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De numero memorabili in summatione progressionis harmonicae naturalis occurrente

Leonhard Euler

DE NVMERO MEMORABILI, IN SVMMATIONE PROGRESSIONIS HARMONICAE NATVRALIS OCCVRRENTE.

Auctore
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§. 1.

Cum olim summationem seriei harmonicae

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

tractassem, eius summam indefinitam sequenti modo expressam deprehendi, vt posito

$$s = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{x},$$

haec summa fit

$$C + lx + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{etc.}$$

vbi litterae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. sunt numeri illi *Bernoulliani* vocati, scilicet $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$, $\mathfrak{D} = \frac{1}{30}$, $\mathfrak{E} = \frac{5}{66}$

$\mathfrak{F} = \frac{691}{2730}$, $\mathfrak{G} = \frac{7}{6}$, $\mathfrak{H} = \frac{3617}{510}$, $\mathfrak{I} = \frac{43867}{798}$, $\mathfrak{K} = \frac{174611}{330}$, $\mathfrak{L} = \frac{854513}{138}$,
$\mathfrak{M} = \frac{236364091}{2730}$, $\mathfrak{N} = \frac{8553103}{2}$, $\mathfrak{O} = \frac{23749461029}{870}$, $\mathfrak{P} = \frac{8615841276005}{14322}$, etc.
tum vero lx denotat logarithmum hyperbolicum numeri x, at littera C, quae per integrationem est ingressa, est certus numerus determinatus ex quouis casu particulari eruendus, quem ex casu $x = 10$ inueni esse;

$$C = 0,5772156649015325,$$

qui numerus eo magis notatu dignus videtur, quod eum nullo adhuc modo ad quampiam mensuram cognitam reuocare mihi quidem licuit.

§. 2. Quod si ergo numerus x accipiatur infinite magnus, tum erit

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{\infty} = C + lx,$$

vnde suspicari licet, istum numerum C esse logarithmum hyperbolicum cuiuspiam numeri notabilis, quem statuamus $= N$, ita vt sit $C = lN$, et summa illus seriei infinitae aequetur logarithmo numeri $N.x$, vnde operae pretium erit in valorem huius numeri N inquirere, quem quidem sufficiet ad quinque vel sex figuras decimales definiuisse, quoniam hinc non difficulter iudicari poterit, num cum quopiam numero cognito conueniat nec ne. Quo igitur hoc facilius praestari possit quaeramus numerum quempiam simpliciorem, cuius logarithmus parum a C discrepet; talis autem deprehenditur $\frac{3}{2} \cdot \frac{6}{5} = \frac{9}{5}$, quippe cuius logarithmus est $= 0,58778$ aliquanto maior quam C, vnde concludimus fore $N < \frac{9}{5}$. Statuamus ergo $N = \frac{9}{5} - \omega$, et cum in genere sit

$l(a - \omega)$

$$l(a-\omega) = l\,a - \frac{\omega}{a} - \frac{\omega^2}{2a^2} - \frac{\omega^3}{3a^3} - \frac{\omega^4}{4a^4} - \text{etc.}$$

hoc casu erit $a = \frac{9}{5}$ hincque $\frac{\omega}{a} = \frac{5\omega}{9}$, pro quo scribamus z, vt fiat $\omega = \frac{9}{5}z$; erit ergo

$$l(\tfrac{9}{5}-\omega) = l\tfrac{9}{5} - z - \tfrac{1}{2}zz - \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 - \text{etc.} = l\,N = C.$$

Quia igitur est $l\frac{9}{5} - C = 0,01057$, erit

$$z + \tfrac{1}{2}zz + \tfrac{1}{3}z^3 + \tfrac{1}{4}z^4 + \text{etc.} = 0,01057,$$

Cum igitur sit $z < 0,01057$, sumamus $z = 0,01000 + y$, erit $zz = 0,0001 + 0,02.y$, id quod pro nostro scopo sufficit. His autem valoribus substitutis prodibit

$$0,01005 + 1,01000.y = 0,01057;$$

vnde deducitur $y = 0,00052$, ideoque $z = 0,01052$; hinc igitur $\omega = 0,01894$, consequenter numerus quaesitus $N = 1,78106$. Totum igitur negotium huc redit, vt investigetur num forte iste numerus N ad quampiam quantitatem cognitam assignabilem teneat rationem.

§. 3. Quoniam autem illum valorem litterae C ex serie, in qua nullus certus ordo elucet, propterea quod numeri *Bernoulliani* secundum legem maxime perplexam progrediuntur, deduxi; haud inutile erit, in seriem magis regularem inquirere, cuius summa ipsi numero C aequalis sit futura, et quae etiam maxime conuergat, vt eius valor etiam hinc definiri possit, id quo eo magis necessarium videtur, quoniam numeri *Bernoulliani* mox adeo increscunt, vt in seriem maxime diuergentem abeant, ideoque dubium merito oriri possit, vtrum valor inuentus pro satis certo haberi queat nec ne.

§. 4. Cum igitur numerus propofitus C reuera aequetur ifti formulae:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \frac{1}{x} - l\,x,$$

denotante x numerum infinitum, haud difficulter perfpicitur, hinc fequentem feriem conficere poffe:

$$\begin{aligned} C = 1 &+ \tfrac{1}{2} - l\tfrac{2}{1} \\ &+ \tfrac{1}{3} - l\tfrac{3}{2} \\ &+ \tfrac{1}{4} - l\tfrac{4}{3} \\ &+ \tfrac{1}{5} - l\tfrac{5}{4} \\ &+ \tfrac{1}{6} - l\tfrac{6}{5} \\ &+ \tfrac{1}{7} - l\tfrac{7}{6} \\ &\text{etc.} \quad \text{etc.} \end{aligned}$$

Manifeftum enim eft his omnibus terminis in vnam fummam collectis ipfam prodire formulam

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{x} - l\,x.$$

Sicque habemus feriem infinitam noftro numero C aequalem, cuius quilibet terminus in genere erit $\frac{1}{n} - l\frac{n}{n-1}$, quae formula erit quafi terminus generalis feriei inuentae.

§. 5. Perpendamus igitur accuratius iftam formulam $\frac{1}{n} - l\frac{n}{n-1}$, et cum fit

$$-l\frac{n}{n-1} = l\frac{n-1}{n} = l\left(1 - \frac{1}{n}\right),$$

per feriem infinitam erit

$$-l\frac{n}{n-1} = -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} - \frac{1}{4n^4} - \frac{1}{5n^5} - \text{etc.}$$

ideoque

$$\frac{1}{n} - l\frac{n}{n-1} = -\frac{1}{2n^2} - \frac{1}{3n^3} - \frac{1}{4n^4} - \frac{1}{5n^5} - \text{etc.},$$

vnde pro numero C inueniendo euolui oportebit fequentem feriem:

$$1 - C = + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \frac{1}{4.2^4} + \frac{1}{5.2^5} + \text{etc.}$$
$$+ \frac{1}{2.3^2} + \frac{1}{3.3^3} + \frac{1}{4.3^4} + \frac{1}{5.3^5} + \text{etc.}$$
$$+ \frac{1}{2.4^2} + \frac{1}{3.4^3} + \frac{1}{4.4^4} + \frac{1}{5.4^5} + \text{etc.}$$
$$+ \frac{1}{2.5^2} + \frac{1}{3.5^3} + \frac{1}{4.5^4} + \frac{1}{5.5^5} + \text{etc.}$$
etc.

§. 6. Defignet nobis breuitatis gratia α fummam feriei reciprocae quadratorum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \text{ etc.},$$

fimilique modo β fummam feriei reciprocae cuborum, δ fummam feriei reciprocae biquadratorum etc., atque valores numericos harum litterarum iam olim fatis exactos (V. Inftit. Calculi differentialis pag. 456.) exhibui: inde igitur erit

$$1 - C = \tfrac{1}{2}(\alpha - 1) + \tfrac{1}{3}(\beta - 1) + \tfrac{1}{4}(\gamma - 1) + \tfrac{1}{5}(\delta - 1) + \tfrac{1}{6}(\varepsilon - 1) + \text{etc.}$$

Eft vero

$$\tfrac{1}{2}(\alpha - 1) = 0,3224670$$
$$\tfrac{1}{3}(\beta - 1) = 0,0673523$$
$$\tfrac{1}{4}(\gamma - 1) = 0,0205808$$
$$\tfrac{1}{5}(\delta - 1) = 0,0073855$$
$$\tfrac{1}{6}(\varepsilon - 1) = 0,0028905$$
$$\tfrac{1}{7}(\zeta - 1) = 0,0011927$$
$$\tfrac{1}{8}(\eta - 1) = 0,0005097$$
$$\tfrac{1}{9}(\theta - 1) = 0,0002231$$
$$\tfrac{1}{10}(\iota - 1) = 0,0000994$$
$$\tfrac{1}{11}(\varkappa - 1) = 0,0000449$$
$$\tfrac{1}{12}(\lambda - 1) = 0,0000205$$
$$\tfrac{1}{13}(\mu - 1) = 0,0000094$$

$$\tfrac{1}{14}(\nu - 1) = 0,0000044$$
$$\tfrac{1}{15}(\xi - 1) = 0,0000020$$
$$\tfrac{1}{16}(o - 1) = 0,0000009$$
$$1 - C = 0,4227831.$$

vnde prodiret $C = 0,5772169$, qui valor autem ob terminos feriei fequentes neglectos diminui debet ad $0,5772164$, vbi in vltima figura tantum octo vnitatibus aberratur.

