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De mirabilibus proprietatibus unciarum, quae in evolutione binomii ad potestatem quamcunqua evecti occurrunt

Leonhard Euler

DE MIRABILIBVS PROPRIETATIBVS VNCIARVM, QVAE IN EVOLVTIONE BINOMII AD POTESTATEM QVAMCVNQVE EVECTI OCCVRRVNT.

Auctore

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Theorema I.

§. 1.

Si pro potestate Binomii ad exponentem n euecti vncias breuitatis gratia litteris α, β, γ, δ, etc. designemus, vt sit

$$\alpha = \frac{n}{1};\ \beta = \frac{n}{1} \cdot \frac{(n-1)}{2};\ \gamma = \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3};\ \delta = \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \cdot \frac{(n-3)}{4};\ \text{etc.}$$

semper erit

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{etc.} = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \cdot \cdot \cdot \frac{4n-2}{n},$$

quae

quae expressio pro casibus, quibus n est numerus fractus, ita per formulam integralem exhiberi potest, vt sit

$$= \frac{2}{\pi} \cdot 2^{2n} \int \frac{x^{2n}\, dx}{\sqrt{(1 - xx)}},$$

hoc integrali ab $x = 0$ vsque ad $x = 1$ extenso, vbi π denotat peripheriam circuli, cuius diameter $= 1$.

Hoc theorema eo magis est notatu dignum, quod vix vlla via directa patet eius veritatem demonstrandi.

Explicatio

pro casibus quibus exponens n est numerus integer positiuus.

§. 2. Quo vis huius theorematis clarius perspiciatur, euoluamus casus simpliciores sequenti modo:

I. Si $n = 1$, erunt vnciae 1, 1, ideoque vi theorematis esse debet

$$1^2 + 1^2 = 2 = \tfrac{2}{1}.$$

II. Si $n = 2$, erunt vnciae 1, 2, 1, ideoque vi theorematis esse debet

$$1^2 + 2^2 + 1^2 = 6 = \tfrac{2}{1} \cdot \tfrac{6}{2}.$$

III. Si $n = 3$, erunt vnciae 1, 3, 3, 1, ideoque vi theorematis esse debet

$$1^2 + 3^2 + 3^2 + 1^2 = 20 = \tfrac{2}{1} \cdot \tfrac{6}{2} \cdot \tfrac{10}{3}.$$

IV. Si $n = 4$, erunt vnciae 1, 4, 6, 4, 1, ideoque vi theorematis esse debet

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70 = \tfrac{2}{1} \cdot \tfrac{6}{2} \cdot \tfrac{10}{3} \cdot \tfrac{14}{4}.$$

V. Si $n=5$, erunt vnciae 1, 5, 10, 10, 5, 1, ideoque vi theorematis esse debet

$$1^2+5^2+10^2+10^2+5^2+1^2=252=\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdot\frac{18}{5}.$$

VI. Si $n=6$, erunt vnciae 1, 6, 15, 20, 15, 6, 1; ideoque vi theorematis esse debet

$$1^2+6^2+15^2+20^2+15^2+6^2+1^2=924=\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdot\frac{18}{5}\cdot\frac{22}{6}.$$

Corollarium.

§. 3. Cum formula $\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\ldots\ldots\frac{4n-2}{n}$ exhibeat maximam vnciam in potestate Binomii ad exponentem $2n$ euecti, theorema nostrum etiam hoc modo enunciari potest:

Si quadrata vnciarum pro potestate exponentis n *in vnam summam colligantur, ea semper aequabitur maximae vnciae in potestate exponentis* 2 n *occurrenti.* Ita pro casibus ante euolutis 2 est maxima vncia pro exponente 2; deinde 6 est maxima vncia pro exponente 4; porro 20 est maxima vncia pro exponente 6; similique modo sequens summa 70 est maxima vncia pro exponente 8. et ita porro.

Explicatio theorematis quo exponens n est numerus fractus.

