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De insignibus proprietatibus unciarum binomii ad uncias quorumvis polynomiorum extensis

Leonhard Euler

DE INSIGNIBVS PROPRIETATIBVS VNCIARVM BINOMII AD VNCIAS QVORVMVIS POLYNOMIORVM EXTENSIS.

Auctore
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Quae non ita pridem demonstraui circa insignes proprietates, quibus vnciae Binomii ad dignitatem quamcunque euecti sunt praeditae, etiam simili modo ad vncias Trinomii et Quadrinomii atque adeo in genere Polynomii cuiusque ad dignitatem quamcunque euecti extendi possunt, id quod in hac dissertatione dilucide ostendere constitui; quae quo facilius intelligi queant, denuo ab vnciis Binomii incipiam, quarum proprietates cum iam satis perspicue a me sint demonstratae, eas hic sine demonstratione succincte ante oculos ponam, quo hoc pacto progressio ad Polynomia clarius perspiciatur: neque enim opus

pus erit, omnia quae sum allaturus, demonstrationibus corroborare, quandoquidem eae omnino similes sunt iis, quae de vnciis Binomii sunt traditae.

I. De vnciis Binomii $(1+z)^n$.

§. 1. Denotet character $\left[\frac{n}{p}\right]$ quamlibet harum vnciarum, hac ratione vt sit

$$(1+z)^n = \left[\tfrac{n}{0}\right] + \left[\tfrac{n}{1}\right]z + \left[\tfrac{n}{2}\right]z^2 + \left[\tfrac{n}{3}\right]z^3 + \left[\tfrac{n}{4}\right]z^4 + \text{ etc.}$$

atque euidens est fore $\left[\frac{n}{0}\right]=1$ et $\left[\frac{n}{1}\right]=n$; tum vero quilibet sequentium terminorum ex praecedente ita definitur, vt sit

$$\left[\tfrac{n}{p+1}\right] = \tfrac{n-p}{p+1}\left[\tfrac{n}{p}\right].$$

Praeterea vero notasse iuuabit, semper fore $\left[\frac{n}{p}\right]=0$, tam quando p est numerus negatiuus quam quando p est numerus maior quam n; id quod intelligendum est, si pro p numeri integri accipiantur. Deinde quia vnciae hae finem versus eodem ordine regrediuntur, quo ab initio progrediuntur, erit $\left[\frac{n}{n}\right]=1$, atque in genere erit $\left[\frac{n}{n-p}\right]=\left[\frac{n}{p}\right]$. Praeterea vero istae vnciae potestatis $(1+z)^n$ cum vnciis sequentis potestatis $(1+z)^{n+1}$ ita cohaerent, vt sit $\left[\frac{n+1}{p+1}\right]=\left[\frac{n}{p}\right]+\left[\frac{n}{p+1}\right]$, vnde sequitur fore

$$\left[\tfrac{n+1}{p+1}\right] = \tfrac{n+1}{p+1}\left[\tfrac{n}{p}\right].$$

§. 2. Quo clarius appareat quales numeri his characteribus quouis casu designentur, sequentem schematismum adiungamus, in quo vnciae potestatum simpliciorum exhibeantur, simulque pro singulis valores tam ipsius n quam ipsius p indicentur:

Valores pro p

n	0,	1,	2,	3,	4,	5,	6
0	1						
1	1 +	1					
2	1 +	2 +	1				
3	1 +	3 +	3 +	1			
4	1 +	4 +	6 +	4 +	1		
5	1 +	5 +	10 +	10 +	5 +	1	
6	1 +	6 +	15 +	20 +	15 +	6 +	1

vnde patet fore exempli caussa $[\frac{6}{4}] = 15$ et $[\frac{5}{3}] = 10$. Constat autem in genere esse

$$[\tfrac{n}{p}] = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-p+1}{p}.$$

