae per  $2 \pi \pi$  divisa producit hanc summationem:

$$\frac{\varphi}{\mu \pi} = \frac{2 \pi \pi}{\pi - \varphi} + \frac{4 \pi \pi}{\pi + \varphi} + \frac{6 \pi \pi}{\pi - \varphi} + \frac{8 \pi \pi}{\pi + \varphi} + \frac{10 \pi \pi}{\pi - \varphi} + \frac{12 \pi \pi}{\pi + \varphi} + \text{etc.}$$

Ac si ponatur  $\varphi = \lambda \pi$ , prohibet

$$\frac{\lambda \pi}{\mu \pi} = \frac{2 \pi \pi}{\pi - \lambda \pi} + \frac{4 \pi \pi}{\pi + \lambda \pi} + \frac{6 \pi \pi}{\pi - \lambda \pi} + \frac{8 \pi \pi}{\pi + \lambda \pi} + \frac{10 \pi \pi}{\pi - \lambda \pi} + \frac{12 \pi \pi}{\pi + \lambda \pi} + \text{etc.}$$

unde si fuerit  $\lambda \pi = -\mu \pi$ , seu  $\lambda = \mu \gamma - 1$ , ob

$$\sin \mu \pi \gamma - 1 = \frac{e^{-\mu \pi} - e^{+\mu \pi}}{2 \gamma - 1}, \text{ erit}$$

atque hinc per differentiationem infinitas alias summationes  
deducere licebit

$$\frac{\mu \pi}{\mu \pi} = \frac{1}{1 + \mu \mu} + \frac{1}{4 + \mu \mu} + \frac{1}{9 + \mu \mu} + \frac{1}{16 + \mu \mu} + \text{etc.}$$

3°. Sit numerator  $P = \varphi$  et fractio  $\frac{\varphi}{\mu \pi}$ .

§. 13. Pro denominatore igitur  $\varphi - i \pi$  numerator  
erit  $1 + i \pi$ , ubi signum superius valet pro  $i$  numero  
pari, inferius vero pro impari. Hinc si loco  $i$  faccassus  
scribamur numeri

$$+ 1 - 1 + 2 - 2 + 3 - 3 + \text{etc.}$$



Est vero  $(\mu V - 1)^n = \mu^{2n}$ , unde erit  $\lambda^n = \mu^{2n+1} V - 1$ ,  
hincque prodit sequens summatio realis :

$$\frac{\mu^{2n+1} \pi}{e^{2n\pi} - e^{-2n\pi}} = \frac{1}{1 + \mu\mu} - \frac{2^{2n+1}}{4 + \mu\mu} + \frac{3^{2n+1}}{9 + \mu\mu} - \frac{4^{2n+1}}{16 + \mu\mu} + \text{etc.}$$

Aliero autem casu, quo  $\psi = 4n - 1$ , prius membrum capi  
debet negative, eritque

$$-\frac{\mu^{2n+1} \pi}{e^{2n\pi} - e^{-2n\pi}} = \frac{1}{1 + \mu\mu} - \frac{2^{2n-1}}{4 + \mu\mu} + \frac{4^{2n-1}}{9 + \mu\mu} - \text{etc.}$$

Hæc autem summationes facile patet veras esse non posse, nisi  
 $\psi$  sit numerus integer impar et quidem positivus.

5. Sit numerator  $P = \phi^2$ , denotante  $\delta$  numerum pa-  
rem positivum quencunque, et fractio  $\frac{\phi^2}{\sin \phi}$ .

§. 15. Pro denominatore ergo  $\phi - i\pi$  numerator  
erit  $+ \frac{2^2 \pi^2}{3}$ , ambiguitate signorum eandem legem tenente,  
Hæc igitur casu ratio signorum perinde se habebit ac casu  
 $P = \phi \phi$ , eritque idcirco.

$$\frac{\sin \psi}{\psi} = \frac{\psi^2}{\pi - \psi} - \frac{\psi^2}{\pi + \psi} + \frac{\psi^2 \pi^2}{2\pi - \psi} + \frac{\psi^2 \pi^2}{2\pi + \psi} + \frac{\psi^2 \pi^2}{3\pi - \phi} + \frac{\psi^2 \pi^2}{3\pi + \phi} + \text{etc.}$$

Quare si ponamus  $\phi = \lambda \pi$  erit hæc series

$$\frac{\lambda^2 \pi^2}{\sin \lambda \pi} = \frac{1}{1 - \lambda} - \frac{1}{1 + \lambda} + \frac{2^2}{2 - \lambda} + \frac{2^2}{2 + \lambda} + \frac{3^2}{3 - \lambda} + \frac{3^2}{3 + \lambda} - \text{etc.}$$

hinc binis terminis in vnum contrahendis fiet

$$\frac{\lambda^2 \pi^2}{2 \sin \lambda \pi} = \frac{\pi}{1 - \lambda \lambda} - \frac{\pi}{4 - \lambda \lambda} + \frac{4^2}{3 - \lambda \lambda} - \frac{4^2}{4 - \lambda \lambda} + \text{etc.}$$

§. 16.

§. 16. Statuamus nunc etiam  $\lambda = \mu V - 1$ , ut sit

$$\sin \lambda \pi = \frac{e^{-\lambda \pi} - e^{+\lambda \pi}}{2 V - 1}$$

Pro valore autem ipsius  $\lambda^2 \pi^2$  duos verum casus evoluti  
oportet, prout fuerit vel  $\delta = 4n$ , vel  $\delta = 4n + 2$ . Prior  
casu, quo  $\delta = 4n$ , erit  $\lambda^{2n} = \mu^{2n}$ , ideoque  $\lambda^{2n+1} = \frac{\mu^{2n+1}}{V - 1}$ ;  
atque hinc orietur ista summatio :

$$\frac{-\mu^{2n+1} \pi}{e^{2n\pi} - e^{-2n\pi}} = \frac{\pi^{2n}}{1 + \mu\mu} - \frac{2^{2n}}{4 + \mu\mu} + \frac{3^{2n}}{9 + \mu\mu} - \frac{4^{2n}}{16 + \mu\mu} + \text{etc.}$$

Pro alio autem casu  $\delta = 4n + 2$  summatio ita se habebit:

$$\frac{+\mu^{2n+1} \pi}{e^{2n\pi} - e^{-2n\pi}} = \frac{\pi^{2n+2}}{1 + \mu\mu} - \frac{2^{2n+2}}{4 + \mu\mu} + \frac{3^{2n+2}}{9 + \mu\mu} - \text{etc.}$$

§. 17. Hæc autem summationes eatenus tantum ve-  
riantur consentaneæ, quatenus pro exponentibus  $\psi$  et  
 $\delta$  numeri integri, prout sunt definiti, accipiantur, nihilque  
impedit quo minus quantumvis magni assumantur. Cum  
enim denominator

$$\sin \phi = \phi - i\phi^2 + i\phi^3 \phi^2 - \text{etc.}$$

ad dimensiones infinitas ipsius  $\phi$  affurgat, dimmodo maxi-  
ma potestas in denominatore non sit infata, resolutio in fractio-  
nes semper ad veritatem perducit. Sin autem exponentes  
illi non essent integri positivi, sed fracti, vel adeo negativi,  
resolutio in fractiones partiales locum plane habere nequit.  
Quamobrem si loco numeratoris  $P$  eiusmodi functiones ip-  
sius  $\phi$  statuamus, quæ etiam ad infinitum dimensionum nu-  
merum adsurgant, cum de summa inuenta non amplius eri-  
mus

V - 1;  
+ etc.  
im capi  
c  
a  
P  
offe, nisi  
um pa-  
 $\frac{\phi^2}{\sin \phi}$   
numerator  
tenente  
ac casu  
 $\frac{2^2 \pi^2}{3}$   
 $3\pi - \phi$   
+ etc.  
+ etc.  
+ etc.  
§. 16.



Verum fieri potest, ut ad fractiones parciales inuenias insuper quasdam partes integrae adici debeant. Huiusmodi igitur casus aliquos euoluamus.

6. Si numerator P = cof. φ et fractio =  $\frac{cof. \phi}{sin. \phi}$ .

§ 18. Cum sit

$$cof. \phi = 1 - \frac{1}{2} \phi^2 + \frac{1}{24} \phi^4 - \frac{1}{720} \phi^6 + \dots etc.$$

potestates ipsius φ in numeratore aequae in infinitum exsurgunt atque in denominatore; unde fieri potest, ut haec fractio partem integram inuolueret, quae cum reperiatur si sumatur φ = ∞, foret ista pars integra =  $\frac{cof. \infty}{sin. \infty} = cof. \infty$ , quae autem in se prioris est indeterminata. Interim tamen, quia eandem casibus euadere potest negatiua atque positiua, medium sumendo valor recte videri potest = 0; ceterum dubium persequentem euolutionem tollitur. Cum pro denominatore φ = iπ fiat A = cof. iπ et C = cof. iπ, erit numerator huius fractionis = 1; hinc ergo nascitur sequens series:

$$\frac{cof. \phi}{sin. \phi} = \phi + \frac{1}{3} \phi^3 + \frac{1}{5} \phi^5 + \frac{1}{7} \phi^7 + \frac{1}{9} \phi^9 + \dots etc. \text{ sine} \\ \text{coag. } \phi = \phi - \frac{1}{3} \phi^3 + \frac{1}{5} \phi^5 - \frac{1}{7} \phi^7 + \frac{1}{9} \phi^9 - \dots etc.$$

Posito igitur φ = λπ, haec series induet hanc formam:

$$\pi \text{ cof. } \lambda \pi = \lambda - \frac{1}{3} \lambda^3 + \frac{1}{5} \lambda^5 - \frac{1}{7} \lambda^7 + \frac{1}{9} \lambda^9 - \frac{1}{11} \lambda^{11} + \dots etc.$$

quae summatio an vera sit per casus inuestigamus. Ac primo quidem si λ denotet numerum integrum, veritas confirmatur; semper enim aliquis feriei terminus sit infinitus; summa vero quoque sit infinita. Summus autem λ =  $\frac{1}{2}$  erit π cof.  $\frac{\pi}{2} = 0$ , ipsa autem series prodiit

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots etc.$$

vbi

omnes parciales diuici debeant.

$$C = \frac{cof. \phi}{sin. \phi}.$$

etc.

rum exsurgunt fractio partem sumatur φ = ∞, e autem in se ia eandem casum medium sumendum denominato- erit numera- sequens series:

$$\frac{1}{1+\phi} + \dots etc. \text{ sine} \\ \frac{1}{1+\phi} - \dots etc.$$

ic formam:

$$- \frac{1}{1-\lambda} + \dots etc.$$

mus. Ac primo rias confirmat fit infinitus; rem λ =  $\frac{1}{2}$  erit

vbi

vbi omnes termini se manifesto destruant. Summus autem insuper λ =  $\frac{1}{2}$ , prodiibique

$$\pi = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots etc.$$

quae est series nouissima Leibniziana. Sicque omne dubium circa veritatem huius summationis euancescit.

