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Uberior evolutio comparationis, quam inter arcus sectionum conicarum instituere licet

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VBERIOR EVOLVTIO COMPARATIONIS QVAM INTER ARCVS SECTIONVM CONICARVM INSTITVERE LICET.

Auctore
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§. 1.

Nouum fere etiamnunc eſt argumentum et minime adhuc ſatis exploratum, quod in omni ſectione conica, ſumto pro lubitu arcu quocunque, ab alio quouis puncto eiusdem curuae ſemper arcum reſcindere liceat, qui ab illo arcu differat quantitate geometrice aſſignabili. Ita ſi in ſectione conica AB pro lubitu accipiatur arcus EF, tum ab alio quocunque puncto M ſemper reſcindi poteſt arcus MN, ita vt differentia inter arcus EF et MN, algebraice aſſignari queat: hocque adeo duplici modo praeſtare licet, prouti a puncto M arcum deſideratum vel antror- Tab. I. Fig. 1.

antrorsum, vti M N, vel retrorsum, vti M n abscindere velimus. Quod si sectio conica fuerit circulus, res ex primis elementis adeo est manifesta, vbi quidem differentia inter binos illos arcus necessario est nulla. Pro parabola autem idem iam dudum a *Bernoulliis* est ostensum; quandoquidem quilibet arcus parabolicus per aggregatum ex quantitate algebraica et logarithmica exprimitur. Quod vero ad Ellipsin et Hyperbolam attinet, quarum rectificationem neque per arcus circulares neque per logarithmos expedire licet, talis comparatio vires Analyseos penitus superare videbatur, donec ab Illustrissimo Comite *Fagnani* prima principia fuere patefacta, quae ad hunc scopum deducerent, et quae deinceps accuratius sum prosecutus, ita vt ista inuestigatio multo latius sit extensa, multoque facilius ad innumeras alias speculationes accommodari queat. Interim tamen operationes, quibus hoc negotium absoluitur, tantopere ab operationibus analyticis solitis recedunt, vt ad singulare calculi genus referendae videantur, cum nequidem veritas istiusmodi comparationum more solito per calculum ostendi possit.

§. 2. Foecundissimum autem hoc argumentum in pluribus Dissertationibus Commentariis Academiae Petropolitanae fusius sum persecutus, atque adeo plures methodos detexi, quae ad eundem finem perducere valeant, quae autem nihilominus ita sunt comparatae, vt tota ista inuestigatio adhuc penitus noua et a vulgari calculo analytico plurimum recedens habenda videatur. Huic eidem argumento etiam sectionem peculiarem in Institutionibus meis Calculi Integralis tribuendam censui, vbi duobus Capitibus hoc argumentum prorsus nouum a pag. 421. vs-

que

que ad pag. 493. ſum complexus, vnde praecipua momenta ad rectificationem ſectionum conicarum ſpectantia depromam, quae in ſequente Theoremate generali ſum comprehenſurus.

Theorema generale.

§. 3. Si character $\Pi : z$ denotet valorem formulae integralis $\int \frac{dz(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}}$, ita ſumtum vt euaneſcat poſito $z = 0$, ſemper ternas huiusmodi formulas: $\Pi : p$, $\Pi : q$, $\Pi : r$ ita inter ſe comparare licet, vt ſit

$$\Pi : p + \Pi : q + \Pi : r$$
$$= \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}(A(pp + qq + rr) - \tfrac{1}{3}Eppqqrr),$$

ſi modo inter quantitates p, q et r iſta relatio ſtabiliatur, vt ſit

$$r = \frac{-p\sqrt{(A(A + Cqq + Eq^4))} - q\sqrt{(A(A + Cpp + Ep^4))}}{A - Eppqq},$$

vnde ſimili modo patet fore

$$p = \frac{-q\sqrt{(A(A + Crr + Er^4))} - r\sqrt{(A(A + Cqq + Eq^4))}}{A - Eqqrr},$$
$$q = \frac{-p\sqrt{(A(A + Crr + Er^4))} - r\sqrt{(A(A + Cpp + Ep^4))}}{A - Epprr}.$$

Dilucidationes.

§. 4. Cum ſit $\Pi : z = \int \frac{dz(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}}$, integrali ita ſumto, vt euaneſcat poſito $z = 0$, patet fore $\Pi : 0 = 0$; tum vero quoniam ſumto z negatiuo valor formulae integralis etiam fit negatiuus, patet fore $\Pi : (-z) = -\Pi : z$, vnde, ſi quantitatum p, q, r vna, veluti p, fuerit negatiua, tum in relatione aſſignata loco $\Pi : p$ ſcribi debet $-\Pi : p$. Ceterum manifeſtum eſt, hanc formulam integralem maxime

xime fore transcendentem, cum neque per logarithmos, neque per quadraturam circuli expediri possit, ita vt ista quantitas $\Pi : z$ per nullas formulas in Analysi receptas exhiberi queat. Paucissimi quidem casus hinc sunt excipiendi, quibus est vel $E = 0$ (hoc enim casu formula per logarithmos vel arcus circulares assignari posset, quod idem eueniret si esset $A = 0$); vel quando quantitas $A + Czz + z^4$ fuerit quadratum, quo casu iterum integratio vt ante succederet; vel denique, si litterae L, M et N ita fuerint comparatae, vt formula proposita algebraïcum accipiat integrale, cuius forma erit $\alpha z \sqrt{(A + Czz + Ez^4)}$. Quia enim eius differentiale est $\frac{\alpha dz (A + 2Czz + 3Ez^4)}{\sqrt{(A + Czz + Ez^4)}}$, si fuerit $L = \alpha A$, $M = 2\alpha C$ et $N = 3\alpha E$, formula $\Pi : z$ vtique huic quantitati algebraicae: $\alpha z \sqrt{(A + Czz + Ez^4)}$ aequabitur.