§. 3. His circa significationem istorum characterum expositis omnes proprietates, quas non ita pridem demonstraui, sequenti formula succincte repraesentari possunt: $\int[\frac{m}{x}][\frac{n}{p+x}] = [\frac{m+n}{n-p}]$, vbi membrum sinistrum denotat summam progressionis, cuius singuli termini sunt producta ex binis vnciis $[\frac{m}{x}]$ et $[\frac{n}{p+x}]$, dum scilicet litterae x successiue omnes valores integri tribuuntur, quamdiu nimirum neuter factorum in nihilum abit; vnde patet incipiendum esse ab $x = 0$, hincque procedendum, donec fiat vel $x = m$ vel $p + x = n$, hoc est vsque ad $x = n - p$, si quidem fuerit $n - p < m$. Huius igitur progressionis summa demonstrata est fore $= [\frac{m+n}{n-p}]$, quae etiam ita exhiberi potest: $[\frac{m+n}{m+p}]$. Hinc igitur sequitur, sumto $p = 0$ fore $\int[\frac{m}{x}] \cdot [\frac{n}{x}] = [\frac{m+n}{n}] = [\frac{m+n}{m}]$ etc. Sumto igitur insuper

$m = n$

$m = n$, erit $\int [\frac{n}{x}]^2 = [\frac{2n}{n}]$, quod eſt illud eximium theorema, quo oſtendi eſſe

$$1 + [\tfrac{n}{1}]^2 + [\tfrac{n}{2}]^2 + [\tfrac{n}{3}]^2 + [\tfrac{n}{4}]^2 + \text{etc.} = [\tfrac{2n}{n}],$$

vbi obſeruaui iſtam ſummam etiam hoc producto exprimi poſſe:

$$\frac{2.\ 6.\ 10. - - - - - 4n - 2}{1.\ 2.\ 3. - - - - - n}.$$

Ita exempli gratia ſi ſumatur $n = 5$, ipſa progreſſio hinc reſultans erit

$$1 + 5^2 + 10^2 + 10^2 + 5^2 + 1 = 252,$$

prior vero expreſſio euadit

$$[\tfrac{2n}{n}] = [\tfrac{10}{5}] = \frac{10.\ 9.\ 8.\ 7.\ 6}{1.\ 2.\ 3.\ 4.\ 5.} = 252,$$

alterum autem productum praebet

$$\frac{2.\ 6.\ 10.\ 14.\ 18}{1.\ 2.\ 3.\ 4.\ 5.} = 252.$$

II. De vnciis Trinomii $(1 + z + zz)^n$.

§. 4. Vtamur iterum eodem charactere $(\frac{n}{p})$ ad vncias huius poteſtatis trinomialis $(1 + z + zz)^n$ deſignandas; vnde probe cauendum erit, ne cum praecedente ſignificatu $[\frac{n}{p}]$ confundantur. Statuamus ſcilicet:

$$(1 + z + zz)^n = 1 + (\tfrac{n}{1})z + (\tfrac{n}{2})zz + (\tfrac{n}{3})z^3 + \text{etc.}$$

ita vt hic quoque ſit $(\frac{n}{0}) = 1$; at quo ſignificatus ſequentium characterum facilius intelligatur, adiungamus ſimilem ſchematiſmum continentem poteſtates ſimpliciores:

Valores

Valores pro p

n	0	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,
1	1+	1+	1										
2	1+	2+	3+	2+	1								
3	1+	3+	6+	7+	6+	3+	1						
4	1+	4+	10+	16+	19+	16+	10+	4+	1				
5	1+	5+	15+	30+	45+	51+	45+	30+	15+	5+	1		
6	1+	6+	21+	50+	90+	126+	141+	126+	90+	50+	21+	6+	1

Hinc igitur patet, ſi fuerit exempli gratia $n = 6$ et $p = 7$, fore $\left(\frac{6}{7}\right) = 126$; ſimilique modo ſi $n = 5$ et $p = 5$, erit $\left(\frac{5}{5}\right) = 51$.