§ 19. Contrahamus binos terminos, primo excepto, in singulos, et obtinebimus

$$\pi \text{ cof. } \lambda \pi = \lambda - \frac{1}{3} \lambda^3 + \frac{1}{5} \lambda^5 - \frac{1}{7} \lambda^7 + \frac{1}{9} \lambda^9 - \frac{1}{11} \lambda^{11} + \dots etc.$$

quae series reducitur ad hanc formam:

$$\frac{1}{2\lambda\lambda} - \frac{\pi \text{ cof. } \lambda \pi}{2\lambda} = \frac{1}{1-\lambda\lambda} + \frac{1}{1-\lambda\lambda} + \frac{\sigma}{\pi-\lambda\lambda} + \frac{\sigma}{\pi-\lambda\lambda} + \dots etc.$$

Quod si hic iterum statnamus λ = μ√-1, ob

$$cof. \mu \pi \sqrt{-1} = \frac{e^{-\mu\pi} + e^{\mu\pi}}{2} \text{ et}$$

$$sin. \mu \pi \sqrt{-1} = \frac{e^{-\mu\pi} - e^{\mu\pi}}{2\sqrt{-1}}$$

haec obtinebuntur summatio:

$$\frac{1}{2\mu\mu} + \frac{\pi(e^{+\mu\pi} + e^{-\mu\pi})}{(e^{+\mu\pi} - e^{-\mu\pi})2\mu} = \frac{1}{1+\mu\mu} + \frac{1}{1+\mu\mu} + \frac{1}{1+\mu\mu} + \dots etc.$$

Nunc autem per se est manifestum, per differentiationem simili modo ut supra inuisitas alias summationes obtineri posse.

II. Summar  $Q = \cos \xi - \cos \phi$ , vt fractio resolunda

§. 20. Cum sit denominator  $Q = \cos \xi - \cos \phi$ , vbi: angulus  $\xi$  vt datus et constans spectatur, is sequentibus casibus evanescit,

$$\phi = \pm \xi, \phi = \pm 2\pi \pm \xi, \phi = \pm 4\pi \pm \xi;$$

$$\phi = \pm 6\pi \pm \xi, \phi = \pm 8\pi \pm \xi; \text{ etc.}$$

ideoque in genere  $\phi = \pm i\pi \pm \xi$  vbi  $i$  denotat omnes numeros pares tam negativos quam positivos; vnde designatores fractionum simplicium quas quaerimus erunt

$$\phi - \xi, \phi + \xi, \phi - 2\pi - \xi, \phi + 2\pi - \xi, \phi + 2\pi + \xi, \text{ etc.}$$

hocque modo omnes fractiones simplices reperiemus, quarum omnium summa aequalis esse debet fractioni propositae

$$\frac{\cos \xi - \cos \phi}{\phi}$$

§. 21. Consideremus nunc primo denominatorem simplicem in genere  $\phi - i\pi - \xi$  ac posito  $\phi = i\pi + \xi$  abeat numerator P in A. Deinde cum ex denominatore fiat  $\frac{A}{\phi} = \text{fn. } \phi$ , erit C = fn. ( $i\pi + \xi$ ) = fn.  $\xi$ , vnde numerator huius fractionis erit  $\frac{A}{\phi} = \frac{A}{\text{fn. } \xi}$ , ideoque fractio hinc nata

$$\frac{\text{fn. } \xi}{\text{fn. } \xi} = 1$$

At vero pro denominatore  $\phi - i\pi + \xi$ , si in. numerore P ponatur  $\phi = i\pi - \xi$ ; prodit quantitas B; ex denominatore autem fiet

$$C = \text{fn. } (i\pi - \xi) = -\text{fn. } \xi,$$

vnde oritur ista fractio:  $-\frac{A}{\text{fn. } \xi}$  Nunc igitur tantum opus est vt loco  $i$  succedat omnes numeri pares tam positivi quam negativi substituatur.

1°.

olunda

$-\cos \phi$ ,  
juendibus

$$\pm \xi;$$

omnes nu-  
lenomina-

$\pi + \xi$ , etc.

us, qua-  
propositae

minatorem

$\pi + \xi$  ab-  
natore fat

numerator  
hinc nata

$-i\pi + \xi$ ,  
quantitas

Nunc igitur  
numeri pa-

1°.

1°. Sit numerator P = 1 et fractio proposita

§. 22. Pro binis igitur formulis generalibus critam A = 1 quam B = 1, vnde istae fractiones generales sunt

$$\frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi} = \frac{\text{fn. } \xi (\phi - i\pi + \xi)}{\phi} = \frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi} + \frac{\text{fn. } \xi (\phi - i\pi + \xi)}{\phi}$$

$$\frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi} = \frac{\text{fn. } \xi (\phi - \xi)}{\phi} + \frac{\text{fn. } \xi (\phi - i\pi)}{\phi} + \frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi}$$

$$+ \frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi} + \frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi} + \frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi} + \text{etc.}$$

$$\frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi} = \frac{\text{fn. } \xi (\phi - \xi)}{\phi} + \frac{\text{fn. } \xi (\phi - i\pi)}{\phi} + \frac{\text{fn. } \xi (\phi - i\pi - \xi)}{\phi} + \text{etc.}$$

§. 23. Quod si ergo fuerit  $\xi = 0$ , erit

$$\frac{1}{\phi} = \frac{1}{\phi} + \frac{1}{\phi} + \frac{1}{\phi} + \frac{1}{\phi} + \frac{1}{\phi} + \frac{1}{\phi} + \text{etc.}$$

Si nunc porro  $\phi = \pi$ , erit haec summatio:

$$\frac{1}{\pi} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048} + \frac{1}{4096} + \frac{1}{8192} + \frac{1}{16384} + \frac{1}{32768} + \frac{1}{65536} + \frac{1}{131072} + \frac{1}{262144} + \frac{1}{524288} + \frac{1}{1048576} + \frac{1}{2097152} + \frac{1}{4194304} + \frac{1}{8388608} + \frac{1}{16777216} + \frac{1}{33554432} + \frac{1}{67108864} + \frac{1}{134217728} + \frac{1}{268435456} + \frac{1}{536870912} + \frac{1}{1073741824} + \frac{1}{2147483648} + \frac{1}{4294967296} + \frac{1}{8589934592} + \frac{1}{17179869184} + \frac{1}{34359738368} + \frac{1}{68719476736} + \frac{1}{137438953472} + \frac{1}{274877906944} + \frac{1}{549755813888} + \frac{1}{1099511627776} + \frac{1}{2199023255552} + \frac{1}{4398046511104} + \frac{1}{8796093022208} + \frac{1}{17592186044416} + \frac{1}{35184372088832} + \frac{1}{70368744177664} + \frac{1}{140737488355328} + \frac{1}{281474976710656} + \frac{1}{562949953421312} + \frac{1}{1125899906842624} + \frac{1}{2251799813685248} + \frac{1}{4503599627370496} + \frac{1}{9007199254740992} + \frac{1}{18014398509481984} + \frac{1}{36028797018963968} + \frac{1}{72057594037927936} + \frac{1}{144115188075855872} + \frac{1}{288230376151711744} + \frac{1}{576460752303423488} + \frac{1}{1152921504606846976} + \frac{1}{2305843009213693952} + \frac{1}{4611686018427387904} + \frac{1}{9223372036854775808} + \frac{1}{18446744073709551616} + \frac{1}{36893488147419103232} + \frac{1}{73786976294838206464} + \frac{1}{147573952589676412928} + \frac{1}{295147905179352825856} + \frac{1}{590295810358705651712} + \frac{1}{1180591620717411303424} + \frac{1}{2361183241434822606848} + \frac{1}{4722366482869645213696} + \frac{1}{9444732965739290427392} + \frac{1}{18889465931478580854784} + \frac{1}{37778931862957161709568} + \frac{1}{75557863725914323419136} + \frac{1}{151115727451828646838272} + \frac{1}{302231454903657293676544} + \frac{1}{604462909807314587353088} + \frac{1}{1208925819614629174706176} + \frac{1}{2417851639229258349412352} + \frac{1}{4835703278458516698824704} + \frac{1}{9671406556917033397649408} + \frac{1}{19342813113834066795298816} + \frac{1}{38685626227668133590597632} + \frac{1}{77371252455336267181195264} + \frac{1}{154742504910672534362390528} + \frac{1}{309485009821345068724781056} + \frac{1}{618970019642690137449562112} + \frac{1}{1237940039285380274899124224} + \frac{1}{2475880078570760549798248448} + \frac{1}{4951760157141521099596496896} + \frac{1}{9903520314283042199192993792} + \frac{1}{19807040628566084398385987584} + \frac{1}{39614081257132168796771975168} + \frac{1}{79228162514264337593543950336} + \frac{1}{158456325028528675187087900672} + \frac{1}{316912650057057350374175801344} + \frac{1}{633825300114114700748351602688} + \frac{1}{1267650600228229401496703205376} + \frac{1}{2535301200456458802993406410752} + \frac{1}{5070602400912917605986812821504} + \frac{1}{10141204801825835211973625643008} + \frac{1}{20282409603651670423947251286016} + \frac{1}{40564819207303340847894502572032} + \frac{1}{81129638414606681695789005144064} + \frac{1}{162259276829213363391578010288128} + \frac{1}{324518553658426726783156020576256} + \frac{1}{649037107316853453566312041152512} + \frac{1}{1298074214633706907132624082305024} + \frac{1}{2596148429267413814265248164610048} + \frac{1}{5192296858534827628530496329220096} + \frac{1}{10384593717069655257060992658440192} + \frac{1}{20769187434139310514121985316880384} + \frac{1}{41538374868278621028243970633760768} + \frac{1}{83076749736557242056487941267521536} + \frac{1}{166153499473114484112975882535043072} + \frac{1}{332306998946228968225951765070086144} + \frac{1}{664613997892457936451903530140172288} + \frac{1}{13292279957849158729038070602803456} + \frac{1}{26584559915698317458076141205606912} + \frac{1}{53169119831396634916152282411213824} + \frac{1}{106338239662793269832304564822427648} + \frac{1}{212676479325586539664609129644855296} + \frac{1}{425352958651173079329218259289710592} + \frac{1}{850705917302346158658436518579421184} + \frac{1}{1701411834604692317316873037158842368} + \frac{1}{3402823669209384634633746074317684736} + \frac{1}{6805647338418769269267492148635369472} + \frac{1}{13611294676837538538534984297270738944} + \frac{1}{27222589353675077077069968594541477888} + \frac{1}{54445178707350154154139937189082955776} + \frac{1}{108890357414700308308279874378165911552} + \frac{1}{217780714829400616616559748756331823104} + \frac{1}{435561429658801233233119497512663646208} + \frac{1}{871122859317602466466238995025327212416} + \frac{1}{174224571863520493293247799005065424832} + \frac{1}{348449143727040986586495598010130849664} + \frac{1}{696898287454081973172991196020261699296} + \frac{1}{1393796574908163946345982392040523398592} + \frac{1}{2787593149816327892691964784081046797184} + \frac{1}{5575186299632655785383929568162093594368} + \frac{1}{11150372599265311570767859136324187188736} + \frac{1}{22300745198530623141535718272648374377472} + \frac{1}{44601490397061246283071436545296747554944} + \frac{1}{89202980794122492566142873090593495109888} + \frac{1}{178405961588244985132285746181186990219776} + \frac{1}{356811923176489970264571492362373980439552} + \frac{1}{713623846352979940529142984724747960879104} + \frac{1}{1427247692705959881058285969449495921758208} + \frac{1}{2854495385411919762116571938898991843516416} + \frac{1}{5708990770823839524233143877797983687032832} + \frac{1}{11417981541647679048466287755595967374065664} + \frac{1}{22835963083295358096932575511191934748131328} + \frac{1}{45671926166590716193865150222383869496262656} + \frac{1}{91343852333181432387730300444767738992525312} + \frac{1}{18268770466636286477546060088953547798505024} + \frac{1}{36537540933272572955092120177907095597010048} + \frac{1}{73075081866545145910184240355814191194020096} + \frac{1}{146150163733090291820368480711628382388040192} + \frac{1}{292300327466180583640736961423256764776080384} + \frac{1}{584600654932361167281473922846513529552160768} + \frac{1}{1169201309864722334562947845693027059104311536} + \frac{1}{2338402619729444669125895691386054118208623072} + \frac{1}{4676805239458889338251791382772108236417246144} + \frac{1}{9353610478917778676503582765544216472834492288} + \frac{1}{1870722095783555735300716553108843294566896576} + \frac{1}{37414441915671114706014331062176865891337931536} + \frac{1}{7482888383134222941202866212435373178275966272} + \frac{1}{14965776766268445882405732424870746356551932544} + \frac{1}{29931553532536891764811464849741492713103885088} + \frac{1}{59863107065073783529622929699482985426207770176} + \frac{1}{119726214130147567059245859398965970852415540352} + \frac{1}{239452428260295134118491718797931941704831080704} + \frac{1}{478904856520590268236983437595863883409662161408} + \frac{1}{957809713041180536473966875191727766819324322816} + \frac{1}{1915619426082361072947933750383455533638648645632} + \frac{1}{3831238852164722145895867500766911067277297291264} + \frac{1}{7662477704329444291791735001533822134554594582528} + \frac{1}{15324955408658888583583470003067644269109191655056} + \frac{1}{30649910817317777167166940006135288538218383310112} + \frac{1}{61299821634635554334333880012270577076436766620224} + \frac{1}{122599643269271108668667760024541154152873533240448} + \frac{1}{24519928653854221733733552004908230830574706648096} + \frac{1}{49039857307708443467467104009816461661149413296192} + \frac{1}{98079714615416886934934208019632923322298926592384} + \frac{1}{1961594292308337738698684160392658466445978531856} + \frac{1}{3923188584616675477397368320785316932891956633712} + \frac{1}{7846377169233350954794736641570633865783913267424} + \frac{1}{15692754338466701909589473283141267731567826534848} + \frac{1}{31385508676933403819178946566282535463135653069696} + \frac{1}{62771017353866807638357893132565070926271306139392} + \frac{1}{125542034707733615276715786265130141852542612278784} + \frac{1}{251084069415467230553431572530260283705085224557568} + \frac{1}{502168138830934461106863145060520567410170449115136} + \frac{1}{1004336277661868922213726290121041134820340898230272} + \frac{1}{2008672555323737844427452580242082269640681664460544} + \frac{1}{4017345110647475688854905160484164539281362928920896} + \frac{1}{8034690221294951377709810320968329078562725857841792} + \frac{1}{1606938044258990275541962064193665815712553171558368} + \frac{1}{3213876088517980551083924128387331631425106343116736} + \frac{1}{6427752177035961102167848256774663262850212666233728} + \frac{1}{1285550435407192220433569651354932652570042533246752} + \frac{1}{2571100870814384440867139302709865305140085066493504} + \frac{1}{5142201741628768881734278605419730610280170132987008} + \frac{1}{10284403483257537763468557210839461220560340265974016} + \frac{1}{20568806966515075526937114421678922441120680531948032} + \frac{1}{41137613933030151053874228843357844882241361063896064} + \frac{1}{8227522786606030210774845768671568976456272212778112} + \frac{1}{16455045573212060421549691537343137952912544255556224} + \frac{1}{32910091146424120843099382674686275905825088511112448} + \frac{1}{65820182292848241686198765349372551811651177022224896} + \frac{1}{13164036458569648337239752669874510322330354404444992} + \frac{1}{26328072917139296674479505339749020644660708808889984} + \frac{1}{52656145834278593348959010679498041289321417617779968} + \frac{1}{105312291668557186697918021358996082578642352235559936} + \frac{1}{210624583337114373395836042717992165157284704471119872} + \frac{1}{421249166674228746791672085435984330314568808942339744} + \frac{1}{84249833334845749358334417087196866062913761788467888} + \frac{1}{168499666669691498716668834174393732125827523576935776} + \frac{1}{336999333339382997433337668348787464251655047153871552} + \frac{1}{6739986666787659948666753366975749285033101143075504} + \frac{1}{13479973333575319897333506733951498570066202286151008} + \frac{1}{26959946667150639794667013467902997140132404572302016} + \frac{1}{53919893334301279589334026935805994280264809144604032} + \frac{1}{107839786668602559178668053871611988560529618289208064} + \frac{1}{215679573337205118357336107743223977121119236578416128} + \frac{1}{4313591466744102367146722154864479542422384731568352} + \frac{1}{8627182933488204734293444309728959084844763463136704} + \frac{1}{1725436586697640946858688861945791816968926892627408} + \frac{1}{3450873173395281893717377723891583633937853785254816} + \frac{1}{6901746346790563787434755447783167267875707570509328} + \frac{1}{1380349269358112757486951089556633453575141514101856} + \frac{1}{2760698538716225514973902179113266907150283028203712} + \frac{1}{5521397077432451029947804358226534142300566056407424} + \frac{1}{1104279415486490205989560871645306828460113211281488} + \frac{1}{2208558830972980411979121743290613676920226422562976} + \frac{1}{4417117661945960823958243486581227353840452845125952} + \frac{1}{8834235323891921647916486973162454707680905690251808} + \frac{1}{17668470647783843295832973946324909415361811380503616} + \frac{1}{35336941295567686591665947892649818830723622761007232} + \frac{1}{70673882591135373183331895785299637661447245522014464} + \frac{1}{14134776518227074636666379157059927532289489104402896$$

§. 24. Ponamus autem in genere  $\xi = a\pi$  et  $\phi = \lambda\pi$ ,  
 ut obtineatur ista summatio:

$$\frac{\pi \sin a\pi}{\pi \cos(\alpha\pi - \cos(\lambda\pi))} = \frac{1}{\lambda - a} + \frac{1}{(\lambda - 2)^2 - a^2} + \frac{1}{(\lambda + 2)^2 - a^2} + \text{etc.}$$

