1785

Investigatio valoris integralis $\int (x^{m-1}\, dx)/(1 - 2x^k \cos\theta + x^{2k})$ a termino $x=0$ ad $x=\infty$ extensi

Leonhard Euler

iae integrales a termino $y=0$ vsque ad $y=1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit:

$$\int\frac{x^{m-1}dx}{(1+x^k)^n}=\frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\int y^{nk-m-1}dy(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}dy(1-y^k)^{\frac{n}{k}-1}}.$$

Vnde si fuerit $m=1$ et $k=2$, sequitur fore

$$\int\frac{dx}{(1+xx)^n}=\frac{\pi}{2}\int\frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}:\int\frac{dy}{\sqrt{(1-yy)}}=\int\frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}$$

Ita si $n=\frac{3}{2}$ erit

$$\int\frac{dx}{(1+xx)^{\frac{3}{2}}}=\int\frac{ydy}{\sqrt{(1-yy)}}$$

cuius veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{1+xx}}$, posterius vero $=1-\sqrt{(1-yy)}$, quae, facto $x=\infty$ et $y=1$, vtique fiunt aequalia. Caeterum pro hac integratione generali notasse iuuabit, exponentem vnitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

IN.

xtendantur, in
repraesentari
$y^k)^{\frac{m}{k}-1}$
$y^k)^{\frac{n}{k}-1}$.
$\int\frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}$
s generatim est
e, facto $x=\infty$
o hac integra-
itate minorem
rum integrali-
IN.

INVESTIGATIO VALORIS INTEGRALIS

$$\int\frac{x^{m-1}dx}{1-2x^k\cos.\theta+x^{2k}}$$

A TERMINO $x=0$ VSQVE AD $x=\infty$ EXTENSI.

§. 1.

Quaeramus primo integrale formulae propositae indefinitum, atque adeo omnes operationes ex primis Analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolui nequit, sit in genere eius factor duplicatus quicunque $1-2x\cos.\omega+xx$; euidens enim est, denominatorem fore productum ex k huiusmodi factoribus duplicatis. Cum igitur, posito hoc factore $=0$, fiat $x=\cos.\omega\pm\sqrt{-1}\sin.\omega$, etiam ipse denominator duplici modo euanescere debebit, siue si ponatur

$$x=\cos.\omega+\sqrt{-1}\sin.\omega,\text{ siue}$$
$$x=\cos.\omega-\sqrt{-1}\sin.\omega.$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, vt sit

$$(\cos.\omega\pm\sqrt{-1}\sin.)^\lambda=\cos.\lambda\omega\pm\sqrt{-1}\sin.\lambda\omega;$$

hinc igitur erit

$$x^k=\cos.k\omega+\sqrt{-1}\sin.k\omega\text{ et}$$
$$x^{2k}=\cos.2k\omega\pm\sqrt{-1}\sin.2k\omega.$$

Sub.

Substituamus ergo hos valores et denominator noster euader

$$1 - 2\cos\theta \cos k\omega + \cos 2k\omega =$$
$$\pm 2\sqrt{-1}\cos\theta \sin k\omega + \sqrt{-1}\sin 2k\omega$$

§. 2. Perspicuum igitur est huius aequationis tam terminos reales quam imaginarios seorsim se mutuo tollere debere, vnde nascuntur hae duae aequationes:

$$\text{I. } 1 - 2\cos\theta \cos k\omega + \cos 2k\omega = 0$$

$$\text{II. } -2\cos\theta \sin k\omega + \sin 2k\omega = 0.$$

Cum igitur sit

$$\sin 2k\omega = 2\sin k\omega \cos k\omega,$$

posterior aequatio induet hanc formam:

$$-2\cos\theta \sin k\omega + 2\sin k\omega \cos k\omega,$$

quae per $2\sin k\omega$ diuisa dat $+\cos k\omega = \cos\theta$, ideoque

$$\cos 2k\omega = \cos 2\theta = \cos\theta^2 - \sin\theta^2 = 2\cos\theta^2 - 1,$$

qui valores in aequatione priore substitui praebent aequationem identicam, ita vt vtrique aequationi satisfiat sumendo $\cos k\omega = \cos\theta$.

§. 3. Pro ω igitur eiusmodi angulum assumi oportet, vt fiat $\cos k\omega = \cos\theta$, vnde quidem statim deducitur $k\omega = \theta$, ideoque $\omega = \frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem cosinum habentes, qui praeter ipsum angulum θ sunt $2\pi \pm \theta$, $4\pi \pm \theta$, $6\pi \pm \theta$ etc. atque adeo in genere $2i\pi + \theta$, denotante i omnes numeros integros, quaesito nostro satisfiet, faciendo $k\omega = 2i\pi + \theta$, vnde colligitur angulus $\omega = \frac{2i\pi \pm \theta}{k}$, sicque pro ω nansisceremus innumerabiles angulos satisfacientes, quorum autem sufficiet tot assumsisse, quot exponens k continet vnitates; successiue igitur angulo ω sequentes tribuamus valores:

noster euader
sin. 2 k ω
aequationis tam
se mutuo tollere
nes:
= 0
= 0.
θ,
cos. θ, ideoque
= 2 cos. θ² − 1,
praebent aequa-
satisfiat sumendo
um assumi opor-
statim deducitur
infiniti dantur an-
ipsum angulum
atque adeo in
numeros integros,
i π + θ, vnde
ω nansisceremus
m autem sufficiet
tot

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \frac{8\pi+\theta}{k},\ \ldots\ \frac{2(k-1)\pi+\theta}{k}$$

Quod si ergo angulo ω successiue singulos istos valores, quorum numerus est $= k$, tribuamus, formula $1 - 2x\cos\omega + xx$ omnes suppeditabit factores duplicatos nostri denominatoris $1 - 2x^k\cos\theta + x^{2k}$, quorum numerus erit $= k$.