§. 4. Quando exponens n est numerus fractus, series vnciarum in infinitum extenditur, vnde earum quadrata etiam constituent seriem infinitam, cuius summa per formulam illam integralem: $\frac{2}{\pi}\cdot 2^{2n}\int\frac{x^{2n}\,dx}{\sqrt{(1-xx)}}$, innotescet,

si-

siquidem hoc integrale ab $x=0$ vsque ad $x=1$ extendatur, id quod vnico exemplo, quo $n=\frac{1}{2}$, ostendisse sufficiet; tum autem erit

$$\alpha=\tfrac{1}{2};\ \beta=-\tfrac{1.1}{2.4};\ \gamma=\tfrac{1.1.3}{2.4.6};\ \delta=-\tfrac{1.1.3.5}{2.4.6.8};\ \varepsilon=\tfrac{1.1.3.5.7}{2.4.6.8.10};\ \text{etc.}$$

quarum igitur valorum quadrata constituent hanc seriem:

$$1+\tfrac{1^2}{2^2}+\tfrac{1^2.1^2}{2^2.4^2}+\tfrac{1^2.1^2.3^2}{2^2.4^2.6^2}+\tfrac{1^2.1^2.3^2.5^2}{2^2.4^2.6^2.8^2}+\tfrac{1^2.1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8^2.10^2}+\text{etc.}$$

cuius ergo summa ex formula integrali:

$$\int\frac{x\,dx}{\sqrt{(1-xx)}},\ \text{extensa ab } x=0 \text{ ad } x=1,$$

est petenda. Est vero

$$\int\frac{x\,dx}{\sqrt{(1-xx)}}=1-\sqrt{(1-xx)},$$

quare facto iam $x=1$ eius valor euadit $=1$. Quocirca summa seriei inuentae erit $=\frac{4}{\pi}$, cuius valor per fractionem decimalem est 1, 273230; atque ad hunc valorem continuo magis appropinquabitur, quo plures termini seriei $1+\alpha^2+\beta^2+\gamma^2+$ etc. actu colligentur; qui calculus quo facilius instituatur, ob $\alpha\alpha=\frac{1}{4}$, notetur esse

$$\beta\beta=\tfrac{1}{16}\alpha\alpha;\ \gamma\gamma=\tfrac{3^2}{6^2}\beta\beta=\tfrac{1}{4}\beta\beta;\ \delta\delta=\tfrac{25}{64}\gamma\gamma;$$
$$\varepsilon\varepsilon=\tfrac{49}{100}\delta\delta;\ \zeta\zeta=\tfrac{81}{144}\varepsilon\varepsilon;\ \eta\eta=\tfrac{121}{196}\zeta\zeta,\ \text{etc.}$$

quo obseruato calculus sequenti modo instituatur:

$$\begin{aligned}
1 &= 1,000000\\
\alpha\alpha &= 0,250000\\
\beta\beta &= 0,015625\\
\gamma\gamma &= 0,003906\\
\delta\delta &= 0,001526\\
\varepsilon\varepsilon &= 0,000748\\
\zeta\zeta &= 0,000420\\
\eta\eta &= 0,000260\\
\theta\theta &= 0,000172\\
\hline
\text{Summa} &= 1,272657.
\end{aligned}$$

 Haec

Haec summa deficit a vero valore hac fractione: 0,000573, quae ergo aequalis censenda est omnibus sequentibus terminis, quos hic praetermisimus, id quod sufficit ad veritatem nostri asserti comprobandam.

§. 5. Si exponenti n alios valores fractos tribuere vellemus, vt $2n$ non amplius foret numerus integer, tunc summa seriei non amplius a quadratura circuli, sed a quadraturis altioribus penderet. Caeterum hic notasse iuuabit, si sumeremus $n = -\frac{1}{2}$, vnde fieret

$$\alpha = -\frac{1}{2}; \ \beta = \frac{1.3}{2.4}; \ \gamma = -\frac{1.3.5}{2.4.6}; \ \delta = \frac{1.3.5.7}{2.4.6.8}; \ \text{etc.}$$

tum summam huius seriei:

$$1 + \frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8^2} + \text{etc.}$$

in infinitum excrescere, quemadmadum etiam nostra formula integralis indicat, quippe quae fit $\frac{1}{\pi}\int\frac{dx}{x\sqrt{(1-xx)}}$. Reperitur vero

$$\int\frac{dx}{x\sqrt{(1-xx)}} = \frac{1}{2}\, l\, \frac{1-\sqrt{(1-xx)}}{1+\sqrt{(1-xx)}} + C,$$

quae constans ita capi debet, vt euanescat posito $x = 0$, ex quo fiet $C = -\frac{1}{2}\, l\, 0$, ideoque $C = \infty$. Statuamus nunc $x = 1$, et iste valor prodibit $= \infty$.