§. 7. Pro eodem autem valore accuratius inueftigando feries multo magis conuergens inueniri poteft. Cum enim fit

$$l\frac{a+1}{a-1} = \frac{2}{a} + \frac{2}{3a^3} + \frac{2}{5a^5} + \frac{2}{7a^7} + \frac{2}{9a^9} + \text{etc.}$$

ob $l\frac{n}{n-1} = l\frac{2n}{2n-2}$, fumatur $a = 2n - 1$ et erit

$$l\frac{n}{n-1} = \frac{2}{2n-1} + \frac{2}{3(2n-1)^3} + \frac{2}{5(2n-1)^5} + \frac{2}{7(2n-1)^7} + \text{etc.}$$

a quo fi fractio $\frac{1}{n}$ auferatur, prodibit terminus generalis noftrae feriei

$$l\frac{n}{n-1} - \frac{1}{n} = \frac{1}{n(2n-1)} + \frac{2}{3(2n-1)^3} + \frac{2}{5(2n-1)^5} + \frac{2}{7(2n-1)^7} + \text{etc.}$$

Quod fi ergo loco n fucceffiue fcribantur numeri 2, 3, 4, 5, etc. fequens orietur feries:

$$\begin{aligned} 1 - C = &+ \frac{1}{2.3} + \frac{2}{3.3^3} + \frac{2}{5.3^5} + \frac{2}{7.3^7} + \frac{2}{9.3^9} + \text{etc.} \\ &+ \frac{1}{3.5} + \frac{2}{3.5^3} + \frac{2}{5.5^5} + \frac{2}{7.5^7} + \frac{2}{9.5^9} + \text{etc.} \\ &+ \frac{1}{4.7} + \frac{2}{3.7^3} + \frac{2}{5.7^5} + \frac{2}{7.7^7} + \frac{2}{9.7^9} + \text{etc.} \\ &+ \frac{1}{5.9} + \frac{2}{3.9^3} + \frac{2}{5.9^5} + \frac{2}{7.9^7} + \frac{2}{9.9^9} + \text{etc.} \\ &\text{etc.} \end{aligned}$$

vbi prima linea verticalis, ob $\frac{1}{n(2n-1)} = \frac{2}{2n-1} - \frac{2}{2n}$, reducitur ad hanc feriem:

$1 =$

$\frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} +$ etc.

cuius summa manifesto est $2\,l\,2 - 1$, qua summa ad alteram partem translata fiet nunc

$$\begin{aligned} 2 - 2\,l\,2 - C = & \tfrac{2}{3.3^3} + \tfrac{2}{5.3^5} + \tfrac{2}{7.3^7} + \tfrac{2}{9.3^9} + \text{etc.} \\ & + \tfrac{2}{3.5^3} + \tfrac{2}{5.5^5} + \tfrac{2}{7.5^7} + \tfrac{2}{9.5^9} + \text{etc.} \\ & + \tfrac{2}{3.7^3} + \tfrac{2}{5.7^5} + \tfrac{2}{7.7^7} + \tfrac{2}{9.7^9} + \text{etc.} \\ & + \tfrac{2}{3.9^3} + \tfrac{2}{5.9^5} + \tfrac{2}{7.9^7} + \tfrac{2}{9.9^9} + \text{etc.} \\ & \text{etc.} \end{aligned}$$

Haec series tantopere conuergit, vt eius summa facile ad multo plures figuras expediri posset quam ante. Hic autem notetur primae seriei summam esse $l\,2 - \frac{2}{3}$; similique modo secundae seriei summa erit $l\,\frac{3}{2} - \frac{2}{5}$, tertiae autem seriei summa $= l\,\frac{4}{3} - \frac{2}{7}$, sequentis seriei summa $= l\,\frac{5}{4} - \frac{2}{9}$, sequentis $= l\,\frac{6}{5} - \frac{2}{11}$, qui ergo valores seorsim computentur. His igitur valoribus, qui in tabulis reperiuntur collectis, superest vt sequentes series in vnam summam colligantur:

$$\begin{aligned} & + \tfrac{2}{3}\left(\tfrac{1}{13^3} + \tfrac{1}{15^3} + \tfrac{1}{17^3} + \tfrac{1}{19^3} + \text{etc.}\right) \\ & + \tfrac{2}{5}\left(\tfrac{1}{13^5} + \tfrac{1}{15^5} + \tfrac{1}{17^5} + \tfrac{1}{19^5} + \text{etc.}\right) \\ & + \tfrac{2}{7}\left(\tfrac{1}{13^7} + \tfrac{1}{15^7} + \tfrac{1}{17^7} + \tfrac{1}{19^7} + \text{etc.}\right) \\ & + \tfrac{2}{9}\left(\tfrac{1}{13^9} + \tfrac{1}{15^9} + \tfrac{1}{17^9} + \tfrac{1}{19^9} + \text{etc.}\right) \end{aligned}$$

id quod per praecepta, quae olim de summatione talium serierum dedi, haud difficulter praestabitur.

§. 8. Quoniam autem de vero valore nostri numeri $C = 0,5772156649015325$, iam vsque ad 16 figuras decimales certi sumus, superfluum foret istum laborem denuo fuscipere, vnde alias feries magis regulares expendamus, quarum fumma huic numero aequetur. Ac primo quidem fimpliciffima feries hoc praeftans ex forma principali

$$C = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \tfrac{1}{x} - l\,x,$$

par hanc refolutionem deducitur:

$$\begin{aligned} C = & \; 1 - l\,2 \\ & + \tfrac{1}{2} - l\,\tfrac{3}{2} \\ & + \tfrac{1}{3} - l\,\tfrac{4}{3} \\ & + \tfrac{1}{4} - l\,\tfrac{5}{4} \\ & + \ldots \\ & \;\;\ldots \\ & + \tfrac{1}{n} - l\,\tfrac{n+1}{n}. \end{aligned}$$

His enim actu collectis vsque ad $\frac{1}{x}$ prodit fumma

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \tfrac{1}{x} - l\,(x+1),$$

quia vero $l\,x$ fupponitur infinitus $l\,(x+1)$ a $l\,x$ non discrepare eft cenfendus.

§. 9. Cum igitur per feriem infinitam fit $l\,\frac{n+1}{n} = l\,1 + \frac{1}{n} = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} -$ etc. erit terminus generalis illius feriei

$$= \tfrac{1}{2n^2} - \tfrac{1}{3n^3} + \tfrac{1}{4n^4} - \tfrac{1}{5n^5} + \text{etc.}$$

vnde nofter numerus C fequenti ferie exprimetur:

$$\begin{aligned} C = & \tfrac{1}{2.1^2} - \tfrac{1}{3.1^3} + \tfrac{1}{4.1^4} - \tfrac{1}{5.1^5} + \text{etc.} \\ & + \tfrac{1}{2.2^2} - \tfrac{1}{3.2^3} + \tfrac{1}{4.2^4} - \tfrac{1}{5.2^5} + \text{etc.} \\ & + \tfrac{1}{2.3^2} - \tfrac{1}{3.3^3} + \tfrac{1}{4.3^4} - \tfrac{1}{5.3^5} + \text{etc.} \end{aligned}$$

$$+\frac{1}{2.4^2}-\frac{1}{3.4^3}+\frac{1}{4.4^4}-\frac{1}{5.4^5}+\text{etc.}$$
$$+\frac{1}{2.5^2}-\frac{1}{3.5^3}+\frac{1}{4.5^4}-\frac{1}{5.5^5}+\text{etc.}$$
etc.