§. 4. Inuentis iam omnibus factoribus duplicatis nostri denominatoris, fractio $\frac{x^{m-1}}{1 - 2x^k\cos\theta + x^{2k}}$ resolui debet in tot fractiones partiales, quarum denominatores sint ipsi isti factores duplicati, quorum numerus est k, ita vt in genere talis fractio partialis habitura sit talem formam:

$$\frac{A + Bx}{1 - 2x\cos\omega + xx}$$

quam insuper resoluamus in binas simplices, etsi imaginarias, et cum sit

$$xx - 2x\cos\omega + 1 = (x - \cos\omega + \sqrt{-1}\sin\omega)(x - \cos\omega - \sqrt{-1}\sin\omega),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x - \cos\omega - \sqrt{-1}\sin\omega} + \frac{g}{x - \cos\omega + \sqrt{-1}\sin\omega},$$

ita vt totum resolutionis negotium huc redeat, vt ambo numeratores f et g determinentur; iis enim inuentis habebitur summa ambarum fractionum

$$= \frac{fx + gx - (f+g)\cos\omega + \sqrt{-1}(f-g)\sin\omega}{xx - 2x\cos\omega + 1},$$

vbi igitur erit

$$B = f + g \text{ et } A = (f-g)\sqrt{-1}\sin\omega - (f+g)\cos\omega.$$

§. 5. Per methodum igitur fractiones quascunque in fractiones simplices resoluendi statuamus

$$\frac{x^{m-1}}{1-2x^k\,\text{cos.}\,\theta+x^{2k}}=\frac{f}{x-\text{cos.}\,\omega-\sqrt{-1}\,\text{sin.}\,\omega}+\text{R},$$

vbi R complectatur omnes reliquas fractiones partiales. Hinc per $x-\text{cos.}\,\omega-\sqrt{-1}\,\text{sin.}\,\omega$ multiplicando habebitur

$$\frac{x^m-x^{m-1}(\text{cos.}\,\omega+\sqrt{-1}\,\text{sin.}\,\omega)}{1-2x^k\,\text{cos.}\,\theta+x^{2k}}=f\text{R}(x-\text{cos.}\,\omega-\sqrt{-1}\,\text{sin.}\,\omega),$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x=\text{cos.}\,\omega+\sqrt{-1}\,\text{sin.}\,\omega$, vt membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra, quia formula $x-\text{cos.}\,\omega-\sqrt{-1}\,\text{sin.}\,\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita vt hinc nihil concludi posse videatur.

§. 6. Hic igitur vtamur regula notissima, et loco tam numeratoris quam denominatoris eorum differentialia scribamus, vnde nostra aequatio accipiet sequentem formam:

$$\frac{mx^{m-1}-(m-1)x^{m-2}(\text{cos.}\,\omega+\sqrt{-1}\,\text{sin.}\,\omega)}{-2kx^{k-1}\,\text{cos.}\,\theta+2kx^{2k-1}}=$$

$$\frac{mx^m-(m-1)x^{m-1}(\text{cos.}\,\omega+\sqrt{-1}\,\text{sin.}\,\omega)}{-2kx^k\,\text{cos.}\,\theta+2kx^{2k}}=f,$$

posito scilicet $x=\text{cos.}\,\omega+\sqrt{-1}\,\text{sin.}\,\omega$. Tum autem erit

$x^m=\text{cos.}\,m\,\omega+\sqrt{-1}\,\text{sin.}\,m\,\omega$ et

$x^{m-1}(\text{cos.}\,\omega+\sqrt{-1}\,\text{sin.}\,\omega)=x^m=\text{cos.}\,m\,\omega+\sqrt{-1}\,\text{sin.}\,m\,\omega$

et pro denominatore

$x^k=\text{cos.}\,k\,\omega+\sqrt{-1}\,\text{sin.}\,k\,\omega$ et

$x^{2k}=\text{cos.}\,2\,k\,\omega+\sqrt{-1}\,\text{sin.}\,2\,k\,\omega$;

vnde

quascunque
sin. ω + R,
artiales. Hinc
ebitur
√−1 sin. ω),
valor ipsi x
ω, vt mem-
tum vero in
sin. ω simul
e tam nume-
ra vt hinc ni-
fima, et loco
differentialia
ntem formam:
) =
) = f,
am autem erit
ω + √−1 sin. m ω
vnde

vnde fit numerator

$x^m=\text{cos.}\,m\,\omega+\sqrt{-1}\,\text{sin.}\,m\,\omega$

et denominator

$-2\,k\,\text{cos.}\,\theta\,\text{cos.}\,k\,\omega+2\,k\,\text{cos.}\,2\,k\,\omega$

$-2\,k\sqrt{-1}\,\text{cos.}\,\theta\,\text{sin.}\,k\,\omega+2\,k\sqrt{-1}\,\text{sin.}\,2\,k\,\omega.$

§. 7. Pro denominatore reducendo recordemur, iam supra inuentum esse $\text{cos.}\,k\,\omega=\text{cos.}\,\theta$, vnde fit $\text{sin.}\,k\,\omega=\text{sin.}\,\theta$, tum vero

$\text{cos.}\,2\,k\,\omega=\text{cos.}\,2\,\theta=2\,\text{cos.}\,\theta^2-1$ et $\text{sin.}\,2\,k\,\omega=2\,\text{sin.}\,\theta\,\text{cos.}\,\theta$,

quibus valoribus adhibitis denominator noster erit

$$2k\,\text{cos.}\,\theta^2-2k+2k\sqrt{-1}\,\text{sin.}\,\theta\,\text{cos.}\,\theta=-2k\,\text{sin.}\,\theta^2+2k\sqrt{-1}\,\text{sin.}\,\theta\,\text{cos.}\,\theta$$
$$=-2\,k\,\text{sin.}\,\theta(\text{sin.}\,\theta-\sqrt{-1}\,\text{cos.}\,\theta),$$

quamobrem hoc valore adhibito habebimus

$$f=\frac{\text{cos.}\,m\,\omega+\sqrt{-1}\,\text{sin.}\,m\,\omega}{2\,k\,\text{sin.}\,\theta(\sqrt{-1}\,\text{cos.}\,\theta-\text{sin.}\,\theta)}.$$

Simul vero hinc sine nouo calculo deducemus valorem g, quippe qui ab f ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g=\frac{\text{cos.}\,m\,\omega-\sqrt{-1}\,\text{sin.}\,m\,\omega}{-2\,k\,\text{sin.}\,\theta(\text{sin.}\,\theta+\sqrt{-1}\,\text{cos.}\,\theta)}.$$

§. 8. Inuentis autem his litteris f et g pro litteris A et C colligemus primo

$$f+g=\frac{\text{cos.}\,\theta\,\text{sin.}\,m\,\omega-\text{sin.}\,\theta\,\text{cos.}\,m\,\omega}{k\,\text{sin.}\,\theta}=\frac{\text{sin.}\,(m\,\omega-\theta)}{k\,\text{sin.}\,\theta},$$

tum vero erit

$$f-g=-\frac{\sqrt{-1}\,\text{cos.}\,(m\,\omega-\theta)}{k\,\text{sin.}\,\theta}.$$

Ex his igitur reperiemus

$$B = \frac{\sin.(m\omega-\theta)}{k\sin.\theta} \text{ et}$$

$$A = \frac{\sin.\omega\cos.(m\omega-\theta)-\cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta} = -\frac{\sin.((m\omega-\theta)-\omega)}{k\sin.\theta};$$

vbi ergo imaginaria ſponte ſe mutuo deſtruxerunt.