§. 5. Quemadmodum hoc argumentum in variis dissertationibus tractaui, in formula integrali numeratorem $L + Mzz + Nz^4$ vlterius per potestates pares quousque libuerit continuare licuisset, eius loco ponendo

$$L + Mzz + Nz^4 + Oz^6 + Pz^8 + Qz^{10} + \text{etc.}$$

verum quia quaelibet potestas ad binas praecedentes facile reduci potest, tali extensione carere poterimus: semper enim statui potest

$$\int \frac{z^{n+4}dz}{\sqrt{(A + Czz + Ez^4)}} = \alpha z^{n+1} \sqrt{((A + Czz + Ez^4))}$$
$$+ \int \frac{dz(\mathfrak{A}z^n + \mathfrak{B}z^{n+2})}{\sqrt{(A + Czz + Ez^4)}}.$$

Erit enim

$\alpha =$

$\alpha = \frac{1}{(n+3)E}$; $\mathfrak{A} = \frac{-(n+1)A}{(n+3)E}$ et $\mathfrak{B} = \frac{-(n+2)C}{(n+3)E}$.
Hinc igitur ſumto $n = 0$ fiet

$$\int \frac{z^4\,dz}{\sqrt{(A + Czz + Ez^4)}} = \frac{1}{E} z \sqrt{((A + Czz + Ez^4))} - \frac{1}{3E} \int \frac{(A + 2Czz)\,dz}{\sqrt{(A + Czz + Ez^4)}},$$

quamobrem hic etiam in noſtra formula integrali terminum Nz^4 omittere potuiſſemus.

§. 6. Cum igitur non obſtante tranſcendentia formulae $\Pi : z$ ternas huiusmodi formulas $\Pi : p$, $\Pi : q$ et $\Pi : r$ ſemper ita inter ſe comparare liceat, vt earum ſumma $\Pi : p + \Pi : q + \Pi : r$ aequetur quantitati algebraicae

$$\frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}} (A(p^2 + q^2 + r^2) - \tfrac{1}{3} E p^2 q^2 r^2),$$

ſi modo inter tres quantitates p, q, r, ea relatio accipiatur, quae in theoremate eſt praeſcripta: haec relatio eo magis eſt notatu digna, quod ternae litterae p, q, r in illam formam aequaliter ingrediantur, ita vt prorſus inter ſe pro lubitu permutari queant. Cum igitur nullae adhuc huiusmodi relationes in Analyſi ſint conſideratae, haec inveſtigatio vtique maxime ardua eſt cenſenda, ac nullum eſt dubium, quin plurima inſuper myſteria analytica altioris indaginis in ſe inuoluat, quae eo magis abſcondita videntur, quod a conſuetis Analyſeos operationibus maxime recedunt.

§. 7. Ternarum autem quantitatum illarum p, q, r binas pro lubitu aſſumere licet, dummodo tertiae is valor tribuatur, qui in theoremate aſſignatus eſt, quae relatio quo concinnius exprimi queat, ſtatuamus breuitatis gratia

$$\sqrt{A(A+Cpp+Ep^4)}=P$$
$$\sqrt{A(A+Cqq+Eq^4)}=Q \text{ et}$$
$$\sqrt{A(A+Crr+Er^4)}=R,$$

tum enim, si binae p et q fuerint datae, erit $r=\frac{-pQ-qP}{A-Eppqq}$; sin autem litterae p et r fuerint datae, erit $q=\frac{-pR-rP}{A-Epprr}$; sin autem binae q et r fuerint datae, erit $p=\frac{-qR-rQ}{A-Eqqrr}$.

§. 8. Pro quouis autem horum casuum etiam plurimum intererit valores litterarum maiuscularum P, Q et R per binas reliquas expressisse. Ponamus igitur binas litteras p et q, ideoque etiam P et Q, esse datas, ita vt sit $r=\frac{-pQ-qP}{A-Eppqq}$; vnde si immediate valorem ipsius R quaerere vellemus, in maximas tricas calculi illaberemur, ad quas euitandas ex tertia relatione, quaeramus valorem ipsius R, qui erit $R=\frac{-rQ-p(A-Eqqrr)}{q}$, vbi si loco r et rr valores substituantur et loco quadratorum P^2 et Q^2 sui valores scribantur, tandem reperietur

$$R=\frac{(ACpq+PQ)(A+Eppqq)+2AAEpq(pp+qq)}{(A-Eppqq)^2}.$$

Simili modo ex datis p et r cum P et R erit

$$Q=\frac{(ACpr+PR)(A+Epprr)+2AAEpr(pp+rr)}{(A-Epprr)^2},$$

ac denique ex datis q et r cum Q et R fiet

$$P=\frac{(ACqr+QR)(A+Eqqrr)+2AAEqr(qq+rr)}{(A-Eqqrr)^2}.$$

§. 9. Cum igitur isti valores tantopere sint complicati, atque adeo duplicem irrationalitatem inuoluant, maxime mirum videbitur, quomodo eos in formulis differentialibus substituere, multo magis autem, quomodo inde ad formulas tractabiles atque adeo integrabiles perueniri queat. Interim tamen hae tantae difficultates haud mediocridiocriter ſubleuabuntur, ſi differentiale quantitatis r ex formula $r = \frac{-pQ - qP}{A - Eppqq}$ euoluamus.

§. 10. Qui labor quo facilius ſuccedat, primo tantum quantitatem p pro variabili habeamus, quandoquidem differentiale ex variabilitate ipſius q oriundum ſponte definitur. Sint igitur ſolae litterae p et P variabiles eritque

$$dr = \frac{-Q\,dp - q\,dP}{A - Eppqq} - \frac{2Epqq\,dp\,(pQ + qP)}{(A - Eppqq)^2},$$

quia igitur:

$$dP = \frac{-ACp\,dp + 2AEp^3\,dp}{\sqrt{A}\,(A + Cpp + Ep^4)},$$

calculo ſubducto reperietur tandem

$$dr = \frac{-dp\,(ACpq + PQ)(A + Eppqq) - 2AAEpq\,dp\,(pp + qq)}{(A - Eppqq)^2\,P},$$

ſimilique modo ob variabilitatem ipſius q colligetur:

$$dr = \frac{-dq\,(ACpq + PQ)(A + Eppqq) - 2AAEpq\,dq\,(pp + qq)}{(A - Eppqq)^2\,Q},$$

quae duae expreſſiones iunctim ſumtae differentiale completum quantitatis r praebebunt.