§. 6. Sin autem statuamus $n = \frac{3}{2}$, vt fiat

$$\alpha = \frac{3}{2}; \ \beta = \frac{3.1}{2.4}; \ \gamma = -\frac{3.1.1}{2.4.6}; \ \delta = \frac{3.1.1.3}{2.4.6.8}; \ \varepsilon = -\frac{3.1.1.3.5}{2.4.6.8.10}; \ \text{etc.}$$

tum summa seriei $1 + \alpha^2 + \beta^2 + \gamma^2 + \text{etc.}$ erit $\frac{16}{\pi}\int\frac{x^3\,dx}{\sqrt{(1-xx)}}$. Est autem

$$\int\frac{x^3\,dx}{\sqrt{(1-xx)}} = \text{const.} - \sqrt{(1-xx)} + \frac{1}{3}(1-xx)^{\frac{3}{2}},$$

vbi capi debet $C = \frac{2}{3}$. Facto nunc $x = 1$, erit summa nostrae seriei $= \frac{32}{3\pi}$.

§. 7.

§. 7. Si exponenti n maiores huiusmodi valores, veluti $\frac{5}{2}$; $\frac{7}{2}$; $\frac{9}{2}$; $\frac{11}{2}$; etc. tribuere velimus, notetur in genere esse

$$\int \frac{x^{i+2}\,dx}{\sqrt{(1-xx)}} = -\frac{x^{i+1}\sqrt{(1-xx)}}{i+2} + \frac{(i+1)}{(i+2)}\int \frac{x^i\,dx}{\sqrt{(1-xx)}}.$$

Vnde si integralia ab $x=0$ ad $x=1$ extendantur, erit

$$\int \frac{x^{i+2}\,dx}{\sqrt{(1-xx)}} = \frac{i+1}{i+2}\int \frac{x^i\,dx}{\sqrt{(1-xx)}}.$$

Quare cum casu $i=1$ sit

$$\int \frac{x\,dx}{\sqrt{(1-xx)}} = 1$$

erit pro sequentibus formulis:

$$\int \frac{x^3\,dx}{\sqrt{(1-xx)}} = \frac{2}{3}$$

$$\int \frac{x^5\,dx}{\sqrt{(1-xx)}} = \frac{2.\,4}{3.\,5}$$

$$\int \frac{x^7\,dx}{\sqrt{(1-xx)}} = \frac{2.\,4.\,6}{3.\,5.\,7}$$

$$\int \frac{x^9\,dx}{\sqrt{(1-xx)}} = \frac{2.\,4.\,6.\,8}{3.\,5.\,7.\,9}$$

etc.

His igitur praenotatis, si ponamus breuitatis gratia

$$1+\alpha^2+\beta^2+\gamma^2+\text{etc.} = S,$$

erit vt sequitur:

I. Pro casu $n=\frac{1}{2}$, erit $S=\frac{4}{\pi}$.

II. Pro casu $n=\frac{3}{2}$, erit $S=\frac{4}{\pi}\cdot\frac{8}{3}$.

III. Pro casu $n=\frac{5}{2}$, erit $S=\frac{4}{\pi}\cdot\frac{8}{3}\cdot\frac{16}{5}$.

IV. Pro casu $n=\frac{7}{2}$, erit $S=\frac{4}{\pi}\cdot\frac{8}{3}\cdot\frac{16}{5}\cdot\frac{24}{7}$.

V. Pro casu $n=\frac{9}{2}$, erit $S=\frac{4}{\pi}\cdot\frac{8}{3}\cdot\frac{16}{5}\cdot\frac{24}{7}\cdot\frac{32}{9}$.

etc. etc.

§. 8.

§. 8. Ex ſuperiori integralium reductione etiam ratio noſtrae formulae integralis in theoremate datae reddi poteſt; cum enim in genere ſit

$$\int \frac{x^{i+2}\,dx}{\sqrt{(1-xx)}} = \frac{i+1}{i+2} \int \frac{x^i\,dx}{\sqrt{(1-xx)}},$$

caſu autem $i = 0$ fiat $\int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2}$, erit vt ſequitur:

$$\int \frac{xx\,dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \cdot \frac{1}{2}.$$
$$\int \frac{x^4\,dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4}.$$
$$\int \frac{x^6\,dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}.$$
$$\int \frac{x^8\,dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}.$$
$$\int \frac{x^{10}\,dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10}.$$

etc. etc.

