



1785

# Observation in aliquot theoremata illustrissimi de la Grange

Leonhard Euler

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Record Created:

2018-09-25

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OBSERVATIONES

IN ALIQUOT

THEOREMATA

ILLVSTR. DE LA GRANGE.

**P**ostquam aliquod Theorema, ex his quae non ita pridem demonstrati, quo ostendi, formulae integralis  $\int \frac{\sqrt{x-ydx}}{T^{\frac{1}{2}}}$ , si post integrationem ponatur  $x = 1$ , valorem esse  $= 1/2$ , cum illustri Domino *de la Grange* communicassem, is novitate huius argumenti permotus, non solum felicissimo successu eius demonstrationem penetravit, sed etiam plurimam alia praecleara inuenta inde deduxit; quorum vberior enucleatio scientiae analyticae maxima incrementa polliceri videntur, ex quo genere aliquot praeclearissima specimina mecum benevole communicavit, quae statim summo studio sum perscrutatus; et quoniam haec materia attentionem mereri videretur, meas meditationes, quae se mihi hac occasione obtulerunt, suis sum expositurus. Cum autem hoc quasi novum Analyticis genus possitissimum in eiusmodi formulis integralibus vertitur, in quibus variabili post integrationem certus valor determinatus tribuitur, ad caediosas verborum ambages evitandas, quas perpetua salium conditionum commemoratio postulare, peculiariter signandi modum adhibebo, quem ante omnia accuratius explicare necesse erit.

Hypo-

Hypothesis.

§. 1. Hac signandi ratione :

$$\int P dx \left[ \begin{array}{l} abx = a \\ adx = b \end{array} \right]$$

delectatur, integrale  $\int P dx$  ita esse assumtum, ut evanescat postquam  $x = a$ , tum vero statim  $x = b$ ; quo pacto manifestum est eius valorem penitus fore determinatum.

Scholion.

§. 2. Quo indoles huius determinationis clarius perspiciantur, quoniam P denotat functionem aliquam ipsius x, Tab. I. eius naturam represententemus linea quadam curva *ix a b c a*, Fig. 1. super axe IO extrinseca, cuius quacunque applicata  $Xx^x$ , abscissae  $IX = x$  respondens, exhibeat ipsam functionem P, huius curvae. Quod si iam capiatur abscissae  $IA = a$ ,  $IB = b$ , quibus respondeant applicatae  $Aa$  et  $Bb$ , formula proposita exprimet aream  $AaBb$ , inter applicatas  $Aa$  et  $Bb$  interceptam. Eodem modo, si alia quaequam abscissa statueretur  $IC = c$  area  $AaCc$  exprimeretur hac formula :

$$\int P dx \left[ \begin{array}{l} abx = a \\ adx = c \end{array} \right];$$

area autem  $BbCc$  ista formula :

$$\int P dx \left[ \begin{array}{l} abx = b \\ adx = c \end{array} \right];$$

tum vero, ab initio I incipiendo, area  $IiAa$  indicabitur per hanc formulam :

$$\int P dx \left[ \begin{array}{l} abx = 0 \\ adx = a \end{array} \right].$$

Euleri Op. Anal. Tom. II.

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vnde

vnde sponte fununt sequentia lemmata ita succincte expressa:

§. 3. *Lemma I.*

$$\int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv b \end{matrix} \right] = -\int P dx \left[ \begin{matrix} abx \equiv b \\ adx \equiv a \end{matrix} \right].$$

Quoniam enim, si  $b$  ut maius spectetur quam  $a$ , formula posterior

$$\int P dx \left[ \begin{matrix} abx \equiv b \\ adx \equiv a \end{matrix} \right]$$

eandem aream  $A a B b$  refert quam prior, sed ordine retrogrado, ista expressio pro negatiua erit habenda, sicque erit quoque

$$\int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv b \end{matrix} \right] + \int P dx \left[ \begin{matrix} abx \equiv b \\ adx \equiv a \end{matrix} \right] = 0.$$

§. 4. *Lemma II.*

$$\int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv b \end{matrix} \right] + \int P da \left[ \begin{matrix} abx \equiv b \\ adx \equiv c \end{matrix} \right] = \int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv c \end{matrix} \right]$$

quemadmodum inspectio figurae manifesto declarat.

§. 5. *Lemma III.*

$$\int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv c \end{matrix} \right] - \int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv b \end{matrix} \right] = \int P dx \left[ \begin{matrix} abx \equiv b \\ adx \equiv c \end{matrix} \right];$$

vbi in binis prioribus formulis idem occurrit terminus  $a quo$ , scilicet  $x \equiv a$ , terminorum vero  $ad quem$ , scilicet  $x \equiv c$  et  $x \equiv b$ , posterior  $x \equiv b$  dar pro tertia formula terminum  $a quo$ , prior vero terminum  $ad quem$ .

§. 6.

preffa:

retro-  
re erit

retro-  
re erit

o.

$\equiv a$   
 $\equiv c$

$\equiv b$   
 $\equiv c$

$a quo$ ,  
 $c$  et  
minum

§. 6.

§. 6. *Lemma IV.*

$$\int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv c \end{matrix} \right] - \int P dx \left[ \begin{matrix} abx \equiv b \\ adx \equiv c \end{matrix} \right] = \int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv b \end{matrix} \right];$$

vbi notetur, binas formulas priores eandem habere terminum  $ad quem$ , scilicet  $x \equiv c$ , terminorum autem  $a quo$  priorem  $x \equiv a$  dare in tertia terminum  $a quo$ , posteriorem vero terminum  $ad quem$ .

§. 7. *Lemma V.*

$$\int P dx \left[ \begin{matrix} abx \equiv a \\ adx \equiv b \end{matrix} \right] + \int P dx \left[ \begin{matrix} abx \equiv b \\ adx \equiv c \end{matrix} \right] + \int P dx \left[ \begin{matrix} abx \equiv c \\ adx \equiv a \end{matrix} \right] = 0.$$

Scholion.

§. 8. His igitur, quae per se sunt maxime perspicua, praemisiss, argumenta praecipua, quae celeb. *de la Grange* mihi perscripsit ordine percurram. Primo autem mentionem insignis paradoxo facit, cuius indolem ipse non satis perspicere fateatur, a quo igitur meas meditationes inchoabo.

Resolutio insignis Paradoxo.

