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Investigatio formulae integralis $\int (x^{m-1} dx)/(1+x^k)^n$ casu, quo post intagrationem statuitur $x=\infty$

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E588

INVESTIGATIO
FORMVLAE INTEGRALIS

$$\int\frac{x^{m-1}\,dx}{(1+x^k)^n}$$

CASV QVO POST INTEGRATIONEM STATVITVR

$x = \infty$.

§. 1.

Iam ſatis notum eſt, huius formulae integrale partim logarithmos, partim arcus circulares complecti, et partes logarithmicas hanc progreſſionem conſtituere:

$$-\tfrac{2}{k}\operatorname{cof.}\tfrac{m\pi}{k}\,l\sqrt{(1-2x\operatorname{cof.}\tfrac{\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\operatorname{cof.}\tfrac{3m\pi}{k}\,l\sqrt{(1-2x\operatorname{cof.}\tfrac{3\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\operatorname{cof.}\tfrac{5m\pi}{k}\,l\sqrt{(1-2x\operatorname{cof.}\tfrac{5\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\operatorname{cof.}\tfrac{7m\pi}{k}\,l\sqrt{(1-2x\operatorname{cof.}\tfrac{7\pi}{k}+xx)}$$
$$-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-$$
$$-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-$$
$$-\tfrac{2}{k}\operatorname{cof.}\tfrac{im\pi}{k}\,l\sqrt{(1-2x\operatorname{cof.}\tfrac{i\pi}{k}+xx)}$$

vbi i denotat numerum imparem non maiorem quam k. Hinc ſi k fuerit numerus par, erit $i = k-1$; ac ſi k fuerit numerus impar, hanc progreſſionem continuari oportet vsque ad $i = k$; eius vero coefficiens duplo minor capi debet, ſeu loco $-\frac{2}{k}$ tantum ſcribi debet $-\frac{1}{k}$, cuius irregularitatis ratio in *Calculo Integrali* eſt expoſita.

§. 2.

ALIS

TVITVR

partim lo-
, et partes

m quam k.
ſi k fuerit
ri oportet
nor capi de-
ıs irregula-

§. 2.

§. 2. Cum hae partes ſponte iam euaneſcant poſito $x = 0$, ſtatuamus ſtatim $x = \infty$, et cum in genere ſit

$$\sqrt{(1-2x\operatorname{cof.}\omega+xx)} = x-\operatorname{cof.}\omega,\ \text{erit}$$

$$l\sqrt{(1-2x\operatorname{cof.}\omega+xx)} = l(x-\operatorname{cof.}\omega) = lx-\tfrac{\operatorname{cof.}\omega}{x} = lx,\ \text{ob}\ \tfrac{\operatorname{cof.}\omega}{x} = 0;$$

omnes ergo illi logarithmi reducuntur ad eandem formam lx, quae multiplicanda eſt per hanc ſeriem:

$$-\tfrac{2}{k}\operatorname{cof.}\tfrac{m\pi}{k}-\tfrac{2}{k}\operatorname{cof.}\tfrac{3m\pi}{k}-\tfrac{2}{k}\operatorname{cof.}\tfrac{5m\pi}{k}\;-\;-\;-\;-\;-\tfrac{2}{k}\operatorname{cof.}\tfrac{im\pi}{k},$$

vbi, vt diximus, i denotat maximum numerum imparem ipſo k non maiorem, hac tamen reſtrictione, vt, ſi k fuerit impar, ideoque $i = k$, vltimum membrum ad dimidium reduci debeat. Quamobrem, ſi huius progreſſionis ſummam inueſtigare velimus, duo caſus erunt conſtituendi: alter quo k eſt numerus par et $i = k-1$, alter vero quo k eſt impar et $i = k$.

Euolutio caſus prioris, quo k eſt numerus par et $i = k-1$.

§. 3. Hoc ergo caſu, poſito $x = \infty$, formula $-\frac{2}{k\,l\,x}$ multiplicatur per hanc Coſinuum ſeriem:

$$\operatorname{cof.}\tfrac{m\pi}{k}+\operatorname{cof.}\tfrac{3m\pi}{k}+\operatorname{cof.}\tfrac{5m\pi}{k}+\operatorname{cof.}\tfrac{7m\pi}{k}+\;-\;-\;-\;-\;+\operatorname{cof.}\tfrac{(k-1)m\pi}{k},$$

cuius ſummam ſtatuamus $= S$. Ducamus hanc ſeriem in $\operatorname{fin.}\frac{m\pi}{k}$, et cum in genere ſit