§. 5. Quoniam hae vnciae pariter ordinem retrogradum ſeruant, earumque vltima eſt $= 1$; euidens eſt fore $\left(\frac{1}{2}\right) = 1$; $\left(\frac{2}{4}\right) = 1$; $\left(\frac{3}{6}\right) = 1$; ſicque porro, ita vt in genere ſit $\left(\frac{n}{2n}\right) = 1$; ergo quia eodem ordine a fine regrediuntur, quo ab initio progrediuntur, erit

$$\left(\frac{n}{2n-1}\right) = \left(\frac{n}{1}\right) \text{ et } \left(\frac{n}{2n-2}\right) = \left(\frac{n}{2}\right),$$

atque in genere $\left(\frac{n}{2n-p}\right) = \left(\frac{n}{p}\right)$. Deinde quia omnes vnciae tam primam antecedentes quam vltimam ſequentes ſunt nullae, valor characteris $\left(\frac{n}{p}\right)$ primo euaneſcet, quoties p fuerit numerus negatiuus, deinde pariter euaneſcet, ſi fuerit $p > 2n$.

§. 6. Porro ex hac tabula perſpicuum eſt ſemper eſſe $\left(\frac{n}{1}\right) = n$: quemadmodum autem ſequentes ſe habeant, non tam facile perſpicitur; at vero oſtendi poteſt, pro qualibet poteſtate ſingulas vncias per binas praecedentes

ope

ope huius formulae determinari:

$$\left(\frac{n}{p+2}\right) = \frac{n-p-1}{p+2}\left(\frac{n}{p+1}\right) + \frac{2p-p}{p+2}\left(\frac{n}{p}\right).$$

Ita si fuerit verbi gratia $n = 5$ et $p = 2$, erit ex tabula modo data

$$\left(\frac{n}{p}\right) = \left(\frac{5}{2}\right) = 15;\ \left(\frac{n}{p+1}\right) = \left(\frac{5}{3}\right) = 30;\ \left(\frac{n}{p+2}\right) = \left(\frac{5}{4}\right) = 45;$$

quia porro est

$$\frac{n-p-1}{p+2} = \frac{2}{4} \text{ et } \frac{2n-p}{p+2} = \frac{8}{4} = 2,$$

vtique erit $45 = \frac{1}{2}.\,30 + 2.\,15$. Simili modo si fuerit $n = 6$ et $p = 5$, erit

$$\left(\frac{n}{p}\right) = \left(\frac{6}{5}\right) = 126,\ \left(\frac{n}{p+1}\right) = \left(\frac{6}{6}\right) = 141, \text{ et } \left(\frac{n}{p+2}\right) = \left(\frac{6}{7}\right) = 126;$$

euidens autem est esse

$$126 = \frac{0}{7}.\,141 + \frac{7}{7}.\,126 = 126.$$

§. 7. Quod nunc porro ad vncias sequentis potestatis $(1 + z + zz)^{n+1}$ attinet, facile ex ipsa formatione apparet, quamlibet earum aequari aggregato ex tribus vnciis potestatis antecedentis, scilicet fore

$$\left(\frac{n+1}{p+2}\right) = \left(\frac{n}{p+2}\right) + \left(\frac{n}{p+1}\right) + \left(\frac{n}{p}\right);$$

vnde si loco $\left(\frac{n}{p+2}\right)$ valor ante inuentus substituatur, reperietur

$$\left(\frac{n+1}{p+2}\right) = \frac{n+1}{p+2}\left(\frac{n}{p+1}\right) + \frac{2n+2}{p+2}\left(\frac{n}{p}\right).$$

Hinc pro potestate septima, ad quam superior tabula non est extensa, singulae vnciae sequenti modo reperientur, statuendo $n = 6$