§. 9. Cum Vir celeb. etiam inuenisset hoc theorema generale

$$\int \frac{x^n - x^m dx}{x} \left[ \begin{matrix} abx \equiv 0 \\ adx \equiv x \end{matrix} \right] = \int \frac{x^n}{m}$$

cuius veritatem non ita pridem pluribus demonstrationibus adduxi, posuit  $x^n \equiv z$  et  $x^m \equiv y$ ; quo facta pars prior  $\int \frac{x^n - y}{x} dx$  transformatur in hanc:  $\int \frac{dz}{z}$ ; simili vero modo al-

C 2

tera

tera  $\int \frac{x^{m-1} dx}{\sqrt{x}}$  in hanc:  $\int \frac{dx}{\sqrt{x}}$ ; unde his partibus formam po-  
fuis sequitur fore

$$\int \frac{dx}{\sqrt{x}} \left[ \begin{matrix} a & z=0 \\ ad & z=1 \end{matrix} \right] - \int \frac{dy}{\sqrt{y}} \left[ \begin{matrix} ab & y=0 \\ ad & y=1 \end{matrix} \right] = \int \frac{dx}{\sqrt{x}}$$

Quare cum hae duae formulae omnino sint similes, atque is-  
dem terminis integrationis contentae, quis non crederet eos  
etiam inter se perfecte fore aequales, siue esse

$$\int \frac{dx}{\sqrt{x}} \left[ \begin{matrix} a & z=0 \\ ad & z=1 \end{matrix} \right] = \int \frac{dy}{\sqrt{y}} \left[ \begin{matrix} ab & y=0 \\ ad & y=1 \end{matrix} \right] ?$$

Invenim tamen vidimus, differentiam inter has formulas esse  
 $\frac{1}{m}$ . Hic igitur se offert quaestio maximi momenti: quemad-  
modum istam manifestam contradictionem dirimere oporteat?

§. 10. Primo autem hic observari convenit, ambas  
quantitates  $y$  et  $z$  certo quodam modo a se invicem pen-  
dere. Cum enim sit  $y = x^m$  et  $z = x^n$ , erit  $y^n = z^m$ , quo  
tamen nexu non impeditur, quo minus, posito siue  $y = c$ ,  
siue  $y = 1$ , etiam fiat  $z = 0$ , siue  $z = 1$ . Invenim tamen hinc  
neutiquam patet, cur ob hanc rationem istae binae formulae:

$$\int \frac{dy}{\sqrt{y}} \left[ \begin{matrix} ab & y=0 \\ ad & y=1 \end{matrix} \right] \text{ et } \int \frac{dx}{\sqrt{x}} \left[ \begin{matrix} a & z=0 \\ ad & z=1 \end{matrix} \right]$$

differe proinde queant; unde haec observatio ad dubium  
solvendum nihil plane conferre videtur.

§. 11. Quin etiam nullo prorufus dubio obnoxia  
videtur haec aequatio multo generalior:

$$\int \frac{dy}{\sqrt{y}} \left[ \begin{matrix} ab & y=a \\ ad & y=b \end{matrix} \right] = \int \frac{dx}{\sqrt{x}} \left[ \begin{matrix} a & z=a \\ ad & z=b \end{matrix} \right];$$

quando-

corum po-

aque is-

nitas esse  
quemad-

it, ambas  
sem pen-

1 dubium

obnoxia

quando-

quandoquidem nihil plane impedit, quo minus loco  $z$  scri-  
bamus  $y$ , vel vicissim; verum plurima phaenomena in ana-  
lysi observata satis luculenter docent, huiusmodi aequalitates  
interdum exceptionem pati, quando valores eundem infiniti.  
Haec autem circumstantia nostro casu vidique locum habet,  
cum formula integralis  $\int \frac{dy}{\sqrt{y}}$  si  $ab = 0$  ad  $y = 1$  extendatur,  
vidique in infinitum excreseat, quod etiam de altera:  $\int \frac{dx}{\sqrt{x}}$ ,  
est tendendum. Si enim fiat  $a = 1$ , applicata nostrae curvae, quae  
est  $\frac{1}{x}$ , manifeste se infinite magna, unde superior aequalitas  
generalis:

$$\int \frac{dy}{\sqrt{y}} \left[ \begin{matrix} ab & y=a \\ ad & y=b \end{matrix} \right] - \int \frac{dx}{\sqrt{x}} \left[ \begin{matrix} ab & z=a \\ ad & z=b \end{matrix} \right] = 0$$

hanc restrictionem potular, nisi vel  $a$  sit  $= 1$ , vel  $b = 1$ ,  
quippe quibus casibus veraque formula sit infinita.

§. 12. His perpenis nullum plane dubium mihi  
quidem superesse videtur, quin in hac circumstantia vera so-  
lutio propofiti paradoxo sit quaerenda, quae scilicet in eo  
versatur, quod sit tam  $\int \frac{dy}{\sqrt{y}} \left[ \begin{matrix} ab & y=0 \\ ad & y=1 \end{matrix} \right] = \infty$ , quam  
 $\int \frac{dx}{\sqrt{x}} \left[ \begin{matrix} a & z=0 \\ ad & z=1 \end{matrix} \right] = \infty$ , ita ut horum inferiorum differentia  
possit aequari quantitati finitae cuiusque, ideoque in se  
spectata prorufus non determinetur; quod autem ista diffi-  
cultas nostro casu sit  $\frac{1}{m}$ , ideoque determinata, inde venit  
quod sit  $y^n = z^m$ .

§. 13. Simile aliquid evenire potest in formulis simi-  
plicioribus, quales sunt  $\int \frac{dx}{\sqrt{x}}$  et  $\int \frac{dy}{\sqrt{y}}$ , quippe quarum valores,  
a termino  $y = 0$  et  $z = 0$  sumti, sunt infiniti, unde, etiam si  
C 3 post

post integrationem idem terminus ad quem statuitur, scilicet  $y = 1$  et  $x = 1$ , tamen hinc nullo modo sequitur, differentiam absolute nihilo aequari, quin potius tanquam indeterminata spectari debeat, cum quidem pro aliis terminis integrationis certo sit

$$\int \frac{dy}{y} \left[ \begin{matrix} ab y = a \\ \text{ad } y = b \end{matrix} \right] = \int \frac{dz}{z} \left[ \begin{matrix} a z = a \\ \text{ad } z = b \end{matrix} \right],$$

dummodo neque  $a$  neque  $b$  fuerit  $= 0$  vel  $= \infty$ .

§. 14. Atque hinc etiam paradoxon propositio pernicus simile proferri potest, quod ita se habet:

$$\int \frac{dz}{z} \left[ \begin{matrix} a z = 0 \\ \text{ad } z = \infty \end{matrix} \right] - \int \frac{dy}{y} \left[ \begin{matrix} ab y = 0 \\ \text{ad } y = \infty \end{matrix} \right] = 1, a,$$

cuius veritas cum in aprico sit posita, si quidem accipiatur  $z = a y$ , etiam paradoxon propositum rite dilutum erit censendum.

### Observationes in hoc Theorema

D. de la Grange.

$$\int \frac{(x^m - x^n) dx}{x} \left[ \begin{matrix} ab x = a \\ \text{ad } x = b \end{matrix} \right] = \int \frac{(b^m - b^n) dy}{y} \left[ \begin{matrix} ab y = m \\ \text{ad } y = n \end{matrix} \right].$$

§. 15. Cum equidem ante aliquod tempus reductiones huiusmodi formularum tractassem, alios terminos integrationis praeterquam ab  $x = 0$  ad  $x = 1$ , non sum contempserim, unde hoc Theorema mihi statim altioris indaginis est visum, atque omnino dignum quod summa cura expendatur. Primum igitur in eius veritatem per series inquirere conatus, quod negotium sequenti modo peregi.

§. 16.

§. 16. Cum sit  
differen-  
termina-  
integra-

§. 16. Pe-

§. 16. Pe-

$$= m \int \frac{dx}{x} \left[ \begin{matrix} ab x = a \\ \text{ad } x = b \end{matrix} \right].$$

§. 16. Pe-

§. 16.