$$\operatorname{fin.}\tfrac{m\pi}{k}\operatorname{cof.}\tfrac{im\pi}{k} = \tfrac{1}{2}\operatorname{fin.}\tfrac{(i+1)m\pi}{k}-\tfrac{1}{2}\operatorname{fin.}\tfrac{(i-1)m\pi}{k},$$

facta hac reductione habebimus

$$S\operatorname{fin.}\tfrac{m\pi}{k} = \tfrac{1}{2}\operatorname{fin.}\tfrac{2m\pi}{k}+\tfrac{1}{2}\operatorname{fin.}\tfrac{4m\pi}{k}+\tfrac{1}{2}\operatorname{fin.}\tfrac{6m\pi}{k}\;-\;-\;-\;-\;+\tfrac{1}{2}\operatorname{fin.}\tfrac{(k-1)m\pi}{k}$$
$$-\tfrac{1}{2}\operatorname{fin.}\tfrac{2m\pi}{k}-\tfrac{1}{2}\operatorname{fin.}\tfrac{4m\pi}{k}-\tfrac{1}{2}\operatorname{fin.}\tfrac{6m\pi}{k}\;-\;-\;-\;-\;-\tfrac{1}{2}\operatorname{fin.}\tfrac{(k-2)m\pi}{k};$$

vbi

vbi omnes termini praeter vltimum manifeſto ſe deſtruunt, ita vt ſit

$$S\ \text{ſin.}\ \tfrac{m\pi}{k} = \tfrac{1}{2}\ \text{ſin.}\ m\pi.$$

Iam vero quia noſtri coefficientes m et k ſupponuntur integri, vtique erit $\text{ſin.}\ m\pi = 0$, ideoque etiam $S = 0$, niſi forte etiam fuerit $\text{ſin.}\ \frac{m\pi}{k} = 0$, qui autem caſus locum habere nequit, quoniam in integratione formulae propoſitae $\frac{x^{m-1}\,dx}{1+x^k}$ ſemper aſſumi ſolet eſſe $m < k$. Hoc igitur modo euictum eſt, caſu quo poſt integrationem ſtatuitur $x = \infty$, omnes partes logarithmicas integralis ſe deſtruere.

Euolutio caſus alterius, quo eſt k numerus impar et $i = k$.

§. 4. Hoc ergo caſu, ſumto $x = \infty$, formula lx multiplicatur per hanc ſeriem:

$$-\tfrac{2}{k}\,\text{coſ.}\,\tfrac{m\pi}{k} - \tfrac{2}{k}\,\text{coſ.}\,\tfrac{3m\pi}{k} - \tfrac{2}{k}\,\text{coſ.}\,\tfrac{5m\pi}{k} - \cdots\cdots - \tfrac{1}{k}\,\text{coſ.}\,\tfrac{im\pi}{k},$$

vbi terminus penultimus eſt $-\frac{2}{k}\,\text{coſ.}\,\frac{(k-2)m\pi}{k}$, pro vltimo vero termino erit $\text{coſ.}\ m\pi = \pm 1$, ſigno ſuperiore valente ſi n ſit numerus par, inferiore ſi impar; quare remoto termino vltimo pro reliquis ponamus

$$\text{coſ.}\,\tfrac{m\pi}{k} + \text{coſ.}\,\tfrac{3m\pi}{k} + \text{coſ.}\,\tfrac{5m\pi}{k} + \cdots\cdots + \text{coſ.}\,\tfrac{(k-2)m\pi}{k} = S$$

ita vt multiplicator ipſius logarithmi x ſit

$$-\tfrac{2S}{k} - \tfrac{1}{k}\,\text{coſ.}\ m\pi.$$

Hinc procedendo vt ante fiet

$$\begin{aligned} S\,\text{ſin.}\,\tfrac{m\pi}{k} = {} & \tfrac{1}{2}\,\text{ſin.}\,\tfrac{2m\pi}{k} + \tfrac{1}{2}\,\text{ſin.}\,\tfrac{4m\pi}{k} + \tfrac{1}{2}\,\text{ſin.}\,\tfrac{6m\pi}{k} \cdots\cdots + \tfrac{1}{2}\,\text{ſin.}\,\tfrac{(k-1)m\pi}{k} \\ & - \tfrac{1}{2}\,\text{ſin.}\,\tfrac{2m\pi}{k} - \tfrac{1}{2}\,\text{ſin.}\,\tfrac{4m\pi}{k} - \tfrac{1}{2}\,\text{ſin.}\,\tfrac{6m\pi}{k} \cdots\cdots - \tfrac{1}{2}\,\text{ſin.}\,\tfrac{(k-3)m\pi}{k}; \end{aligned}$$

vbi

e deſtruunt,

ponuntur in- S = 0, niſi us locum ha- ae propoſitae loc igitur mo- ſtatuitur $x = \infty$, ere.