Quodſi iam exponenti n ſucceſſiue numeros integros, 1, 2, 3, 4, etc. tribuamus, indeque concludamus valorem noſtrae ſeriei

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \text{etc.} = S = \frac{2}{\pi} \cdot 2^{2n} \int \frac{x^{2n}\,dx}{\sqrt{(1-xx)}},$$

reperiemus pro S hos valores:

I. Pro caſu $n = 1$ erit $S = 2$.

II. Pro caſu $n = 2$ erit $S = \frac{2}{1} \cdot \frac{6}{2}$.

III. Pro caſu $n = 3$ erit $S = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3}$.

IV. Pro caſu $n = 4$ erit $S = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4}$.

V. Pro caſu $n = 5$ erit $S = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5}$.

etc. etc.

Vnde

Vnde patet pro quouis exponente integro n fore

$$S = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \ldots \ldots \frac{4n-2}{n},$$

prorsus vti in theoremate statuimus.

Scholion.

§. 9. Quemadmodum hic pro casibus, quibus exponens n est numerus fractus, summam nostrae seriei per formulam integralem repraesentauimus, quae iam inuoluit peripheriam circuli π, ita etiam pluribus modis valor eiusdem summae S per alias formulas integrales exprimi potest, quarum aliquas hic adiungamus. Prima scilicet est

$$\text{I. } S = \frac{2}{n \int x^{n-1} dx (1-x)^{n-1}}.$$

$$\text{II. } S = \frac{1}{n \int x^{n} dx (1-x)^{n-1}}.$$

$$\text{III. } S = \frac{1}{(2n+1) \int x^{n} dx (1-x)^{n}}.$$

Vbi quidem, vt ante, has formulas integrales a termino $x = 0$ vsque ad $x = 1$ extendi oportet. Ita pro casu $n = 1$ prima harum formarum praebet $S = \frac{2}{\int dx} = \frac{2}{1}$; secunda vero formula dat $S = \frac{1}{\int x dx} = \frac{2}{1}$; tertia porro formula dat $S = \frac{1}{3 \int x dx (1-x)}$. Est vero

$$\int x dx (1-x) = \tfrac{1}{2} x x - \tfrac{1}{3} x^3 = \tfrac{1}{6},$$

ex quo fit $S = \frac{2}{1}$.

Consideremus hic etiam casum $n = \frac{1}{2}$, ac prima istarum formularum praebet $S = \frac{4}{\int \frac{dx}{\sqrt{(x - xx)}}}$. Posito autem hic $x = yy$, fit

$$\int \frac{dx}{\sqrt{(x - xx)}} = 2 \int \frac{dy}{\sqrt{(1 - yy)}} = \frac{2\pi}{2} = \pi,$$

sicque erit $S = \frac{4}{\pi}$, prorsus vti supra est inuentum. Contemplemur adhuc casum $n = 3$ ac prima harum formularum dabit $S = \frac{2}{3\int xx\,dx\,(1-x)^2}$. Est vero

$$\int xx\,dx\,(1 - x)^2 = \frac{1}{30},$$

hincque ergo erit $S = 20$, vt supra.

Theorema II.

§. 10. Manentibus litteris α, β, γ, δ, etc. vnciis pro potestate exponentis n, si simili modo litterae α', β', γ', δ', etc. denotent vncias pro potestate exponentis n'; hincque formetur ista series:

$$1 + \alpha\alpha' + \beta\beta' + \gamma\gamma' + \delta\delta' + \text{etc.}$$

eius summa aequabitur isti producto:

$$\frac{n + n'}{1} \cdot \frac{n + n' - 1}{2} \cdot \frac{n + n' - 2}{3} \cdot \frac{n + n' - 3}{4} \cdots \frac{n' + 1}{n},$$

quae eadem summa etiam per sequentes formulas integrales exprimi potest:

siue per $\frac{1}{n\int x^{n'}\,dx\,(1 - x)^{n-1}}$,

siue per $\frac{n + n'}{nn'\int x^{n'-1}\,dx\,(1 - x)^{n-1}}$,

siue per $\frac{1}{(n + n' + 1)\int x^{n'}\,dx\,(1 - x)^{n}}$,

vbi

vbi integralia ab $x = 0$ vsque ad $x = 1$ ſunt extendenda.