§. 16. Cum sit  
 $x^m = a^m x + \frac{(a^2 - 1)x^2}{2} + \frac{(a^3 - 1)x^3}{6} + \frac{(a^4 - 1)x^4}{24} + \text{etc.}$   
 $x^n = a^n x + \frac{(a^2 - 1)x^2}{2} + \frac{(a^3 - 1)x^3}{6} + \frac{(a^4 - 1)x^4}{24} + \text{etc.}$

Hanc ergo seriem ducamus in  $\frac{dx}{x}$ , et quia in genere

$$\int (1+x)^\lambda \frac{dx}{x} \left[ \begin{matrix} ab x = a \\ \text{ad } x = b \end{matrix} \right] = \frac{(1+b)^\lambda - (1+a)^\lambda}{\lambda}$$

formulae ad similitram partem scripae valor per hanc seriem infinitam exprimentur:

$$\frac{(a^2 - 1)(b - 1a)}{1 \cdot 2} + \frac{(a^3 - 1)(b^2 - 1a^2)}{1 \cdot 2 \cdot 3} + \frac{(a^4 - 1)(b^3 - 1a^3)}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

§. 17. Simili modo pro formula ad dextram posita per seriem infinitam erit

$$b^m - a^m = y \frac{(1b - 1a)}{1} + y^2 \frac{(1b^2 - 1a^2)}{1 \cdot 2} + y^3 \frac{(1b^3 - 1a^3)}{1 \cdot 2 \cdot 3} + \text{etc.}$$

quae ergo ducatur in  $\frac{dy}{y}$ , et quia in genere est

$$\int y^\lambda \frac{dy}{y} \left[ \begin{matrix} ab y = m \\ \text{ad } y = n \end{matrix} \right] = \frac{n^{\lambda+1} - m^{\lambda+1}}{\lambda+1},$$

valor istius formulae per seriem hanc infinitam exprimentur:

$$\frac{(n - m)(1b - 1a)}{1} + \frac{(n^2 - m^2)(1b^2 - 1a^2)}{1 \cdot 2} + \frac{(n^3 - m^3)(1b^3 - 1a^3)}{1 \cdot 2 \cdot 3} + \text{etc.}$$

Quia igitur haec series cum praecedente perfecte congruit, veritas theorematis firmiter est euista.

§. 18. Verum hinc nequaquam perficitur, quomodo sagacissimus auctor ad hoc Theorema sit perductus, quam obrem, rebus probe perceptis, viam inveni, ex iisdem principis, quibus antehac sum usus, ad eandem formulas perueniendi. Inchoandum autem est ab hac forma simplicissima:

$$\int x^\lambda \frac{dx}{x} \left[ \begin{matrix} abx \equiv a \\ \text{ad } x \equiv b \end{matrix} \right] = \frac{b^\lambda - a^\lambda}{\lambda},$$

vbi vtrunque per  $d\lambda$  multiplicans demon-  
stratur, et cum, vti iam passim demonstra-  
tum reperitur, sit

$$\int d\lambda \int x^\lambda \frac{dx}{x} = \int x^\lambda d\lambda,$$

quacri tantum debet hoc integrale:  $\int x^\lambda d\lambda$ , spectata quanti-  
tate  $x$  vt constans, ita vt sola  $\lambda$  sit variabilis. Est vero

$$\int x^\lambda d\lambda = \frac{x^\lambda}{\lambda} + C,$$

quemadmodum ex elementis calculi exponentialis liquet. Hic  
vero cardo rei in hoc versatur, vt situd integrale certa lege  
definiatur, quam deinceps etiam in altera parte observari  
oportet. Statuamus ergo talia integralia ita capi, vt emanentem  
positio  $\lambda = 0$ , erique

$$\int x^\lambda d\lambda = \frac{x^\lambda - 1}{\lambda},$$

quo pacto pro sinistra parte habebimus

$$\int d\lambda \int x^\lambda \frac{dx}{x} = \int \frac{dx}{x} \left( \frac{x^\lambda - 1}{\lambda} \right).$$

§. 19. Pro parte autem dextra habebimus

$$\int x^\lambda (b^\lambda - a^\lambda),$$

qua formula eadem lege integrata, vt factio  $\lambda = 0$  prodeat  
nihilum, hunc valorem more hic recepto representare licebit:

$$\int \frac{d\lambda}{y} (b^\lambda - a^\lambda) \left[ \begin{matrix} ab y \equiv 0 \\ \text{ad } y \equiv \lambda \end{matrix} \right]$$

Hic

Hic enim nil aliud fecimus, nisi quod pro  $\lambda$  scripsimus  $y$ ,  
et facta integration loco  $y$  eius valorem  $\lambda$  relictis assu-  
mus, sicque affecturi sumus sequentem formulam:

$$\int (x^\lambda - 1) \frac{dx}{x} \left[ \begin{matrix} abx \equiv a \\ \text{ad } x \equiv b \end{matrix} \right] = \int \frac{d\lambda}{y} (b^\lambda - a^\lambda) \left[ \begin{matrix} ab y \equiv 0 \\ \text{ad } y \equiv \lambda \end{matrix} \right],$$

quam tanquam Theorema vtilissimum spectare licet.

§. 20. Vt ergo huius Theorematis nunciemur re-  
quentes reductiones:

$$\int (x^m - 1) \frac{dx}{x} \left[ \begin{matrix} abx \equiv a \\ \text{ad } x \equiv b \end{matrix} \right] = \int \frac{d\lambda}{y} (b^\lambda - a^\lambda) \left[ \begin{matrix} ab y \equiv 0 \\ \text{ad } y \equiv n \end{matrix} \right] \text{ et}$$

$$\int (x^m - 1) \frac{dx}{x} \left[ \begin{matrix} abx \equiv a \\ \text{ad } x \equiv b \end{matrix} \right] = \int \frac{d\lambda}{y} (b^\lambda - a^\lambda) \left[ \begin{matrix} ab y \equiv 0 \\ \text{ad } y \equiv m \end{matrix} \right];$$

quare si formula posterior a priore subtrahatur, erit

$$\int (x^m - x^n) \frac{dx}{x} \left[ \begin{matrix} abx \equiv a \\ \text{ad } x \equiv b \end{matrix} \right] = \int \frac{d\lambda}{y} (b^\lambda - a^\lambda) \left[ \begin{matrix} ab y \equiv 0 \\ \text{ad } y \equiv n \end{matrix} \right] \\ - \int \frac{d\lambda}{y} (b^\lambda - a^\lambda) \left[ \begin{matrix} ab y \equiv 0 \\ \text{ad } y \equiv m \end{matrix} \right];$$

verum ista formula ad dextram posita per reductionem in  
lemmate 3.º offensam reuocatur ad hanc formam simpli-  
cioram:

$$\int \frac{d\lambda}{y} (b^\lambda - a^\lambda) \left[ \begin{matrix} ab y \equiv m \\ \text{ad } y \equiv n \end{matrix} \right];$$

vnde patet, hoc modo ipsam hoc insigne Theorema etiam  
ex nostris principijs inuestigari posse.

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§. 21.