erus impar

, formula lx

$-\frac{1}{k}\,\text{coſ.}\,\frac{im\pi}{k}$,

, pro vltimo eriore valente quare remoto

$\frac{-2)m\pi}{k} = S$

$+\frac{1}{2}\,\text{ſin.}\,\frac{(k-1)m\pi}{k}$

$-\frac{1}{2}\,\text{ſin.}\,\frac{(k-3)m\pi}{k}$;

vbi

vbi iterum omnes termini praeter vltimum ſe mutuo tollunt, ita vt hinc prodeat

$$S\ \text{ſin.}\ \tfrac{m\pi}{k} = \tfrac{1}{2}\ \text{ſin.}\ \tfrac{(k-1)m\pi}{k} = \tfrac{1}{2}\ \text{ſin.}\left(m\pi - \tfrac{m\pi}{k}\right);$$

at vero eſt

$$\text{ſin.}\left(m\pi - \tfrac{m\pi}{k}\right) = \text{ſin.}\ m\pi\ \text{coſ.}\ \tfrac{m\pi}{k} - \text{coſ.}\ m\pi\ \text{ſin.}\ \tfrac{m\pi}{k},$$

vbi notetur eſſe $\text{ſin.}\ m\pi = 0$, ob m numerum integrum; habebimus ergo

$$S\ \text{ſin.}\ \tfrac{m\pi}{k} = \mp\tfrac{1}{2}\ \text{coſ.}\ m\pi\ \text{ſin.}\ \tfrac{m\pi}{k},\ \text{ſiue}\ S = -\tfrac{1}{2}\,\text{coſ.}\ m\pi,$$

conſequenter multiplicator ipſius lx erit

$$= \tfrac{1}{k}\,\text{coſ.}\ m\pi - \tfrac{1}{k}\,\text{coſ.}\ m\pi = 0,$$

ſicque manifeſtum eſt, ſiue k ſit numerus par ſiue impar, omnia membra logarithmica in noſtro integrali ſe mutuo deſtruere, ſiquidem poſt integrationem ſtatuamus $x = \infty$, quemadmodum hic ſemper ſupponimus.

§. 5. Conſideremus nunc etiam partes a circulo pendentes, ex quibus integrale noſtrae formulae componitur. Hae autem partes ſequentem progreſſionem conſtituere ſunt compertae:

$$\begin{aligned} & \tfrac{2}{k}\,\text{ſin.}\,\tfrac{m\pi}{k}\,\text{A tang.}\,\frac{x\,\text{ſin.}\,\frac{\pi}{k}}{1-x\,\text{coſ.}\,\frac{\pi}{k}} + \tfrac{2}{k}\,\text{ſin.}\,\tfrac{3m\pi}{k}\,\text{A tang.}\,\frac{x\,\text{ſin.}\,\frac{3\pi}{k}}{1-x\,\text{coſ.}\,\frac{3\pi}{k}} \\ & + \tfrac{2}{k}\,\text{ſin.}\,\tfrac{5m\pi}{k}\,\text{A tang.}\,\frac{x\,\text{ſin.}\,\frac{5\pi}{k}}{1-x\,\text{coſ.}\,\frac{5\pi}{k}} + \tfrac{2}{k}\,\text{ſin.}\,\tfrac{7m\pi}{k}\,\text{A tang.}\,\frac{x\,\text{ſin.}\,\frac{7\pi}{k}}{1-x\,\text{coſ.}\,\frac{7\pi}{k}} \\ & + \cdots\cdots\cdots\cdots + \tfrac{2}{k}\,\text{ſin.}\,\tfrac{im\pi}{k}\,\text{A tang.}\,\frac{x\,\text{ſin.}\,\frac{i\pi}{k}}{1-x\,\text{coſ.}\,\frac{i\pi}{k}} \end{aligned}$$

vbi in vltimo membro eſt vel $i = k - 1$, vel $i = k$; prius ſcilicet valet ſi i eſt numerus par, poſterius ſi impar.

§. 6.