§. 10. Quod ſi iam vt ſupra litterae α, β, γ, δ etc. denotent ſummas ſerierum reciprocarum quadratorum, cuborum, biquadratorum et altiorum poteſtatum, per eas noſter numerus C ita exprimetur:

$$C=\tfrac{1}{2}\alpha-\tfrac{1}{3}\beta+\tfrac{1}{4}\gamma-\tfrac{1}{5}\delta+\tfrac{1}{6}\varepsilon-\tfrac{1}{7}\zeta+\text{etc.}$$

Supra autem iam inuenimus hanc ſeriem easdem litteras inuoluentem:

$$1-C=\tfrac{1}{2}(\alpha-1)+\tfrac{1}{3}(\beta-1)+\tfrac{1}{4}(\gamma-1)+\tfrac{1}{5}(\delta-1)+\text{etc.}$$

quae ambae ſeries eo magis ſunt notatu dignae, quod diverſo modo per easdem litteras α, β, γ, δ, valorem ipſius C exhibeant.

§. 11. Variis igitur modis hae duae ſeries inuicem combinari poterunt, vnde concluſiones omni attentione dignas deriuare licebit. Ac primo quidem illae ſeries inuicem additae producent ſequentem ſummationem:

$$1=\alpha-\tfrac{1}{2}-\tfrac{1}{3},+\tfrac{1}{2}\gamma-\tfrac{1}{4}-\tfrac{1}{5},+\tfrac{1}{3}\varepsilon-\tfrac{1}{6}-\tfrac{1}{7},+\tfrac{1}{4}\eta-\tfrac{1}{8}-\tfrac{1}{9},+\text{etc.}$$

vbi tantum ſummae poteſtatum parium occurrunt, quae, vti inueni, per poteſtates peripheriae circuli π exhiberi poſſunt, cum ſit

$$\alpha=\frac{\pi^2}{6},\ \gamma=\frac{\pi^4}{90},\ \varepsilon=\frac{\pi^6}{945},\ \eta=\frac{\pi^8}{9450},\ \iota=\frac{\pi^{10}}{93555},\ \text{etc.}$$

Quare cum iſtius ſeriei prorſus ſingularis ſumma ſit $=1$, operae pretium erit, primores ſaltem terminos per fractiones decimales euoluere. Erit igitur

 $\alpha-$

$$\alpha - \tfrac{1}{2} - \tfrac{1}{3} = 0,8116007335$$
$$\tfrac{1}{2}\gamma - \tfrac{1}{4} - \tfrac{1}{5} = 0,0911616168$$
$$\tfrac{1}{3}\varepsilon - \tfrac{1}{6} - \tfrac{1}{7} = 0,0295905446$$
$$\tfrac{1}{4}\eta - \tfrac{1}{8} - \tfrac{1}{9} = 0,0149082279$$
$$\tfrac{1}{5}\iota - \tfrac{1}{10} - \tfrac{1}{11} = 0,0092898241$$

quorum quinque terminorum ſumma iam eſt

$$= 0,9565509469,$$

ita vt reliquorum omnium ſumma producere debeat

$$0,0434490531.$$

§. 12. Haec ſummatio eo magis eſt memorabilis, quod nullae adhuc huiusmodi ſeries in Analyſi ſint conſideratae. Hic autem probe notari neceſſe eſt, terminos iſtius ſeriei ita diſponi debere, quemadmodum ex combinatione binarum praecedentium ſerierum ſunt nati, ita vt ſinguli termini ex tribus conſtent partibus. Si enim verbi gratia omnes partes negatiuas ad ſiniſtram transponere vellemus, prodiret hac aequatio:

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}+\text{etc.} = \alpha+\tfrac{1}{2}\gamma+\tfrac{1}{3}\varepsilon+\tfrac{1}{4}\eta+\tfrac{1}{5}\iota+\text{etc.}$$

vnde nihil plane cognoſcere liceret, propterea quod ex vtraque parte haberetur quantitas infinita; vbi quoniam primi termini ſerierum α, γ, ε, etc. ſunt vnitates, ex his ſolis pro dextra parte oritur ſeries

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\text{etc.},$$

quae ipſa ſeries ad ſiniſtram reperitur. Neque tamen hinc ſequitur reliquos terminos dextri membri nihilo fore aequales, quoniam manifeſto ſeries ad ſiniſtram poſita duplo plures continet terminos quam occurrunt in parte dextra.

§. 13. Nunc etiam ambas feries pro C et $1-C$ fupra §. 10. expofitas alteram ab altera fubtrahamus, ac prodibit fequens feries non minus notatu digna:

$$2C-1=+\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{2}{3}\beta,+\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{2}{5}\delta,+\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{2}{7}\zeta,+\tfrac{1}{8}+\tfrac{1}{9}-\tfrac{2}{9}\theta,+\text{etc.}$$

vbi tantum fummae poteftatum imparium occurrunt. Novimus autem effe

$$2C-1=0,1544313298.$$

Videamus igitur quinam valores numerici ex quinque primoribus terminis nafcantur, et cum fit

$$\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{2}{3}\beta=0,0319620645$$
$$\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{2}{5}\delta=0,0352288980$$
$$\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{2}{7}\zeta=0,0214240158$$
$$\tfrac{1}{8}+\tfrac{1}{9}-\tfrac{2}{9}\theta=0,0134425793$$
$$\tfrac{1}{10}+\tfrac{1}{11}-\tfrac{2}{11}\varkappa=0,0090010567$$

quoniam horum quinque terminorum fumma tantum eft 0,1110586143, haec feries neutiquam apta eft cenfenda ad verum valorem ipfius C explorandum, fiquidem adhuc effet incognitus.

§. 14. Ad hunc modum etiam euoluamus expresfionem §. 7. inuentam, quae fecundum lineas verticales hoc modo repraefentetur:

$$\begin{aligned}2-2\,l\,2-C=&+\tfrac{2}{3}\left(\tfrac{1}{3^3}+\tfrac{1}{5^3}+\tfrac{1}{7^3}+\tfrac{1}{9^3}+\tfrac{1}{11^3}+\text{etc.}\right)\\&+\tfrac{2}{5}\left(\tfrac{1}{3^5}+\tfrac{1}{5^5}+\tfrac{1}{7^5}+\tfrac{1}{9^5}+\tfrac{1}{11^5}+\text{etc.}\right)\\&+\tfrac{2}{7}\left(\tfrac{1}{3^7}+\tfrac{1}{5^7}+\tfrac{1}{7^7}+\tfrac{1}{9^7}+\tfrac{1}{11^7}+\text{etc.}\right)\\&+\tfrac{2}{9}\left(\tfrac{1}{3^9}+\tfrac{1}{5^9}+\tfrac{1}{7^9}+\tfrac{1}{9^9}+\tfrac{1}{11^9}+\text{etc.}\right)\\&+\text{etc.}\end{aligned}$$

vbi

vbi cum tantum poteſtates impares numerorum imparium occurrant, hae ſeries per litteras ſupra aſſumtas β, δ, ζ, θ etc. exprimantur, quandoquidem nouimus eſſe.

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \text{etc.} = \frac{7}{8}\beta$$

$$1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} + \text{etc.} = \frac{31}{32}\delta$$

$$1 + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{9^7} + \text{etc.} = \frac{127}{128}\zeta$$

$$1 + \frac{1}{3^9} + \frac{1}{5^9} + \frac{1}{7^9} + \frac{1}{9^9} + \text{etc.} = \frac{511}{512}\theta$$

etc. etc.

His igitur valoribus ſubſtitutis habebimus hanc ſeriem:

$$2 - 2l2 - C = \frac{2}{3}\cdot\frac{7}{8}\beta - \frac{2}{3} + \frac{2}{5}\cdot\frac{31}{32}\delta - \frac{2}{5} + \frac{2}{7}\cdot\frac{127}{128}\zeta - \frac{2}{7} + \frac{2}{9}\cdot\frac{511}{512}\theta - \frac{2}{9} + \text{etc.}$$

Modo ante autem inuenimus eſſe

$$2C - 1 = \frac{1}{2} + \frac{1}{3} - \frac{2}{3}\beta, + \frac{1}{4} + \frac{1}{5} - \frac{2}{5}\delta, + \frac{1}{6} + \frac{1}{7} - \frac{2}{7}\zeta, + \frac{1}{8} + \frac{1}{9} - \frac{2}{9}\theta, + \text{etc.}$$

quae ſeries ad modo inuentam addita praebet

$$1 - 2l2 + C = \frac{1}{2} - \frac{1}{3} - \frac{2}{3\cdot 2^3}\beta, + \frac{1}{4} - \frac{1}{5} - \frac{2}{5\cdot 2^5}\delta, + \frac{1}{6} - \frac{1}{7} - \frac{2}{7\cdot 2^7}\zeta + \text{etc.}$$

vbi omnes fractiones abſolutae conſtituunt hanc ſeriem:

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \text{etc.}$$

cuius ſumma quia eſt finita $= 1 - l2$, eius loco tuto hunc valorem ſcribere licet, hincque peruenietur ad ſequentem ſummationem:

$$l2 - C = \frac{1}{3\cdot 2^2}\beta + \frac{1}{5\cdot 2^4}\delta + \frac{1}{7\cdot 2^6}\zeta + \frac{1}{9\cdot 2^8}\theta + \text{etc.}$$

quae ſeries tantopere conuergit, vt ex ea haud difficulter valor noſtri numeri C elici queat.