Explicatio.

pro caſibus, quibus exponentes n et n' ſunt numeri integri poſitiui.

§. 11. Quo exempla huius theorematis clarius ob oculos ponamus, quoniam caſus, quo $n' = n$, iam in primo theoremate ſunt euoluti, differentiam inter hos exponentes n et n' ſtatuamus primo $= 1$, vt ſit $n' = n + 1$, et percurramus ſequentes caſus:

I. Sit $n = 1$ - - - - - - 1, 1
$n' = 2$ - - - - - - 1, 2, 1

erit ſeries 1, + 2 + 0 = 3.

Cum igitur ſit $n + n' = 3$, productum datum euadit $\frac{3}{1}$, vti requiritur.

II. Sit $n = 2$ - - - - - - 1 + 2 + 1
$n' = 3$ - - - - - - 1 + 3 + 3 + 1

hinc ſeries 1 + 6 + 3 + 0 = 10

verum ob $n + n' = 5$ productum illud ſit $\frac{5}{1} \cdot \frac{4}{2}$.

III. Sit $n = 3$ - - - - - 1 + 3 + 3 + 1
$n' = 4$ - - - - - 1 + 4 + 6 + 4 + 1

ergo ſeries 1 + 12 + 18 + 4 + 0 = 35

at ob $n + n' = 7$ noſtrum productum euadit $= \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}$.

IV. Sit $n = 4$ - - - - $1 + 4 + 6 + 4 + 1$
$n' = 5$ - - - - $1 + 5 + 10 + 10 + 5 + 1$

ergo series $1 + 20 + 60 + 40 + 5 + 0 = 126$

at ob $n + n' = 9$ nostrum productum erit $= \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}$.

V. Sit $n = 5$ - - - - $1 + 5 + 10 + 10 + 5 + 1$
$n' = 6$ - - - - $1 + 6 + 15 + 20 + 15 + 6 + 1$

ergo series $1 + 30 + 150 + 200 + 75 + 6 + 0 = 462$

hinc ob $n + n' = 11$ nostrum productum erit $\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$.

etc.

§. 12. Statuamus nunc $n' = n + 2$ et productum exhibitum fiet

$$\frac{2n+2}{1} \cdot \frac{2n+1}{2} \cdot \frac{2n}{3} \cdot \ldots \ldots \frac{n+1}{n}.$$

Hinc igitur percurramus sequentes casus:

I. Sit $n = 1$ - - - - $1 + 1$
$n' = 3$ - - - - $1 + 3 + 3 + 1$

ergo series $1 + 3 + 0 = 4$

productum autem nostrum fit $\frac{4}{1}$.

II. Sit $n = 2$ - - - - $1 + 2 + 1$
$n' = 4$ - - - - $1 + 4 + 6 + 4 + 1$

ergo series $1 + 8 + 6 + 0 = 15$

at productum nostrum $= \frac{6 \cdot 5}{1 \cdot 2} = 15$.

III. Sit $n = 3$ - - - - $1 + 3 + 3 + 1$
$n' = 5$ - - - - $1 + 5 + 10 + 01 + 5 + 1$

ergo series $1 + 15 + 30 + 01 + 0 = 56$

at productum nostrum fit $= \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$.

IV.

IV. Sit $n = 4$ - - - - $1 + 4 + 6 + 4 + 1$
$n' = 6$ - - - - $1 + 6 + 15 + 20 + 15 + 6$

ergo series $1 + 24 + 90 + 80 + 15 + 0 = 210$
at productum nostrum erit $= \frac{10.\,9.\,8.\,7}{1.\,2.\,3.\,4}$.

V. Sit $n = 5$ - - - - $1 + 5 + 10 + 10 + 5 + 1$
$n' = 7$ - - - - $1 + 7 + 21 + 35 + 35 + 21 + 7$

ergo series $1 + 35 + 210 + 350 + 175 + 21 + 0 = 792$
at productum nostrum fiet $= \frac{12.\,11.\,10.\,9.\,8}{1.\,2.\,3.\,4.\,5} = 792$.

etc. etc.

Explicatio
pro casibus, quibus alter exponens n' est numerus fractus.