§. 9. Inuentis his valoribus A et B inueſtigari oportet integrale partiale $\int \frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$, vbi, cum denominatoris differentiale ſit

$$2xdx - 2dx\cos.\omega = 2dx(x-\cos.\omega),$$

ſtatuamus

$$A+Bx = B(x-\cos.\omega)+C, \text{ eritque } C = A+B\cos.\omega,$$

hinc igitur erit

$$C = \frac{\cos.\omega\sin.(m\omega-\theta)-\sin.((m\omega-\theta)-\omega)}{k\sin.\theta}.$$

Quia vero

$-\sin.(m\omega-\theta-\omega) = -\sin.(m\omega-\theta)\cos.\omega+\cos.(m\omega-\theta)\sin.\omega$, erit

$$C = \frac{\sin.\omega\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Hac ergo forma adhibita formula integranda $\frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$ diſcerpatur in has duas partes:

$$\frac{B(x-\cos.\omega)dx}{1-2x\cos.\omega+xx} + \frac{Cdx}{1-2x\cos.\omega+xx}.$$

Hic igitur prioris partis integrale manifeſto eſt

$$B\,l\sqrt{(1-2x\cos.\omega+xx)},$$

alterius vero partis facile patet integrale per arcum circuli expreſſum iri, cuius tangens ſit $\frac{x\sin.\omega}{1-x\cos.\omega}$. Ad hoc integrale inueniendum ponamus

$$\int \frac{Cdx}{1-2x\cos.\omega+xx} = D.\ A\text{tang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

et

; runt. B inueſtigari , cum deno- ω), $= A+B\cos.\omega$, $-\theta)\sin.\omega$, erit $\frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$ di- ſt er arcum cir- Ad hoc inte- et

et ſumtis differentialibus, quia $d.\ A\text{tang.}\,t$ aequale eſt $\frac{dt}{1+tt}$, habebimus

$$\frac{Cdx}{1-2x\cos.\omega+xx} = D.\frac{dx\sin.\omega}{1-2x\cos.\omega+xx}$$

vnde manifeſto ſit

$$D = \frac{C}{\sin.\omega} = \frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

§. 10. Subſtituamus igitur loco B et D valores modo inuentos et ex ſingulis factoribus denominatoris

$$1 - 2x^k\cos.\theta + x^{2k},$$

quorum forma eſt $1-2x\cos.\omega+xx$, oritur pars integralis conſtans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}\,l\sqrt{(1-2x\cos.\omega+xx)} + \frac{\cos.(m\omega-\theta)}{k\sin.\theta}A\text{tang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

quae euaneſcit ſumto $x = 0$. In hac igitur forma tantum opus eſt vt loco ω ſucceſſiue ſcribamus valores ſupra indicatos, ſcilicet:

$$\omega = \frac{\theta}{k},\ \frac{2\pi-\theta}{k},\ \frac{4\pi-\theta}{k},\ \frac{6\pi+\theta}{k},\ \text{etc.}$$

donec perueniatur ad $\frac{2(k-1)\pi+\theta}{k}$; tum enim ſumma omnium harum formarum praebebit totum integrale indefinitum formulae propoſitae.

§. 11. Poſtquam igitur integrale indefinitum eliciumus, nihil aliud ſupereſt, niſi vt in eo faciamus $x = \infty$, quo facto pars logarithmica, ob

$$\sqrt{(1-2x\cos.\omega+xx)} = x-\cos.\omega,$$

erit $B\,l(x-\cos.\omega)$. Eſt vero

$$l(x-\cos.\omega) = l\,x - \frac{\cos.\omega}{x} = l\,x, \text{ ob } \frac{\cos.\omega}{k} = 0;$$

quam-

quamobrem facto $x = \infty$ quaelibet pars logarithmica habebit hanc formam: $\frac{2\cos.(m\omega-\theta)}{k f^{n}.\theta} l x$. Deinde pro partibus a circulo pendentibus, facto $x = \infty$ fit

$$\frac{x\,\sin.\omega}{1 - x\cos.\omega} = -\text{tang.}\,\omega = \text{tang.}(\pi - \omega),$$

sicque arcus, cuius haec est tangens, erit $= \pi - \omega$, hincque pars circularis quaecunque fiet $\frac{2\sin.(m\omega-\theta)}{k f^{n}.\theta}(\pi - \omega)$.

§. 12. Cum quilibet valor anguli ω in genere hanc habeat formam: $\frac{2 i \pi + \theta}{k}$, erit angulus

$$m\omega - \theta = \frac{2 i m \pi - \theta(k-m)}{k} \text{ et } \pi - \omega = \frac{\pi(k-2i) - \theta}{k}.$$

Ponamus breuitatis gratia

$$\frac{\theta(k-m)}{k} = \zeta \text{ et } \frac{m\pi}{k} = \alpha, \quad \text{vt fit } m\omega - \theta = 2 i \alpha - \zeta,$$

vbi loco i scribi debent successiue numeri 0, 1, 2, 3, etc. vsque ad $k - 1$. Hinc igitur si omnes partes logarithmicas in vnam summam colligamus, ea ita repraesentari poterit:

$$\frac{l x}{k f^{n}.\theta}(-\sin.\zeta + \sin.(2\alpha - \zeta) + \sin.(4\alpha - \zeta) + \sin.(6\alpha - \zeta)$$
$$+ \sin.(8\alpha - \zeta) - - - - + \sin.(2(k-1)\alpha - \zeta));$$

vbi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet, totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