§. 15. Operae pretium igitur erit hanc ſeriem accuratius euoluere; quod quo facilius fieri poſſit, quia omnes litterae β, δ, ζ, θ, etc. vnitatem continent: hae vni-

vnitates seorsim sumtae praebent hanc seriem:

$$\frac{1}{3.2^2}+\frac{1}{5.2^4}+\frac{1}{7.2^6}+\frac{1}{9.2^8}+\text{etc.}$$

cuius summa est $l\,3-1$, quo valore hic introducto erit

$$1-l\tfrac{3}{2}-C=\frac{1}{3.2^2}(\beta-1)+\frac{1}{5.2^4}(\delta-1)+\frac{1}{7.2^6}(\zeta-1)+\frac{1}{9.2^8}(\theta-1)+\text{etc.}$$

Hic pro parte sinistra est

$$1-l\tfrac{3}{2}=0,5945348918918356;$$

pro parte dextra autem valores singulorum terminorum reperiuntur sequentes:

$$\frac{1}{3.4}(\beta-1)=0,0168380752632995$$
$$\frac{1}{5.4^2}(\delta-1)=0,0004615969388358$$
$$\frac{1}{7.4^3}(\zeta-1)=0,0000186367798808$$
$$\frac{1}{9.4^4}(\theta-1)=0,0000008716982752$$
$$\frac{1}{11.4^5}(\varkappa-1)=0,0000000438732781$$
$$\frac{1}{13.4^6}(\mu-1)=0,0000000023047504$$
$$\frac{1}{15.4^7}(\xi-1)=0,0000000001244638$$
$$\frac{1}{17.4^8}(\pi-1)=0,0000000000068551$$
$$\frac{1}{19.4^9}(\sigma-1)=0,0000000000003831$$

Pro reliquis $0,0000000000000225$

$$\text{Summa}=0,0173192269900443$$
$$1-l\tfrac{3}{2}=0,5945348918918356$$
$$C=0,5772156649017913$$

qui valor olim inuento, ob leuissimos errores non euitandos, conformis est censendus.

§. 16. Praeter has autem series, quas pro determinando numero C hactenus dedimus, innumerabiles alias reperire licet. Cum enim in genere statui possit

$$C = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \ldots\ldots + \frac{1}{x+p} - l(x+q),$$

quia numerus x infinite magnus assumi debet, semper idem valor hinc resultabit, quicunque numeri loco p et q, modo sint finiti, accipiantur. Quod si iam breuitatis gratia ponamus

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots\ldots + \tfrac{1}{p} = \Pi, \text{ erit}$$

$$C = \Pi + \frac{1}{1+p} + \frac{1}{2+p} + \frac{1}{3+p} + \ldots + \frac{1}{x+p} - l(x+q).$$

Haec iam forma in sequentem seriem resolui potest.

$$\begin{aligned} \Pi - l\,q &+ \frac{1}{1+p} - l\frac{q+1}{q} \\ &+ \frac{1}{2+p} - l\frac{q+2}{q+1} \\ &+ \frac{1}{3+p} - l\frac{q+3}{q+2} \\ &\quad - - \quad - - \\ &\quad - - \quad - - \\ &+ \frac{1}{n+p} - l\frac{q+n}{q+n-1} \end{aligned}$$

vnde igitur innumerabiles diuersas series infinitas exhibere licet, quarum omnium summa eadem est, scilicet $= C$.

§. 17. Quo autem hae series magis conuergant, talem relationem inter q et p assumi conueniet, vt, quando $l\frac{q+n}{q+n-1}$ in seriem resoluitur, eius primus terminus aequalis euadat ipsi $\frac{1}{n+p}$. Tres autem habentur potissimum modi hunc logarithmum in seriem conuertendi; primus oritur ex serie generali

$$l(1+z) = z - \tfrac{1}{2}zz + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \text{etc.}$$

vbi posito $1+z=\frac{q+n}{q+n-1}$, fit $z=\frac{1}{q+n-1}$; hoc ergo casu sumi conueniet $p=q-1$. Sin autem hac resolutione vti velimus:

$$l\frac{1}{1-z}=z+\tfrac{1}{2}zz+\tfrac{1}{3}z^3+\tfrac{1}{4}z^4+\text{etc.},$$

facto $\frac{1}{1-z}=\frac{q+n}{q+n-1}$, fit $z=\frac{1}{q+n}$, quo ergo casu capi debebit $p=q$. Tertius modus desumitur ex hac resolutione:

$$l\frac{1+z}{1-z}=2z+\tfrac{2}{3}z^3+\tfrac{2}{5}z^5+\tfrac{2}{7}z^7+\text{etc.}$$

posito igitur $\frac{1+z}{1-z}=\frac{q+n}{q+n-1}$ erit $z=\frac{1}{2(q+n)-1}$, quo casu capi debet $p=q-\frac{1}{2}$. Hoc igitur casu, vt p fiat numerus integer, pro q fractio assumi debet formae $m+\frac{1}{2}$, tum enim fiet $p=m$. At si q fuerit numerus integer, p vtique erit fractio $=q-\frac{1}{2}$; cum autem valor ipsius Π hoc casu non pateat, eum ante omnia inuestigari oportet.

§. 18. Hunc in finem consideremus sequentem seriem cum suis indicibus:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1, & 1+\frac{1}{2}, & 1+\frac{1}{2}+\frac{1}{3}, & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \end{array}$$

vbi indici generali n respondebit terminus

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\ldots+\tfrac{1}{n},$$

qui si dicatur $=N$, erunt sequentes termini, indicibus $n+1$, $n+2$, $n+3$ respondentes isti: $N+\frac{1}{n+1}$, $N+\frac{1}{n+1}+\frac{1}{n+2}$, $N+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}$, vbi ergo nulla difficultas occurrit, quoties n fuerit numerus integer. Ponamus igitur indici $\frac{1}{2}$ respondere terminum z, quippe ad cuius inuentionem praesens institutum reducitur, qui si fuerit inuentus, sequentes termini hoc modo progredientur:

$$\begin{array}{cccc} \frac{1}{2} & 1+\frac{1}{2} & 2+\frac{1}{2} & 3+\frac{1}{2} \\ z, & z+\frac{2}{3}, & z+\frac{2}{3}+\frac{2}{5}, & z+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}; \end{array}$$

ſicque in genere indici $n+\frac{1}{2}$ reſpondebit terminus,

$$z+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+ \ldots \ldots \frac{2}{2n+1}.$$

Quare ſi numerus n capiatur infinite magnus, quo caſu termini indicibus n et $n+1$ reſpondentes non amplius a ſe inuicem diſcrepant, iis etiam terminus medius, indici $n+\frac{1}{2}$ reſpondens, aequalis fieri debet, atque ex hoc principio ſequens aequatio conficitur:

$$z+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\frac{2}{9}+ \ldots +\frac{2}{2n+1}$$
$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ \ldots +\frac{1}{n},$$

vbi ex vtraque parte terminorum numerus eſt idem, vnde ſingulis terminis a ſiniſtra ad dextram translatis et debite interpolatis prodibit

$$z=1-\frac{2}{3}+\frac{1}{2}-\frac{2}{5}+\frac{1}{3}-\frac{2}{7}+\frac{1}{4}-\frac{2}{9}+\frac{1}{5}-\frac{2}{11}+\text{etc.}$$

in infinitum, ſicque valor ipſius z per hanc ſeriem infinitam exprimitur, ex qua, cum ſit

$$\frac{1}{2}z=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\text{etc.}$$

manifeſto erit $\frac{1}{2}z=1-l\,2$, ideoque $z=2-2\,l\,2$.

§. 19. Quod ſi igitur p fuerit fractio formae $n+\frac{1}{2}$, reſpondentes valores litterae Π ſequenti modo ſe habebunt:

p	Π
$\frac{1}{2}$	$2 - 2\,l\,2$
$1 + \frac{1}{2}$	$2 - 2\,l\,2 + \frac{2}{3}$
$2 + \frac{1}{2}$	$2 - 2\,l\,2 + \frac{2}{3} + \frac{2}{5}$
$3 + \frac{1}{2}$	$2 - 2\,l\,2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7}$
etc.	etc.
$m + \frac{1}{2}$	$2 - 2\,l\,2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \dots \dots \frac{2}{2m+1}$.