§. 6. Cum etiam omnia haec membra euaneſcant poſito $x = 0$, faciamus pro inſtituto noſtro $x = \infty$. In genere igitur fiet

$$\text{A tang.}\,\frac{x\,\text{ſin.}\,\frac{i\pi}{k}}{1 - x\,\text{coſ.}\,\frac{i\pi}{k}} = \text{A tang.}\left(-\,\text{tang.}\,\frac{i\pi}{k}\right).$$

Eſt vero

$$-\,\text{tang.}\,\frac{i\pi}{k} = +\,\text{tang.}\,\frac{(k-i)\pi}{k},$$

ex quo hic arcus fit $= \frac{(k-i)\pi}{k}$. Hinc ergo loco i ſcribendo ſucceſſiue numeros 1, 3, 5, 7 etc. iſtae partes noſtri integralis quaeſiti erunt

$$\frac{2(k-1)\pi}{kk}\,\text{ſin.}\,\frac{1m\pi}{k} + \frac{2(k-3)\pi}{kk}\,\text{ſin.}\,\frac{3m\pi}{k} + \frac{2(k-5)\pi}{kk}\,\text{ſin.}\,\frac{5m\pi}{k}$$
$$+\frac{2(k-7)\pi}{kk}\,\text{ſin.}\,\frac{7m\pi}{k} + \frac{2(k-9)\pi}{kk}\,\text{ſin.}\,\frac{9m\pi}{k} + - - - - \frac{2(k-i)\pi}{kk}\,\text{ſin.}\,\frac{im\pi}{k}$$

vbi caſu, quo k eſt numerus par, progredi oportet vsque ad $i = k - 1$: ac ſi k ſit numerus impar, vsque ad $i = k$.

§. 7. Statuamus breuitatis gratia

$$(k-1)\,\text{ſin.}\,\frac{m\pi}{k} + (k-3)\,\text{ſin.}\,\frac{3m\pi}{k} + (k-5)\,\text{ſin.}\,\frac{5m\pi}{k} + \dots\dots$$
$$+ (k-i)\,\text{ſin.}\,\frac{im\pi}{k} = S$$

ita vt integrale quaeſitum ſit $\frac{2\pi S}{kk}$, quandoquidem partes logarithmicae ſe mutuo deſtruxerunt. Multiplicemus nunc vtrinque per $2\,\text{ſin.}\,\frac{m\pi}{k}$, et cum in genere ſit

$$2\,\text{ſin.}\,\frac{m\pi}{k}\,\text{ſin.}\,\frac{im\pi}{k} = \text{coſ.}\,\frac{(i-1)m\pi}{k} - \text{coſ.}\,\frac{(i+1)m\pi}{k},$$

facta ſubſtitutione erit

membra euaneſcant
noſtro $x = \infty$. In

tang. $\frac{i\pi}{k}$).

rgo loco i ſcriben-
ae partes noſtri in-

π ſin. $\frac{5m\pi}{k}$

- $\frac{2(k-i)\pi}{kk}$ ſin. $\frac{im\pi}{k}$

di oportet vsque ad
vsque ad $i = k$.

5) ſin. $\frac{5m\pi}{k}$ + ...

andoquidem partes
Multiplicemus nunc
e ſit

coſ. $\frac{(i+1)m\pi}{k}$,

2 S

$$2\,S\,\text{ſin.}\,\frac{m\pi}{k} = (k-1)\,\text{coſ.}\,\frac{0m\pi}{k} + (k-3)\,\text{coſ.}\,\frac{2m\pi}{k} + (k-5)\,\text{coſ.}\,\frac{4m\pi}{k} - - - -$$
$$-(k-1)\,\text{coſ.}\,\frac{2m\pi}{k} - (k-3)\,\text{coſ.}\,\frac{4m\pi}{k} - (k-5)\,\text{coſ.}\,\frac{6m\pi}{k} - - - -$$
$$- - - - + (k-i)\,\text{coſ.}\,\frac{(i-1)m\pi}{k}$$
$$- (k-i)\,\text{coſ.}\,\frac{(i+1)m\pi}{k}$$

quae ſeries manifeſto contrahitur in ſequentem:

$$2S\,\text{ſin.}\,\frac{m\pi}{k} = (k-1) - 2\,\text{coſ.}\,\frac{2m\pi}{k} - 2\,\text{coſ.}\,\frac{4m\pi}{k} - 2\,\text{coſ.}\,\frac{6m\pi}{k} - - - - 2\,\text{coſ.}\,\frac{(i-1)m\pi}{k}$$
$$- (k-i)\,\text{coſ.}\,\frac{(i+1)m\pi}{k}$$

vbi, primo et vltimo membro ſublatis, regularem termini intermedii conſtituunt ſeriem, pro cuius valore inueſtigando ponamus

$$T = \text{coſ.}\,\frac{2m\pi}{k} + \text{coſ.}\,\frac{4m\pi}{k} + \text{coſ.}\,\frac{6m\pi}{k} + - - - - - + \text{coſ.}\,\frac{(i-1)m\pi}{k},$$

ita vt ſit

$$2\,S\,\text{ſin.}\,\frac{m\pi}{k} = k - 1 - 2\,T - (k-i)\,\text{coſ.}\,\frac{(i+1)m\pi}{k}.$$

Hic autem iterum conuenit duos caſus perpendere, prout k fuerit par vel impar.