§. 13. Ad hoc ostendendum ponamus

$$S = -\sin.\zeta + \sin.(2\alpha - \zeta) + \sin.(4\alpha - \zeta) + - - - - + \sin.(2(k-1)\alpha - \zeta)$$

multiplicemus vtrinque per $2\sin.\alpha$, et cum sit

$$2\sin.\alpha\,\sin.\phi = \cos.(\alpha - \phi) - \cos.(\alpha + \phi)$$

huius reductionis ope obtinebimus sequentem expressionem:

rithmica habe-
partibus a cir-
$\pi - \omega$, hinc-
$(\pi - \omega)$.
in genere hanc
$\frac{\pi(k-2i)-\theta}{k}$.
$\theta = 2 i \alpha - \zeta$,
1, 2, 3, etc. vs-
logarithmicas
sentari poterit:
$+\sin.(6\alpha - \zeta)$
$(k-1)\alpha - \zeta))$;
facile suspicari
redigi. Verum
se est.
us
$(2(k-1)\alpha - \zeta)$
fit
$+ \phi)$
expressionem:
2 S

$$2 S \sin.\alpha = \cos.(\alpha + \zeta) - \cos.(\alpha - \zeta) + \cos.(3\alpha - \zeta)$$
$$+ \cos.(\alpha - \zeta) \quad - \cos.(3\alpha - \zeta)$$
$$+ \cos.(5\alpha - \zeta) - - - - - \cos.((2k-1)\alpha - \zeta)$$
$$- \cos.(5\alpha - \zeta) - - - - - -$$

vnde deletis terminis se mutuo destruentibus habebitur

$$2 S \sin.\alpha = \cos.(\alpha + \zeta) - \cos.(2k - 1)\alpha - \zeta).$$

§. 14. Ponamus hos duos angulos, qui soli sunt relicti, $\alpha + \zeta = p$ et $(2k - 1)\alpha - \zeta = q$, eritque eorum summa $p + q = 2\alpha k$. Quia porro est $\alpha = \frac{m\pi}{k}$, erit $p + q = 2 m \pi$, hoc est multiplo totius circuli peripheriae, ob m numerum integrum. Quare cum sit $q = 2 m \pi - p$, erit $\cos.q = \cos.p$; vnde patet summam inuentam nihilo esse aequalem, sicque manifestum est, omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x = \infty$ se mutuo destruere.

§. 15. Progrediamur igitur ad partes circulares, quarum forma generalis, vt vidimus, est $\frac{2\sin.(m\omega-\theta)}{k f^{n}.\theta}(\pi - \omega)$, quae posito $\alpha = \frac{m\pi}{k}$ et $\zeta = \frac{\theta(k-m)}{k}$ fit

$$\frac{2\sin.(2 i \alpha - \zeta)}{k f^{n}.\theta}\left(\pi - \frac{2 i \pi - \theta}{k}\right) = \frac{2\sin.(2 i \alpha - \zeta)}{k f^{n}.\theta}\left(\pi - \frac{2 i \pi}{k} - \frac{\theta}{k}\right).$$

Hic ponatur porro $\frac{\pi}{k} = \beta$ et $\pi - \frac{\theta}{k} = \gamma$, vt forma generalis fit $\frac{2\sin.(2 i \alpha - \zeta)}{k f^{n}.\theta}(\gamma - 2 i \beta)$. Quare si loco i scribamus ordine valores 0, 1, 2, 3, 4, vsque ad $k - 1$, omnes partes circulares hanc progressionem constituent:

$$\frac{2}{k f^{n}.\theta}(\gamma \sin.\zeta + (\gamma - 2\beta)\sin.(2\alpha - \zeta) + (\gamma - 4\beta)\sin.(4\alpha - \zeta)$$
$$- - - - - - (\gamma - 2(k-1)\beta)\sin.(2(k-1)\alpha - \zeta)).$$

Ponamus igitur

$$S = \gamma \operatorname{cof.} \zeta + (\gamma - 2\beta) \operatorname{cof.} (2\alpha - \zeta) + (\gamma - 4\beta) \operatorname{cof.} (4\alpha - \zeta)$$
$$- - - - (\gamma - 2(k-1)\beta) \operatorname{cof.} (2(k-1)\alpha - \zeta)$$

vt fumma omnium partium circularium fit $\frac{S}{k \sin. \theta}$, quae ergo praebebit valorem quaefitum formulae integralis propofitae, cafu quo poft integrationem ftatuitur $x = \infty$, ita vt totum negotium in inueftigando valore ipfius S verfetur.

§. 16. Hunc in finem multiplicemus vtrinque per $2 \operatorname{fin.} \alpha$, et cum in genere fit

$$2 \operatorname{fin.} \alpha \operatorname{cof.} \phi = \operatorname{fin.} (\alpha + \phi) - \operatorname{fin.} (\phi - \alpha)$$

hac reductione in fingulis terminis facta perueniemus ad hanc aequationem:

$$2S \operatorname{fin.} \alpha = \gamma \operatorname{fin.} (\alpha + \zeta) + \gamma \operatorname{fin.} (\alpha - \zeta) + (\gamma - 2\beta) \operatorname{fin.} (3\alpha - \zeta)$$
$$- (\gamma - 2\beta) \operatorname{fin.} (\alpha - \zeta) - (\gamma - 4\beta) \operatorname{fin.} (3\alpha - \zeta)$$
$$+ (\gamma - 4\beta) \operatorname{fin.} (5\alpha - \zeta) - - - - + (\gamma - 2(k-1)\beta) \operatorname{fin.} (2k-1)\alpha - \zeta)$$
$$- (\gamma - 6\beta) \operatorname{fin.} (7\alpha - \zeta) - - - - -$$

vbi praeter primum et vltimum terminum omnes reliqui contrahi poffunt, ita vt prodeat

$$2S \operatorname{fin.} \alpha = \gamma \operatorname{fin.} (\alpha + \zeta) + 2\beta \operatorname{fin} (\alpha - \zeta) + 2\beta \operatorname{fin.} (3\alpha - \zeta) + 2\beta \operatorname{fin.} (5\alpha - \zeta)$$
$$- - - - - - - - . \operatorname{fin.} ((2k-1)\alpha - \zeta) +$$

§. 17. Iam pro hac ferie fummanda ponamus porro

$$T = 2 \operatorname{fin.} (\alpha - \zeta) + 2 \operatorname{fin.} (3\alpha - \zeta) + 2 \operatorname{fin.} (5\alpha - \zeta) + - - - + 2 \operatorname{fin.} (2k-3)\alpha - \zeta)$$

vt habeamus

2 S

) cof. (4α − ζ)
) α − ζ)
, quae ergo
alis propofitae,
, ita vt totum
fetur.
vtrinque per
)
perueniemus ad
. (3α − ζ)
. (3α − ζ)
)β) fin. (2k−1)α − ζ)
omnes reliqui
) + 2β fin. (5α − ζ)
ponamus porro
+ 2 fin. (2k−3)α − ζ)
2 S

$$2S \operatorname{fin.} \alpha = \gamma \operatorname{fin.} (\alpha + \zeta) + (\gamma - 2(k-1)\beta) \operatorname{fin.} (2k-1)\alpha - \zeta) + \beta T.$$