Si $p = -2 \mid \left(\frac{7}{0}\right) = \frac{7}{0}\cdot\left(\frac{6}{-1}\right) + \frac{14}{0}\cdot\left(\frac{6}{-2}\right) = 1$ vti per se constat.
Si $p = -1 \mid \left(\frac{7}{1}\right) = \frac{7}{1}\cdot\left(\frac{6}{0}\right) + \frac{14}{1}\cdot\left(\frac{6}{-1}\right) = 7$
Si $p = 0 \mid \left(\frac{7}{2}\right) = \frac{7}{2}\cdot\left(\frac{6}{1}\right) + \frac{14}{2}\cdot\left(\frac{6}{0}\right) = 28$
Si $p = 1 \mid \left(\frac{7}{3}\right) = \frac{7}{3}\cdot\left(\frac{6}{2}\right) + \frac{14}{3}\cdot\left(\frac{6}{1}\right) = 77$
Si $p = 2 \mid \left(\frac{7}{4}\right) = \frac{7}{4}\cdot\left(\frac{6}{3}\right) + \frac{14}{4}\cdot\left(\frac{6}{2}\right) = 161$
Si $p = 3 \mid \left(\frac{7}{5}\right) = \frac{7}{5}\cdot\left(\frac{6}{4}\right) + \frac{14}{5}\cdot\left(\frac{6}{3}\right) = 266$
Si $p = 4 \mid \left(\frac{7}{6}\right) = \frac{7}{6}\cdot\left(\frac{6}{5}\right) + \frac{14}{6}\cdot\left(\frac{6}{4}\right) = 357$

id quod egregie conuenit cum naturali harum progressionum continuatione.

§. 8. Reliquae proprietates, quarum demonstratio simili modo expediri potest, quo pro Binomio est factum, respiciunt seriem, cuius terminus generalis ita exprimitur: $\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right)$, qui scilicet est productum ex duobus quibusuis characteribus huius generis, vnde singuli termini formantur, si loco x ordine scribantur valores 0, 1, 2, 3, 4, etc. vsque ad $x = 2m$, vel vsque ad $x = 2n - p$, quorum duorum terminorum minor valet. Huius autem seriei demonstrari potest summam esse $\int\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right) = \left(\frac{m+n}{2n-p}\right)$, quae summa etiam ita exprimitur: $\left(\frac{m+n}{2m+p}\right)$. Ita si sumamus $m = 3$, series ex factore $\left(\frac{m}{x}\right)$ orta erit

$$1 + 3 + 6 + 7 + 6 + 3 + 1.$$

Deinde si sumamus $n = 3$ et $p = 2$, alter factor $\left(\frac{3}{2+x}\right)$ dat hanc seriem: $6 + 7 + 6 + 3 + 1$, cuius termini in superiores sigillatim multiplicati praebent hanc progressionem:

$$6 + 21 + 36 + 21 + 6 = 90;$$

at

at vero formula $\left(\frac{m+n}{2n-p}\right)$ fit $=\left(\frac{8}{4}\right)$, cuius valor ex tabula ſupra allata reperietur fore $=90$.

§. 9. Hinc igitur ſi ſumamus $p=0$ prodibit iſta ſummatio: $\int\left(\frac{m}{x}\right)\left(\frac{n}{x}\right)=\left(\frac{m+n}{2m}\right)$, vel etiam $=\left(\frac{m+n}{2n}\right)$. Quare ſi porro capiamus $m=n$, erit $\int\left(\frac{n}{x}\right)^2=\left(\frac{2n}{2n}\right)$, cuius veritatem per ſequentia exempla exploremus:

Si	erit
$n=0$	$1^2=1=\left(\frac{0}{0}\right)=1$
$n=1$	$1^2+1^2+1^2=3=\left(\frac{2}{2}\right)=3$
$n=2$	$1^2+2^2+3^2+2^2+1^2=19=\left(\frac{4}{4}\right)=19$
$n=3$	$1+3^2+6^2+7^2+6^2+3^2+1^2=141=\left(\frac{6}{6}\right)=141.$

Hoc igitur modo, quoniam tabula noſtra non vlterius eſt extenſa, valorem aſſignare poterimus characteris $\left(\frac{8}{8}\right)$, ſumto ſcilicet $n=4$; erit enim:

$$\left(\frac{8}{8}\right)=1^2+4^2+10^2+16^2+19^2+16^2+10^2+4^4+1^2$$

ſiue

$$\left(\frac{8}{8}\right)=19^2+2\left(1^2+4^2+10^2+16^2\right)=1107.$$

III. De vnciis quadrinomii $(1+z+zz+z^3)^n$.

§. 10. Vtamur hic etiam eodem charactere $\left(\frac{n}{p}\right)$ ad ſingulas vncias huius poteſtatis euolutae exprimendas, ita vt ſit

$$(1+z+zz+z^3)^n=1+\left(\frac{n}{1}\right)z+\left(\frac{n}{2}\right)zz+\left(\frac{n}{3}\right)z^3\ \ldots\ \left(\frac{n}{p}\right)z^p$$

vbi iterum manifeſto eſt $\left(\frac{n}{0}\right)=1$; at quo ſignificatus ſequentium characterum facilius intelligatur, adiungamus ſimilem ſchematiſmum continentem poteſtates ſimpliciores:

Valores pro p

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	1+	1+	1+	1										
2	1+	2+	3+	4+	3+	2+	1							
3	1+	3+	6+	10+	12+	12+	10+	6+	3+	1				
4	1+	4+	10+	20+	31+	40+	44+	40+	31+	20+	10+	4+	1	
5	1+	5+	15+	35+	65+	101+	135+	155+	155+	135+	101+	65+	35+	15
6	1+	6+	21+	56+	120+	216+	336+	456+	546+	580+	546 etc.			

Hinc igitur patet, ſi fuerit verbi gratia $n = 6$ et $p = 7$, fore $(\frac{6}{7}) = 456$. Similique modo ſi fuerit $n = 5$ et $p = 10$ erit $(\frac{5}{10}) = 101$.

§. 11. Nunc ex ordine retrogrado pro vltimis cuiusque ordinis terminis erit $(\frac{1}{3}) = 1$; $(\frac{6}{2}) = 1$; $(\frac{3}{9}) = 1$, atque adeo in genere $(\frac{n}{3n}) = 1$. Ex eodem porro principio ſequitur fore $(\frac{n}{3n-p}) = (\frac{n}{p})$; hincque intelligitur, valorem formulae $(\frac{n}{p})$ in nihilum eſſe abiturum, tam ſi fuerit $p < 0$ quam ſi fuerit $p > 3n$, quod quidem de numeris integris eſt intelligendum.

§. 12. Caeterum hic itidem euidens eſt eſſe $(\frac{n}{1}) = n$; pro ſequentibus autem notandum eſt, quemlibet terminum per ternos praecedentes ita determinari vt ſit

$$\left(\frac{n}{p+3}\right) = \frac{n-p-2}{p+3}\left(\frac{n}{p+2}\right) + \frac{2n-p-1}{p+3}\left(\frac{n}{p+1}\right) + \frac{3n-p}{p+3}\left(\frac{n}{p}\right).$$

Ita ſi fuerit $n = 5$ et $p = 2$, erit

$$\left(\frac{5}{5}\right) = \frac{1}{5}\left(\frac{5}{4}\right) + \frac{7}{5}\left(\frac{5}{3}\right) + \frac{13}{5}\left(\frac{5}{2}\right);$$

euidens autem eſt ob $(\frac{5}{2}) = 15$; $(\frac{5}{3}) = 35$; $(\frac{5}{4}) = 65$ et

$(\frac{5}{5})$

$\left(\frac{5}{5}\right) = 101$ fore

$$101 = \tfrac{1}{5}(75 + 7 \cdot 35 + 13 \cdot 15) = 101.$$

§. 13. Sequentis autem poteſtatis, cuius exponens $= n + 1$, ſingulae vnciae per ſuperiores ita determinantur, vt ſit

$$\left(\frac{n+1}{p+3}\right) = \left(\frac{n}{p+3}\right) + \left(\frac{n}{p+2}\right) + \left(\frac{n}{p+1}\right) + \left(\frac{n}{p}\right),$$

vbi ſi loco $\left(\frac{n}{p+3}\right)$ valor ante inuentus ſubſtituatur prodibit:

$$\left(\frac{n+1}{p+3}\right) = \frac{n+1}{p+3}\left(\frac{n}{p+2}\right) + \frac{2n+2}{p+3}\left(\frac{n}{p+1}\right) + \frac{3n+3}{p+3}\left(\frac{n}{p}\right),$$