Quod si iam  $a$  fuerit quantitas imaginaria, siue  $a = \beta\sqrt{-1}$ ,  
 summatio haec erit:

$$\frac{\pi (\beta^2 - e^{\beta\pi})}{2\cos(\beta^2 + e^{\beta\pi}) - 2\cos(\lambda\pi)} = \frac{1}{\lambda + \beta\beta} + \frac{1}{(\lambda - 2)^2 + \beta\beta} + \frac{1}{(\lambda + 2)^2 + \beta\beta} + \frac{1}{(\lambda - 4)^2 + \beta\beta} + \frac{1}{(\lambda + 4)^2 + \beta\beta} + \text{etc.}$$

§. 25. Hinc si proponatur haec fractio in seriem  
 resoluenda:  $\frac{1}{a^2 - \cos^2 \theta}$ , siue  $\frac{1}{a^2 - \cos^2 \lambda\pi}$ , duos casus considerari oportet,  
 prout  $a$  fuerit vel vitare minor vel maior. Sit  $a < 1$ ,  
 ut feri queat  $a = \cos \alpha\pi$ , unde sit  $\alpha = \frac{\lambda \cos \alpha}{\pi}$ , atque in-  
 venio  $\alpha$  reperitur

$$\frac{1}{a^2 - \cos^2 \lambda\pi} = \frac{1}{\pi^2 (\lambda^2 - a^2)} + \frac{1}{(\lambda - 2)^2 - a^2} + \frac{1}{(\lambda + 2)^2 - a^2} + \frac{1}{(\lambda - 4)^2 - a^2} + \text{etc.}$$

Sin autem fuerit  $a > 1$ , quaerit debet  $\beta$ , ut fiat

$$\frac{\beta^2 \pi^2 + e^{-\beta\pi}}{2} = a.$$

Hinc ergo fiet  $e^{\pm i\beta\pi} + 1 = 2ae^{\beta\pi}$ , unde radice extrahâ re-  
 peritur  $\beta^2 \pi = a + \sqrt{(aa - 1)}$  hincque  $e^{\beta\pi} = a - \sqrt{(aa - 1)}$   
 unde porro fiet

$$\beta \pi = i(a + \sqrt{(aa - 1)}), \text{ ergo}$$

$$\beta = \frac{i}{\pi} i(a + \sqrt{(aa - 1)}).$$

Invenio igitur hoc numero  $\beta$  potissima formula præbet  
 hanc seriem:

$$\frac{\pi \sqrt{(aa - 1)}}{\beta(a - \cos(\lambda\pi))} = \frac{1}{\lambda - a} + \frac{1}{(\lambda - 2)^2 + \beta\beta} + \frac{1}{(\lambda + 2)^2 + \beta\beta} + \frac{1}{(\lambda - 4)^2 + \beta\beta} + \text{etc.}$$

et  $\phi = \lambda\pi$ ,

$$\frac{1}{\lambda - a} + \frac{1}{(\lambda - 2)^2 + \beta\beta} + \frac{1}{(\lambda + 2)^2 + \beta\beta} + \text{etc.}$$

in seriem  
 vari oportet  
 sit  $a < 1$ ,  
 atque in-

$$\frac{1}{\lambda - a} + \frac{1}{(\lambda - 2)^2 + \beta\beta} + \frac{1}{(\lambda + 2)^2 + \beta\beta} + \text{etc.}$$

extrahâ re-  
 $\sqrt{(aa - 1)}$

a præbet  
 $\frac{1}{\lambda - a} + \frac{1}{(\lambda - 2)^2 + \beta\beta} + \text{etc.}$   
 confi-

consequenter habebimus

$$\frac{1}{\pi \sqrt{(a^2 - 1)}} \left( \frac{1}{\lambda - a} + \frac{1}{(\lambda - 2)^2 + \beta\beta} + \frac{1}{(\lambda + 2)^2 + \beta\beta} + \frac{1}{(\lambda - 4)^2 + \beta\beta} + \text{etc.} \right)$$

casu autem medio, quo  $a = 1$ , fit  $\omega = 0$ ; cum vero po-  
 natur  $a = 1 - \infty$ , eritque

$$A \cos(\pi - \omega) = A \sin \sqrt{(2\omega - \omega\omega)} = \sqrt{(2\omega - \omega\omega)}$$

Est vero etiam

$$\sqrt{(1 - a^2)} = \sqrt{(2\omega - \omega\omega)},$$

unde pro hoc casu seriet summatio erit

$$\frac{1}{\pi \cos \lambda\pi} = \frac{1}{\pi} \left( \frac{1}{\lambda} + \frac{1}{(\lambda - 2)^2} + \frac{1}{(\lambda + 2)^2} + \frac{1}{(\lambda - 4)^2} + \frac{1}{(\lambda + 4)^2} + \text{etc.} \right)$$

Cum igitur sit

$$1 - \cos \lambda\pi = 2 \sin \frac{1}{2} \lambda\pi^2,$$

habebimus hanc summationem:

$$\frac{\pi \pi}{4 \sin \frac{1}{2} \lambda\pi^2} = \frac{1}{\lambda} + \frac{1}{(\lambda - 2)^2} + \frac{1}{(\lambda + 2)^2} + \frac{1}{(\lambda - 4)^2} + \text{etc.}$$

quae series iam supra § 23 est inuenta.

2°. Sit nunc  $P = \sin \phi$  et fractio propolita  $\frac{\sin \phi}{\cos \xi - \cos \phi}$ .

§. 26. Cum igitur sit  $P = \sin \phi$  summo,  $\phi = i\pi + \xi$   
 erit  $A = \sin(\pi + \xi) = \sin \xi$ ; at posito  $\phi = i\pi - \xi$  prodit  
 $B = -\sin \xi$ ; hinc binæ fractiones inde resultantis erunt

$$\frac{1}{\phi - i\pi - \xi} + \frac{1}{\phi - i\pi + \xi} = \frac{2\phi - 2i\pi}{(\phi - i\pi)^2 - \xi^2}$$

Quare si loco  $i$  succedant omnes eius valores scribamus, nan-  
 ciscitur sequentem seriem:

$$\frac{1}{\cos \xi - \cos \phi} = \frac{1}{\phi - \xi} + \frac{1}{\phi - i\pi - \xi} + \frac{1}{\phi - i2\pi - \xi} + \frac{1}{\phi - i3\pi - \xi} + \frac{1}{\phi - i4\pi - \xi} + \text{etc.}$$

siue  $\frac{f_m \cdot \phi}{a - \cos \phi} = \frac{\phi}{\phi - \frac{1}{2}\pi} + \frac{\phi - \frac{1}{2}\pi}{\phi - \frac{3}{2}\pi} + \frac{\phi - \frac{3}{2}\pi}{\phi - \frac{5}{2}\pi} + \dots$

§. 27. Hinc si fuerit  $\xi = 0$ , erit

$$\frac{f_m \cdot \phi}{a - \cos \phi} = \frac{\phi}{\phi - \frac{1}{2}\pi} + \frac{\phi - \frac{1}{2}\pi}{\phi - \frac{3}{2}\pi} + \frac{\phi - \frac{3}{2}\pi}{\phi - \frac{5}{2}\pi} + \dots$$

cuius igitur seriei su. ra est  $\frac{1}{2} \cot \frac{1}{2} \phi$ . Hinc si ponamus  $\phi = \lambda \pi$ , erit

$$\frac{1}{2} \pi \cot \frac{1}{2} \lambda \pi = \lambda + \frac{1}{\lambda - 2} + \frac{1}{\lambda - 4} + \frac{1}{\lambda - 6} + \dots$$

et contrahendis binis terminis

$$\frac{1}{2} \pi \cot \frac{1}{2} \lambda \pi = \lambda + \frac{\lambda^2}{\lambda^2 - 4} + \frac{\lambda^2}{\lambda^2 - 16} + \dots$$

hincque

$$\frac{1}{2 \lambda \lambda} \frac{\pi \cot \frac{1}{2} \lambda \pi}{4 \lambda} = \frac{1}{4 - \lambda \lambda} + \frac{1}{16 - \lambda \lambda} + \frac{1}{36 - \lambda \lambda} + \dots$$

Quod si hic loco  $\lambda$  scribamus  $2 \lambda$ , habebimus

$$\frac{1}{4 \lambda \lambda} \frac{\pi \cot \lambda \pi}{4 \lambda} = \frac{1}{4 - \lambda \lambda} + \frac{1}{16 - \lambda \lambda} + \frac{1}{36 - \lambda \lambda} + \dots$$

siue

$$\frac{1}{2 \lambda \lambda} \frac{\pi \cot \lambda \pi}{4 \lambda} = \frac{1}{4 - \lambda \lambda} + \frac{1}{16 - \lambda \lambda} + \frac{1}{36 - \lambda \lambda} + \dots$$

quae series est plane eadem, quam supra § 19 invenimus.

§. 28. Ponamus nunc ut supra  $\xi = a \pi$  et  $\phi = \lambda \pi$ ,

ut obtineatur sequens series :

$$\frac{\pi f_m \cdot \lambda \pi}{a \cos \lambda \pi - \cos a \pi} = \frac{\lambda}{\lambda - a \alpha} + \frac{\lambda + \pi}{\lambda - \pi - a \alpha} + \frac{\lambda + 2\pi}{\lambda - 2\pi - a \alpha} + \dots$$

Sin autem hic ponatur  $a = \beta \gamma - 1$ , ista series sequentem induet formam:

$$\frac{\pi f_m \cdot \lambda \pi}{\beta^2 \pi + \gamma^2 \pi - 2 \cos \lambda \pi} = \frac{\lambda}{\lambda + 2} + \frac{\lambda - 2}{(\lambda - 2)^2 + \beta \beta} + \frac{\lambda}{\lambda + 2} + \frac{\lambda - 4}{(\lambda - 4)^2 + \beta \beta} + \dots$$

§. 29.