§. 11. Hic autem imprimis notari meretur, quod in vtraque formula differentiali pro dp et dq numerator prorſus idem prodierit, atque adeo ille penitus cum valore pro R ſupra inuento congruat (vide §. 8). Hoc igitur valore ſubſtituto completum differentiale quantitatis r erit

$$dr = -\frac{R\,dp}{P} - \frac{R\,dq}{Q}, \text{ ita vt fit}$$

$$\frac{dr}{R} = -\frac{dp}{P} - \frac{dq}{Q}.$$

Hinc igitur loco P, Q, R ſuos valores ſubſtituendo et per $\sqrt{A}$ multiplicando, erit

$$\frac{dp}{\sqrt{(A + Cpp + Ep^4)}} + \frac{dq}{\sqrt{(A + Cqq + Eq^4)}} + \frac{dr}{\sqrt{(A + Crr + Er^4)}} = 0,$$

 vnde

vnde sequitur fore:

$$\int \frac{dp}{\sqrt{(A+Cpp+Ep^4)}} + \int \frac{dq}{\sqrt{(A+Cqq+Eq^4)}} + \int \frac{dr}{\sqrt{(A+Crr+Er^4)}} = 0,$$

siquidem singula integralia ita capiantur, vt euanescant posito $p = 0$, $q = 0$ et $r = 0$.

§. 12. Hac insigni proprietate inuenta, inquiramus porro, quemadmodum inde principalis relatio inter formulas $\Pi : p$, $\Pi : q$ et $\Pi : r$ ostendi queat; quod quo facilius fieri possit in numeratoribus formularum nostrarum integralium sumamus $N = 0$, atque ostendi oportebit, istam aequationem integralem semper locum habere:

$$\int \frac{dp(L+Mpp)}{P} + \int \frac{dq(L+Mqq)}{Q} + \int \frac{dr(L+Mrr)}{R} = \frac{Mpqr}{A}.$$

Quod si iam loco $\frac{dr}{R}$ scribamus $-\frac{dp}{P} - \frac{dq}{Q}$, aequatio ista hanc induet formam:

$$M\int \frac{dp(pp-rr)}{P} + M\int \frac{dq(qq-rr)}{Q} = \frac{Mpqr}{A},$$

siue ad differentialia descendendo ostendi debet, hanc aequationem veritati esse contentaneam:

$$\frac{dp(pp-rr)}{P} + \frac{dq(qq-rr)}{Q} = \frac{pqdr}{A} + \frac{prdq}{A} + \frac{qrdp}{A}.$$

Quod si ergo in dextra parte scribamus loco dr valorem $-\frac{Rdp}{P} - \frac{Rdq}{Q}$, demonstrandum est fore

$$\frac{dp(pp-rr)}{P} + \frac{dq(qq-rr)}{Q} = \frac{qdp(rP-pR)}{AP} + \frac{pdq(rQ-qR)}{AQ}$$

siue

$$\frac{dp(App-Arr-qrP+pqR)}{AP} + \frac{dq(Aqq-Arr-prQ+pqR)}{AQ} = 0.$$

§. 13. Cum igitur haec aequalitas subsistere debeat, quicunque valores binis variabilibus p et q tribuantur, necesse est vt vtraque pars seorsim nihilo aequetur; quocirca

circa ostendi debet fore tam

$$App - Arr - qrP + pqR = 0$$

quam

$$Aqq - Arr - prQ + pqR = 0.$$

Harum aequationum posterior a priore subtracta relinquit

$$A(pp - qq) - r(qP - pQ) = 0,$$

vbi si loco r valor substituatur, fit

$$A(pp - qq) + \frac{qqPP - ppQQ}{A - Eppqq} = 0.$$

Est vero

$$qqPP - ppQQ = (qq - pp)(AA - AEppqq),$$

vnde aequalitas manifesto patet. Tantum igitur superest, vt veritas alterutrius doceatur. Supra autem vidimus esse

$$R = -\frac{rQ - p(A - Eqqrr)}{q},$$

qui valor in priore aequatione substitutus praebet,

$$-Arr - r(qP + pQ) + Eppqqrr = 0,$$

deinde vero quia

$$qP + pQ = -r(A - Eppqq),$$

hoc valore substituto resultat

$$-Arr + rr(A - Eppqq) + Eppqqrr = 0,$$

cuius veritas est manifesta.

§. 14. Hoc igitur modo ex nostris formulis veritatem Theorematis generalis pro casu $N = 0$ per multas quidem ambages ob oculos posuimus. Facile autem intelligitur, si etiam litterae N rationem habere vellemus, demonstrationem difficillimis calculis fore inuolutam, quos vix quisquam esset superaturus, nisi iam ante de veritate

afferti

afferti nostri fuisset conuictus. Tanto magis igitur nostrum Theorema omni attentione et admiratione dignum est censendum, quod per consueta Analyseos artificia vix vlla via pateat eius demonstrationem in genere concinnandi, multo minus has sublimes veritates a priori inuestigandi.

Applicatio.

ad sectiones conicas.

Tab. I. Fig. 2. §. 15. Consideremus igitur semiellipsin ACB, cuius centrum sit in O, ac ponatur semiaxis transuersus $AO = BO = a$ et semiaxis coniugatus $OC = c$. Tum vero ducta applicata quacunque $Zz = z$ denotet nostra formula $\Pi : z$, arcum ellipsis AZ illi applicatae respondentem; vnde patet, si fuerit $z = 0$ fore etiam $\Pi z = 0$, at sumta $Zz = OC = c$, erit $\Pi : c = AC$, scilicet quadranti elliptico aequale. Hinc autem intelligitur, eidem applicatae Zz innumerabiles respondere arcus ellipticos; praeter minimum enim AZ ipsi conuenient arcus $4\Pi : c + AZ$, item $8\Pi : c + AZ$, $12\Pi : c + AZ$. Praeterea vero, quia ex altera parte etiam datur talis applicata $Z'z'$, ei quoque conueniet arcus $AZ' = 2\Pi : c - AZ$; similique modo etiam $6\Pi : c - AZ$, $10\Pi : c - AZ$ etc., sicque ista formula $\Pi : z$ erit functio infinitiformis ipsius z, scilicet in Ellipsi; nam in Hyperbola omnes isti valores, praeter vnum vel duos, euadent imaginarii.