ſiue erit

$$\left(\frac{n+1}{p+3}\right) = \frac{n+1}{p+3}\left[\left(\frac{n}{p+2}\right) + 2\left(\frac{n}{p+1}\right) + 3\left(\frac{n}{p}\right)\right];$$

hinc ſi loco $p+3$ ſcribamus p fiet

$$\left(\frac{n+1}{p}\right) = \frac{n+1}{p}\left[\left(\frac{n}{p-1}\right) + 2\left(\frac{n}{p-2}\right) + 3\left(\frac{n}{p-3}\right)\right].$$

Ita ſi ſumamus $n = 5$ et $p = 6$ habebimus:

$$\left(\frac{6}{6}\right) = \tfrac{6}{6}\left[\left(\frac{5}{5}\right) + 2\left(\frac{5}{4}\right) + 3\left(\frac{5}{3}\right)\right],$$

ideoque ſubſtitutis valoribus ex tabula ſuperiore erit

$$\left(\frac{6}{6}\right) = 101 + 2 \cdot 65 + 3 \cdot 35 = 336,$$

eſt vero vtique $\left(\frac{6}{6}\right) = 336$.

§. 14. Reliquae proprietates redeunt ad ſummationem feriei, cuius terminus generalis eſt productum $\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right)$. Si enim loco x ordine ſcribantur numeri $0, 1, 2, 3, 4,$ etc. donec ad terminos euaneſcentes perveniatur, erit $\int \left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right) = \left(\frac{m+n}{3n-p}\right)$, quae ſumma etiam eſt $= \left(\frac{m+n}{3m+p}\right)$. Hinc ſi $p = 0$ orietur haec ſummatio:

$$\left(\frac{m}{0}\right)\left(\frac{n}{0}\right) + \left(\frac{m}{1}\right)\left(\frac{n}{1}\right) + \left(\frac{m}{2}\right)\left(\frac{n}{2}\right) + \left(\frac{m}{3}\right)\left(\frac{n}{3}\right) + \text{etc.} = \left(\frac{m+n}{3n}\right)$$

ſiue etiam $=\left(\frac{m+n}{3m}\right)$. Quare ſi inſuper fuerit $m=n$, orietur haec ſummatio:

$$\left(\frac{n}{0}\right)^2+\left(\frac{n}{1}\right)^2+\left(\frac{n}{2}\right)^2+\left(\frac{n}{3}\right)^2+\left(\frac{n}{4}\right)^2+\text{etc.}=\left(\frac{2n}{3n}\right).$$

Ita ſi fuerit $n=1$ erit

$$1^2+1^2+1^2+1^2=\left(\frac{2}{3}\right)=4.$$

Deinde ſi $n=2$ erit

$$1^2+2^2+3^2+4^2+3^2+2^2+1^2=\left(\frac{4}{6}\right)=44.$$

Porro ſi $n=3$ erit

$$1^2+3^2+6^2+10^2+12^2+12^2+10^2+6^2+3^2+1=\left(\frac{6}{9}\right)=580.$$

IV. De vnciis Polynomii cuiuſcunque $(1+z+zz+z^3\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot+z^\lambda)^n$.

§. 16. Facta euolutione huius poteſtatis deſignet iſte character $\left[\frac{p}{n}\right]$ coefficientem poteſtatis z^p, ita vt ſit

$$(1+z+zz+\ \cdot\ \cdot\ \cdot\ \cdot+z^\lambda)^n$$
$$=1+\left[\frac{n}{1}\right]z+\left[\frac{n}{2}\right]zz+\left[\frac{n}{3}\right]z^3+\left[\frac{n}{4}\right]z^4\ \cdot\ \cdot+\left[\frac{n}{p}\right]z^p$$

vnde quia vltimus terminus huius poteſtatis eſt $z^{\lambda n}$, erit $\left[\frac{n}{\lambda n}\right]=1$; quamobrem manifeſtum eſt, ſi fuerit vel p numerus negatiuus vel numerus maior quam λn, perpetuo fore $\left[\frac{n}{p}\right]=0$; deinde quia hi coefficientes pariter ordine retrogrado gaudent, erit $\left[\frac{n}{\lambda n-p}\right]=\left[\frac{n}{p}\right]$. Porro perſpicuum eſt fore vt ſupra $\left[\frac{n}{0}\right]=1$ et $\left[\frac{n}{1}\right]=n$. Caeterum in genere nullam formulam pro ſingulis poteſtatibus adiicere licet.