§. 20. Vtamur resolutione secunda logarithmorum, vbi erat $p = q$ et

$$l\frac{n+q}{n+q-1} = \frac{1}{n+q} + \frac{1}{2(n+q)^2} + \frac{1}{3(n+q)^3} + \frac{1}{4(n+q)^4} + \text{etc.}$$

vnde pro nostro numero C nanciscimur hanc seriem:

$$\begin{aligned}\Pi - lq &- \frac{1}{2(q+1)^2} - \frac{1}{3(q+1)^3} - \frac{1}{4(q+1)^4} - \frac{1}{5(q+1)^5} - \text{etc.}\\ &- \frac{1}{2(q+2)^2} - \frac{1}{3(q+2)^3} - \frac{1}{4(q+2)^4} - \frac{1}{5(q+2)^5} - \text{etc.}\\ &- \frac{1}{2(q+3)^2} - \frac{1}{3(q+3)^3} - \frac{1}{4(q+3)^4} - \frac{1}{5(q+3)^5} - \text{etc.}\\ &\text{etc.}\end{aligned}$$

vbi est

$$\Pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \dots + \frac{1}{q},$$

vnde patet, quo maior numerus q accipiatur, hanc seriem eo magis futuram esse conuergentem.

§. 21. Si quis autem voluerit valorem nostri numeri C accuratius definire, ne opus quidem erit logarithmos, qui in singulis terminis occurrunt, in series resoluere. Quin etiam non necesse est certam relationem inter binos numeros p et q statuere, sed vtrumque pro arbitrio accipere licebit, ita vt, postquam valor primi membri $\Pi - lq$ fuerit expeditus, existente

$$\Pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots \dots + \frac{1}{p},$$

 totum

totum negotium reducatur ad ſummationem huius ſeriei infinitae:

$$\left(\frac{1}{p+1} - l\,\frac{q+1}{q}\right) + \left(\frac{1}{p+2} - \frac{q+2}{q+1}\right) + \left(\frac{1}{p+3} - l\,\frac{q+3}{q+2}\right) + \text{etc.}$$

cuius loco conſideremus hanc ſeriem generalem ſummandam:

$$S = X + X' + X'' + X'' + \text{etc.},$$

vbi X ſit functio quaecunque ipſius x, ſequentes vero termini oriantur, ſi loco x ſucceſſiue ſcribantur valores $x+1$, $x+2$, $x+3$, quem in finem loco litterae q ſcribatur x, et quia differentia inter p et q eſt data, ſtatuatur $p = x + a - 1$, ita vt ſit

$$X = \frac{1}{x+a} - l\,\frac{x+1}{x} \text{ et } X' = \frac{1}{x+a+1} - l\,\frac{x+2}{x+1},$$

et ita porro. Sicque ſumto pro x numero quocunque valor noſtri numeri C erit $C = \Pi - l\,q + S$, ſiue

$$C = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots \ldots \frac{1}{x+a-1} - l\,x + S,$$

vbi ambo numeri x et a arbitrio noſtro permittuntur, quo non obſtante in ſequente euolutione numerum x vt variabilem tractare licebit.

§. 22. Nunc igitur loco x ſcribamus $x+1$, et valor ipſius S abeat tum in S', ita vt ſit

$$S' = X' + X'' + X''' + \text{etc.}$$

eritque $S' - S = -X$. Verum cum S ſit certa functio ipſius x, erit per reductionem notiſſimam:

$$S = S + \frac{dS}{dx} + \frac{ddS}{1.2\,dx^2} + \frac{d^3S}{1.2.3\,dx^3} + \text{etc.},$$

vnde naſcitur iſta aequatio:

$$X + \frac{dS}{dx} + \frac{ddS}{1.2\,dx^2} + \frac{d^3S}{1.2.3\,dx^3} + \frac{d^4S}{1.2.3.4\,dx^4} + \text{etc.} = 0,$$

ex cuius duobus prioribus membris, siquidem sequentia continuo decrescere spectemus, concluditur fore propemodum $dS = -X\,dx$ ideoque $S = -\int X\,dx$, quod integrale ita capi debet, vt si esset $x = \infty$ fieret $S = 0$, quandoquidem sumto $x = \infty$, foret vtique $C = \Pi - l\,q$.

§. 23. Erit ergo $-\int X\,dx$ primus terminus novae seriei, per quam litteram S exprimere nobis est propositum, atque ex forma aequationis facile intelligitur statui debere

$$S = -\int X\,dx + \alpha X + \beta\frac{dX}{dx} + \gamma\frac{ddX}{dx^2} + \delta\frac{d^3X}{dx^3} + \varepsilon\frac{d^4X}{dx^4} + \text{etc.}$$

vnde fit

$$\begin{aligned} \frac{dS}{dx} &= -X + \frac{\alpha\,dX}{dx} + \frac{\beta\,ddX}{dx^2} + \frac{\gamma\,d^3X}{dx^3} + \frac{\delta\,d^4X}{dx^4} + \text{etc.} \\ \frac{ddS}{dx^2} &= -\frac{dX}{dx} + \frac{\alpha\,ddX}{dx^2} + \frac{\beta\,d^3X}{dx^3} + \frac{\gamma\,d^4X}{dx^4} + \frac{\delta\,d^5X}{dx^5} + \text{etc.} \\ \frac{d^3S}{dx^3} &= -\frac{ddX}{dx^2} + \frac{\alpha\,d^3X}{dx^3} + \frac{\beta\,d^4X}{dx^4} + \frac{\gamma\,d^5X}{dx^5} + \text{etc.} \\ &\text{etc.} \end{aligned}$$

quibus valoribus substitutis ac debite ordinatis obtinebitur sequens aequatio:

$$\begin{array}{llllll} 0 = X & + \frac{\alpha\,dX}{dx} & + \frac{\beta\,ddX}{dx^2} & + \frac{\gamma\,d^3X}{dx^3} & + \frac{\delta\,d^4X}{dx^4} & + \text{etc.} \\ \;\;\; -X & -\frac{1}{2} & +\frac{\alpha}{2} & +\frac{\beta}{2} & +\frac{\gamma}{2} & \\ & & -\frac{1}{6} & +\frac{\alpha}{6} & +\frac{\beta}{6} & \\ & & & -\frac{1}{24} & +\frac{\alpha}{24} & \\ & & & & -\frac{1}{120} & \\ & & & & \text{etc.} & \end{array}$$

vnde sequentes determinationes oriuntur:

$$\alpha = \tfrac{1}{2},\ \beta = -\tfrac{1}{2}\alpha + \tfrac{1}{6},\ \gamma = -\tfrac{1}{2}\beta - \tfrac{1}{6}\alpha + \tfrac{1}{24},\ \delta = -\tfrac{1}{2}\gamma - \tfrac{1}{6}\beta - \tfrac{1}{24}\alpha + \tfrac{1}{120},$$

atque hinc

$$\alpha = \tfrac{1}{2},\ \beta = -\tfrac{1}{12},\ \gamma = 0,\ \delta = \tfrac{1}{720},\ \text{etc.}$$

§. 24.

§. 24. Hoc autem modo determinatio litterarum α, β, γ, δ, etc. nimis fit molesta, vnde vt hunc laborem subleuemus, consideremus sequentem seriem, vbi iidem coefficientes occurrant, eique deriuatae subscribantur hoc modo:

$$
\begin{array}{rl}
v = & -1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.} \\
+\frac{1}{2} v z = & \quad -\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma + \frac{1}{2}\delta + \text{etc.} \\
+\frac{1}{6} v z^2 = & \quad\quad -\frac{1}{6} + \frac{1}{6}\alpha + \frac{1}{6}\beta + \frac{1}{6}\gamma + \text{etc.} \\
+\frac{1}{24} v z^3 = & \quad\quad\quad -\frac{1}{24} + \frac{1}{24}\alpha + \frac{1}{24}\beta + \text{etc.} \\
+\frac{1}{120} v z^4 = & \quad\quad\quad\quad -\frac{1}{120} + \frac{1}{120}\alpha + \text{etc.} \\
\text{etc.} & \quad\quad\quad\quad \text{etc.}
\end{array}
$$

His igitur seriebus in vnam summam collectis, ob

$$\alpha - \tfrac{1}{2} = 0,\ \beta + \tfrac{1}{2}\alpha - \tfrac{1}{6} = 0,\ \gamma + \tfrac{1}{2}\beta + \tfrac{1}{6}\alpha - \tfrac{1}{24} = 0, \text{ etc.}$$

nascetur:

$$v\left(1 + \tfrac{1}{2}z + \tfrac{1}{6}z^2 + \tfrac{1}{24}z^3 + \tfrac{1}{120}z^4 + \tfrac{1}{720}z^5 + \text{etc.}\right) = -1.$$

Cum igitur sit

$$e^z = 1 + z + \tfrac{1}{2}zz + \tfrac{1}{6}z^3 + \tfrac{1}{24}z^4,$$

euidens est quantitatem v multiplicari per $\frac{e^z - 1}{z}$, ita vt sit $\frac{v(e^z - 1)}{z} = -1$, vnde fit $v = \frac{-z}{e^z - 1}$; sicque tantum opus est vt hinc valor ipsius v in seriem resoluatur, quippe quae conuenire debet cum serie assumta, sicque sponte valores litterarum α, β, γ, etc. se manifestabunt.