Euolutio caſus prioris, quo k eſt numerus par et $i = k - 1$.

§. 8. Hoc ergo caſus habebimus

$$T = \text{coſ.}\,\frac{2m\pi}{k} + \text{coſ.}\,\frac{4m\pi}{k} + \text{coſ.}\,\frac{6m\pi}{k} + - - - - \text{coſ.}\,\frac{(k-2)m\pi}{k}.$$

Multiplicemus denuo per $2\,\text{ſin.}\,\frac{m\pi}{k}$, et per reductiones ſupra indicatas habebimus

$$2\,T\,\text{ſin.}\,\frac{m\pi}{k} = \text{ſin.}\,\frac{3m\pi}{k} + \text{ſin.}\,\frac{5m\pi}{k} + \text{ſin.}\,\frac{7m\pi}{k} + - - - - - \text{ſin.}\,\frac{(k-1)m\pi}{k}$$
$$-\,\text{ſin.}\,\frac{m\pi}{k} - \text{ſin.}\,\frac{3m\pi}{k} - \text{ſin.}\,\frac{5m\pi}{k} - - - - - \text{ſin.}\,\frac{(k-1)m\pi}{k};$$

deletis igitur terminis ſe mutuo tollentibus erit

$$2\,T\,\text{ſin.}\frac{m\pi}{k} = -\,\text{ſin.}\frac{m\pi}{k} + \text{ſin.}\frac{(k-1)m\pi}{k}.$$

Eſt vero

$$\text{ſin.}\frac{(k-1)m\pi}{k} = \text{ſin.}\left(m\pi - \frac{m\pi}{k}\right) = \text{ſin.}\,m\pi\,\text{coſ.}\frac{m\pi}{k} - \text{coſ.}\,m\pi\,\text{ſin.}\frac{m\pi}{k},$$

vbi $\text{ſin.}\,m\pi = 0$, quamobrem fiet $2\,T = -1 - \text{coſ.}\,m\pi$.

§. 9. Inuento valore pro T colligitur fore

$$2\,S\,\text{ſin.}\frac{m\pi}{k} = k, \text{ ideoque } S = \frac{k}{2\,\text{ſin.}\frac{m\pi}{k}}$$

Denique vero ipſe valor formulae noſtrae integralis, quem quaerimus, erit $\frac{2\pi S}{kk}$, et nunc manifeſtum eſt, integrale noſtrae formulae, caſu quo S eſt numerus par, fore $\frac{\pi}{k\,\text{ſin.}\frac{m\pi}{k}}$, ſiquidem poſt integrationem ſtatuatur $x = \infty$.

Euolutio alterius caſus, quo k eſt numerus impar et $i = k$.

§. 10. Hoc ergo caſu eſt

$$T = \text{coſ.}\frac{2m\pi}{k} + \text{coſ.}\frac{4m\pi}{k} + \text{coſ.}\frac{6m\pi}{k} + \text{-----} + \text{coſ.}\frac{(k-1)m\pi}{k},$$

quae ſeries multiplicata per $2\,\text{ſin.}\frac{m\pi}{k}$ producet vt ante

$$2\,T\,\text{ſin.}\frac{m\pi}{k} = \text{ſin.}\frac{3m\pi}{k} + \text{ſin.}\frac{5m\pi}{k} + \text{ſin.}\frac{7m\pi}{k} + \text{----} + \text{ſin.}\frac{km\pi}{k}$$
$$-\,\text{ſin.}\frac{m\pi}{k} - \text{ſin.}\frac{3m\pi}{k} - \text{ſin.}\frac{5m\pi}{k} - \text{------} - \text{ſin.}\frac{(k-2)m\pi}{k},$$

vnde deletis terminis ſe mutuo tollentibus reperietur

$$2\,T\,\text{ſin.}\frac{m\pi}{k} = -\,\text{ſin.}\frac{m\pi}{k} + \text{ſin.}\,m\pi$$

ideoque

$$2\,T = -1 + \frac{\text{ſin.}\,m\pi}{\text{ſin.}\frac{m\pi}{k}} = 1, \text{ ob } \text{ſin.}\,m\pi = 0,$$

hincque

$\text{ſ.}\,m\pi\,\text{ſin.}\frac{m\pi}{k}$,
$-\,\text{coſ.}\,m\pi$.
fore
ralis, quem
grale noſtrae
$\frac{\pi}{\text{n.}\frac{m\pi}{k}}$, ſiqui-
us impar et
$\text{oſ.}\frac{(k-1)m\pi}{k}$,
vt ante
$- + \text{ſin.}\frac{km\pi}{k}$
$- - \text{ſin.}\frac{(k-2)m\pi}{k}$,
rietur
$\tau = 0$,
hincque

hincque porro fiet

$$2\,S\,\text{ſin.}\frac{m\pi}{k} = k;$$

quare cum valor integralis quaeſitus ſit $\frac{2\pi S}{kk}$. erit etiam hoc caſu integrale noſtrum $= \frac{\pi}{k\,\text{ſin.}\frac{m\pi}{k}}$, prorſus vti praecedente caſu. Hinc ergo deducimus ſequens

Theorema.