$\frac{1}{2} \pi + \dots$

$\frac{1}{2} \pi + \dots$

i ponamus

$\frac{1}{2} \pi + \dots$

$\frac{1}{2} \pi + \dots$

$\frac{1}{2} \pi + \dots$

$\frac{1}{2} \pi + \dots$

$\frac{1}{2} \pi + \dots$

et  $\phi = \lambda \pi$ ,

$\frac{1}{2} \pi + \dots$

$\frac{1}{2} \pi + \dots$

$\frac{1}{2} \pi + \dots$

§. 29.

§. 29. Quod si igitur propofita fuerit haec fractio: siue  $\frac{f_m \cdot \lambda \pi}{a - \cos \lambda \pi}$  iterum duos calus euolui conuenit, alterum quo  $a < 1$ , alterum quo  $a > 1$ . Priore quidem casu, quo  $a < 1$ , statuitur  $\cos a \pi = a$ , vnde fit  $a = \frac{1}{\text{Arc cos } a}$ , quo inuenio erit

$$\frac{f_m \cdot \lambda \pi}{a - \cos \lambda \pi} = \frac{1}{2} \left( \frac{\lambda}{\lambda - a \alpha} + \frac{\lambda - 2}{\lambda - \pi - a \alpha} + \frac{\lambda + 2}{\lambda + \pi - a \alpha} + \dots \right)$$

Sin autem  $a > 1$ , quaeri debet  $\beta$ , ita ut sit ut ante

$$\beta = \frac{1}{2} (a + \sqrt{a^2 - 1})$$

quo valore inuenio erit

$$\frac{f_m \cdot \lambda \pi}{a - \cos \lambda \pi} = \frac{1}{2} \left( \frac{\lambda}{\lambda - \beta \alpha} + \frac{\lambda - \beta}{\lambda - \beta \pi - \alpha \alpha} + \frac{\lambda + \beta}{\lambda + \beta \pi - \alpha \alpha} + \dots \right)$$

Sin autem fuerit  $a = 1$ , cum sit tam  $a = 0$  quam  $\beta = 0$ , eademque series resulat, quam supra ex casu  $\xi = 0$  eliquimus. Hinc ergo si sumatur  $\lambda = \frac{1}{2}$  prohibet series

$$\frac{\pi}{a \cos \alpha \pi} = \frac{1}{1 - a \alpha} - \frac{1}{1 - \pi - a \alpha} + \frac{1}{1 + \pi - a \alpha} - \dots$$

vel etiam haec :

$$\frac{\pi}{\beta \pi - \gamma^2 \pi} = \frac{2}{1 + 4 \beta \beta} + \frac{6}{9 + 4 \beta \beta} + \frac{10}{25 + 4 \beta \beta} + \frac{14}{49 + 4 \beta \beta} + \dots$$

Ceterum per se intelligitur, per differentiationem plurimas alias series formari posse.

### III. Sic fractio resoluenda $\frac{\cos \phi - \cos a \pi}{a \cos \lambda \pi - \cos a \pi}$

§. 30. Ante omnia igitur hic quaeri debet, quibusnam casibus iste denominator euanescat. Cum igitur in genere sit  $\cos \phi = \cos (i \pi + \phi)$ , denotante  $i$  numerum parum, similique modo  $\cos 2 \phi = \cos (i' \pi + 2 \phi)$ , habebimus  $i \pi \pm \phi = i' \pi \pm 2 \phi$ , vnde ob ambiguitatem signorum sequentes calus eruntur:  $\phi = i \pi$ ,  $\phi = \frac{1}{2} \pi$ . Hic autem pro-

2

b: est observandum; casus priores his occurrere, seu factores hinc natos  $\phi - i\pi$  bis esse collocandos, ita ut factor denominatoris sit  $(\phi - i\pi)^2$ . Quod cum minus clare appareat, ita ostendamus: quoniam in genere est

$$\text{col. } a - \text{col. } b = 2 \text{ fin. } \frac{a+b}{2}, \text{ fin. } \frac{b-a}{2}$$

erit noster denominator  $2 \text{ fin. } \frac{1}{2} \phi \text{ fin. } \frac{1}{2} \phi$ , qui igitur evanescit tam quando  $\text{fin. } \frac{1}{2} \phi = 0$  quam quando  $\text{fin. } \frac{1}{2} \phi = \pi$ . Fit autem  $\text{fin. } \frac{1}{2} \phi = 0$  quoties  $\phi = i\pi$ ; denotante  $i$  omnes numeros integros, ideoque  $\phi = 2i\pi$ . Similique modo  $\text{fin. } \frac{1}{2} \phi = \pi$ , si  $\frac{1}{2} \phi = i\pi$  ideoque  $\phi = 2i\pi$ , quae posterior formula, quoties  $i$  est numerus integer, priores casus suppleat; sicque manifestum est, in factoribus occurrere omnia quadrata  $(\phi - i\pi)^2$ . Reliqui vero factores  $\phi - 2i\pi$ , quando  $i$  per 3 non est divisibile, erunt simplices.

§. 31. Cum igitur formula  $(\phi - 2i\pi)^2$  sit factor nostri denominatoris  $\text{col. } \phi - \text{col. } 2 \phi$ , secundum regulam pro huiusmodi casibus statuantur

$$\frac{\text{col. } \phi - 2i\pi}{\phi - 2i\pi} = \frac{\beta}{\phi - 2i\pi} + R,$$

vbi R complectitur omnes reliquas fractiones. Nunc veritatem multiplicemus per  $(\phi - 2i\pi)^2$  et habebimus

$$\frac{(\phi - 2i\pi)^2}{\phi - 2i\pi} = \alpha + \beta(\phi - 2i\pi) + R(\phi - 2i\pi)^2$$

Faciamus  $\phi = 2i\pi$  fietque  $\alpha = \frac{(\phi - 2i\pi)^2}{\phi - 2i\pi}$ , cuius fractionis numerator et denominator evanescens, hinc differentialibus sub finibus fiet  $\alpha = \frac{2(\phi - 2i\pi)}{\phi - 2i\pi}$ . Vbi cum numerator et denominator iterum evanescant, denno eorum loco differentialia scribantur erique  $\alpha = \frac{2}{\phi - 2i\pi}$ . Nunc igitur posueruntur  $\phi = 2i\pi$  reperietur  $\alpha = \frac{1}{i}$ .

§. 32.

occurrere, seu factores in minus clare appareat, ita ut factor denominatoris sit  $(\phi - 2i\pi)^2$ . Quod cum minus clare appareat, ita ostendamus: quoniam in genere est

$$\text{col. } \phi - \text{col. } 2 \phi = 2 \text{ fin. } \frac{\phi - 2\phi}{2}, \text{ fin. } \frac{2\phi - \phi}{2}$$

qui igitur evanescit tam quando  $\text{fin. } \frac{1}{2} \phi = 0$ . Fit autem  $\text{fin. } \frac{1}{2} \phi = 0$  quoties  $\phi = 2i\pi$ ; denotante  $i$  omnes numeros integros, ideoque  $\phi = 2i\pi$ . Similique modo  $\text{fin. } \frac{1}{2} \phi = \pi$ , si  $\frac{1}{2} \phi = i\pi$  ideoque  $\phi = 2i\pi$ , quae posterior formula, quoties  $i$  est numerus integer, priores casus suppleat; sicque manifestum est, in factoribus occurrere omnia quadrata  $(\phi - 2i\pi)^2$ . Reliqui vero factores  $\phi - 4i\pi$ , quando  $i$  per 3 non est divisibile, erunt simplices.

§. 31. Cum igitur formula  $(\phi - 2i\pi)^2$  sit factor nostri denominatoris  $\text{col. } \phi - \text{col. } 2 \phi$ , secundum regulam pro huiusmodi casibus statuantur

$$\frac{\text{col. } \phi - 2i\pi}{\phi - 2i\pi} = \frac{\beta}{\phi - 2i\pi} + R,$$

vbi R complectitur omnes reliquas fractiones. Nunc veritatem multiplicemus per  $(\phi - 2i\pi)^2$  et habebimus

$$\frac{(\phi - 2i\pi)^2}{\phi - 2i\pi} = \alpha + \beta(\phi - 2i\pi) + R(\phi - 2i\pi)^2$$

Faciamus  $\phi = 2i\pi$  fietque  $\alpha = \frac{(\phi - 2i\pi)^2}{\phi - 2i\pi}$ , cuius fractionis numerator et denominator evanescens, hinc differentialibus sub finibus fiet  $\alpha = \frac{2(\phi - 2i\pi)}{\phi - 2i\pi}$ . Vbi cum numerator et denominator iterum evanescant, denno eorum loco differentialia scribantur erique  $\alpha = \frac{2}{\phi - 2i\pi}$ . Nunc igitur posueruntur  $\phi = 2i\pi$  reperietur  $\alpha = \frac{1}{i}$ .

§. 32.

§. 32. Jam in aequatione

$$\frac{(\phi - 2i\pi)^2}{\phi - 2i\pi} = \alpha + \beta(\phi - 2i\pi) + R(\phi - 2i\pi)^2$$

terminus  $\alpha = \frac{1}{i}$  ad alteram partem transferatur et ad eandem denominationem reducat et resultabit haec aequatio:

$$\frac{(\phi - 2i\pi)^2}{\phi - 2i\pi} - \frac{1}{i}(\text{col. } \phi - \text{col. } 2 \phi) = \beta(\phi - 2i\pi) + R(\phi - 2i\pi)^2$$

unde per  $\phi - 2i\pi$  dividendo fiet

$$\frac{(\phi - 2i\pi)^2 - \frac{1}{i}(\text{col. } \phi - \text{col. } 2 \phi)}{\phi - 2i\pi} = \beta + R(\phi - 2i\pi)$$

Quod si iam statuantur  $\phi = 2i\pi$ ,  $\beta$  aequabitur fractioni, cuius tam numerator quam denominator ter evanescit, ita ut triplici differentiatione sit opus.