§. 25. Quo igitur hinc valorem ipsius v commode in seriem conuertamus, quia illa aequatio nobis praebet $e^z =$

$e^z = \frac{v - rz}{v}$; ſtatuamus $v = u + \frac{1}{2}z$, vt prodeat

$$e^z = \frac{u - \frac{1}{2}z}{u + \frac{1}{2}z} = \frac{2u - z}{2u + z};$$

hinc erit $z = l(2u - z) - l(2u + z)$ et differentiando $dz = \frac{4(zdu - udz)}{4uu - zz}$, quam ergo formulam viciſſim ita integrari oportet, vt poſito $z = 0$ fiat $u = 1$. Statuatur nunc $u = sz$, ac ſumto $z = 0$ fiet $s = -\infty$; tum autem erit $dz = \frac{4ds}{4ss - 1}$, ex qua iam aequatione eiusmodi ſeriem pro s quaeri oportet, vt ſumto $z = 0$ fiat $s = \infty$.

§. 27. Cum igitur hinc habeamus $4ss - 1 = \frac{4ds}{dz}$, pro s fingamus hanc ſeriem:

$$2s = \frac{A}{z} + Bz + Cz^3 + Dz^5 + Ez^7 + \text{etc.}$$

vnde fit

$$\frac{2ds}{dz} = -\frac{A}{zz} + B + 3Czz + 5Dz^4 + 7Ez^6 + \text{etc.},$$

tum vero

$$\begin{aligned} 4ss = \frac{AA}{zz} + 2AB &+ 2ACzz + 2ADz^4 + 2AEz^6 + \text{etc.} \\ +BB &\qquad\quad + 2BC \quad\; + 2BD \quad + \text{etc.} \\ & \qquad\qquad\qquad\quad\;\, + CC \end{aligned}$$

quibus ſeriebus ſubſtitutis aequatio $4ss - 1 - \frac{4ds}{dz} = 0$ ſuppeditat hanc expreſſionem:

$$\left.\begin{array}{l} \frac{2A}{zz} - 2B - 6Czz - 10Dz^4 - 14Ez^6 - 18Fz^8 - \text{etc.} \\ + AA + 2AB + 2AC + 2AD + 2AE + 2AF + \text{etc.} \\ \quad - 1 \;\; + BB \;\; + 2BC + 2BD + 2BE + \text{etc.} \\ \qquad\qquad\qquad\qquad + CC \;\; + 2CD + \text{etc.} \end{array}\right\} = 0.$$

 Hinc

Hinc igitur ex primis terminis fit $A = -2$; ex sequentibus porro deducitur:

$$2B = 2AB - 1$$
$$6C = +2AC + BB$$
$$10D = 2AD + 2BC$$
$$14E = 2AE + 2BD + CC$$
$$18F = 2AF + 2BE + 2CD$$
$$22G = 2AG + 2BF + 2CE + DD$$

etc.

§. 28. Cum igitur sit $A = -2$, sequentes determinationes obtinebuntur:

$$B = -\tfrac{1}{6}, \qquad E = \tfrac{2BD + CC}{18},$$
$$C = \tfrac{BB}{10}, \qquad F = \tfrac{2BE + 2CD}{22},$$
$$D = \tfrac{2BC}{14}, \qquad G = \tfrac{2BF + 2CE + DD}{26}, \text{ etc.}$$

Quo igitur has aequationes simpliciores reddamus, statuamus $B = 2\mathfrak{A}$, $C = 2\mathfrak{B}$, $D = 2\mathfrak{C}$, $E = 2\mathfrak{D}$, etc. tum enim determinationes inuentae dabunt:

$$\mathfrak{A} = -\tfrac{1}{12}, \qquad \mathfrak{E} = \tfrac{2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}}{11},$$
$$\mathfrak{B} = \tfrac{\mathfrak{A}\mathfrak{A}}{5}, \qquad \mathfrak{F} = \tfrac{2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}}{13},$$
$$\mathfrak{C} = \tfrac{2\mathfrak{A}\mathfrak{B}}{7}, \qquad \mathfrak{G} = \tfrac{2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{E} + 2\mathfrak{C}\mathfrak{D}}{15},$$
$$\mathfrak{D} = \tfrac{2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}}{9}, \qquad \mathfrak{H} = \tfrac{2\mathfrak{A}\mathfrak{G} + 2\mathfrak{B}\mathfrak{F} + 2\mathfrak{C}\mathfrak{E} + \mathfrak{D}\mathfrak{D}}{17},$$

quos valores multo facilius definire licebit quam superiores α, β, γ, δ.

§. 29. Inuentis autem valoribus harum litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. primo erit:

$$s = -\tfrac{1}{z} + \mathfrak{A}z + \mathfrak{B}z^3 + \mathfrak{C}z^5 + \mathfrak{D}z^7 + \mathfrak{E}z^9 + \text{etc.}$$

tum vero hinc erit $u = sz$ atque $v = u + \frac{1}{2}z$, vnde pro v hanc adepti fumus feriem:

$$v = -1 + \tfrac{1}{2}z + \mathfrak{A}z^2 + \mathfrak{B}z^4 + \mathfrak{C}z^6 + \mathfrak{D}z^8 + \text{etc.}$$

vbi perfpicitur omnes poteftates impares ipfius z praeter primam hic deficere. Cum igitur pofuerimus

$$v = -1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \zeta z^6 + \text{etc.}$$

comparatione inftituta patet fore

$$\alpha = \tfrac{1}{2},\ \beta = \mathfrak{A},\ \gamma = 0,\ \delta = \mathfrak{B},\ \varepsilon = 0,\ \zeta = \mathfrak{C},\ \eta = 0, \text{etc.}$$

His igitur valoribus introductis fumma generalis S fupra inueftigata fiet:

$$S = -\int X\,dx + \tfrac{1}{2}X + \mathfrak{A}\frac{dX}{dx} + \mathfrak{B}\frac{d^3X}{dx^3} + \mathfrak{C}\frac{d^5X}{dx^5} + \mathfrak{D}\frac{d^7X}{dx^7} + \text{etc.}$$

§. 30. Quoniam hic valor ipfius $\mathfrak{A}$ eft negatiuus $= -\frac{1}{12}$, fequens vero littera $\mathfrak{B}$ fit pofitiua, fequens $\mathfrak{C}$ iterum negatiua et ita deinceps alternatim, huius conditionis vt ftatim rationem habeamus, fimulque hos numeros ad eos, qui *Bernoulliani* vocari folent, perducamus, ponamus

$$2s = -\tfrac{2}{z} - Az + Bz^3 - Cz^5 + Dz^7 - Ez^9 + \text{etc.}$$

et facta euolutione reperitur

$$A = \tfrac{1}{6},\quad B = \tfrac{AA}{10},\quad C = \tfrac{2AB}{14},\quad D = \tfrac{2AC + BB}{18},$$

$$E = \tfrac{2AD + 2BC}{22},\quad F = \tfrac{2AE + 2BD + CC}{26},\quad G = \tfrac{2AF + 2BE + 2CD}{30},\ \text{etc.}$$

tum vero erit

$$v = -1 + \tfrac{1}{2}z - \tfrac{1}{2}Azz + \tfrac{1}{2}Bz^4 - \tfrac{1}{2}Cz^6 + \tfrac{1}{2}Dz^8 - \text{etc.}$$

 quae

quae feries fi comparetur cum affumta, fiet

$$\alpha = \tfrac{1}{2},\ \beta = -\tfrac{1}{2}A,\ \gamma = 0,\ \delta = \tfrac{1}{2}B,$$
$$\varepsilon = 0,\ \zeta = -\tfrac{1}{2}C,\ \eta = 0, \text{ etc.}$$