§. 13. Sufficiat hic sumsisse $n' = -\frac{1}{2}$, vnde series vnciarum

$$1 + \alpha' + \beta' + \gamma' + \delta' + \text{etc. erit}$$
$$1 - \frac{1}{2} + \frac{1.\,3}{2.\,4} - \frac{1.\,3.\,5}{2.\,4.\,6} + \frac{1.\,3.\,5.\,7}{2.\,4.\,6.\,8} - \text{etc.}$$

His igitur terminis singulatim in seriem

$$1 + \alpha + \beta + \gamma + \delta + \text{etc.}$$

ductis, orietur ista series:

$$1 - \frac{1}{2}\alpha + \frac{1.\,3}{2.\,4}\beta - \frac{1.\,3.\,5}{2.\,4.\,6}\gamma + \frac{1.\,3.\,5.\,7}{2.\,4.\,6.\,8}\delta - \text{etc.}$$

cuius ergo summa aequabitur huic producto:

$$\frac{n - \frac{1}{2}}{1} \cdot \frac{n - \frac{3}{2}}{2} \cdot \frac{n - \frac{5}{2}}{3} \cdots \frac{\frac{1}{2}}{n}$$

siue

siue

$$\frac{2n-1}{2} \cdot \frac{2n-3}{4} \cdot \frac{2n-5}{6} \cdot \frac{2n-7}{8} \cdot \ldots \ldots \frac{1}{2n}$$

quod quomodo eueniat sequentibus casibus examinemus:

I. Sit $n = 1$, hincque $\alpha = 1$, $\beta = 0$, $\gamma = 0$, etc. vnde nostra series erit $1 - \frac{1}{2} = \frac{1}{2}$; at vero nostrum productum erit $= \frac{1}{2}$.

II. Sit $n = 2$, erit $\alpha = 2$; $\beta = 1$; $\gamma = 0$; etc. vnde nostra series prodit $= 1 - \frac{2}{2} + \frac{1.1}{2.4} = \frac{3}{8}$; at vero productum nostrum euadit $= \frac{3}{8}$.

III. Sit $n = 3$, ideoque $\alpha = 3$; $\beta = 3$; $\gamma = 1$; $\delta = 0$ etc. vnde series prodit $= 1 - \frac{3}{2} + \frac{3.3}{2.4} - \frac{3.5}{2.4.6} = \frac{5}{16}$. at vero productum nostrum euadit $= \frac{5}{16}$.

IV. Sit $n = 4$, ideoque $\alpha = 4$; $\beta = 6$; $\gamma = 4$; $\delta = 1$; $\varepsilon = 0$; etc. vnde series erit

$$1 - \frac{1}{2} . 4 + \frac{3}{8} . 6 - \frac{5}{16} . 4 + \frac{35}{128} . 1 = \frac{35}{128},$$

productum autem nostrum erit

$$\frac{7}{2} . \frac{5}{4} . \frac{3}{6} . \frac{1}{8} = \frac{35}{128}.$$

Explicatio
pro casibus, quibus ambo exponentes sunt numeri fracti.

§. 14. Sufficiat hic solum casum euoluisse, quo $n = \frac{1}{2}$ et $n' = -\frac{1}{2}$. Hic igitur pro $n = \frac{1}{2}$ series vnciarum erit:

$$1 + \frac{1}{2} - \frac{1.1}{2.4} + \frac{1.1.3}{2.4.6} - \frac{1.1.3.5}{2.4.6.8} + \text{etc.}$$

verum pro exponente $n' = -\frac{1}{2}$ series vnciarum erit:

$$1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \frac{1.3.5.7}{2.4.6.8} - \text{etc.}$$

Ex

Ex his igitur binis feriebus combinatis orietur feries in theoremate commemorata:

$$1 - \frac{1}{2^2} - \frac{1^2 \cdot 3}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \text{etc.}$$

quae ergo feries in infinitum excurrit; et quoniam n non eft numerus integer, producto theorematis vti non licet; quam ob rem ad formulas integrales in theoremate exhibitas erit recurrendum, quarum prima pro fumma huius feriei praebet $\frac{2}{\int \frac{dx}{\sqrt{(x-xx)}}}$;

fecunda forma dat $\frac{-40}{\int \frac{dx}{x\sqrt{(x-xx)}}}$;

tertia autem forma dat $\frac{1}{\int \frac{dx\sqrt{(1-x)}}{\sqrt{x}}}$.

Vbi quidem haec integralia a termino $x = 0$ vsque ad terminum $x = 1$ funt extendenda, quae quia evndem valorem producere debent, fecundam formulam hic praetermitti conueniet.