§. 16. Pro arcu igitur AZ analytice exprimendo, vocetur abscissa $Oz = v$ et cum sit ex natura ellipsis

$\frac{vv}{aa} + \frac{zz}{cc} = 1$, erit

$v =$

$v = \frac{a}{c}\sqrt{(cc - zz)}$, hincque

$dv = -\frac{azdz}{c\sqrt{(cc - zz)}}$,

vnde colligitur elementum arcus AZ

$$\sqrt{(dv^2 + dz^2)} = dz\sqrt{1 + \frac{aazz}{cc(cc - zz)}} = dz\sqrt{\frac{c^4 + (aa - cc)zz}{cc(cc - zz)}},$$

quocirca habebimus

$$\Pi : z = \int \frac{dz\sqrt{(c^4 + (aa - cc)zz}}{c\sqrt{(cc - zz)}}.$$

§. 17. Cum igitur in genere posuissemus

$$\Pi : z = \int \frac{dz(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}},$$

ante omnia nostram formulam ad eandem formam reducamus, dum scilicet eius numeratorem et denominatorem multiplicamus per $\sqrt{(c + (aa - cc)zz)}$, tum autem prodibit

$$\Pi : z = \int \frac{dz(c^4 + (aa - cc)zz)}{c\sqrt{(cc - zz)(c^4 + (aa - cc)zz)}},$$

vnde patet, pro hoc casu fore $L = c^4$, $M = aa - cc$ et $N = 0$; deinde vero $A = c^8$, $C = c^4(aa - 2cc)$ et $E = -cc(aa - cc)$ vnde, si vt supra breuitatis gratia ponamus

$$Z = \sqrt{A(A + Czz + Ez^4)}, \text{ erit}$$

$$Z = c^5\sqrt{(cc - zz)(c^4 + (aa - cc)zz)}.$$

His igitur formulis eodem modo vti conueniet, vti in genere est monstratum.

§. 18. Quo has formulas concinniores reddamus, loco litterae c introducamus semiparametrum ellipsis, qui sit $= b$, et cum sit $cc = ab$, fiet primo

$$Z = a^3bb\sqrt{b(ab - zz)(abb + (a - b)zz)},$$

hinc-

hincque fiet ipsa formula

$$\Pi : z = \int \frac{dz \sqrt{(abb + (a-b)zz)}}{\sqrt{b(ab - zz)}}.$$

Praeterea vero erit $L = aabb$, $M = a(a-b)$, $A = a^4 b^4$, $C = a^3 bb(a - 2b)$ et $E = -aab(a-b)$ Loco semiaxis transuersi a insuper introducamus excentricitatem, quae sit $= n$, et quia ex elementis constat esse $a = \frac{b}{1-nn}$: hoc valore substituto fiet

$$Z = \frac{b^5}{(1-nn)^3} \sqrt{\frac{(b^3 - b(1-nn)zz)(b^3 + bnnzz)}{1-nn}}.$$

Vel potius hanc totam reductionem a principio repetamus, et cum sit

$$\Pi : z = \int \frac{dz \sqrt{(bb + nnzz)}}{\sqrt{(bb - (1-nn)zz)}},$$

hac ad formam generalem reducta fit

$$\Pi : z = \int \frac{dz(bb + nnzz)}{\sqrt{(bb + nnzz)(bb - (1-nn)zz)}},$$

ideoque comparatio praebet $L = bb$, $M = nn$, $N = 0$; $A = b^4$, $C = bb(2nn - 1)$ et $E = -nn(1-nn)$; tum vero erit

$$Z = bb\sqrt{((bb + nnzz)(bb - (1-nn)zz))}.$$

Atque nunc haec formula aeque valet pro omnibus sectionibus conicis. Quando enim $n < 1$, habebitur ellipsis; casu $n = 1$ parabola; at si $n > 1$ prodit hyperbola; pro circulo autem erit $n = 0$.

§. 8. Statuantur nunc ternae applicatae p, q, r indeque deriuantur valores deriuati

$$P = bb\sqrt{(bb + nnpp)(bb - (1-nn)pp)};$$
$$Q = bb\sqrt{(bb + nnqq)(bb - (1-nn)qq)};$$
$$R = bb\sqrt{(bb + nnrr)(bb - (1-nn)rr)};$$

tum

tum vero ex binis p et q tertia r ita determinetur, vt fit

$$r = \frac{-pQ - qP}{b^4 + nn(1 - nn)ppqq}, \text{ eritque}$$

$$R = \frac{(b^6(2nn-1)pq + PQ)(b^4 + nn(1-nn)ppqq - 2b^8 nn(1-nn)pq(pp+qq)}{(b^4 + nn(1-nn)ppqq)^2}$$

quibus positis habebitur sequens comparatio ternorum arcuum:

$$\Pi : p + \Pi : q + \Pi : r = \frac{nnpqr}{bb},$$

vbi binos arcus $\Pi : p$ et $\Pi : q$ pro lubitu assumere licet; hinc enim semper assignari poterit tertius $\Pi : r$, vt omnium summa fiat quantitas algebraica, dummodo notetur, horum arcuum semper vnum duosue fore negatiuos, cum sit $\Pi : (-z) = -\Pi : z$.

Translatio formularum praecedentium ad alterutrum focum sectionis conicae.