Iam multiplicemus, vt hactenus, per fin. α, et cum fit

$$2 \operatorname{fin.} \alpha \operatorname{fin.} \phi = \operatorname{cof.} (\phi - \alpha) - \operatorname{cof.} (\phi + \alpha),$$

facta hac reductione nancifcimur

$$T \operatorname{fin.} \alpha = + \operatorname{cof.} \zeta + \operatorname{cof.} (2\alpha - \zeta) + \operatorname{cof.} (4\alpha - \zeta) + - - - + \operatorname{cof.} (2(k-2)\alpha - \zeta)$$
$$- \operatorname{cof.} (2\alpha - \zeta) - \operatorname{cof.} (4\alpha - \zeta) - \operatorname{cof.} (6\alpha - \zeta) - - - - - \operatorname{cof.} (2(k-1)\alpha - \zeta)$$

vnde deletis terminis, quae fe mutuo deftruunt, remanebit tantum ifta expreffio:

$$T \operatorname{fin.} \alpha = \operatorname{cof.} \zeta - \operatorname{cof.} (2(k-1)\alpha - \zeta).$$

Cum igitur fit $\alpha = \frac{m\pi}{k}$ erit

$$2(k-1)\alpha = 2m\pi - \frac{2m\pi}{k},$$

cuius loco fcribere licet $-\frac{2m\pi}{k}$, vnde ob $\zeta = \frac{\theta(k-m)}{k}$ erit

$$T \operatorname{fin.} \alpha = \operatorname{cof.} \frac{\theta(k-m)}{k} - \operatorname{cof.} \left(\frac{2m\pi + \theta(k-m)}{k}\right).$$

§. 18. Nunc vero notetur in genere effe

$$\operatorname{cof.} p - \operatorname{cof.} q = 2 \operatorname{fin.} \frac{q+p}{2} \operatorname{fin.} \frac{q-p}{2},$$

quare cum fit

$$p = \frac{\theta(k-m)}{k} \text{ et } q = \frac{2m\pi + \theta(k-m)}{k}, \text{ erit}$$
$$\frac{q+p}{2} = \frac{m\pi + \theta(k-m)}{k} \text{ et } \frac{q-p}{2} = \frac{m\pi}{k},$$

vnde fequitur fore

$$T \operatorname{fin.} \alpha = 2 \operatorname{fin.} \left(\frac{m\pi + \theta(k-m)}{k}\right) \operatorname{fin.} \frac{m\pi}{k}, \text{ ideoque}$$
$$T = 2 \operatorname{fin.} \left(\frac{m\pi + \theta(k-m)}{k}\right), \text{ ob } \alpha = \frac{m\pi}{k}.$$

§. 19. Hoc igitur valore T inuento reperiemus porro

$$2S \operatorname{fin.} \alpha = \gamma \operatorname{fin.} (\alpha + \zeta) + (\gamma - 2(k-1)\beta) \operatorname{fin.} (2k-1)\alpha - \zeta)$$
$$+ 2\beta \operatorname{fin.} \left(\frac{m\pi + \theta(k-m)}{k}\right)$$

 quae

quae ob $\frac{m\pi+\theta(k-m)}{k}=\alpha+\zeta$ reducitur ad hanc formam:

$$2\,S\,\text{ſin.}\,\alpha=(\gamma+2\beta)\,\text{ſin.}(\alpha+\zeta)+(\gamma-2(k-1)\beta)\,\text{ſin.}((2k-1)\alpha-\zeta),$$

quae ita repraeſentari poteſt:

$$2\,S\,\text{ſin.}\,\alpha=(\gamma+2\beta)(\text{ſin.}(\alpha+\zeta)+\text{ſin.}((2k-1)\alpha-\zeta))-2k\beta\,\text{ſin.}((2k-1)\alpha-\zeta),$$

vbi pro parte priore, ob

$$\text{ſin.}\,p+\text{ſin.}\,q=2\,\text{ſin.}\,\tfrac{p+q}{2}\,\text{coſ.}\,\tfrac{p-q}{2},\ \text{erit}$$

$$\tfrac{p+q}{2}=\alpha k\ \text{et}\ \tfrac{p-q}{2}=(k-1)\alpha-\zeta;$$

vnde pars ipſa prior fit

$$2(\gamma+2\beta)\,\text{ſin.}\,\alpha k\,\text{coſ.}((k-1)\alpha-\zeta),$$

vbi cum ſit $\alpha k=m\pi$, erit ſin. $\alpha k=0$, ita vt tantum ſuperſit

$$2\,S\,\text{ſin.}\,\alpha=-2\beta k\,\text{ſin.}((2k-1)\alpha-\zeta)\ \text{hincque}$$

$$S=-\frac{\beta k\,\text{ſin.}((2k-1)\alpha-\zeta)}{\text{ſin.}\,\alpha}.\ \text{Eſt vero}$$

$$(2k-1)\alpha-\zeta=2m\pi-\frac{m\pi}{k}-\frac{\theta(k-m)}{k};$$

omiſſo termino $2m\pi$ erit igitur

$$S=+\frac{\pi\,\text{ſin.}\,\frac{m\pi+\theta(k-m)}{k}}{\text{ſin.}\,\frac{m\pi}{k}}$$

ideoque valor quaeſitus erit

$$\frac{S}{k\,\text{ſin.}\,\theta}=+\frac{\pi\,\text{ſin.}\,\frac{m\pi+\theta(k-m)}{k}}{k\,\text{ſin.}\,\theta\,\text{ſin.}\,\frac{m\pi}{k}}$$

quae forma reducitur ad hanc:

$$\frac{\pi\,\text{ſin.}\,\frac{m(\pi-\theta)+\theta}{k}}{k\,\text{ſin.}\,\theta\,\text{ſin.}\,\frac{m\pi}{k}}.$$

§. 20.

ad hanc formam:
)β) ſin.(2k−1)α−ζ),
)−2kβ ſin.((2k−1)α−ζ),
q, erit
−ζ;
)α−ζ),
, ita vt tantum ſu-
)α−ζ) hincque
−θ(k−m)/k;
§. 20.

§. 20 Contemplemur hic ante omnia caſum quo $\theta=\frac{\pi}{2}$, et formula integralis propoſita abit in hanc:

$$\int\frac{x^{m-1}\,dx}{1+x^{2k}}$$

cuius ergo valor, ſi poſt integrationem ponatur $x=\infty$, euadet

$$=\frac{\pi\,\text{ſin.}\left(\frac{\pi}{2}+\frac{m\pi}{2k}\right)}{k\,\text{ſin.}\,\frac{m\pi}{k}}=\frac{\pi\,\text{coſ.}\,\frac{m\pi}{2k}}{k\,\text{ſin.}\,\frac{m\pi}{k}}.$$

Quia igitur eſt

$$\text{ſin.}\,\frac{m\pi}{k}=2\,\text{ſin.}\,\frac{m\pi}{2k}\,\text{coſ.}\,\frac{m\pi}{2k},$$

prodibit iſte valor $=\frac{\pi}{2k\,\text{ſin.}\,\frac{m\pi}{2k}}$, qui valor egregie conuenit cum eo, quem non ita pridem pro formula $\int\frac{x^{m-1}\,dx}{1+x^k}$ aſſignauimus, ſi quidem loco k ſcribatur $2k$.