§. 16. Lex autem, qua termini huius ſeriei:

$$\left[\frac{n}{0}\right]+\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{4}\right]+\text{etc.}$$

ex

ex praecedentibus determinantur, sequenti formula includi potest:

$$\left[\frac{n}{p+\lambda}\right] = \frac{n+1-p-\lambda}{p+\lambda}\left[\frac{n}{p+\lambda-1}\right] + \frac{2(n+1)-p-\lambda}{p+\lambda}\left[\frac{n}{p+\lambda-2}\right]$$
$$+ \frac{3(n+1)-p-\lambda}{p+\lambda}\left[\frac{n}{p+\lambda-3}\right] \cdot \cdot \cdot \cdot + \frac{\lambda(n+1)-p-\lambda}{p+\lambda}\left[\frac{n}{p}\right],$$

quae expressio, si loco $p+\lambda$ scribamus simpliciter p, induet hanc formam:

$$\left[\frac{n}{p}\right] = \frac{n+1-p}{p}\left[\frac{n}{p-1}\right] + \frac{2(n+1)-p}{p}\left[\frac{n}{p-2}\right] + \frac{3(n+1)-p}{p}\left[\frac{n}{p-3}\right]$$
$$+ \frac{4(n+1)-p}{p}\left[\frac{n}{p-4}\right] + \cdot \cdot \cdot \cdot \frac{\lambda(n+1)-p}{p}\left[\frac{n}{p-\lambda}\right].$$

Huius formulae ope singuli seriei termini facile formari poterunt: quia enim nouimus esse $\left[\frac{n}{0}\right] = 1$, antecedentes vero omnes $= 0$, pro formatione singulorum terminorum habebimus:

$$\left[\frac{n}{1}\right] = \frac{n}{1}\left[\frac{n}{0}\right]$$
$$\left[\frac{n}{2}\right] = \frac{n-1}{2}\left[\frac{n}{1}\right] + \frac{2n}{2}\left[\frac{n}{0}\right]$$
$$\left[\frac{n}{3}\right] = \frac{n-2}{3}\left[\frac{n}{2}\right] + \frac{2n-1}{3}\left[\frac{n}{1}\right] + \frac{3n}{3}\left[\frac{n}{0}\right]$$
$$\left[\frac{n}{4}\right] = \frac{n-3}{4}\left[\frac{n}{3}\right] + \frac{2n-2}{4}\left[\frac{n}{2}\right] + \frac{3n-1}{4}\left[\frac{n}{1}\right] + \frac{4n}{4}\left[\frac{n}{0}\right]$$
$$\left[\frac{n}{5}\right] = \frac{n-4}{5}\left[\frac{n}{4}\right] + \frac{2n-3}{5}\left[\frac{n}{3}\right] + \frac{3n-2}{5}\left[\frac{n}{2}\right] + \frac{4n-1}{5}\left[\frac{n}{1}\right] + \frac{5n}{5}\left[\frac{n}{0}\right].$$

Hic autem probe obseruandum est, has formulas non vltra λ terminos continuari debere, quandoquidem pro formula generali $\left[\frac{n}{p}\right]$ vltimum membrum vidimus esse

$$= \frac{\lambda(n+1)-p}{p}\left[\frac{n}{p-\lambda}\right].$$

§. 17. Quodsi porro hinc ad potestatem sequentem, cuius exponens est $n+1$, progredi velimus, vbi potestatis z^p coefficiens est $\left[\frac{n+1}{p}\right]$ ex ipsa formatione harum potestatum manifestum est fore

$\left[\frac{n+1}{p}\right]$

$$\left[\frac{n+1}{p}\right] = \left[\frac{n}{p}\right] + \left[\frac{n}{p-1}\right] + \left[\frac{n}{p-2}\right] + \left[\frac{n}{p-3}\right] \dots + \left[\frac{n}{p-\lambda}\right].$$

Quodſi iam hic loco $\left[\frac{n}{p}\right]$ ſcribamus valorem ante exhibitum, prodibit ſequens aequatio:

$$\left[\frac{n+1}{p}\right] = \frac{n+1}{p}\left[\frac{n}{p+1}\right] + \frac{2(n+1)}{p}\left[\frac{n}{p-2}\right] + \frac{3(n+1)}{p}\left[\frac{n}{p-3}\right] \dots + \frac{\lambda(n+1)}{p}\left[\frac{n}{p-\lambda}\right],$$

vbi terminorum numerus itidem eſt $= \lambda$.