His igitur litteris A, B, C, D, etc. introductis fumma generalis erit:

$$S = -\int X\,dx + \tfrac{1}{2}X - \tfrac{1}{2}A\frac{dX}{dx} + \tfrac{1}{2}B\frac{d^3X}{dx^3} - \tfrac{1}{2}C\frac{d^5x}{dx^5} + \text{etc.}$$

Si porro faciamus $A = 2\mathfrak{A}$, $B = 2\mathfrak{B}$, $C = 2\mathfrak{C}$, etc. relationes inter has litteras ita fe habebunt:

$$\mathfrak{A} = \tfrac{1}{12}, \qquad \mathfrak{E} = \frac{2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}}{11},$$
$$\mathfrak{B} = \frac{\mathfrak{A}\mathfrak{A}}{5}, \qquad \mathfrak{F} = \frac{2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}}{13},$$
$$\mathfrak{C} = \frac{2\mathfrak{A}\mathfrak{B}}{7}, \qquad \mathfrak{G} = \frac{2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{E} + 2\mathfrak{C}\mathfrak{D}}{15},$$
$$\mathfrak{D} = \frac{2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}}{9}, \qquad \text{etc.}$$

tum vero erit

$$S = -\int X\,dx + \tfrac{1}{2}X - \mathfrak{A}\frac{dX}{dx} + \mathfrak{B}\frac{d^3X}{dx^3} - \mathfrak{C}\frac{d^5X}{dx^5} + \text{etc.}$$

Ex his formulis iam intelligitur, iftos numeros affines effe illis, qui in poteftates reciprocas pares ingrediuntur; erit enim

$$\mathfrak{A} = \tfrac{1}{2}\cdot\tfrac{1}{6},\ \mathfrak{B} = \tfrac{1}{2^3}\cdot\tfrac{1}{90},\ \mathfrak{C} = \tfrac{1}{2^5}\cdot\tfrac{1}{945},\ \mathfrak{D} = \tfrac{1}{2^7}\cdot\tfrac{1}{9450},\ \mathfrak{E} = \tfrac{1}{2^9}\cdot\tfrac{1}{93555};$$

quos numeros vltra trigefimam poteftatem olim iam euolutos dedi.

§. 31. Hinc igitur pro noftro inftituto fequens theorema vniuerfale proponamus.

Theorema.

Si propofita fuerit feries in infinitum excurrens:

$$S = X + X' + X'' + X''' + \text{etc.},$$

tum

tum eius ſumma ſequenti modo exprimetur:

$$S = -\int X\,dx + \tfrac{1}{2}X - \tfrac{1}{2.6}\cdot\tfrac{dX}{dx} + \tfrac{1}{2^3.90}\cdot\tfrac{d^3X}{dx^3} - \tfrac{1}{2^5.945}\cdot\tfrac{d^5X}{dx^5}$$
$$+ \tfrac{1}{2^7.9450}\cdot\tfrac{d^7X}{dx^7} - \tfrac{1}{2^9.93555}\cdot\tfrac{d^9X}{dx^9} + \text{etc.}$$

vbi ſequentes coefficientes depromere licet ex Introductione mea in Analyſin Infinitorum pag. 131. Hic autem notetur, integrale $\int X\,dx$ ita capi debere, vt euaneſcat poſito $x = \infty$.

Applicatio
ad noſtrum caſum quo $X = \frac{1}{x+a} - l\frac{x+1}{x}$.

§. 32. Cum igitur ſit $X = \frac{1}{x+a} + l\,x - l\,(x+1)$, erit primo pro formula integrali

$$\int X\,dx = l(x+a) + x\,l\,x - (x+1)\,l\,(x+1) + C,$$

quae conſtans C quoniam ita accipi debet, vt integrale euaneſcat facto $x = \infty$, iſtud integrale in hanc formam transfundatur:

$$\int X\,dx = -x\,l\,\frac{x+a}{x} + l\,\frac{x+a}{x+1} + C,$$

quae expreſſio facto $x = \infty$ fit

$$\int X\,dx = -\infty\,l\,\frac{\infty+1}{\infty} + l\,1 + C = 0,$$

vbi quia

$$l\,\frac{\infty+1}{\infty} = l\,1 + \frac{1}{\infty} = \frac{1}{\infty} - \frac{1}{2\infty^2} + \frac{1}{3\infty^3},$$

erit $\infty\,l\,\frac{\infty+1}{\infty} = 1$, ideoque iſtud integrale $C - 1 = 0$, ergo conſtans $C = 1$, quamobrem primum membrum noſtrae expreſſionis eſt

$$\int X\,dx = -x\,l\,\frac{x+1}{x} + l\,\frac{x+a}{x+1} + 1,$$

vnde pro binis prioribus membris habebimus

$$-\int X\,dx + \tfrac{1}{2}X = (x - \tfrac{1}{2})\,l\,\frac{x+1}{x} - l\,\frac{x+a}{x+1} + \frac{1}{2(x+a)} - 1.$$

§. 33. Reliqua membra noſtrae expreſſionis nulla laborant difficultate, ſiquidem per differentiationem continuam reperitur:

$$\frac{dX}{dx} = -\frac{1}{(x+a)^2} + \frac{1}{x} - \frac{1}{x+1};$$

$$\frac{d^3X}{dx^3} = 1.\,2\left(-\frac{3}{(x+a)^4} + \frac{1}{x^3} - \frac{1}{(x+1)^3}\right);$$

$$\frac{d^5X}{dx^5} = 1.\,2.\,3.\,4\left(-\frac{5}{(x+a)^6} + \frac{1}{x^5} - \frac{1}{(x+1)^5}\right);$$

$$\frac{d^7X}{dx^7} = 1\ldots\ldots 6\left(-\frac{7}{(x+a)^8} + \frac{1}{x^7} - \frac{1}{(x+1)^7}\right);$$

$$\frac{d^9X}{dx^9} = 1\ldots\ldots 8\left(-\frac{9}{(x+a)^{10}} + \frac{1}{x^9} - \frac{1}{(x+1)^9}\right);$$

etc. etc.

§. 34. Ex his igitur ſumma noſtrae ſeriei S colligitur fore

$$\begin{aligned}
S = {} & \left(x - \tfrac{1}{2}\right) l\,\frac{x+1}{x} - l\,\frac{x+a}{x+1} + \frac{1}{2(x+a)} \\
& - \frac{1}{1.2.3}\cdot\frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+a)^2}\right) \\
& + \frac{1}{3.4.5}\cdot\frac{1}{6}\left(\frac{1}{x^3} - \frac{1}{(x+1)^3} - \frac{3}{(x+a)^4}\right) \\
& - \frac{1}{5.6.7}\cdot\frac{1}{6}\left(\frac{1}{x^5} - \frac{1}{(x+1)^5} - \frac{5}{(x+a)^6}\right) \\
& + \frac{1}{7.8.9}\cdot\frac{3}{10}\left(\frac{1}{x^7} - \frac{1}{(x+1)^7} - \frac{7}{(x+a)^8}\right) \\
& - \frac{1}{9.10.11}\cdot\frac{5}{6}\left(\frac{1}{x^9} - \frac{1}{(x+1)^9} - \frac{9}{(x+a)^{10}}\right) \\
& + \frac{1}{11.12.13}\cdot\frac{691}{210}\left(\frac{1}{x^{11}} - \frac{1}{(x+1)^{11}} - \frac{11}{(x+a)^{12}}\right) \\
& - \frac{1}{13.14.15}\cdot\frac{35}{2}\left(\frac{1}{x^{13}} - \frac{1}{(x+1)^{13}} - \frac{13}{(x+a)^{14}}\right). \\
& + \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

vbi fractiones ſecundo loco poſitae $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{3}{10}$, $\frac{5}{6}$, $\frac{691}{210}$, etc. ſunt numeri Bernoulliani vocati, qui vlterius ita progrediuntur: $\frac{35}{2}$, $\frac{3617}{30}$, $\frac{43867}{42}$, $\frac{1222277}{110}$, $\frac{854513}{6}$, $\frac{1181820455}{546}$, $\frac{76977927}{2}$, $\frac{23749461029}{30}$, $\frac{8615841276005}{462}$, $\frac{84802531453387}{170}$, $\frac{90219075042845}{6}$.

§. 35.