§. 21. Euoluamus etiam caſum quo $\theta=\pi$ et formula noſtra integralis $\int\frac{x^{m-1}\,dx}{(1+x^k)^2}$, cuius ergo, facto $x=\infty$, valor erit

$$\frac{\pi\,\text{ſin.}\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{k\,\text{ſin.}\,\theta\,\text{ſin.}\,\frac{m\pi}{k}}=\frac{\pi}{k\,\text{ſin.}\,\frac{m\pi}{k}}\cdot\frac{\text{ſin.}\,\frac{m(\pi-\theta)}{k}+\theta}{\text{ſin.}\,\theta}.$$

Huius autem poſterioris fractionis, caſu $\theta=\pi$, tam numerator quam denominator euaneſcit; quare, vt eius verus valor eruatur, loco vtriusque eius differentiale ſcribamus, quo facto iſta fractio abibit in hanc: $\frac{d\theta\left(1-\frac{m}{k}\right)\text{coſ.}\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{d\theta\,\text{coſ.}\,\theta}$, cuius valor facto $\theta=\pi$ nunc manifeſto eſt $1-\frac{m}{k}$; ſicque valor in-

integralis quaesitus erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k\,\text{sin.}\frac{m\pi}{k}}$, prorsus vti in superiore dissertatione inuenimus.

§. 22. Quo autem valorem generalem inuentum commodiorem reddamus, ponamus $\pi-\theta=\eta$, fietque $\text{sin.}\theta=\text{sin.}\eta$ et $\text{cos.}\theta=-\text{cos.}\eta$; tum vero erit angulus

$$\frac{m(\pi-\theta)}{k}+\theta=\frac{m\pi}{k}+\pi-\eta,$$

cuius sinus est $\text{sin.}\left(1-\frac{m}{k}\right)\eta$, vnde valor quaesitus nostrae formulae erit $\frac{\pi\,\text{sin.}\left(1-\frac{m}{k}\right)\eta}{k\,\text{sin.}\eta\,\text{sin.}\frac{m\pi}{k}}$, atque hinc tandem sequens adepti sumus

Theorema.

§. 23. *Si haec formula integralis:*

$$\int\frac{x^{m-1}\,dx}{1+2x^k\,\text{cos.}\eta+x^{2k}}$$

a termino $x=0$ *vsque ad terminum* $x=\infty$ *extendatur, eius valor erit* $=\frac{\pi\,\text{sin.}\left(1-\frac{m}{k}\right)\eta}{k\,\text{sin.}\eta\,\text{sin.}\frac{m\pi}{k}}$, *siue cum sit*

$$\text{sin.}\left(1-\frac{m}{k}\right)\eta=\text{sin.}\eta\,\text{cos.}\frac{m\eta}{k}-\text{cos.}\eta\,\text{sin.}\frac{m\eta}{k},$$

iste valor etiam hoc modo exprimi potest:

$$\frac{\pi\,\text{cos.}\frac{m\eta}{k}}{k\,\text{sin.}\frac{m\pi}{k}}-\frac{\pi\,\text{sin.}\frac{m\eta}{k}}{k\,\text{tang.}\eta\,\text{sin.}\frac{m\pi}{k}}.$$

orsus vti in
n inuentum
e $\text{sin.}\theta=\text{sin.}\eta$
sitis nostrae
idem sequens
'endatur, eius
n. $\frac{m\eta}{k}$,
§. 24

§. 24. Consideremus nunc alio modo hanc formulam integralem: $\int\frac{x^{m-1}\,dx}{1+2x^k\,\text{cos.}\eta+x^{2k}}$, cuius valor a termino $x=0$ vsque ad $x=1$ ponatur $=P$, eiusdem vero valor ab $x=1$ vsque ad $x=\infty$ ponatur $=Q$, ita vt $P+Q$ exhibere debeat ipsum valorem ante inuentum. Nunc vero pro valore Q inueniendo ponamus $x=\frac{1}{y}$ et formula nostra ita repraesentata:

$$\frac{x^m}{1+2x^k\,\text{cos.}\eta+x^{2k}}\frac{dx}{x},$$

ob $\frac{dx}{x}=-\frac{dy}{y}$ induet hanc formam:

$$-\int\frac{y^{-m}}{1+2y^{-k}\,\text{cos.}\eta+y^{-2k}}\frac{dy}{y}=-\int\frac{y^{2k-m-1}\,dy}{y^{2k}+2y^k\,\text{cos.}\eta+1}$$

cuius valor a termino $y=1$ vsque ad $y=0$ extendi debet. Commutatis igitur his terminis habebimus

$$Q=+\int\frac{y^{2k-m-1}\,dy}{y^{2k}+2y^k\,\text{cos.}\eta+1}$$

a termino $y=0$ vsque ad $y=1$.

§. 25. Quia in vtraque forma pro P et Q eadem conditio integrationis praescribitur, a termino 0 vsque ad 1, nihil impedit quo minus in posteriore loco y scribamus x, vnde pro $P+Q$ habebimus hanc formam integralem:

$$\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\,\text{cos.}\eta+x^{2k}}\,dx,$$

cuius valor, a termino $x=0$ vsque ad $x=1$ extensus, aequabitur huic expressioni: $\frac{\pi\,\text{sin.}\left(1-\frac{m}{k}\right)\eta}{k\,\text{sin.}\eta\,\text{sin.}\frac{m\pi}{k}}$. Comparatis igitur his binis formulis integralibus nanciſcemur ſequens Theorema notatu maxime dignum.

Theorema.

§. 26. *Haec formula integralis:*

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos.\eta + x^{2k}} dx,$$

a termino $x = 0$ *vsque ad terminum* $x = 1$ *extenſa, aequalis eſt huic formulae integrali:* $\int \frac{x^{m-1} dx}{1 + 2x^k \cos.\eta + x^{2k}}$, *a termino* $x = 0$ *vsque ad terminum* $x = \infty$ *extenſae: vtriusque enim valor erit* $\frac{\pi \sin.(1 - \frac{m}{k})\eta}{k \sin.\eta \sin.\frac{m\pi}{k}}$.