§. 20. Sit nunc F alteruter focus nostrae ellipsis seu sectionis conicae in genere, qui quidem vertici A sit propior; atque ex elementis constat, posito angulo $AFZ = \Phi$, tum fore distantiam $FZ = \frac{lb}{1 + n \cos. \Phi}$, vnde colligitur applicata $Zz = z = \frac{b \sin. \Phi}{1 + n \cos. \Phi}$, ita vt nunc sit arcus

$$AZ = \Pi : z = \Pi : \frac{b \sin. \Phi}{1 + n \cos. \Phi},$$

qui ergo, cum nunc spectetur vt functio anguli Φ, designetur hoc charactere: $AZ = \Gamma : \Phi$, ita vt sit

$$\Pi : z = \Pi : \frac{b \sin. \Phi}{1 + n \cos. \Phi} = \Gamma : \Phi.$$

Videamus igitur quomodo iste arcus per angulum Φ exprimatur; constat autem posita distantia $FZ = v$, fore arcum $AZ = \int \sqrt{(dv^2 + vv\, d\Phi^2)}$, quare cum sit

$v =$

$v = \frac{b}{1 + n \, cos. \Phi}$, erit

$d v = \frac{n b d \Phi \, sin. \Phi}{(1 + n \, cos. \Phi)^2}$, vnde fit

$d v^2 = \frac{n n b b d \Phi^2 \, sin. \Phi^2}{(1 + n \, cos. \Phi)^4}$, cui si addatur

$v v d \Phi^2 = \frac{b b d \Phi^2}{1 + n \, cos. \Phi^2}$, erit summa

$= \frac{b b d \Phi^2 (1 + 2n \, cos. \Phi + n n)}{(1 + n \, cos. \Phi)^4}$,

sicque erit arcus

$$A Z = \Pi : z = \Gamma : \Phi = \int \frac{b d \Phi}{1 + n \, cos. \Phi)^2} \sqrt{(1 + 2 n \, cos. \Phi + n n)}$$

$$= b \int \frac{d \Phi \sqrt{(1 + n n + 2n \, cos. \Phi)}}{(1 + n \, cos. \Phi)^2}.$$

Hinc autem porro colligetur.

$$Z = b^4 \sqrt{\left(1 + \frac{n n \, sin. \Phi^2}{(1 + n \, cos. \Phi)^2}\right) \left(1 + \frac{(n n - 1) \, sin. \Phi^2}{(1 + n \, cos. \Phi)^2}\right)},$$

siue

$$Z = \frac{b^4}{(1 + n \, cos. \Phi)^2} \sqrt{(1 + nn + 2n \, cos. \Phi)(+ nn + 2n \, cos. \Phi + cos. \Phi^2)}$$

siue

$$Z = \frac{b^4 (n + cos. \Phi) \sqrt{(1 + n n + 2n \, cos. \Phi)}}{(1 + n \, cos. \Phi)^2}.$$

§. 21. Quod si iam in calculum introducamus ternas applicatas p, q et r, quibus respondeant anguli ad focum ζ, η et θ, ita vt sit

$$p = \frac{b \, sin. \zeta}{1 + n \, cos. \zeta}, \quad q = \frac{b \, sin. \eta}{1 + n \, cos. \eta} \text{ et } r = \frac{b \, cos. \theta}{1 + n \, cos. \theta},$$

tum vero

$$P = \frac{b^4 (n + cos. \zeta) \sqrt{(1 + n n + 2 n \, cos. \zeta)}}{(1 + n \, cos. \zeta)^2},$$

$$Q = \frac{b^4 (n + cos. \eta) \sqrt{(1 + n n + 2 n \, cos. \eta)}}{(1 + n \, cos. \eta)^2},$$

$$R = \frac{b^4 (n + cos. \theta) \sqrt{(1 + n n + 2 n \, cos. \theta)}}{(1 + n \, cos. \theta)^2};$$

hinc iam, si inter ternas applicatas p, q, r relatio supra indicata statuatur, haec arcuum comparatio obtinebitur:

$$\Gamma : \zeta + \Gamma : \eta + \Gamma : \theta = \frac{n n b \, sin. \zeta . \, sin. \eta . \, sin. \theta}{(1 + n \, cos. \zeta)(1 + n \, cos. \eta)(1 + n \, cos. \theta)}.$$

§. 22.

§. 22. Relatio autem inter litteras p, q, r ſtabilienda ad noſtros angulos traducta erat

$$r(b^4 + nn(1 - nn)ppqq) = -pQ - qP,$$

cuius membrum ſiniſtrum facta ſubſtitutione induet hanc formam:

$$\frac{b^5 \text{ſin.}\theta\left((1 + 2n(\text{coſ.}\zeta + \text{coſ.}\eta) + nn(1 + 4\text{coſ.}\zeta\,\text{coſ.}\eta + \text{coſ.}\zeta^2\,\text{coſ.}\eta^2)\right) + 2n^3\text{coſ.}\zeta\,\text{coſ.}\eta(\text{coſ.}\zeta + \text{coſ.}\eta) + n^4(\text{coſ.}\zeta^2 + \text{coſ.}\eta^2 - 1)}{(1 + n\,\text{coſ.}\theta)(1 + n\,\text{coſ.}\zeta)^2(1 + n\,\text{coſ.}\eta)^2}$$

membrum vero dextrum ad hanc formam reducitur:

$$-\frac{b^5 \text{ſin.}\zeta(n + \text{coſ.}\eta)\sqrt{(1 + nn + 2n\,\text{coſ.}\eta)}}{(1 + n\,\text{coſ.}\zeta)(1 + n\,\text{coſ.}\eta)^2} - \frac{b^5 \text{ſin.}\eta(n + \text{coſ.}\zeta)\sqrt{(1 + nn + 2n\,\text{coſ.}\zeta)}}{(1 + n\,\text{coſ.}\eta)(1 + n\,\text{coſ.}\zeta)^2}.$$

Hic quidem vtrinque per b^5 diuidi poteſt, neque tamen hinc patet, quomodo angulus θ ex binis reliquis angulis ζ et η definiri queat.

Digreſſio ad Parabolam.