§. 35. Hactenus numerum x tanquam variabilem spectauimus, nunc autem facta euolutione ambos numeros x et a pro lubitu assumere licet, indeque semper idem valor pro numero nostro C resultabit, quippe qui erit

$$C = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{x + a - 1} - lx + S,$$

atque series S eo promptius conuerget, quo maiores numeri loco x et a accipiantur. Inter binos autem numeros x et a eiusmodi relationem assumi conueniet, vt formula $\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+a)^2}$ proxime euanescat. Veluti si fuerit $x = 10$, haec formula: $\frac{1}{110} - \frac{1}{(10+a)^2}$, fit minima, si fuerit vel $a = 0$ vel $a = 1$; omnium autem minima fiet sumto $a = \frac{1}{2}$, tum enim iste valor euadet $\frac{1}{110} - \frac{4}{441}$; vnde patet, perpetuo expedire vt sumatur $a = \frac{1}{2}$. Quamobrem si fuerit x numerus integer, primum membrum erit

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \ldots + \frac{1}{x - \frac{1}{2}},$$

cuius seriei valor quemadmodum inueniri debeat supra est ostensum: erit scilicet

$$2 - 2l2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \ldots \ldots + \frac{2}{2x - 1}.$$

Hoc igitur modo nostra expressio maxime conuerget, et dummodo numerus x modice magnus accipiatur, pauci termini sufficient ad valorem numeri C satis exacte eruendum.

§. 36. Postquam igitur valorem nostri numeri C per series infinitas expressimus, videamus, annon etiam per formulas finitas integrales exhiberi queat, vnde tutius concludi poterit ad quodnam genus quantitatum iste numerus sit referendus. Ac primo quidem series indefini-

ta

ta $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n}$, ex euolutione huius formulae integralis nafcitur: $\int \frac{(1 - x^n)\,dx}{1 - x}$, fiquidem poft integrationem ftatuatur $x = 1$, vbi iam numerus n tam fractus quam integer effe poteft, ficque habebimus

$$C = \int \frac{(1 - x^n)\,dx}{1 - x} - l\,n,$$

fiquidem loco n numerus infinitus accipiatur. Inueftigemus ergo etiam formulam integralem, quae pofito $x = 1$ indefinite nobis praebeat $l\,n$. Cum autem pofito $x = 1$ fiat $\frac{1 - x^n}{1 - x} = n$, erit $l\frac{1 - x^n}{1 - x} = l\,n$; vnde patet, fi n denotet numerum infinite magnum, tum fore

$$C = \int \frac{(1 - x^n)\,dx}{1 - x} - l\,\frac{1 - x^n}{1 - x},$$

vbi fcilicet ftatuitur $x = 1$.

§. 37. Eft vero porro

$$l(1 - x^n) = - n \int \frac{x^{n-1}\,dx}{1 - x^n} \text{ et } l(1 - x) = - \int \frac{dx}{1 - x},$$

quibus valoribus fubftitutis per meras formulas integrales habebimus:

$$C = \int \frac{(1 - x^n)\,dx}{1 - x} + n \int \frac{x^{n-1}\,dx}{1 - x^n} - \int \frac{dx}{1 - x},$$

quae expreffio reducitur ad hanc formam:

$$C = - \int \frac{x^n\,dx}{1 - x} + n \int \frac{x^{x-1}\,dx}{1 - x^n},$$

ita vt iam C aequetur differentiae harum duarum formularum

larum integralium, siquidem post integrationem statuatur $x = 1$, exponens vero n capiatur infinite magnus; vnde patet, hanc formulam eo propius verum valorem formulae C esse exhibituram, quo maior numerus n assumatur.

§. 38. Verum etiam has formulas ab exponentibus infinite magnis liberare licebit, statuendo $x^n = z$, et quoniam casu $x = 1$ fit etiam $z = 1$, in formulis integralibus hinc oriundis capi debebit $z = 1$. Quare cum hinc fit $n x^{n-1} d x = d z$, tum vero $x = z^{\frac{1}{n}}$, hincque

$$d x = \tfrac{1}{n} z^{\frac{1}{n} - 1} d z = \frac{z^{\frac{1}{n}} d z}{n z},$$

nostrae formulae abibunt in sequentes:

$$C = -\tfrac{1}{n} \int \frac{z^{\frac{1}{n}} d z}{1 - z^{\frac{1}{n}}} + \int \frac{d z}{1 - z}.$$

Notum autem est esse $l z = n (z^{\frac{1}{n}} - 1)$, existente $n = \infty$, quo valore in priore parte substituto fit

$$C = \int \frac{d z}{l z} + \int \frac{d z}{1 - z},$$

si modo post integrationem ponatur $z = 1$, ita vt haec formula iam penitus ab infinito sit liberata, quae per unicum signum summatorium ita exhiberi potest:

$$C = \int d z \left(\tfrac{1}{1 - z} + \tfrac{1}{l z} \right);$$

 sicque

ſicque tota quaeſtio circa naturam numeri C eo reducitur, vt inueſtigetur valor iſtius formulae integralis:

$$\int dz\left(\frac{1}{1-z}+\frac{1}{lz}\right),$$

a termino $z=0$ vsque ad $z=1$ extenſus, quae forma vtique omnem attentionem meretur. Euidens autem eſt, priorem partem huius formulae

$$\int\frac{dz}{1-z}=-l(1-z),$$

hoc caſu fieri $+\infty$. Deinde etiam oſtendi, alteram partem $\int\frac{dz}{lz}$ infinitum negatiuum praebere; ex quo intelligitur, ambas formulas coniunctas valorem finitum determinatum producere poſſe.

§. 39. Ceterum quoniam ſupra nonnullas eximias propietates ad numerum C ſpectantes deteximus, operae pretium erit, eas hic coniunctim aſpectui expoſuiſſe. Notetur ergo litteras α, β, γ, δ, etc. deſignare ſummas ſequentium ſerierum:

$$\alpha=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}$$
$$\beta=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}$$
$$\gamma=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}$$
$$\delta=1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\text{etc.}$$
etc. etc.

quibus valoribus conſtitutis ſequentia theoremata ſunt inventa:

I. $1-C=\frac{1}{2}(\alpha-1)+\frac{1}{3}(\beta-1)+\frac{1}{4}(\gamma-1)+\frac{1}{5}(\delta-1)+\text{etc.}$

II. $C=\frac{1}{2}\alpha-\frac{1}{3}\beta+\frac{1}{4}\gamma-\frac{1}{5}\delta+\frac{1}{6}\varepsilon-\frac{1}{7}\zeta+\text{etc.}$

III. $1=\left(\alpha-\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{2}\gamma-\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{3}\varepsilon-\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{4}\eta-\frac{1}{8}-\frac{1}{9}\right)+\text{etc.}$

IV.

$$\text{IV. } 2C - 1 = \left(\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{2}{3}\beta\right) + \left(\tfrac{1}{4} + \tfrac{1}{5} - \tfrac{2}{5}\delta\right) + \left(\tfrac{1}{6} + \tfrac{1}{7} - \tfrac{2}{7}\zeta\right) + \left(\tfrac{1}{8} + \tfrac{1}{9} - \tfrac{2}{9}\theta\right) + \text{etc.}$$

$$\text{V. } 2 - 2\,l2 - C = \left(\tfrac{2}{3}\cdot\tfrac{7}{8}\beta - \tfrac{2}{3}\right) + \left(\tfrac{2}{5}\cdot\tfrac{31}{32}\delta - \tfrac{2}{5}\right) + \left(\tfrac{2}{7}\cdot\tfrac{127}{128}\zeta - \tfrac{2}{7}\right) + \left(\tfrac{2}{9}\cdot\tfrac{511}{512}\theta - \tfrac{2}{9}\right) + \text{etc.}$$

$$\text{VI. } 1 - 2\,l2 + C = \left(\tfrac{1}{2} - \tfrac{1}{3} - \tfrac{2}{3\cdot 2^3}\beta\right) + \left(\tfrac{1}{4} - \tfrac{1}{5} - \tfrac{2}{5\cdot 2^5}\delta\right) + \left(\tfrac{1}{6} - \tfrac{1}{7} - \tfrac{2}{7\cdot 2^7}\zeta\right) + \text{etc.}$$

$$\text{VII. } l\,2 - C = \tfrac{1}{3\cdot 2^2}\beta + \tfrac{1}{5\cdot 2^4}\delta + \tfrac{1}{7\cdot 2^6}\zeta + \tfrac{1}{9\cdot 2^8}\theta + \tfrac{1}{11\cdot 2^{10}}\varkappa + \text{etc.}$$

$$\text{VIII. } 1 - l\tfrac{3}{2} - C = \tfrac{1}{3\cdot 2^2}(\beta - 1) + \tfrac{1}{5\cdot 2^4}(\delta - 1) + \tfrac{1}{7\cdot 2^6}(\zeta - 1) + \tfrac{1}{9\cdot 2^8}(\theta - 1) + \text{etc.}$$

Hic ſcilicet vbique eſt $C = 0,5772156649015325$ ſiue

$$C = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots \ldots + \tfrac{1}{n} - l\,n$$

ſumto pro n numero infinito.