§. 27. Quod ſi hanc fractionem: $\frac{\sin.\eta}{1 + 2x^k \cos.\eta + x^{2k}}$ in ſeriem infinitam euoluamus, quae ſit,

$$\sin.\eta + A x^k + B x^{2k} + C x^{3k} + D x^{4k} + E x^{5k} \text{ etc.}$$

per denominatorem multiplicando perueniemus ad hanc expreſſionem infinitam:

$$\begin{array}{lllllll} \sin.\eta = \sin.\eta + & A x^k & + B x^{2k} & + C x^{3k} & + D x^{4k} & + E x^{5k} & + F x^{6k} + \text{etc.} \\ & +2\sin.\eta\cos.\eta & +2A\cos.\eta & +2B\cos.\eta & +2C\cos.\eta & +2D\cos.\eta & +2E\cos.\eta + \text{etc.} \\ & & + \sin.\eta & + A & + B & + C & + D \text{ etc.} \end{array}$$

vnde ſingulis terminis ad nihilum reductis reperiemus

1° $A + 2 \sin.\eta \cos.\eta = 0$, hincque $A = -\sin.2\eta$.
2° $B + 2A\cos.\eta + \sin.\eta = 0$, vnde fit $B = \sin.3\eta$.
3° $C + 2B\cos.\eta + A = 0$, vnde fit $C = -\sin.4\eta$.
4° $D + 2C\cos.\eta + B = 0$, vnde fit $D = \sin.5\eta$
etc. etc.

ita

Theore-
ſa, aequalis
, a termino
vsque enim
in. η
cof. η + x
x etc.
l hanc ex-
x + F x + etc.
of.η+2Ecof.η+ etc.
+ D etc.
iemus
= − fin. 2 η.
= fin. 3 η.
− fin. 4 η.
= fin. 5 η
ita

ita vt noſtra fractio $\frac{\sin.\eta}{1 + 2x^k \cos.\eta + x^{2k}}$ reſoluatur in hanc ſeriem:

$$\sin.\eta - x^k \sin.2\eta + x^{2k} \sin.3\eta - x^{3k} \sin.4\eta + x^{4k} \sin.5\eta \text{ etc.}$$

§. 28. Multiplicemus nunc hanc ſeriem per $x^{m-1} dx + x^{2k-m-1} dx$ et poſt integrationem faciamus $x = 1$, vt obtineamus valorem huius formulae:

$$\sin.\eta \int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos.\eta + x^{2k}} dx$$

pro caſu $x = 1$, hocque modo perueniemus ad geminas ſequentes ſeries:

$$\frac{\sin.\eta}{m} - \frac{\sin.2\eta}{m+k} + \frac{\sin.3\eta}{m+2k} - \frac{\sin.4\eta}{m+3k} + \frac{\sin.5\eta}{m+4k} - \text{etc.}$$
$$\frac{\sin.\eta}{2k-m} - \frac{\sin.2\eta}{3k-m} + \frac{\sin.3\eta}{4k-m} - \frac{\sin.4\eta}{5k-m} + \frac{\sin.5\eta}{6k-m} - \text{etc.}$$

Aggregatum igitur harum duarum ſeriesum iunctim ſumtarum aequabitur huic valori: $\frac{\pi \sin.(1 - \frac{m}{k})\eta}{k \sin.\frac{m\pi}{k}}$, vnde ſubiungamus adhuc iſtud Theorema:

Theorema.

§. 29. *Si* η *denotet angulum quemcunque, litterae vero* m *et* k *pro lubitu accipiantur, ex iisque binae ſequentes ſeries formentur:*

$$P = \frac{\sin.\eta}{m} - \frac{\sin.2\eta}{m+k} + \frac{\sin.3\eta}{m+2k} - \frac{\sin.4\eta}{m+3k} + \frac{\sin.5\eta}{m+4k} - \text{etc.}$$
$$Q = \frac{\sin.\eta}{2k-m} - \frac{\sin.2\eta}{3k-m} + \frac{\sin.3\eta}{4k-m} - \frac{\sin.4\eta}{5k-m} + \frac{\sin.5\eta}{6k-m} - \text{etc.}$$

neutrius quidem ſumma exhiberi poteſt, vtriusque autem iunctim ſumtae ſumma erit

P +

$$P + Q = \frac{\pi \sin. (1 - \frac{m}{k}) \eta}{k \sin. \frac{m\pi}{k}}.$$

Corollarium.

§. 30. Quod si ergo angulum η infinite paruum capiamus, vt fiat

$$\sin. \eta = \eta;\ \sin. 2\eta = 2\eta;\ \sin. 3\eta = 3\eta;\ \text{etc.}$$

quia in formula summae fiet

$$\sin. (1 - \tfrac{m}{k}) \eta = (1 - \tfrac{m}{k}) \eta;$$

si vtrinque per η diuidamus, obtinebimus sequentem seriem geminatam:

$$\frac{1}{m} - \frac{2}{m+k} + \frac{3}{m+2k} - \frac{4}{m+3k} + \frac{5}{m+4k} - \text{etc.}$$
$$\frac{1}{2k-m} - \frac{2}{3k-m} + \frac{3}{4k-m} - \frac{4}{5k-m} + \frac{5}{6k-m} - \text{etc.}$$

cuius ergo summa erit $(1 - \frac{m}{k}) \frac{\pi}{k \sin. \frac{m\pi}{k}}$; vbi notetur, ambas istas series non incongrue in hanc simplicem contrahi posse:

$$\frac{2k}{m(2k-m)} - \frac{8k}{(m+k)(3k-m)} + \frac{18k}{(m+2k)(4k-m)} - \frac{32k}{(m+3k)(5k-m)} + \text{etc.}$$

vbi numeratores sunt numeri quadrati duplicati.

§. 31. Formulae autem, quarum valores hactenus inuenimus, multo concinnius et elegantius exprimi possunt, si loco exponentis m scribamus $k - n$, tum enim in valore integrali inuento fit $(1 - \frac{m}{k}) \eta = \frac{n\eta}{k}$; at vero pro denominatore fiet $\frac{m\pi}{k} = \pi - \frac{n\pi}{k}$, cuius sinus erit $\sin. \frac{n\pi}{k}$; sicque nostra formula inuenta hanc induet formam: $\frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$, quae ergo exprimet valorem huius formulae integralis:

... η infinite paruum

$3\eta = 3\eta$; etc.

... mus sequentem seriem

... $\frac{5}{m+4k}$ — etc.

... $\frac{5}{6k-m}$ — etc.