§. 11. *Si haec formula differentialis:* $\frac{x^{m-1}\,dx}{1 + x^k}$, *ita integretur, vt, poſito* $x = 0$, *integrale euaneſcat, tum vero ſtatuatur* $x = \infty$, *valor inde reſultans ſemper erit* $\frac{\pi}{k\,\text{ſin.}\frac{m\pi}{k}}$, *ſiue* k *ſit numerus par, ſiue impar.* Huius Theorematis demonſtratio ex praecedentibus eſt manifeſta.

§. 12. In euolutione huius formulae aſſumſimus eſſe $m < k$, quia alioquin membra logarithmica ſe non deſtruiſſent; at vero ne hac quidem limitatione nunc amplius eſt opus. Caſu enim quo foret $m = k$, integrale formulae $\frac{x^{m-1}\,dx}{1 + x^k}$ eſſet $\frac{1}{k}\,l(1 + x^k)$, quod facto $x = \infty$ fieret etiam ∞; verum hoc idem indicat, noſtrum integrale eſſe $\frac{\pi}{k\,\text{ſin}\,\pi} = \infty$. Dummodo ergo m non fuerit maius quam k, noſtra formula veritati ſemper eſt conſentanea.

§. 13. Quin etiam ne quidem neceſſe eſt vt exponentes m et k ſint numeri integri, dummodo non fuerit $m > k$; ſi enim fuerit $m = \frac{\mu}{\lambda}$ et $k = \frac{\kappa}{\lambda}$, erit valor per no-

ſtram

ſtram formulam $\frac{\pi\lambda}{\kappa\,\text{ſin.}\frac{\mu\pi}{\nu}}$, cuius veritas ita oſtenditur. Quia hoc caſu formula integranda eſt $\int\frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\kappa}{\lambda}}}\frac{dx}{x}$, ſtatuatur $x=y^{\lambda}$, erit $\frac{dx}{x}=\frac{\lambda dy}{y}$ et formula fiet

$$\int\frac{y^{\mu}}{1+y^{\kappa}}\frac{\lambda dy}{y}=\lambda\int\frac{y^{\mu-1}dy}{1+y^{\kappa}}$$

cuius valor vtique erit $\frac{\lambda\pi}{\lambda\,\text{ſin.}\frac{\mu\pi}{\kappa}}$.

Alia demonſtratio Theorematis.

§. 14. Denotet P valorem integralis $\int\frac{x^{m}}{1+x^{k}}\frac{dx}{x}$ a termino $x=0$ vsque ad $x=1$; at Q valorem eiusdem integralis a termino $x=1$ vsque ad $x=\infty$, ita vt P + Q praebeat eum ipſum valorem, qui in theoremate continetur. Nunc pro valore Q inueniendo ſtatuatur $x=\frac{1}{y}$, vnde fit $\frac{dx}{x}=-\frac{dy}{y}$, fietque

$$Q=\int\frac{y^{-m}}{1+y^{-k}}\cdot\frac{-dy}{y}=-\int\frac{y^{k-m}}{1+y^{k}}\cdot\frac{dy}{y}$$

a termino $y=1$ vsque ad $y=0$. Hinc igitur commutatis terminis erit $Q=+\int\frac{y^{k-m}}{1+y^{k}}\frac{dy}{y}$, a termino $y=0$ vsque ad $y=1$. Iam quia hoc integrali expedito littera y ex calculo egreditur, loco y ſcribere licebit x, ita vt ſit

$$Q=\int\frac{x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x},$$

quo

..tur. Quia
, ſtatuatur
$\int\frac{x^{m}}{1+x^{k}}\frac{dx}{x}$
..m eiusdem
.. vt P + Q
continetur.
.., vnde fit
commutatis
.. vsque ad
y ex calcu-
..it
quo

quo facto habebimus

$$P+Q=\int\frac{x^{m}+x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x}$$

a termino $x=0$ vsque ad terminum $x=1$. Verum non ita pridem demonſtraui, valorem huius formulae integralis intra terminos $x=0$ et $x=1$ contentum eſſe $=\frac{\pi}{k\,\text{ſin.}\frac{m\pi}{k}}$. Hinc igitur naſcitur ſequens Theorema non minus notatu dignum.