§. 23. Quoniam igitur non patet, quomodo in genere ex binis angulorum ζ et η tertium determinari oporteat, hanc inueſtigationem ad Parabolam transferamus, ponendo $n = 1$; tum autem membrum illud ſiniſtrum abit in $\frac{\text{ſin.}\theta}{1 + \text{coſ.}\theta} = \text{tang.}\frac{1}{2}\theta$: membrum autem dextrum euadit

$$-\frac{\text{ſin.}\zeta\sqrt{(2 + 2\,\text{coſ.}\eta)}}{(1 + \text{coſ.}\zeta)(1 + \text{coſ.}\eta)} - \frac{\text{ſin.}\eta\sqrt{(2 + 2\,\text{coſ.}\zeta)}}{(1 + \text{coſ.}\zeta)(1 + \text{coſ.}\eta)}$$

$$= -\frac{\text{tang.}\frac{1}{2}\zeta}{\text{coſ.}\frac{1}{2}\eta} - \frac{\text{tang.}\frac{1}{2}\eta}{\text{coſ.}\frac{1}{2}\zeta},$$

ita vt aequatio noſtra prodierit

$$\text{tang.}\tfrac{1}{2}\theta = -\frac{\text{tang.}\frac{1}{2}\zeta}{\text{coſ.}\frac{1}{2}\eta} - \frac{\text{tang.}\frac{1}{2}\eta}{\text{coſ.}\frac{1}{2}\zeta} = -\frac{\text{ſin.}\frac{1}{2}\zeta - \text{ſin.}\frac{1}{2}\eta}{\text{coſ.}\frac{1}{2}\zeta\,\text{coſ.}\frac{1}{2}\eta}.$$

§. 24.

§. 24. Quod quo clarius appareat notetur esse

$$p = b \text{ tang. } \tfrac{1}{2}\zeta, \quad q = b \text{ tang. } \tfrac{1}{2}\eta, \quad r = b \text{ tang. } \tfrac{1}{2}\theta;$$

praeterea vero

$$P = \frac{b^4}{\text{cos. } \frac{1}{2}\zeta}, \quad Q = \frac{b^4}{\text{cos. } \frac{1}{2}\eta}, \quad R = \frac{b^4}{\text{cos. } \frac{1}{2}\theta}.$$

Cum igitur etiam pro hoc casu prodeat $R = pq + PQ$ erit

$$\frac{1}{\text{cos. } \frac{1}{2}\theta} = \frac{-1 + \text{sin. } \frac{1}{2}\zeta \text{ sin. } \frac{1}{2}\eta}{\text{cos. } \frac{1}{2}\zeta \text{ cos. } \frac{1}{2}\eta}:$$

ante autem inuenimus,

$$\text{tang. } \tfrac{1}{2}\theta = \frac{-\text{sin. } \frac{1}{2}\zeta - \text{sin. } \frac{1}{2}\eta}{\text{cos. } \frac{1}{2}\zeta \text{ cos. } \frac{1}{2}\eta},$$

vnde haec aequatio per illam diuisa praebet

$$\text{sin. } \tfrac{1}{2}\theta = \frac{-\text{sin. } \frac{1}{2}\zeta - \text{sin. } \frac{1}{2}\eta}{1 + \text{sin. } \frac{1}{2}\zeta \text{ sin. } \frac{1}{2}\eta}, \text{ siue}$$

$$\text{sin. } \tfrac{1}{2}\theta + \text{sin. } \tfrac{1}{2}\zeta + \text{sin. } \tfrac{1}{2}\eta + \text{sin. } \tfrac{1}{2}\zeta \text{ sin. } \tfrac{1}{2}\eta \text{ sin. } \tfrac{1}{2}\theta = 0,$$

in qua aequatione terni anguli ζ, η, θ sunt permutabiles, quemadmodum rei natura postulat, quae proprietas in valore primo inuento non tam erat manifesta.

§. 25. Quod si ergo terni anguli ζ, η, θ, ita a se inuicem pendeant, vt sit

$$\text{sin. } \tfrac{1}{2}\zeta + \text{sin. } \tfrac{1}{2}\eta + \text{sin. } \tfrac{1}{2}\theta + \text{sin. } \tfrac{1}{2}\zeta \text{ sin. } \tfrac{1}{2}\eta \text{ sin. } \tfrac{1}{2}\theta = 0,$$

tum in parabola terni arcus his angulis ζ, η, θ respondentes semper ita erunt comparati, vt sit

$$\Gamma : \zeta + \Gamma : \eta + \Gamma : \theta = b \text{ tang. } \tfrac{1}{2}\zeta \text{ tang. } \tfrac{1}{2}\eta \text{ tang. } \tfrac{1}{2}\theta.$$

Hinc si dati fuerint bini anguli ζ et η, tertius θ ope formulae primum inuentae facillime definitur, qua erat

tang.

$$\text{tang.} \tfrac{1}{2}\theta = \frac{-\text{sin.} \tfrac{1}{2}\zeta - \text{sin.} \tfrac{1}{2}\eta}{\text{cos.} \tfrac{1}{2}\zeta \,\text{cos.} \tfrac{1}{2}\eta},$$

quae expressio per meros factores ita exhiberi potest:

$$\text{tang.} \tfrac{1}{2}\theta = \frac{-2\,\text{sin.} \frac{\zeta-\eta}{4} \,\text{cos.} \frac{\zeta-\eta}{4}}{\text{sin.} \tfrac{1}{2}\zeta \,\text{cos.} \tfrac{1}{2}\eta};$$

vnde patet, si anguli ζ et η fuerint positiui, tertium θ necessario fieri negatiuum, siue arcum ipsi respondentem negatiue capi debere. Ceterum patet, si vnus horum angulorum, veluti ζ, euanescat, tum fore $\text{sin.}\tfrac{1}{2}\theta + \text{sin.}\tfrac{1}{2}\eta = 0$, siue summam duorum reliquorum nihilo aequari, siue alterum alterius fieri negatiuum.

Problema

In quadrante Elliptico A O C, *sumto pro lubitu arcu* A Q, *ab altero termino* C *abscindere arcum* C R, *qui illum arcum* A Q *superet quantitate algebraica.* Tab. I. Fig. 3.

Solutio.