§. 21. Hoc autem Theoremate generalissimo vir ingeniosissimus est vius ad Theorema meum demonstrandum, quæ ostendi esse

$$\int (x^n - x^m) \frac{dx}{x} \left[ abx = 0 \right] = \frac{n}{m};$$

tantum enim opus erat, ut caperetur  $a = 0$  et  $b = 1$ , quæ passio formula ad dextram posita integralis abit in

$$\int \frac{dy}{y} \left[ aby = m \right],$$

cuius valor manifeste fit  $ln - lm = \frac{n}{m}$ , quæ est nova demonstratio mei Theorematum, cuiusmodi quidem dudum plures alias dederam.

### Oberationes in Theorema

D: de la Grange.

$$\int \frac{x^n - x^m dx}{(1+x^r)^l} \left[ abx = 0 \right] = l \left( \frac{\text{tag. } \frac{(n+1)\pi}{2r}}{\text{tag. } \frac{(m+1)\pi}{2r}} \right).$$

§. 22. Quia hic ambo exponentes  $m$  et  $n$  neque a se inuicem neque ab exponente  $r$  pendunt, manifestum est, pro utraque potestate  $x^m$  et  $x^n$  seorsum integrale talem formam habere debere:

$$\int \frac{x^n dx}{(1+x^r)^l} = \text{tag. } \frac{(m+1)\pi}{2r} + C \text{ et}$$

$$\int \frac{x^m dx}{(1+x^r)^l} = \text{tag. } \frac{(n+1)\pi}{2r} + C.$$

Si enim posterior forma a priorè subtrahatur, constans C

alissimo vir ostendendum,

$b = 1$ , quæ in

est nova deducendum plu-

2

et  $n$  neque manifestum integrale talem

et

constans C

ex calculo egreditur, et ipsum integrale propostum restat. Hic igitur plurimum inereit valorem istius constans C determinasse.

§. 23. Inter formulas integrales, quarum valores pro casu, quo post integrationem variabiles infinita stantur, ex primis principis calculi integralis assignari, reperitur ista:

$$\int \frac{x^{k+n} dx}{1+x^k} \left[ abx = 0 \right] = \frac{\pi}{2k} \cot \frac{\pi}{2k} = \frac{\pi}{2k} \text{fin. } \frac{(k+n)\pi}{2k}$$

vbi autem assumitur, exponentem  $n$  non maiorem capi quam  $k$ . Quod si iam hic exponentis  $n$  ut variabilis præterit, sgeta ipsa  $x$  ut constans, et vrinque per  $dn$  multiplicetur deouoque integratur, formula finitæ erit

$$\int dn \int \frac{x^{k+n} dx}{1+x^k} = \int \frac{dx}{x(1+x^k)} \int x^{k+n} dn,$$

vbi posteriorem integrale fit

$$\int x^{k+n} dn = \frac{x^{k+n}}{1+x^k} + C.$$

Ut autem hoc integrale determinetur, constancem ita determinemus, ut id euaneſcat postò  $n = 0$ , vnde obtinetur

$$\int x^{k+n} dn = \frac{x^{k+n} - x^k}{1+x^k},$$

ita ut formula integralis ad finitram posita futura sit

$$\int \frac{x^{k+n} - x^k dx}{1+x^k} \left[ abx = 0 \right].$$

§. 24. Pro parte dextra autem habebimus hoc integrale:  $\int \frac{\pi dn}{2k \text{fin. } \frac{(k+n)\pi}{2k}}$ , eiam ita sumendum, ut euaneſcat postò

D 2

postò  $n = c$ . Hunc in finem statuimus angulum  $\frac{(k+n)\pi}{k} = \phi$ ,  
 et quia hinc erit  $d\phi = \frac{\pi}{k} dx$ , formula nostra integranda erit  
 $\int_{\tan \phi}^{d\phi}$ , cuius integrale per regulas notas in genere est:

$$I \text{ tag. } \phi + C = I \text{ tag. } \frac{(k+n)\pi}{k} x + C,$$

quod, factò  $n = 0$ , abire in  $I \text{ tag. } \frac{\pi}{4} + C$ . Quare cum  $\text{tag. } \frac{\pi}{4} = 1$   
 et  $I = 0$ , evidens est constantem  $C$  fore  $= 0$ , ita vt in-  
 tegrale hoc qualesiam sit  $I \text{ tag. } \frac{(k+n)\pi}{k}$ . Hinc ergo affectui  
 sumus istam reductionem generalem:

$$\int \frac{x^{k+n} - x^k dx}{1 + x^{2k}} \left[ \begin{matrix} ab \ x = 0 \\ ad \ x = \infty \end{matrix} \right] = I \text{ tag. } \frac{(k+n)\pi}{4k},$$

vbi autem probe notari oportet, exponentes  $m$  et  $n$  maiores  
 capi non licere quam  $k$ .

§. 25. Cum igitur, loco  $n$  alium numerum  $m$  su-  
 mendo, simili modo sit:

$$\int \frac{x^{k+m} - x^k dx}{1 + x^{2k}} \left[ \begin{matrix} ab \ x = 0 \\ ad \ x = \infty \end{matrix} \right] = I \text{ tag. } \frac{(k+m)\pi}{4k},$$

subtrahatur ista formula a precedente, et obtinebitur ista:

$$\int \frac{x^{k+n} - x^{k+m} dx}{1 + x^{2k}} \left[ \begin{matrix} ab \ x = 0 \\ ad \ x = \infty \end{matrix} \right] = I \left( \text{tag. } \frac{(k+n)\pi}{4k} - \text{tag. } \frac{(k+m)\pi}{4k} \right),$$

quae manifestò cum formula propositioni congruit, si modo loco  
 $k+n-1$  scribatur  $n$  et  $m$  loco  $k+m-1$ , ac loco expo-  
 nentis  $2k$  scribatur  $r$ , cum enim manifestò fiat:

$$\int \frac{x^r - x^m dx}{(1+x^r)^2} \left[ \begin{matrix} ab \ x = 0 \\ ad \ x = \infty \end{matrix} \right] = I \left( \text{tag. } \frac{(r+n)\pi}{2r} - \text{tag. } \frac{(r+m)\pi}{2r} \right).$$

§. 26.

§. 26. Quoniam ista analysis nos perduxit ad hanc  
 formam:

$$\int \frac{x^{k+n} - x^k dx}{1 + x^{2k}} \left[ \begin{matrix} ab \ x = 0 \\ ad \ x = \infty \end{matrix} \right] = I \text{ tang. } \frac{(k+n)\pi}{4k},$$

hic maximi momenti erit observasse, semper fore

$$\int \frac{x^k dx}{1 + x^{2k}} \left[ \begin{matrix} ab \ x = 0 \\ ad \ x = \infty \end{matrix} \right] = 0$$

id quod ira ostendere possum: Ponatur  $x^k = z$ , erit

$$x^{k-1} dx = \frac{dz}{k} \text{ et } I x = \frac{I z}{k},$$

seque ista formula induet hanc formam:  $\int \frac{dz}{(1+z^2)^{1/2}}$  vbi ter-  
 mini integrationis etiamnum sunt  $z = 0$  et  $z = \infty$ . Fiat  
 porro  $z = \text{tang. } \phi$ , vnde termini integrationis erunt  $\phi = 0$   
 et  $\phi = \frac{\pi}{2}$ ; hinc autem ob  $d\phi = \frac{dz}{1+z^2}$  nascetur ista formula:

$$\int \frac{d\phi}{\text{tang. } \phi} \left[ \begin{matrix} a \ \phi = 0 \\ ad \ \phi = \frac{\pi}{2} \end{matrix} \right]$$

cuius valorem in nihilum abire offendi debet.