Theorema.

§. 15. *Valor huius formulae integralis:*

$$\int\frac{x^{m}+x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x}$$

intra terminos $x=0$ *et* $x=1$ *contentus, aequalis eſt valori iſtius integralis:* $\int\frac{x^{m}}{1+x^{k}}\cdot\frac{dx}{x}$, *intra terminos* $x=0$ *et* $x=\infty$ *contento.*

§. 16. His expenſis formulam integralem in titulo propoſitam aggrediamur, et quo eam ad formam hactenus tractatam reducamus, in ſubſidium vocemus ſequentem reductionem:

$$\int\frac{x^{m-1}dx}{(1+x^{k})^{\lambda+1}}=\frac{Ax^{m}}{(1+x^{k})^{\lambda}}+B\int\frac{x^{m-1}dx}{(1+x^{k})^{\lambda}},$$

vnde facta differentiatione prodit ſequens aequatio:

$$\frac{x^{m-1}dx}{(1+x^{k})^{\lambda+1}}=\frac{mAx^{m-1}dx}{(1+x^{k})^{\lambda}}-\frac{\lambda kAx^{m+k-1}dx}{(1+x^{k})^{\lambda+1}}+\frac{Bx^{m-1}dx}{(1+x^{k})^{\lambda}},$$

quae

quae aequatio, per $x^{m-1}dx$ diuiſa ac per $(1+x^k)^\lambda$ multiplicata, terminum negatiuum a dextra ad ſiniſtram transponendo, erit

$$\frac{1+\lambda k A x^k}{1+x^k} = m\mathrm{A}+\mathrm{B}$$

quae aequatio manifeſto ſubſiſtere nequit, niſi ſit $\lambda k \mathrm{A} = 1$, ſiue $\mathrm{A} = \frac{1}{\lambda k}$, vnde erit $1 = m\mathrm{A}+\mathrm{B} = \frac{m}{\lambda k}+\mathrm{B}$, ſicque erit $\mathrm{B} = 1-\frac{m}{\lambda k}$.

§. 17. Inuentis his valoribus pro litteris A et B, primum aſſumimus, integralia ita capi, vt euaneſcant poſito $x=0$; tum vero poſito $x=\infty$, quia exponens n minor ſupponitur quam k, membrum abſolutum littera A affectum ſponte euaneſcit, ita vt hoc caſu $x=\infty$ fiat

$$\int\frac{x^{m-1}\,dx}{(1+x^k)^{\lambda+1}} = \left(1-\frac{m}{\lambda k}\right)\int\frac{x^{m-1}\,dx}{(1+x^k)^\lambda}.$$

Quod ſi iam primo capiamus $\lambda=1$, quia antè inuenimus pro eodem caſu $x=\infty$ eſſe

$$\int\frac{x^{m-1}\,dx}{1+x^k} = \frac{\pi}{k\,\text{ſin.}\,\frac{m\pi}{k}},$$

habebimus valorem iſtius integralis

$$\int\frac{x^{m-1}\,dx}{(1+x^k)^2} = \left(1-\frac{m}{k}\right)\frac{\pi}{k\,\text{ſin.}\,\frac{m\pi}{k}},$$

ſi quidem integrale etiam a termino $x=0$ vsque ad terminum $x=\infty$ extendatur.

§. 18. Quod ſi iam ſimili modo ponamus $\lambda=2$, reperietur pro iisdem terminis integrationis

$\int x$

1 x^k)^λ mul-
ſiniſtram trans-
ſi ſit λ k A = 1,
m/λk + B, ſicque
litteris A et B,
euaneſcant poſito
ponens n minor
tera A affectum
x
)^λ.
antè inuenimus
vsque ad ter-
namus λ = 2,
∫ x

$$\int\frac{x^{m-1}\,dx}{(1+x^k)^3} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\frac{\pi}{k\,\text{ſin.}\,\frac{m\pi}{k}};$$

eodem modo ſi litterae λ continuo maiores valores tribuantur, reperientur ſequentes integralium formae omni attentione dignae:

$$\int\frac{x^{m-1}dx}{(1+x^k)^4} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\frac{\pi}{k\,\text{ſin.}\,\frac{m\pi}{k}}$$

$$\int\frac{x^{m-1}dx}{(1+x^k)^5} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\frac{\pi}{k\,\text{ſin.}\,\frac{m\pi}{k}}$$

$$\int\frac{x^{m-1}dx}{(1+x^k)^6} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\left(1-\frac{m}{5k}\right)\frac{\pi}{k\,\text{ſin.}\,\frac{m\pi}{k}}$$

etc. etc.