Prima autem differentiatio dabit:

$$\beta = \frac{2(\phi - 2i\pi) + \frac{1}{i}(\text{fin. } \phi - 2 \text{ fin. } 2 \phi)}{2(\phi - 2i\pi) - (\phi - 2i\pi)(\text{fin. } \phi - 2 \text{ fin. } 2 \phi)}$$

Secunda differentiatio dabit:

$$\beta = \frac{2 + \frac{1}{i}(\text{col. } \phi - 4 \text{ col. } 2 \phi)}{2(\phi - 2i\pi) + 4 \text{ fin. } 2 \phi - (\phi - 2i\pi)(\text{col. } \phi - 4 \text{ col. } 2 \phi)}$$

Tertia denique differentiatio dabit:

$$\beta = \frac{-\frac{1}{i}(\text{fin. } \phi - 8 \text{ fin. } 2 \phi)}{-3 \text{ col. } \phi + 12 \text{ col. } 2 \phi + (\phi - 2i\pi)(\text{fin. } \phi - 8 \text{ fin. } 2 \phi)}$$

Nunc autem factio  $\phi = 2i\pi$  numerator quidem iterum evanescit, denominator vero evadit 9, ita ut sit  $\beta = 0$ .

§. 33. At vero ille valor pro  $\beta$  sine differentiatione factus erui potest ponendo  $\phi = 2i\pi + \omega$ , existens  $\omega$  infinite parvo; tum autem erit

Q 3

col

cof.  $\Phi = \text{cof. } \omega$  et cof.  $2\Phi = \text{cof. } 2\omega$ ;

aequatio autem fiet

$$\frac{\omega}{\text{cof. } \omega} = \frac{1}{2} + \beta \omega + R \omega \omega.$$

Nunc ambos cofus proxime exhibeamus vsque ad quartam potestatem ipsius  $\omega$  procedendo, et cum fit

cof.  $\omega = 1 - \frac{1}{2}\omega\omega + \frac{1}{24}\omega^4$  et

cof.  $2\omega = 1 - 2\omega\omega + \frac{1}{6}\omega^4$ , erit

cof.  $\omega - \text{cof. } 2\omega = \frac{1}{2}\omega\omega - \frac{1}{6}\omega^4 = \frac{1}{6}\omega\omega(1 - \frac{1}{2}\omega\omega)$ ,

quo valore substituto habebimus

$$\frac{1}{3(1 - \frac{1}{2}\omega\omega)} = \frac{1}{2}(1 + \frac{1}{2}\omega\omega) = \frac{1}{2} + \beta\omega + R\omega\omega;$$

hincque fit  $\beta = \frac{1}{2}\omega$ ; sicque factio  $\omega = 0$  erit etiam  $\beta = 0$ .

§. 34. Hanc obrem pro denominatoris factore quadrato  $(\Phi - 2i\pi)^2$  ob  $\alpha = \frac{1}{2}$  fractio inde nata erit  $\frac{1}{\Phi - 2i\pi}$ . Pro reliquis autem factoribus simplicibus  $\Phi - i\pi$  statuanus

$$\frac{1}{\text{cof. } \Phi - \text{cof. } 2\Phi} = \frac{\alpha}{\Phi - i\pi} + R,$$

quae aequatio multiplicetur per  $\Phi - i\pi = \omega$ , vt prodeat

$$\frac{\omega}{\Phi - 2i\pi} = \alpha + R\omega.$$

Vbi noceat numerum  $i$  non esse per 3 diuisibilem, vnde  $\frac{2i\pi}{3}$  sequentes angulos exprimet:

at anguli  $\frac{2i\pi}{3}, \frac{4i\pi}{3}, \frac{8i\pi}{3}, \frac{10i\pi}{3}, \frac{14i\pi}{3}$  quorum angulorum cofus est idem  $-\frac{1}{2}$ , finus autem horum angulorum sunt fin.  $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}$ , vbi signum superius valet, si  $i$  sit  $3\pi + 1$ , inferius vero si fuerit  $i = 3\pi + 2$ . At vero fin.  $\frac{1}{2}$  semper est  $-\frac{\sqrt{3}}{2}$ , vbi iterum signum superius valet.

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1 - 1/2 ω ω,

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ata erit  $\frac{1}{\Phi - 2i\pi}$   
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um superius valet

si  $i = 3\pi + 1$ , inferius vero si  $i = 3\pi + 2$ . Haecque regula semper valet, siue  $n$  sit numerus posituus sine negatiuus.

§. 35. His praenotatis erit

cof.  $\Phi = -\frac{1}{2}\text{cof. } \omega + \frac{\sqrt{3}}{2}\text{fin. } \omega$  et

cof.  $2\Phi = -\frac{1}{2}\text{cof. } 2\omega + \frac{\sqrt{3}}{2}\text{fin. } 2\omega,$

vnde vero proxime habebimus

cof.  $\Phi = -\frac{1}{2}(1 - 2\omega\omega) + \frac{\sqrt{3}}{2}\omega\omega$  et

cof.  $2\Phi = -\frac{1}{2}(1 - 2\omega\omega) + \frac{\sqrt{3}}{2}2\omega\omega,$

vbi perpeno signa superiora valent si  $i = 3\pi + 1$ , inferiora autem si  $i = 3\pi + 2$ . Hinc igitur erit noscer denominator

cof.  $\Phi - \text{cof. } 2\Phi = -\frac{1}{2}\omega\omega + \frac{\sqrt{3}}{2}\omega\omega$

vnde fit  $\frac{1}{-\frac{1}{2}\omega + \frac{\sqrt{3}}{2}\omega} = \alpha$ . Posito igitur  $\omega = 0$  erit

$\alpha = \frac{1}{\sqrt{3}}$ , ita vt ex factore  $\Phi - \frac{2i\pi}{3}$  nascatur ista fractio:

$$\frac{1}{\sqrt{3}} \frac{2}{3\sqrt{3}} \left( \frac{1}{\Phi - \frac{2i\pi}{3}} \right) = \frac{1}{\sqrt{3}} \frac{2}{(3\Phi - 2i\pi)\sqrt{3}}$$

§. 36. Euoluanus igitur primo omnes certminos facti ex factoribus geminatis  $(\Phi - 2i\pi)^2$  natos, et cum numerator fuffet  $\frac{1}{3}$ , si loco  $i$  succedant omnes scribamus numeros integros tam posituos quam negatiuos, series orietur sequens:

$$\frac{1}{3\sqrt{3}} \left( \frac{1}{\Phi - 2i\pi} + \frac{1}{\Phi + 2i\pi} + \frac{1}{\Phi - 4i\pi} + \frac{1}{\Phi + 4i\pi} + \frac{1}{\Phi - 6i\pi} + \frac{1}{\Phi + 6i\pi} + \dots \right)$$



Jam sit  $\phi = i\pi$ , et quoniam hoc casu numerator ac denominator nostrae fractionis evanescunt, statamus  $\phi - i\pi = \alpha$ , erique

$$\text{fn. } \phi = \text{fn. } (i\pi + \omega) = \text{fn. } i\pi \cos \omega + \text{fn. } \omega \cos i\pi = \pm \text{fn. } \omega$$

ob  $\text{fn. } i\pi = 0$  et  $\cos i\pi = \pm 1$ ; vbi signum superius valet si  $i$  sit numerus par, inferius vero si impar, quod tamen discrimen hic non in casum venit, cum sit  $\text{fn. } \phi = \text{fn. } \omega$ . Hinc igitur erit

$$\frac{\omega \omega}{\text{fn. } \omega} = \alpha + \beta \omega + R \omega \omega.$$

Cum igitur sit

$$\text{fn. } \omega = \omega - i\omega^2 = \omega (1 - i\omega \omega), \text{ erit}$$

$$\frac{1}{(1 - i\omega \omega)^2} = 1 + i\omega \omega + \alpha + \beta \omega + R \omega \omega$$

unde fit  $\text{fn. } \alpha = 1$ . Tum vero aequatio erit  $i\omega = \beta + R\omega$ , sique factio  $\omega = 0$  sit  $\beta = 0$ , consequenter ex denominatoris factore  $(\phi - i\pi)^2$  oritur haec factio  $\frac{\omega}{\phi - i\pi^2}$ .

§. 41. Tribuantur nunc ipsi  $i$  omnes valores debiti ac repetatur haec series:

$$\frac{1}{\text{fn. } \phi^2 - \phi} + \frac{1}{(\phi - i\pi)^2} + \frac{1}{(\phi - i\pi)^2} + \frac{1}{(\phi - i\pi)^2} + \frac{1}{(\phi - i\pi)^2} + \text{etc.}$$

quae quidem series deduci possit ex § 18, vbi invenimus  $\frac{\omega}{\text{fn. } \phi} = \phi + \frac{1}{\phi - i\pi} + \frac{1}{\phi + i\pi} + \frac{1}{\phi - i\pi} + \frac{1}{\phi + i\pi} + \text{etc.}$ , unde per differentiationem signis mutatis ea ipsa oritur series quam hic invenimus.

§. 42. Quod si factio fuisset propofita  $\frac{\omega}{\text{fn. } \phi^2}$  et eodem modo resolutio institueretur, ob

col

numerator ac denominator  $\phi - i\pi = \alpha$ ,

$$i\pi = \pm \text{fn. } \omega$$

num superius valet par, quod tamen sit  $\text{fn. } \phi = \text{fn. } \omega$ .