... $\frac{m\pi}{k}$; vbi notetur, am-
... mplicem contrahi posse:
... $- \frac{32k}{(m+3k)(5k-m)} + \text{etc.}$
... duplicati.

... rum valores hactenus
... tius exprimi possunt, si
... tum enim in valore
... ; at vero pro deno-
... us erit $\sin. \frac{n\pi}{k}$; sicque
... formam: $\frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$,
... ormulae integralis:

$$\int \frac{x^{k-n-1} dx}{1 + 2x^k \cos. \eta + x^{2k}},$$

ab $x = 0$ vsque ad $x = \infty$, vt et huius formulae:

$$\int \frac{x^{k-n-1} + x^{k+n-1} dx}{1 + x^k \cos. \eta + x^{2k}},$$

a termino $x = 0$ vsque ad terminum $x = 1$; et quia vtriusque valor est $\frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$, perspicuum est eum manere eundem, etsi loco n scribatur $-n$, ex quo prior formula ita repraesentari poterit:

$$\int \frac{x^{k \pm n - 1}}{1 + x^k \cos. \eta + x^{2k}};$$

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

§. 32. Ponendo $m = k - n$ etiam series nostra geminata pulchriorem accipiet faciem; habebitur enim

$$\frac{\sin. \eta}{k-n} - \frac{\sin. 2\eta}{2k-n} + \frac{\sin. 3\eta}{3k-n} - \frac{\sin. 4\eta}{4k-n} + \text{etc.}$$
$$\frac{\sin. \eta}{k+n} - \frac{\sin. 2\eta}{2k+n} + \frac{\sin. 3\eta}{3k+n} - \frac{\sin. 4\eta}{4k+n} + \text{etc}$$

cuius ergo summa erit $\frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \frac{n\pi}{k}}$. Tum vero si hae geminae series in vnam contrahantur et vtrinque per $2k$ diuidatur, obtinebitur sequens summatio memoratu digna:

$$\frac{\pi \sin. \frac{n\eta}{k}}{2kk \sin. \frac{n\pi}{k}} = \frac{\sin. \eta}{kk - nn} - \frac{2 \sin. 2\eta}{4kk - nn} + \frac{3 \sin. 3\eta}{9kk - nn} - \frac{4 \sin. 4\eta}{16kk - nn} + \text{etc.}$$

§. 33.

§. 33. Quod si haec postrema series differentietur, sumendo solum angulum η variabilem, ob

$d.\,\text{sin.}\,\frac{n\eta}{k} = \frac{n\,d\eta}{k}\,\text{cos.}\,\frac{n\eta}{k}$ habebimus

$$\frac{\pi n\,\text{cos.}\,\frac{n\eta}{k}}{2k^2\,\text{sin.}\,\frac{n\pi}{k}} = \frac{\text{cos.}\,\eta}{kk-nn} - \frac{4\,\text{cos.}\,2\eta}{4kk-nn} + \frac{9\,\text{cos.}\,3\eta}{9kk-nn} - \frac{16\,\text{cos.}\,4\eta}{16kk-nn} + \text{etc.}$$

Vnde si sumatur $\eta = 0$, orietur ista summatio:

$$\frac{\pi n}{2k^2\,\text{sin.}\,\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.}$$

Sin autem sumatur $\eta = 90° = \frac{\pi}{2}$, erit $\text{cos.}\,\eta = 0$, $\text{cos.}\,2\eta = -1$, $\text{cos.}\,3\eta = 0$, $\text{cos.}\,4\eta = +1$ etc. vnde nascitur sequens series:

$$\frac{n\pi\,\text{cos.}\,\frac{n\pi}{2k}}{2k^2\,\text{sin.}\,\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

Quia autem $\text{sin.}\,\frac{n\pi}{k} = 2\,\text{sin.}\,\frac{n\pi}{2k}\,\text{cos.}\,\frac{n\pi}{2k}$, erit eiusdem seriei summa $\frac{n\pi}{4k^2\,\text{sin.}\,\frac{n\pi}{2k}}$.

§. 34. At si series illa §. 32 exhibita in $d\eta$ ducatur et integretur, ob

$\int d\eta\,\text{sin.}\,\frac{n\eta}{k} = -\frac{k}{n}\,\text{cos.}\,\frac{n\eta}{k}$, erit

$$C - \frac{\pi\,\text{cos.}\,\frac{n\eta}{k}}{2nk\,\text{sin.}\,\frac{n\pi}{k}} = -\frac{\text{cos.}\,\eta}{kk-nn} + \frac{\text{cos.}\,2\eta}{4kk-nn} - \frac{\text{cos.}\,3\eta}{9kk-nn} + \frac{\text{cos.}\,4\eta}{16kk-nn} + \text{etc.}$$

Vt autem hic constantem addendam C definiamus, sumamus $\eta = 0$, fietque

$$C - \frac{\pi}{2nk\,\text{sin.}\,\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.}$$

quare

s differentietur,
of. 4η / k-nn + etc.
o:
16 / kk-nn + etc.
, cos. 2η = -1,
ascitur sequens
64 / 64kk-nn + etc.
dem seriei sum-
a in dη duca-
cos. 4η / 16kk-nn + etc.
mus, sumamus
1 / nn + etc.
quare

quare si huius seriei summa aliunde pateat, constans C definiri poterit. Series autem haec in sequentem geminatam resolui potest:

$$2nC - \frac{\pi}{k\,\text{sin.}\,\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$

$$- \frac{1}{k-n} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} - \text{etc.}$$

§. 35. Cum igitur in *Introductione in Analysin Infinitorum* pag. 142. ad hanc peruenissem seriem:

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.}$$

$$= \frac{\pi}{2kn\,\text{sin.}\,\frac{n\pi}{k}} - \frac{1}{2nn}$$

(hic scilicet loco litterarum ibi adhibitarum m et n scripsi n et k) hoc valore adhibito nostra aequatio erit

$$C - \frac{\pi}{2nk\,\text{sin.}\,\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk\,\text{sin.}\,\frac{n\pi}{k}},$$

vnde fit $C = \frac{1}{2nn}$. Hinc ergo habebimus istam summationem:

$$\frac{\pi\,\text{cos.}\,\frac{n\eta}{k}}{2nk\,\text{sin.}\,\frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\text{cos.}\,\eta}{kk-nn} - \frac{\text{cos.}\,2\eta}{4kk-nn}$$

$$+ \frac{\text{cos.}\,3\eta}{9kk-nn} - \frac{\text{cos.}\,4\eta}{16kk-nn} + \text{etc.}$$

quae series vtique omni attentione digna videtur.