§. 27. Ad hoc demonstrandum statuitur axis III<sup>2</sup> Tab. I.  
 super quo ab initio I summa abscissa indefinita I p =  $\phi$ , ap-  
 plicata sit  $= \frac{1}{\text{tang. } \phi}$ . Quod si ergo hic axis I II in O bi-  
 lincetur, vt fit IO =  $\frac{\pi}{4}$ , in hoc puncto applicata erit

$$= I \frac{1}{\text{tang. } \frac{\pi}{4}} = \infty.$$

Iam ab hoc puncto O vtrinque capiuntur intervalla aequa-  
 lia O p = O q =  $\omega$ , et pro puncto p erit  $\phi = \frac{\pi}{4} - \omega$ , sic-  
 que in hoc puncto p applicata erit  $\frac{1}{\text{tang. } (\frac{\pi}{4} - \omega)}$ ; est ve-

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FO

§. 26.



to tang.  $(\frac{1}{2} - \omega) = \cot(\frac{1}{2} + \omega)$ , quare cum sit  $\text{sect} = -\text{tang}$ . applicata in hoc puncto  $p$  erit  $\frac{-1}{\text{tang}(\frac{1}{2} + \omega)}$ ; at quia est

$$1q = (\frac{1}{2} + \omega), \text{ erit applicata in puncto } q = \frac{+1}{\text{tang}(\frac{1}{2} + \omega)}$$

si que aequalis est applicatae in  $p$ , sed in contrarium ver- gens. Ita si applicata sursum directa fuerit  $q Q_n$  in puncto  $p$  eadem applicata deorsum erit directa  $p P = q Q$ .

§. 28. Quod si ergo talis curva super axe  $II = \frac{1}{2}$  existuat, ita vt abscissae  $\phi$  respondeat applicata  $\frac{1}{\text{tang}\phi}$ , haec curva ex diabus portionibus inter se perfecte aequalibus constabit, circa punctum medium  $O$  ita dispositis, vt curva sinistra sit  $IPM$  in infinitum descendens ad asymp- totam  $O\pi$ , pars autem dextra similimodo a  $II$  finitior- sum sursum ascendet ad asymptotam  $O\pi$ . Quare cum formula integralis  $\int \frac{dx}{1 + x^2} = \phi = 0$  ad  $\phi = \frac{1}{2}$  extensa, exprimat totius huius curvae ab  $I$  vsque ad  $II$  protractae arcum, euidentis erit, totam hanc arcum ad nihilum redigi, quia portio eius negativae sumenda perfecte similis est portioni positivae su- mendae.

§. 29. Sic igitur per demonstrationem omnino fir- gularem euidentem est, semper esse:

$$\int \frac{x^k dx}{1 + x^{2k}} \Big|_{ad\ x=0}^{ad\ x=\infty} = 0,$$

quod certe est Theorema in hoc genere maxime notatu dignum. Quod si ergo cum illustri  $D. de la Grange$  sta-

tuanus  $2k = r$ , erit  $\int \frac{x^{r-1} dx}{(1+x^2)^k} = 0$ ; praeterea vero pro nostra

notra formula §. 24. exhibita, ob

$$\int \frac{x^k dx}{(1+x^2)^k} \Big|_{ad\ x=0}^{ad\ x=\infty} = 0,$$

deducitur istud Theorema omnino notabile:

$$\int \frac{x^{k+n} dx}{1 + x^{2k}} \Big|_{ad\ x=0}^{ad\ x=\infty} = \text{tang.} \frac{(k+n)\pi}{4k},$$

quod more  $D. de la Grange$  ita proponi potest:

$$\int \frac{x^k dx}{(1+x^2)^k} \Big|_{ad\ x=0}^{ad\ x=\infty} = \text{tang.} \frac{(n-1)\pi}{2k}$$

si que patet constantem illam supra §. 22. a nobis inductam reuera nihil acquari.

§. 30. Quoniam Demonstratio huius Theorematis methodo facta inuicem inuitur, eius veritatem per series asser- disse iuuabit. Ad hoc autem valorem formulae

$$\int \frac{x^{\lambda-1} dx}{(1+x^2)^k} \Big|_{ad\ x=0}^{ad\ x=\infty}$$

in duas partes diuisi necesse est (scilicet loco  $x$  ferbendo  $\lambda - 1$ ), quae sint

$$P = \int \frac{x^{\lambda-1} dx}{(1+x^2)^k} \Big|_{ad\ x=1}^{ad\ x=\infty} \text{ et}$$

$$Q = \int \frac{x^{\lambda-1} dx}{(1+x^2)^k} \Big|_{ad\ x=0}^{ad\ x=1}$$

ita vt  $P + Q$  exprimat valorem quem quaerimus. Nunc in postferiore parte loco  $x$  ferbamus  $\frac{1}{x}$ , si que

$$Q = \int \frac{x^{-\lambda} dx}{(1+x^2)^k} \Big|_{ad\ x=0}^{ad\ x=1} = \int \frac{x^{2-\lambda} dx}{(1+x^2)^k} \Big|_{ad\ x=0}^{ad\ x=1} \text{ et}$$

et commutatis terminis integrationis

$$Q = - \int \frac{x^{\lambda-1} dx}{1+x^2} \left[ \begin{matrix} a & x=0 \\ b & x=1 \end{matrix} \right].$$

Nunc autem loco  $x$  scribamus  $x$ , quia termini integrationis vrinque sunt iidem, erit

$$P + Q = \int \frac{x^{\lambda-1} - x^{2r-\lambda} dx}{1+x^2} \left[ \begin{matrix} ab & x=0 \\ ab & x=1 \end{matrix} \right]$$

cuius ergo valor formulae propositae est aequalis.

§. 31. Iam fractionem  $\frac{1}{1+x^r}$  in seriem infinitam

conuertamus

$$1 - x^r + x^{2r} - x^{3r} + x^{4r} - \dots$$

cuius singuli termini in  $\frac{dx}{x}$  ( $x^\lambda - x^{r-\lambda}$ ) ducti producent

$$\frac{dx}{x} (x^\lambda - x^{r-\lambda}) - \frac{dx}{x} (x^{2\lambda} - x^{2r-\lambda}) + \frac{dx}{x} (x^{3\lambda} - x^{3r-\lambda}) - \dots$$

Cum autem per Theorema principale in hoc genere fit

$$\int \frac{dx}{x} (x^a - x^b) \left[ \begin{matrix} ab & x=0 \\ ad & x=1 \end{matrix} \right] = \frac{a^d}{b^d},$$

singulis membris hoc modo integratis prodibit

$$P + Q = \frac{1}{r-\lambda} - \frac{1}{2r-\lambda} + \frac{1}{3r-\lambda} - \frac{1}{4r-\lambda} + \dots$$