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$$+ R \omega \omega$$

erit  $i\omega = \beta + R\omega$ , er ex denominatoris factore  $\frac{\omega}{\phi - i\pi^2}$ .

Por

nnes valores de-

$$\frac{1}{\text{fn. } \phi} + \frac{1}{(\phi - i\pi)^2} + \text{etc.}$$

18, vbi invenimus

$$\phi + \frac{1}{\phi - i\pi} + \text{etc.}$$

ea ipsa oritur se-

propofita  $\frac{\omega}{\text{fn. } \phi^2}$  et

col

$$\cos (i\pi + \omega) = \pm \cos \omega \text{ itaque} \\ \cos \phi = \cos \omega = 1 - \omega \omega$$

quoniam secundae potestates ipsius  $\omega$  non in computum veniunt, numerator foret vt in casu praecedente  $= 1$ , itaque eadem plane series prodisset, id quod vique foret absurdum. Supra autem iam animadvertimus, huiusmodi resolutiones veritati non esse consentaneas, nisi quantitas variabilis  $\phi$  in numeratore pauciores habeat dimensiones quam in denominatore, quia aequum praeter sentem fractionum partes integre essent accessituras, id quod hoc casu manifeste evenit, cum sit  $\frac{\omega}{\text{fn. } \phi} = \frac{\omega}{\text{fn. } \phi} - 1$ , ita vt pars integra hoc casu sit  $= -1$ .

V. Sit factio resoluenda  $= \frac{\omega}{\text{fn. } \phi^2}$ .

§. 43. Pro hoc casu poni oportebit

$$\frac{\omega}{\text{fn. } \phi^2} = \frac{\alpha}{(\phi - i\pi)^2} + \frac{\beta}{(\phi - i\pi)^2} + \frac{\gamma}{\phi - i\pi} + R.$$

Ponamus nunc iterum  $\phi = i\pi + \omega$ , et cum sit

$$\frac{\omega}{\text{fn. } \phi} = \pm \frac{1}{\omega} (1 + i\omega \omega)$$

(vbi ratio signorum legem supra datam servet) haec resolute aequatio, postquam per  $\omega^2$  fuerit multiplicata:

$$+ \frac{\omega^2 (1 + i\omega \omega)}{\omega^2} = \alpha + \beta \omega + \gamma \omega \omega + R \omega^2 = 1 + i\omega \omega;$$

unde manifesto fit  $\alpha = \pm 1$ , cum vero  $\beta + \gamma \omega + R \omega \omega = i\omega$  sique erit  $\beta = 0$  et  $\gamma = \pm 1$ ; Hoc igitur modo ex denominatore cubico  $(\phi - i\pi)^2$  nascitur haec duas fractiones:  $\frac{\omega}{\phi - i\pi^2} + \frac{1}{\phi - i\pi^2}$ .

R 2

§. 44.



§. 44. Tribuamus igitur litterae  $i$  successitue omnes valores tam positivos quam negativos atque obtinemus sequentem resolutionem:

$$\frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i} = \frac{\Phi^i}{(\Phi^i - \pi)^2} + \frac{\Phi^i}{(\Phi^i + \pi)^2} + \frac{\Phi^i}{(\Phi^i - \pi)^2} - \text{etc.}$$

$$+ \frac{\Phi^i}{2(\Phi^i - \pi)} - \frac{\Phi^i}{2(\Phi^i + \pi)} + \frac{\Phi^i}{2(\Phi^i - \pi)} - \text{etc.}$$

Hic observasse iuvabit, infactorem seriem iam supra: in primo exemplo esse inveniam; vnde insignimus fore summam huius seriei  $= \frac{1}{\sqrt{-1}} \Phi^i$ ; quamobrem series superior cuborum sola aequabitur huic formulae:  $\frac{1}{\sqrt{-1}} \Phi^i - \frac{1}{2\sqrt{-1}} \Phi^i$ .

§. 45. Egrege hoc quoque convenit cum principis supra stabilitis, ex quibus per differentiationem continuo alias novas series ertere docuimus. Cum enim sit

$$\frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i} = \frac{\Phi^i}{(\Phi^i - \pi)^2} + \frac{\Phi^i}{(\Phi^i + \pi)^2} + \frac{\Phi^i}{(\Phi^i - \pi)^2} + \text{etc.}$$

hinc deducitur differentiendo

$$\frac{-\text{col. } \Phi^i}{\sqrt{-1}} = \frac{1}{\Phi^i} + \frac{1}{(\Phi^i - \pi)^2} + \frac{1}{(\Phi^i + \pi)^2} - \frac{1}{(\Phi^i - \pi)^2} - \frac{1}{(\Phi^i + \pi)^2} + \text{etc.}$$

atque hinc denno differentiendo

$$\frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i} = \frac{1}{\Phi^i} - \frac{1}{(\Phi^i - \pi)^2} - \frac{1}{(\Phi^i + \pi)^2} + \frac{1}{(\Phi^i - \pi)^2} + \frac{1}{(\Phi^i + \pi)^2} + \text{etc.}$$

quae reducitur ad hanc formam:  $\frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i} = \frac{1}{\Phi^i}$ , id quod egregie convenit cum valore praecedente.

VI. Sit fractio proposita resoluenda  $= \frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i - \sqrt{-1} \Phi^i}$

§. 46. Denominator iste tang.  $\Phi - \text{fn. } \Phi$  manifesto evanescit casibus quibus  $\Phi = i\pi$ , denotante  $i$  omnes numeros integros tam positivos quam negativos, vnde fractiones simplices, quarum denominatores continent istam factorem  $\Phi - i\pi$ , evadunt infiniti casu  $\Phi = i\pi$ , dum reliquae fractiones

est omnes nebulinus se-

$$\frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i} = \text{etc.}$$

in supra in priore summam cuborum

est principium continuo sit

$$+ \text{etc.}$$

$$\frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i} + \text{etc.}$$

Hic quod egregie

re manifeste omnes nuda fractiones factorem aliquae fractiones

est manifeste omnes nuda fractiones factorem aliquae fractiones

iones retinent valorem finitum. Haecque consideratio nobis aperit novam methodum, omnes fractiones simplices inuestigandi. Pro quovis enim tali factore evanescente quae-ratur valor ipsius fractionis propositae, qui cum fiat infinitus, ei aequales esse debebunt si seriei termini, qui eodem casu evadunt infiniti. Hanc ob rem statui oportebit  $\Phi = i\pi$ ; denotante  $\omega$  angulum infinite parvum. Quo factio fractio proposita fiet circa quaedam functionem  $\omega$ , quam secundum eius dimensiones evolvi conveniet.

§. 47. Hanc igitur ideam sequentes, duos casus distinguere debemus, prouti  $i$  fuerit numerus vel par vel impar, quoniam prioro casu sit  $\text{fn. } \Phi = \text{fn. } \omega$ , posteriore vero casu sit  $\text{fn. } \Phi = -\text{fn. } \omega$ , dum utroque casu manet

$$\text{tang. } \Phi = \text{tang. } \omega.$$

Sit igitur primo  $i$  numerus impar et erit casu  $\Phi = i\pi$  nostra fractio  $\frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i - \sqrt{-1} \Phi^i}$ . Est vero proxime tang.  $\omega = \omega + \frac{1}{2} \omega^3$  et  $\text{fn. } \omega = \omega - \frac{1}{2} \omega^3$ , vnde ista fractio fiet

$$\frac{1}{2\omega + \frac{1}{2}\omega^3} = \frac{1}{2\omega(1 + \frac{1}{4}\omega^2)} = \frac{1}{2\omega} (1 - \frac{1}{4}\omega^2)$$

Haec iam expressio sponte praebet has duas fractiones  $\frac{1}{2\omega} - \frac{1}{8}\omega$ , vnde ob  $\omega = \Phi - i\pi$  pro isto factore oritur haec fractio simplex  $\frac{1}{2(\Phi - i\pi)}$ , quia altera pars evanescit. Quare si nunc loco  $i$  ordine scribamus numeros impares, sequentem fractionum seriem adificemur:

$$\frac{1}{2(\Phi - \pi)} + \frac{1}{2(\Phi + \pi)} + \frac{1}{2(\Phi - 3\pi)} + \frac{1}{2(\Phi + 3\pi)} + \frac{1}{2(\Phi - 5\pi)} + \text{etc.}$$

§. 48. Sit nunc etiam  $i$  numerus par, vnde sit tang.  $\Phi = \text{tang. } \omega$  et  $\text{fn. } \Phi = \text{fn. } \omega$ , hinc fractio nostra erit  $\frac{1}{\sqrt{-1}} \frac{\Phi^i}{\Phi^i - \sqrt{-1} \Phi^i}$ , vbi, facta evolutione  $i$  vni termini  $\omega$  se

te tollunt, ita vt in hoc denominatore infima potestas ipsius  $\omega$  futura sit  $\omega^i$ . Atque ob hanc causam approximationem vltimus continuari oportet quam casu precedente. Hunc in finem loco tang.  $\omega$  scribamus  $\frac{jm. \omega}{jst. \omega}$ , vt fractio nostra sit  $\frac{jm. \omega - \frac{est. \omega}{jm. \omega} \omega}{jst. \omega}$ . Cum iam sit

fin.  $\omega$  cof.  $\omega = \frac{1}{2}$  fin.  $2 \omega$ , erit per series

fin.  $\omega = \omega - \frac{1}{2} \omega^2 + \frac{1}{12} \omega^3$  et

fin.  $2 \omega = 2 \omega - \frac{1}{2} \omega^2 + \frac{1}{12} \omega^3$

vnde totus denominator erit

$1 + \frac{1}{2} \omega^2 - \frac{1}{2} \omega^2 = 1 - \frac{1}{2} \omega \omega$

numerator vero est cof.  $\omega = 1 - \frac{1}{2} \omega \omega$ , vnde tota fractio nostra erit