§. 18. Quodſi iam hinc pariter ſeriem formemus, cuius terminus generalis ſit productum $\left[\frac{m}{x}\right]\left[\frac{n}{p+x}\right]$, cuius ergo ipſi termini ſinguli reperiuntur, ſi loco x ordine ſcribantur numeri 0, 1, 2, 3, 4, etc. donec perueniatur ad terminos euaneſcentes, id quod eueniet, quando x vel vltra λm vel vltra $\lambda n - p$ augetur; tum totius ſeriei ſumma erit $= \left[\frac{m+n}{\lambda n - p}\right]$, vel etiam $\left[\frac{m+n}{\lambda m + p}\right]$, quae ſummatio ita repraeſentari poterit:

$$\int \left[\frac{m}{x}\right]\left[\frac{n+1}{p+x}\right] = \left[\frac{m+n}{\lambda n - p}\right] = \left[\frac{m+n}{\lambda m + p}\right],$$

ipſa autem progreſſio his conſtabit terminis:

$$\left[\frac{m}{0}\right]\left[\frac{n}{p}\right] + \left[\frac{m}{1}\right]\left[\frac{n}{p+1}\right] + \left[\frac{m}{2}\right]\left[\frac{u}{p+2}\right] + \left[\frac{m}{3}\right]\left[\frac{n}{p+3}\right] + \text{etc.}$$

Quodſi ergo fuerit $p = 0$ et $m = n$, ſinguli ſeriei termini fiunt quadrata, ſcilicet habebitur iſta ſeries:

$$\left[\frac{n}{0}\right]^2 + \left[\frac{n}{1}\right]^2 + \left[\frac{n}{2}\right]^2 + \left[\frac{n}{3}\right]^2 + \left[\frac{n}{4}\right]^2 + \text{etc.}$$

cuius ſumma erit $= \left[\frac{2n}{\lambda n}\right]$. Hoc ſcilicet caſu ambae formulae in vnam coaleſcunt.

§. 19. Hoc igitur modo inſignes illae proprietates, quas non ita pridem ſuper vnciis Binomii demonſtraui, extendi poſſunt tam ad trinomia et quadrinomia quam ad Polynomia cuiuscunque ordinis; et quemadmodum easdem

dem has proprietates pro Binomio ostendi etiam locum habere, quando exponentes m et n sunt vel numeri negatiui, vel adeo numeri fracti, eadem proprietas pro Polynomiis quibusque locum habere deprehenditur; quod quidem non tam facile perspicitur, propterea quod euolutio potestatis

$$(1 + z + z^2 + z^3 \ldots \ldots + z^\lambda)^n,$$

quando exponens n est vel numerus negatiuus vel fractus, multo magis euadit perplexa, neque etiam interpolationem admittit, quemadmodum id pro casu Binomii praestare licuit, vbi si $\left[\frac{n}{p}\right]$ designet vnciam potestatis z^p, quae ex euolutione dignitatis $(1 + z)^n$ oritur, ostendi si breuitatis gratia statuatur $l\frac{1}{x} = u$, semper fore

$$\left[\frac{n}{p}\right] = \frac{\int u^n\, dx}{\int u^p\, dx \int u^{n-p}\, dx},$$

si scilicet haec integralia ab $x = 0$ vsque ad $x = 1$ extendantur. Pro interpolatione enim monstraui esse

$$\int \frac{dx}{\sqrt{u}} = \sqrt{\pi} \text{ et } \int dx \sqrt{u} = \tfrac{1}{2}\sqrt{\pi},$$

denotante scilicet π peripheriam circuli, cuius diameter $= 1$.