§. 32. Omnes hos logarithmos in vnicum compingere licebit, ratione habita signi cuiusque, hocque modo reperitur fore

$$P + Q = \frac{1}{r-\lambda} - \frac{1}{2r-\lambda} + \frac{1}{3r-\lambda} - \frac{1}{4r-\lambda} + \frac{1}{5r-\lambda} - \frac{1}{6r-\lambda} + \dots$$

At

integrations

infinitam

ducunt

$$+ \lambda - x^{2r-\lambda}$$

nerve fit

+ etc.

in compingere modo re-

$$\frac{1}{r-\lambda} - \frac{1}{2r-\lambda} + \dots$$

At

At vero in *Introductione in Analysis Insuperiorum* pag. 147. ostendi esse

$$\text{tang. } \frac{m\pi}{2n} = \frac{m}{2n} - \frac{m^3}{2n^3} + \frac{m^5}{2n^5} - \frac{m^7}{2n^7} + \dots$$

quae series manifesto inuentam transformatur, statimendo  $m = \lambda$  et  $n = r$ , ita vt nunc sit  $P + Q = \frac{1}{r-\lambda} + \text{tang. } \frac{\lambda}{2r}$ , profus vt supra est inuentum.

Additamentum.

§. 33. In differentiatione Aetorum Tomo V. parte I. infera, vnde desumfi hoc theorema:

$$\int \frac{x^{k+n} dx}{1+x^{2k}} \left[ \begin{matrix} ab & x=0 \\ ad & x=\infty \end{matrix} \right] = \frac{\pi}{2k} \text{col. } \frac{n}{2k}.$$

simul occurrunt sequentia:

$$\int \frac{x^{k-n} + x^{k+n} dx}{1+x^{2k}} \left[ \begin{matrix} ab & x=0 \\ ad & x=\infty \end{matrix} \right] = \frac{\pi}{k} \text{col. } \frac{n}{2k}$$

$$\int \frac{x^{k-n} + x^{k+n} dx}{1+x^{2k}} \left[ \begin{matrix} ab & x=0 \\ ad & x=1 \end{matrix} \right] = \frac{\pi}{2k} \text{col. } \frac{n}{2k}$$

$$\int \frac{x^{k-n} - x^{k+n} dx}{1-x^{2k}} \left[ \begin{matrix} ab & x=0 \\ ad & x=\infty \end{matrix} \right] = \frac{\pi}{k} \text{tang. } \frac{n}{2k}$$

$$\int \frac{x^{k-n} - x^{k+n} dx}{1-x^{2k}} \left[ \begin{matrix} ab & x=0 \\ ad & x=1 \end{matrix} \right] = \frac{\pi}{2k} \text{tang. } \frac{n}{2k}$$

$$\int \frac{x^{k-n} + x^{k+n} dx}{1+2k \text{col. } \eta + x^{2k}} \left[ \begin{matrix} ab & x=0 \\ ad & x=\infty \end{matrix} \right] = \frac{2\pi}{k} \text{fin. } \frac{n}{2k}$$

$$\int \frac{x^{k-n} + x^{k+n} dx}{1+2k \text{col. } \eta + x^{2k}} \left[ \begin{matrix} ab & x=0 \\ ad & x=1 \end{matrix} \right] = \frac{k \text{fin. } \eta \text{fin. } \frac{n}{2k}}{k}$$

Euleri Op. Anal. Tom. II.

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$$\int \frac{x^{k \pm n - 1} dx}{1 + 2x^k \cos \eta + x^{2k}} = \frac{2 \pi \sin \frac{n \eta}{k}}{k \sin \eta \sin \frac{n \eta}{k}}$$

quas formulas ergo simili modo tractare operae pretium erit.

§. 34. Incipiamus igitur a formula

$$\int \frac{x^{k-n} + x^{k+n} dx}{1 + x^{2k}} = \frac{\pi}{2k \cos \frac{n \eta}{k}} \left[ \frac{abx=0}{adx=1} \right]$$

quia praecedens cum formula iam tractata prioribus connexerit, quae si ducatur in  $dx$  et ita integratur, ut integrale evanescat posito  $n=0$ , quoniam est

$$\int x^{k-n} dx = \frac{x^{k-n+1}}{k-n+1} \text{ et } \int x^{k+n} dx = \frac{x^{k+n+1}}{k+n+1}$$

cum vero, ut ante vidimus,

$$\int \frac{dx}{1+x^{2k}} \cos \frac{n \eta}{k} = \text{tang. } \frac{(k+n)\pi}{4k}$$

probitur haec integratio:

$$\int \frac{x^{k+n} - x^{k-n} dx}{(1+x^{2k})^{3/2}} = \text{tang. } \frac{(k+n)\pi}{4k}$$

qui ergo valor prioribus convenit cum eo, quem pro formula

$$\int \frac{x^{k+n} dx}{1+x^{2k}} \left[ \frac{abx=0}{adx=\infty} \right]$$

invenimus.

§. 35. Simili modo tractemus sequentem formam:

$$\int \frac{x^{k-n} - x^{k+n} dx}{1-x^{2k}} \left[ \frac{abx=0}{adx=\infty} \right] = \frac{\pi}{k} \text{ tang. } \frac{n \eta}{2k}$$

quas

quae, ducta in  $dx$  et ut supra integrata, praebet a parte sinistra

$$\int \frac{x^{k-n} - x^{k+n} dx}{1-x^{2k}} = \frac{\pi}{k} \left[ \frac{abx=0}{adx=\infty} \right]$$

a parte autem dextra

$$\int \frac{\pi dx}{k} \text{ tang. } \frac{n \eta}{2k} = \frac{\pi}{k} \text{ tang. } \frac{n \eta}{2k}$$

Ad hoc integrandum fiat  $\frac{\pi}{k} = \phi$ , erique  $\frac{\pi n \eta}{2k} = a \phi$ , sicque formula integranda erit

$$\int \frac{dx}{\cos \phi} = -a \cos \phi + C = a \cos \frac{\pi n \eta}{2k} + C$$

Hae igitur aetate, effectus debet a  $1 + C = 0$ , ideoque constans  $C = 0$ , quocirca haec integratio nobis suppediat sequentem formulam:

$$\int \frac{x^{k-n} - x^{k+n} dx}{1-x^{2k}} = a \cos \frac{n \eta}{2k}$$

sequens autem formula  $\left[ \frac{abx=0}{adx=1} \right]$  singulari evolutione non indiget, cum eius valor sit huius sensus.

§. 36. Evoluamus casum quo  $k=2$  et  $n=1$ , et ex parte sinistra habebimus

$$-\int \frac{(1-x)^2 dx}{1-x^4} = -\int \frac{(1-x)}{(1+x)(1+x^2)} \frac{dx}{1+x^2} \left[ \frac{abx=0}{adx=\infty} \right];$$

at vero ex dextra parte:  $a \cos \frac{\eta}{2} = a \cos \frac{\eta}{2} = a \cos \frac{\eta}{2} = a \cos \frac{\eta}{2}$  resolvitur in has duas:  $\frac{1}{1+x^2} = \frac{1}{2} \left( \frac{1}{1+x^2} + \frac{1}{1-x^2} \right)$  resolvitur in has duas:  $\frac{1}{1+x^2} = \frac{1}{2} \left( \frac{1}{1+x^2} + \frac{1}{1-x^2} \right)$  unde formula nostra resolvitur in has duas:

$$\int \frac{dx}{1+x^2} = \int \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

Sed

Sed ex formâ generali

$$\int \frac{x^{n-1} dx}{(1+x^2)^{1/2}} = \text{tang.} \frac{\lambda \pi}{2n}$$

utriusque valor in infinitum exercebit, sicque nihil impedit, quo minus differentia = 1/2.