$\frac{1 - \frac{1}{2} \omega \omega}{1 - \frac{1}{2} \omega \omega} = \frac{1 - \frac{1}{2} \omega \omega}{\frac{1}{2} \omega^2}$ ;

hincque partes resulantantes erunt  $\frac{2}{3} - \frac{1}{2} \omega$ , quae ambae casu  $\omega = 0$  sunt infinitae. Facile autem patet, si approximationem vltimus extendissemus, in sequenti termino litteram  $\omega$  iam in numeratorem transfuram fuisse. Scribatur igitur  $\Phi - i \pi$  loco  $\omega$ , et partes ex hoc factore denominatoris oritunde erunt  $\frac{2}{3} - \frac{1}{2} \Phi - i \pi$ , vnde loco  $i$  successu omnes numeros pares scribendo ista prohibet series geminata:

$\frac{\Phi^2 + (\Phi - i \pi)^2}{2(\Phi - i \pi)} + \frac{\Phi^2 + (\Phi - i \pi)^2}{2(\Phi - i \pi)^2} + \frac{\Phi^2 + (\Phi - i \pi)^2}{2(\Phi - i \pi)^3} + \frac{\Phi^2 + (\Phi - i \pi)^2}{2(\Phi - i \pi)^4} + \text{etc.}$

§. 49. Iungamus igitur has series ex utroque casu deductas et fractio propolita  $\frac{jm. \omega}{jst. \omega} \Phi - \frac{1}{2} \frac{jm. \omega}{jst. \omega}$  resolui reperitur in ternas series sequentes:

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= tota fractio

re ambae casu approximatione litteram  $\omega$  scribatur igitur denominatoris successu omnes geminata:  $\frac{2}{3} - \frac{1}{2} \Phi - i \pi$  + etc.

utroque casu solui reperitur

1 2

$\frac{1}{2(\Phi - i \pi)} + \frac{1}{2(\Phi + i \pi)} + \frac{1}{2(\Phi - i \pi)^2} + \frac{1}{2(\Phi + i \pi)^2} + \frac{1}{2(\Phi - i \pi)^3} + \frac{1}{2(\Phi + i \pi)^3} + \text{etc.}$   
 $-\frac{1}{2\Phi} + \frac{1}{2(\Phi - i \pi)} - \frac{1}{2(\Phi + i \pi)} - \frac{1}{2(\Phi - i \pi)^2} + \frac{1}{2(\Phi + i \pi)^2} - \frac{1}{2(\Phi - i \pi)^3} + \frac{1}{2(\Phi + i \pi)^3} + \text{etc.}$

§. 50. Quilibet hic facile sentiet, istam methodum non parum antecellere illi, qua ante vti sumus, quandoquidem hoc modo statim fractiones ex quilibet denominatoris factore orundas nasci sumus, neque opus fuerat earum numeratores per litteras indefinitas designare. Praeterea etiam hac ratione non opus erat sollicite inquirere, quomodo singuli factores simplices in denominatore contineantur, siquidem nostra methodus hoc sponne declarat.

§. 51. In huiusmodi autem scribendis generalibus, vbi quorundam terminorum denominatorum certo casu enascuntur ideoque hi termini in infinitum exerebant, quacri soleat, his terminis sublati, quanta futura sit summa reliquorum terminorum. Ita pro casu quo  $i$  est numerus impar, terminus  $\frac{2}{3} - \frac{1}{2} \Phi - i \pi$  sit infinitus casu  $\Phi = i \pi$ . Hoc igitur termino deleto quaeritur, quanta futura sit summa reliquorum terminorum casu  $\Phi = i \pi$ . Ad hanc quaestionem solvendam ponatur  $\Phi - i \pi = \omega$ , atque ex § 47 patet fore

$\frac{2}{3} - \frac{1}{2} \omega = \frac{2}{3} - \frac{1}{2} \omega + R$

vbi R complectitur omnes reliquos terminos, quorum summa desideratur casu  $\Phi = i \pi$ . Transferatur igitur terminus  $\frac{2}{3} - \frac{1}{2} \omega$  in alteram partem ac statim elucet fore

$R = -\frac{1}{2} \omega = 0$  ob  $\omega = 0$ ,

ita vt omisso termino illo infinito summa omnium reliquorum casu  $\Phi = i \pi$  semper sit 0.

§. 52.

§. 52. Quando autem  $i$  est numerus par, eadem conclusio locum habebit, ad quod ostendendum necesse est approximationem adhibere vltimus continuare. Tum autem erit numerator

$$\cos \omega = 1 - \frac{1}{2} \omega \omega + \frac{1}{24} \omega^4;$$

pro denominatore vero

$$\sin \omega = \omega - \frac{1}{6} \omega^3 + \frac{1}{120} \omega^5 - \frac{1}{325} \omega^7 \text{ et}$$

$$\sin 2\omega = 2\omega - \frac{2}{3} \omega^3 + \frac{16}{315} \omega^5 - \frac{224}{315} \omega^7,$$

unde fit ipse denominator

$$\frac{1}{2} \omega^2 - \frac{1}{6} \omega^4 + \frac{1}{240} \omega^6 = \frac{1}{2} \omega^2 (1 - \frac{1}{6} \omega \omega + \frac{1}{24} \omega^2);$$

hinc factor posterior in numeratorem translatus praebebit

$$1 + \frac{1}{2} \omega \omega + \frac{1}{24} \omega^4$$

$$\frac{1 - \frac{1}{2} \omega \omega - \frac{1}{24} \omega^4}{\frac{1}{2} \omega^2}$$

quae aequari debet toti feriei posito  $\Phi = i\pi$ , hoc est terminis inueniendis  $(\frac{\Phi-i}{\pi})^2 - \frac{1}{24}(\frac{\Phi-i}{\pi})^4$  cum omnibus reliquis R, unde elicitur  $R = -\frac{1}{120} \omega = 0$ ; unde patet etiam his casibus summam omnium reliquorum esse = 0.

§. 53. Quod si ergo summam  $\Phi = 0$  et terminos in infinitum excrecentes deleamus, termini remanentes erunt

$$-\frac{1}{2\pi} + \frac{1}{4\pi} - \frac{1}{6\pi} + \frac{1}{8\pi} - \frac{1}{10\pi} + \frac{1}{12\pi} - \text{etc.}$$

$$+\frac{1}{4\pi} - \frac{1}{8\pi} + \frac{1}{12\pi} - \frac{1}{16\pi} + \frac{1}{20\pi} - \frac{1}{24\pi} + \text{etc.}$$

$$-\frac{1}{8\pi} + \frac{1}{16\pi} - \frac{1}{24\pi} + \frac{1}{32\pi} - \frac{1}{40\pi} + \frac{1}{48\pi} - \text{etc.}$$

ubi omnes termini manifesto se tollunt, id quod etiam omnibus reliquis casibus, quibus ponitur  $\Phi = i\pi$ , contingit. §. 54.

rus par, eadem idem necesse est uare. Tum au-

acus praebebit

$\pi$ , hoc est terminis reliquis R, unde iam his casibus

) = 0 et termini remanentes

$$\text{etc.}$$

$$\text{etc.}$$

$$\frac{1}{16\pi} - \text{etc.}$$

quod etiam omnibus casibus, quibus  $\Phi = i\pi$ , contingit. §. 54.

§. 54. Sin autem binos terminos contiguos considerauerimus, hae series prodibunt:

$$-\frac{1}{2\pi} + \frac{\Phi}{\pi} - \frac{\Phi^2}{2\pi^2} + \frac{\Phi^3}{3\pi^3} - \frac{\Phi^4}{4\pi^4} + \frac{\Phi^5}{5\pi^5} - \frac{\Phi^6}{6\pi^6} + \frac{\Phi^7}{7\pi^7} - \text{etc.}$$

$$+\frac{\Phi}{2\pi} + \frac{\Phi^2}{4\pi^2} + \frac{\Phi^3}{8\pi^3} + \frac{\Phi^4}{16\pi^4} + \frac{\Phi^5}{32\pi^5} + \frac{\Phi^6}{64\pi^6} + \frac{\Phi^7}{128\pi^7} + \text{etc.}$$

quarum ferierum summa est  $\frac{1}{2\pi} \frac{\Phi - \Phi^2}{1 - \Phi}$ . Quod si nunc hic ponamus  $\Phi = 0$ , siue  $\Phi = \omega$ , quoniam omnes termini sunt distinctibiles per  $\Phi = 0$ , eorumque autem summa inuenta est  $-\frac{1}{2\pi} \omega$ , si vtriusque per  $\omega$  diuidamus, summa erit  $-\frac{1}{2\pi}$  ipsae autem series euadent

$$-\frac{1}{2\pi} + \frac{1}{4\pi} - \frac{1}{8\pi} + \frac{1}{16\pi} - \frac{1}{32\pi} + \frac{1}{64\pi} - \text{etc.}$$

$$-\frac{1}{16\pi} + \frac{1}{32\pi} - \frac{1}{64\pi} + \frac{1}{128\pi} - \frac{1}{256\pi} + \frac{1}{512\pi} - \text{etc.}$$

§. 55. Mutatis igitur signis et reductis terminis ad formam simplicissimam imperrabimus hanc summationem:

$$\frac{1}{2\pi} = \frac{1}{\pi} (1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \text{etc.})$$

$$+\frac{1}{2\pi} (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \text{etc.})$$

Notum autem est esse

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \text{etc.} = \frac{\pi}{2}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \text{etc.} = \frac{\pi}{2}$$

unde haec aequalitas manifesto in oculos incurrit.