§. 19. Quare ſi littera n denotet numerum quemcunque integrum, pro formula in titulo expreſſa, ſi eius integrale a termino $x=0$ vsque ad $x=\infty$ extendatur, eius valor ſequenti modo ſe habebit:

$$\left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\cdots\left(1-\frac{m}{(n-1)k}\right)\frac{\pi}{k\,\text{ſin.}\,\frac{m\pi}{k}}$$

qui ergo conueniet huic formulae integrali:

$$\int\frac{x^{m-1}\,dx}{(1+x^k)^n}.$$

§. 20. Hic quidem neceſſario pro n alii numeri praeter integros accipi non licet: at vero per methodum interpolationum, quae fuſius iam paſſim eſt explicata, hanc integrationem etiam ad caſus, quibus exponens n eſt numerus fractus, extendere licet. Quod ſi enim ſequentes formu-

lae

hae integrales a termino $y = 0$ vsque ad $y = 1$ extendantur, in genere valor noſtrae formulae propoſitae ita repraeſentari poterit:

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^n} = \frac{\pi}{k\,\text{ſin.}\frac{m\pi}{k}} \cdot \frac{\int y^{nk-m-1}\,dy\,(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}\,dy\,(1-y^k)^{\frac{n}{k}-1}}.$$

Vnde ſi fuerit $m = 1$ et $k = 2$, ſequitur fore

$$\int \frac{dx}{(1+xx)^n} = \frac{\pi}{2}\int \frac{y^{2(n-1)}\,dy}{\sqrt{(1-yy)}} : \int \frac{dy}{\sqrt{(1-yy)}} = \int \frac{y^{2(n-1)}\,dy}{\sqrt{(1-yy)}}$$

Ita ſi $n = \frac{3}{2}$ erit

$$\int \frac{dx}{(1+xx)^{\frac{3}{2}}} = \int \frac{y\,dy}{\sqrt{(1-yy)}}$$

cuius veritas ſponte elucet, quia integrale prius generatim eſt $\frac{x}{\sqrt{1+xx}}$, poſterius vero $= 1 - \sqrt{(1-yy)}$, quae, facto $x = \infty$ et $y = 1$, vtique fiunt aequalia. Caeterum pro hac integratione generali notaſſe iuuabit, exponentem vnitate minorem accipi non poſſe, quia alioquin valores amborum integralium in infinitum excreſcerent.

IN;

tendantur, in
repraeſentari
$y^k)^{\frac{m}{k}-1}$
$y^k)^{\frac{n}{k}-1}$.
$\int \frac{y^{2(n-1)}\,dy}{\sqrt{(1-yy)}}$
s generatim eſt
e, facto $x = \infty$
o hac integra-
itate minorem
rum integrali-
IN;

INVESTIGATIO
VALORIS INTEGRALIS

$$\int \frac{m^{m-1}\,dx}{1 - 2x^k\,\text{coſ.}\,\theta + x^{2k}}$$

A TERMINO $x = 0$ VSQVE AD $x = \infty$ EXTENSI.

§. 1.

Quaeramus primo integrale formulae propoſitae indefinitum, atque adeo omnes operationes ex primis Analyſeos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales ſimplices reſolui nequit, ſit in genere eius factor duplicatus quicunque $1 - 2x\,\text{coſ.}\,\omega + xx$; euidens enim eſt, denominatorem fore productum ex k huiusmodi factoribus duplicatis. Cum igitur, poſito hoc factore $= 0$, fiat $x = \text{coſ.}\,\omega \pm \sqrt{-1}\,\text{ſin.}\,\omega$, etiam ipſe denominator duplici modo euaneſcere debebit, ſiue ſi ponatur

$$x = \text{coſ.}\,\omega + \sqrt{-1}\,\text{ſin.}\,\omega, \text{ ſiue}$$
$$x = \text{coſ.}\,\omega - \sqrt{-1}\,\text{ſin.}\,\omega.$$

Conſtat autem omnes poteſtates harum formularum ita commode exprimi poſſe, vt ſit

$$(\text{coſ.}\,\omega \pm \sqrt{-1}\,\text{ſin.})^\lambda = \text{coſ.}\,\lambda\omega \pm \sqrt{-1}\,\text{ſin.}\,\lambda\omega;$$

hinc igitur erit

$$x^k = \text{coſ.}\,k\omega + \sqrt{-1}\,\text{ſin.}\,k\omega \text{ et}$$
$$x^{2k} = \text{coſ.}\,2k\omega \pm \sqrt{-1}\,\text{ſin.}\,2k\omega.$$

Sub-