§. 37. Quod si hic in posteriore formula statuerimus  $x \cdot x = z$ , ea abibit in hanc:  $\int \frac{1}{(1+z/z)^{1/2}}$ , quae priori omnino est similis atque sub iisdem terminis integrationis continetur. Hic igitur iterum occurrit Paradoxon prioris simile III, quod ab Ill. de la Grange fuit memoratum: duas scilicet hic habentur formae prioris pares:  $\int \frac{dx}{(1+x^2)^{1/2}}$  et  $\int \frac{dx}{(1+x^2)^{1/2}}$ , quarum utraque a termino 0 ad  $\infty$  integrari oportet, nihil tamen minus earum differentia non est nulla, sed utriusque manifeste est ita, quod utriusque integralis valor in infinitum exercebit.

§. 38. Quod si binas potremas formulas eodem modo tractare et per  $d^n$  multiplicatas integrare velimus, a parte sinistra refertur ista formula integralis:

$$\int \frac{x^{k+n} - x^{k-n}}{(1+x^2)^{1/2}} \frac{dx}{x} \left[ \text{ab } x = 0 \text{ ad } x = \infty \right];$$

pro dextra autem parte nanciscimur hanc formulam integalem:

$$\int \frac{2 \pi d^n \text{fn.} \frac{1}{2} \text{fn.} \frac{1}{2}}{k \text{fn.} \frac{1}{2} \text{fn.} \frac{1}{2}}$$

a termino  $n = 0$  extendendam. Verum, haec integratio nullo modo succedit, si enim ponamus  $\frac{1}{2} \pi = \phi$ , fiet  $\frac{1}{2} \pi = \phi$ , ponendo

que nihil

a statua-  
riori om-  
nis, con-  
n: duas  
x:  $\int \frac{dx}{(1+x^2)^{1/2}}$ ,  
ortet, ni-  
sed utri-  
xi in eo  
infinitum

s eodem  
limus, a

iam ince-

ario nullo  
ponendo

Ponendo  $\frac{1}{2} \pi = \alpha$ , unde formula integranda erit  $\int_{\frac{1}{2} \pi}^{\alpha} \int_{\text{im. } \phi}^{\text{im. } \alpha} d\phi \text{im. } \alpha \phi$ , cuius valor aliter nisi per signum summatorum exprimi non potest, sicque nulla concinna Theoremata hinc derivare licet.

§. 39. Quemadmodum autem hic, exponentem  $n$  ut variabellem spectando, transformationes per integrationem infinitam, ita etiam differentiatio egregias transformationes suppeditabit, quod argumentum unica formula principali illustrasse sufficet. Consideremus scilicet hanc formulam:

$$\int \frac{x^{k+n} dx}{1+x^2} \frac{dx}{x} \left[ \text{ab } x = 0 \text{ ad } x = \infty \right] = \frac{\pi}{2k} \text{cof.} \frac{n\pi}{2k}$$

quae, summo exponente  $n$  ut solo variabili, continuo differentietur, ubi notandum est esse  $d \cdot x^{k+n} = x^{k+n-1} d^n / x$ . At vero pro formula  $\frac{\pi}{2k} \text{cof.} \frac{n\pi}{2k}$  ferbamus litteram  $\nu$ , quae ergo spectanda erit tanquam functio ipsius  $n$ , cuius ergo differentia cuiusque ordinis sunt in nostra potestate. Hinc igitur sequentes reductiones consequentur:

$$\begin{aligned} \int \frac{x^{k+n} dx}{1+x^2} \frac{dx}{x} &= \frac{d\nu}{dn} \text{ sine} \\ \int \frac{x^{k+n-1} dx}{1+x^2} \frac{dx}{x} \left[ \text{ab } x = 0 \text{ ad } x = \infty \right] &= \frac{d\nu}{dn} \\ \int \frac{x^{k+n-2} dx}{1+x^2} \frac{dx}{x} \left[ \text{ab } x = 0 \text{ ad } x = \infty \right] &= \frac{d^2 \nu}{dn^2} \\ \int \frac{x^{k+n-3} dx}{1+x^2} \frac{dx}{x} \left[ \text{ab } x = 0 \text{ ad } x = \infty \right] &= \frac{d^3 \nu}{dn^3} \\ \int \frac{x^{k+n-4} dx}{1+x^2} \frac{dx}{x} \left[ \text{ab } x = 0 \text{ ad } x = \infty \right] &= \frac{d^4 \nu}{dn^4} \end{aligned}$$

$\int x$

$$\int \frac{x^{n+1-1} dx (1/x)^n}{1+x^2} \left[ \begin{matrix} ab \neq 0 \\ ad \neq \infty \end{matrix} \right] = \frac{d^2 y}{dx^2}$$

§ 40. Cum igitur hinc totum negotium ad differentialia continua ipsius  $y$  reducat, ea sequenti modo commo-  
dissime reperire licebit. Cum enim sit

$$y = \frac{\pi}{2k} \operatorname{cof.} \frac{2\pi}{k} \operatorname{erit} y \operatorname{cof.} \frac{2\pi}{k} = \frac{\pi}{2k},$$

hincque continuo differentiendo obichimus sequentes for-  
mulas:

$$\begin{aligned} \frac{dy}{dx} \operatorname{cof.} \frac{2\pi}{k} - \frac{\pi}{2k} y \operatorname{fin.} \frac{2\pi}{k} &= 0 \\ \frac{d^2 y}{dx^2} \operatorname{cof.} \frac{2\pi}{k} - \frac{2\pi}{k} \frac{dy}{dx} \operatorname{fin.} \frac{2\pi}{k} - \frac{\pi^2}{k^2} y \operatorname{cof.} \frac{2\pi}{k} &= 0 \\ \frac{d^3 y}{dx^3} \operatorname{cof.} \frac{2\pi}{k} - \frac{3\pi}{k} \frac{d^2 y}{dx^2} \operatorname{fin.} \frac{2\pi}{k} - \frac{3\pi^2}{k^2} \frac{dy}{dx} \operatorname{cof.} \frac{2\pi}{k} + \frac{\pi^3}{k^3} y \operatorname{fin.} \frac{2\pi}{k} &= 0 \\ \frac{d^4 y}{dx^4} \operatorname{cof.} \frac{2\pi}{k} - \frac{4\pi}{k} \frac{d^3 y}{dx^3} \operatorname{fin.} \frac{2\pi}{k} - \frac{6\pi^2}{k^2} \frac{d^2 y}{dx^2} \operatorname{cof.} \frac{2\pi}{k} + \frac{4\pi^3}{k^3} \frac{dy}{dx} \operatorname{fin.} \frac{2\pi}{k} \\ &+ \frac{\pi^4}{k^4} y \operatorname{cof.} \frac{2\pi}{k} = 0 \end{aligned}$$

etc. etc.  
unde singula differentialia aliora ex inferioribus formari  
possunt.