§. 26. Sint huius Ellipsis semiaxes vt supra $OA = a$ et $OC = c$, et cum sit arcus $CR = AC - AR$, requiritur vt fiat $AC - AR - AQ$ quantitas algebraica. Ducantur ad axem OA perpendicula Qq et Rr, quae vocentur $Qq = q$ et $Rr = r$, quae respectu formularum supra inuentarum capi debent negatiua, quia arcus respondentes AQ et AR hic negatiue capiuntur. Cum igitur arcus $\Pi : p$ hic sit quadrans AC, erit $p = c$ $A = c^4$, $C = c^4(aa - 2cc)$, $E = -cc(aa - cc)$; pro applicata quacunque z vero erit formula respondens

$$Z = c^5 \sqrt{(cc - zz)(c^4 + (aa - cc)zz)},$$

vnde

vnde pro caſu $z = c$ fiet $Z = 0$, quocirca pro praeſenti caſu, vbi $p = c$, erit $P = 0$. Deinde autem ſi loco q ibi ſcribatur $-q$, fiet

$$Q = c^5 \sqrt{(cc - qq)}\,(c^4 + (aa - cc)\,qq).$$

§. 27. Sumtis autem litteris q et r negatiuis, cum in genere inuenerimus

$$r = \frac{-pQ - qP}{A - Epppqq}, \text{ ob } p = c \text{ et } P = 0 \text{ fiet}$$

$$-r = \frac{-cQ}{A - Epppqq}, \text{ ideoque}$$

$$r = \frac{cc\sqrt{(cc - qq)}\,(c^4 + (aa - cc)\,qq)}{c^4 + (aa - cc)\,qq},$$

quo valore inuento erit differentia arcuum $CR - AQ$ ſiue

$$\Pi : c - \Pi : q - \Pi : r = \frac{M}{\sqrt{A}} \cdot pqr = \frac{aa - cc}{c^3} \cdot qr;$$

quamobrem ſi loco r valorem inuentum ſubſtituamus, habebimus

$$CR - AQ = \frac{(aa - cc)\,q\sqrt{(cc - qq)}\,(c^4 + (aa - cc)\,qq)}{c\,(c^4 + (aa - cc)\,qq)}.$$

Hic igitur quantitas q arbitrio noſtro eſt relicta, vnde arcum AQ pro lubitu aſſumere licet, hincque punctum R, ſeu applicata $Rr = r$, ita eſt determinata, vt differentia arcuum $CR - AQ$ fiat algebraica; formulae autem inuentae manifeſto reducuntur ad has ſimpliciores:

$$r = \frac{cc\sqrt{(cc - qq)}}{\sqrt{(c^4 + (aa - cc)\,qq)}},$$

et differentia arcuum

$$CR - AQ = \frac{(aa - cc)\,q\sqrt{(cc - qq)}}{c\sqrt{(c^4 + (aa - cc)\,qq)}},$$

vbi notetur eſſe arcum

$$AQ = \int \frac{dq\sqrt{(c^4 + (aa - cc)\,qq)}}{c\sqrt{(cc - qq)}}.$$

§. 28.

§. 28. Quoniam puncta Q et R inter se permutari possunt, siquidem est $CR - AQ = CQ - AR$, hanc permutabilitatem etiam valor pro r inuentus ostendit. Sumtis enim quadratis obtinebitur ista aequatio:

$$c^6 - c^4(qq + rr) - (aa - cc)qqrr = 0,$$

quae manifesto reducitur ad hanc formam concinniorem:

$$(cc - qq)(cc - rr) = \frac{aaqqrr}{cc};$$

vnde si statuamus $qr = uu$, vt sit $qqrr = u^4$, ex hac aequatione erit

$$qq + rr = cc - \frac{(aa - cc)}{c^4}u^4,$$

quare, si $2qr = 2uu$ siue addatur siue subtrahatur, colligitur fore

$$q + r = \sqrt{cc + 2uu - \frac{(aa - cc)u^4}{c^4}} \text{ et}$$

$$q - r = \sqrt{cc - 2uu - \frac{(aa - cc)u^4}{c^4}};$$

vnde sumto u pro lubitu ambae quantitates q et r simili modo exprimuntur. Hoc modo etiam facile effici potest, vt ambo puncta Q et R congruant; facto enim $q - r = 0$ fiet $uu = -\frac{c^4 \pm ac^3}{aa - cc}$; erit ergo vel

$$uu = \frac{c^3}{a + c}, \text{ vel } uu = -\frac{c^3}{a - c};$$

tum autem erit

$$qq = \frac{c^3}{a + c}, \text{ vel } qq = -\frac{c^3}{a - c},$$

quorum valorum positiuum sumi oportet. Quia autem q superare nequit c, prior tantum valor locum habere potest, quo est $qq = \frac{c^3}{a + c}$.

§. 29. Conueniant igitur ambo haec puncta in puncto U, ita vt sit applicata $Uu = \frac{c\sqrt{c}}{\sqrt{a+c}}$, tum vero erit ar- Tab. I. Fig. 4.

arcuum differentia

$$CU - AU = \frac{aa - cc}{a + c} = a - c,$$

ita vt haec differentia aequetur ipsi differentiae axium OA et OC. Hinc igitur erit $AO + AU = CO + CU$, vbi manifestum est, si esset $a = c$, tum punctum U in medium arcus AC incidere. Ad hoc punctum U clarius intelligendum quaeramus etiam interuallum Ou, et cum sit

$$\frac{Ou^2}{aa} + \frac{Uu^2}{cc} = 1, \text{ erit } Ou^2 = aa - \frac{aac}{a+c} = \frac{a^3}{a+c},$$

vnde patet fore $\frac{Uu}{Ou} = \frac{c\sqrt{c}}{a\sqrt{a}}$, quae ergo est tangens anguli AOU.