§ 41. Quo autem hac operationes magis subleuen-  
tur, statim brevitatis gratia  $\frac{\pi}{k} = \alpha$ , ut sit  $y = \frac{\operatorname{cof.} \alpha}{\operatorname{cof.} \alpha}$ , ac-  
que singula differentialia ex superioribus aequationibus se-  
quenti modo determinabuntur:

$$\begin{aligned} \frac{dy}{dx} &= \alpha y \operatorname{tang.} \alpha \\ \frac{d^2 y}{dx^2} &= 2 \alpha \frac{dy}{dx} \operatorname{tang.} \alpha + \alpha^2 y \\ \frac{d^3 y}{dx^3} &= 3 \alpha^2 \frac{d^2 y}{dx^2} \operatorname{tang.} \alpha + 3 \alpha \alpha^2 \frac{dy}{dx} - \alpha^3 y \operatorname{tang.} \alpha \end{aligned}$$

$$\begin{aligned} \frac{d^4 y}{dx^4} &= 4 \alpha^3 \frac{d^3 y}{dx^3} \operatorname{tang.} \alpha + 6 \alpha^2 \frac{d^2 y}{dx^2} - 4 \alpha^2 \frac{dy}{dx} \operatorname{tang.} \alpha - \alpha^4 y \\ \frac{d^5 y}{dx^5} &= 5 \alpha^4 \frac{d^4 y}{dx^4} \operatorname{tang.} \alpha + 10 \alpha^3 \frac{d^3 y}{dx^3} - 10 \alpha^3 \frac{d^2 y}{dx^2} \operatorname{tang.} \alpha - 5 \alpha^4 \frac{dy}{dx} \\ &+ \alpha^5 y \operatorname{tang.} \alpha \end{aligned}$$

etc. etc.  
Quod si brevitatis gratia insuper statim tang.  $\alpha = t$ ,  
et praecedentes valores in sequentibus substituiamus, repe-  
riemus:

$$\begin{aligned} \frac{dy}{dx} &= \alpha y t \\ \frac{d^2 y}{dx^2} &= \alpha \alpha y (x^2 t + 1) \\ \frac{d^3 y}{dx^3} &= \alpha^2 y (6 t^2 + 5 t) \\ \frac{d^4 y}{dx^4} &= \alpha^3 y (24 t^3 + 28 t^2 + 5) \\ \frac{d^5 y}{dx^5} &= \alpha^4 y (120 t^4 + 180 t^3 + 61 t^2) \\ \frac{d^6 y}{dx^6} &= \alpha^5 y (720 t^5 + 1320 t^4 + 652 t^3 + 61 t^2) \end{aligned}$$

§ 42. Ex consideratione harum expressionum faci-  
lis erui potest operatio, cuius ope ex quolibet earum ex-  
pressionum sequens colligi potest. Sit enim pro differenti-  
ordinis indeliniti

$$\frac{d^{\lambda} y}{d x^{\lambda}} = \alpha^{\lambda} y \cdot P$$

at pro ordine sequente

$$\frac{d^{\lambda+1} y}{d x^{\lambda+1}} = \alpha^{\lambda+1} y \cdot Q$$

et quoniam vidimus valorem ipsius  $P$  talem habere formam:

$$P = A t^{\lambda} + B t^{\lambda-1} + C t^{\lambda-2} + D t^{\lambda-3} \dots$$

atque valor ipsius  $Q$  ex sequentibus binis seriebus erit com-  
positus :

$$Q = (\lambda + 1) A \lambda^{\lambda+1} + (\lambda - 2) B \lambda^{\lambda-1} + (\lambda - 3) C \lambda^{\lambda-2} + (\lambda - 5) D \lambda^{\lambda-4} \text{ etc.}$$

$$+ \lambda A \lambda^{\lambda-1} + (\lambda - 2) B \lambda^{\lambda-2} + (\lambda - 4) C \lambda^{\lambda-3} + \text{etc.}$$

unde patet hanc determinationem ita repræsentari posse, ut sit

$$Q = \frac{t \lambda^{\lambda+1}}{d t} + \frac{d P}{d t}$$

§. 43. Hæc vero formulæ, quæ ex cognito valore  $P$  sequens  $Q$  derivatur, etiam ex ipsâ natura rei sequenti modo ostendi possunt. Cum per hypothèsin sit

$$\frac{d^{\lambda} v}{d n^{\lambda}} = \alpha^{\lambda} v P,$$

erit differentiando

$$\frac{d^{\lambda+1} v}{d n^{\lambda}} = \alpha^{\lambda} P d v + \alpha^{\lambda} v d P;$$

initio autem vidimus esse  $\frac{d^{\lambda} v}{d n^{\lambda}} = \alpha v t$ , siue  $d v = \alpha v t d n$   
quo valore substituto fit

$$\frac{d^{\lambda+1} v}{d n^{\lambda+1}} = \alpha^{\lambda+1} v P t + \alpha^{\lambda} v \frac{d P}{d n};$$

cum vero assumimus  $t = \text{rang. } \alpha n$ , unde differentiando fit  
 $\alpha d n = \frac{d t}{t}$ , quo valore in postremo termino substituto ob-  
tinebuntur

$$\frac{d^{\lambda+1} v}{d n^{\lambda+1}} = \alpha^{\lambda+1} v P t + \alpha^{\lambda+1} v \frac{d P (1+tD)}{d t} = \alpha^{\lambda+1} v \left( P t + \frac{d P (1+tD)}{d t} \right)$$

quæ forma manifeste reducitur ad hanc :

$$\frac{d^{\lambda+1} v}{d n^{\lambda+1}} = \alpha^{\lambda+1} v \frac{t \cdot d P t + d P}{d t},$$

ita

quod erit com-

$$(\lambda - 3) D \lambda^{\lambda-3} \text{ etc.}$$

$$(\lambda - 4) C \lambda^{\lambda-4} \text{ etc.}$$

nam posse, ut sit

et cognito valore

rei sequenti

ut

$$d v = \alpha v t d n$$

differentiando fit  
no substituto ob-

$$\left( P t + \frac{d P (1+tD)}{d t} \right)$$

ita

ita ut sit

$$Q = \frac{t \lambda^{\lambda+1} + d P}{d t} = P t + \frac{d P (1+tD)}{d t};$$

unde intelligitur, si sumatur  $t t + x = 0$ , quo facto in no-  
stris formulis signa terminorum alterabuntur, et omnia hæc  
 $t$ , fieri  $Q = P$ ; unde patet, hoc casu omnes formulas supe-  
riores eundem valorem esse adepturas, id quod etiam ex  
formulis supra exhibitis manifestum est, ex quibus erit  $2 - 1 = 1$ ;  
 $6 - 5 = 1$ ;  $24 - 28 + 5 = 1$ ;  $120 - 150 + 61 = 1$ ;  
 $720 - 1320 + 662 - 61 = 1$ ; etc. unde insignè criterium  
obtinetur, verum formulæ istæ recte sint per calculum de-  
rictæ.