§. 30. Quia in Ellipsi ambo semiaxes a et c sunt permutabiles, quemadmodum arcus AQ definitur per applicatam $Qq = q$, simili modo permutatis axibus arcus CR definietur per applicatam $Rs = Or$. Posita igitur $Rs = s$ erit per formulam integralem arcus

Tab. I. Fig. 3.

$$CR = \int \frac{ds\sqrt{(a^4 - (aa - cc)ss)}}{a\sqrt{(aa - ss)}},$$

sicque erit

$$\int \frac{ds\sqrt{(a^4 - (aa - cc)ss)}}{a\sqrt{(aa - ss)}} - \int \frac{dq\sqrt{c^4 + (aa - cc)qq}}{c\sqrt{(cc - qq)}}$$
$$= \frac{(aa - cc)qr}{c^3} = \frac{(aa - cc)q\sqrt{(cc - qq)}}{c\sqrt{(c^4 + (aa - cc)qq)}}.$$

Videamus igitur quomodo s se habeat respectu q; primo autem erit $\frac{ss}{aa} + \frac{rr}{cc} = 1$, vnde fit

$$ss = aa - \frac{aa}{cc}rr = \frac{a^4 qq}{c^4 + (aa - cc)qq},$$

consequenter

$$c^4 ss + (aa - cc)qqss - a^4 qq = 0;$$

vnde patet, permutatis litteris a et c etiam permutari q et s, vti rei natura postulat.

§. 31.

§. 31. Hinc igitur colligimus iſtud Theorema analyticam:

Theorema.

Si capiatur $s = \frac{aaq}{\sqrt{(c^4 + (aa-cc)qq)}}$, erit differentia iſtarum formularum integralium ſemper algebraica:

$$\int \frac{ds\sqrt{(a^4-(aa-cc)ss)}}{a\sqrt{(aa-ss)}} - \int \frac{dq\sqrt{(c^4+(aa-cc)qq)}}{c\sqrt{(cc-qq)}} = \frac{(aa-cc)q\sqrt{(cc-qq)}}{c\sqrt{(c^4+(aa-cc)qq)}}.$$

§. 32. Operae igitur pretium erit per euolutionem calculi hanc egregiam reductionem oſtendiſſe. Primo igitur cum ſit

$$s = \frac{aaq}{\sqrt{(c^4+(aa-cc)qq)}}, \text{ erit}$$

$$\sqrt{(aa-ss)} = \frac{ac\sqrt{(cc-qq)}}{\sqrt{(c^4+(aa-cc)qq)}} \text{ et}$$

$$\sqrt{(a^4-(aa-cc)ss)} = \frac{aacc}{\sqrt{(c^4+(aa-cc)qq)}},$$

vnde ſit pro prima formula integrali

$$\frac{\sqrt{(a^4-(aa-cc)ss)}}{a\sqrt{(aa-ss)}} = \frac{c}{\sqrt{(cc-qq)}}.$$

Deinde vero reperitur

$$ds = \frac{aac^4dq}{(c^4+(aa-cc)qq)^{\frac{3}{2}}};$$

hinc igitur formularum integralium prior erit

$$\int \frac{ds\sqrt{(a^4-(aa-cc)ss)}}{a\sqrt{(aa-ss)}} = \int \frac{aac^6dq}{c(c^4+(aa-cc)qq)^{\frac{3}{2}}\sqrt{(cc-qq)}}$$

ab hac igitur ſi ſubtrahatur altera $\int \frac{dq\sqrt{(c^4+(aa-cc)qq)}}{c\sqrt{(cc-qq)}}$; differentiam integrabilem eſſe oportet. Facta autem reductione ad communem denominatorem haec differentia ſit:

$$\int \frac{(aa-cc)dq(c^6-2c^4qq-(aa-cc)q^4)}{c(c^4+(aa-cc)qq)^{\frac{3}{2}}\sqrt{(c-q)qc}}$$

cuius

cuius integrale ergo esse debet $\frac{(aa-cc)q\sqrt{(cc-qq)}}{c\sqrt{(c^4+(aa-cc)qq)}}$, quod tentanti mox patebit. Nullum autem est dubium, quin iste casus, si probe perpendatur, largum campum sit aperturus huiusmodi inuestigationes adcuratius excolendi.

§. 33. Solutio autem istius problematis elegantius sequenti modo adornari potest. Cum sit $Qq = q$, erit $Oq = \frac{a}{c}\sqrt{(cc-qq)}$, similique modo ob $Rs = s$ erit $Os = \frac{c}{a}\sqrt{(aa-ss)}$; quare cum inter q et s ista inuenta sit aequatio:

$$s = \frac{aaq}{\sqrt{(c^4+(aa-cc)qq)}}, \text{ erit}$$

$$ccss(cc-qq) = aaqq(aa-ss) \text{ ideoque}$$

$$\frac{cs}{\sqrt{(aa-ss)}} = \frac{aq}{\sqrt{(cc-qq)}}, \text{ siue } \frac{cc}{a}\cdot\frac{Rs}{Os} = \frac{aa}{c}\cdot\frac{Qq}{Oq}.$$

Hinc si duci intelligantur rectae OQ et OR et vocentur anguli $AOQ = \phi$ et $COR = \psi$, erit $\frac{cc}{a}$ tang. $\psi = \frac{aa}{c}$ tang. ϕ, siue hi anguli ita sunt comparati, vt sit tang. ψ : tang. $\phi = a^3 : c^3$, sicque ex angulo ϕ pro lubitu assumto facile definitur angulus ψ.

§. 34. Deinde cum inuenta sit arcuum differentia

$$CR - AQ = \frac{(aa-cc)q\sqrt{(cc-qq)}}{c\sqrt{(c^4+(aa-cc)qq)}}, \text{ ob}$$

$$\sqrt{(c^4+(aa-cc)qq)} = \frac{aaq}{s} \text{ erit}$$

$$CR - AQ = \frac{(aa-cc)s\sqrt{(cc-qq)}}{aac} = \frac{s}{c}\sqrt{(cc-qq)} - \frac{c}{aa}s\sqrt{(cc-qq)}$$

$$= \frac{s\sqrt{(cc-qq)}}{c} - \frac{q\sqrt{(aa-ss)}}{a} = qs\left(\frac{\sqrt{(cc-qq)}}{cq} - \frac{\sqrt{(aa-ss)}}{as}\right)$$

quae expressio ob tang. $\phi = \frac{cq}{a\sqrt{(cc-qq)}}$ et tang. $\psi = \frac{as}{c\sqrt{(aa-ss)}}$ ad hanc formam reducitur: $qs\left(\frac{\cot.\phi}{a} - \frac{\cot.\psi}{c}\right)$.

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