§. 15. Euoluamus igitur formulam primam $\frac{2}{\int \frac{dx}{\sqrt{(x-xx)}}}$, pro qua ftatuamus $x = yy$, vt prodeat $\frac{1}{\int \frac{dy}{\sqrt{(1-yy)}}}$. Notum autem eft effe pro terminis affignatis $\int \frac{dy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$, quam ob rem valor noftrae feriei erit $\frac{2}{\pi}$. Tertia autem formula, quae erat $\frac{1}{\int \frac{dx\sqrt{(1-x)}}{\sqrt{x}}}$, pofito $x = yy$ fiet

$$\int \frac{dx}{\sqrt{x}} \sqrt{(1-x)} = 2\int dy \sqrt{(1-yy)}$$

at

at vero haec formula $\int dy\sqrt{(1-yy)}$ exprimit aream quadrantis, cuius radius $=1$, quae cum sit $=\frac{1}{4}\pi$, erit summa nostrae seriei $\frac{2}{\pi}$, vt ante.

§. 16. Hinc igitur patet, summam seriei inuentae esse $=\frac{2}{\pi}$. Quare si seriem breuitatis gratia ita repraesentemus:

$$1-A-B-C-D-F-\text{etc.}=\frac{2}{\pi},\ \text{erit}$$

$$A+B+C+D\ \text{etc.}=1-\frac{2}{\pi}.$$

Supra autem vidimus esse proxime $\frac{4}{\pi}=1,273230$, vnde fieri debet

$$A+B+C+D+\text{etc.}=0,363385;$$

hic autem est

$$A=\frac{1}{4};\ B=\frac{1.3}{16}A;\ C=\frac{3.5}{36}B;\ D=\frac{5.7}{64}C;$$
$$E=\frac{7.9}{100}D;\ F=\frac{9.11}{144}E;\ \text{etc.}$$

Euoluamus igitur singulos hos factores in fractionibus decimalibus, eritque

$$\begin{aligned} A &= 0,250000\\ B &= 0,046875\\ C &= 0,019531\\ D &= 0,010680\\ E &= 0,006729\\ F &= 0,004626\\ \hline \text{Summa} &= 0,338441 \end{aligned}$$

quae adhuc deficit a veritate quantitate 0,024944; quod mirum non est, cum sequentes termini praetermissi; quia enim continuo minus decrescunt, facile tantum discrimen parere possunt.

Scho-

Scholion.

§. 17. Quae hactenus sunt allata et per complura exempla illustrata, veritatem nostrorum theorematum satis comprobare videntur, etiamsi nulla demonstratio directa proferri posset. Quanquam autem nulla via directa patere videtur, istam veritatem perscrutandi, tamen duplici modo ad completam demonstrationem pertingere licet, quorum alter ipsa natura vnciarum innititur, alter vero ex calculo probabilitatum peti potest. Priorem igitur demonstrandi modum hic dilucide exponamus, qui simul nobis innumerabilia alia theoremata affinia patefaciet.

Definitio.

§. 18. Huiusmodi charactere: $[\frac{p}{q}]$, designabimus productum ex q fractionibus formatum, quarum numeratores, a littera superiori p incipientes, continuo vnitate decrescant, denominatores vero ab vnitate incipientes continuo per vnitatem crescant; vnde intelligitur, istum characterem $[\frac{p}{q}]$ designare istud productum more solito expressum:

$$\frac{p}{1} \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \frac{p-3}{4} \cdot \ldots \ldots \cdot \frac{p-q+1}{q}.$$

Corollarium 1.

§. 19. Hac iam ratione vncias singularum potestatum, quas supra litteris α, β, γ, etc. repraesentauimus, sequenti modo satis succincte et eleganter exhibere licebit, quippe quae pro exponente n erunt:

$$\frac{n}{1} = \left[\frac{n}{1}\right]$$
$$\frac{n(n-1)}{1.\ 2} = \left[\frac{n}{2}\right]$$
$$\frac{n(n-1)(n-2)}{1.\ 2.\ 3} = \left[\frac{n}{3}\right]$$
$$\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} = \left[\frac{n}{4}\right]$$
$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5} = \left[\frac{n}{5}\right]$$

etc. etc.

Atque hinc intelligitur, cum pro quauis poteſtate vnciarum omnium prima ſemper ſit vnitas, fore iſto nouo repraeſentandi modo $\left[\frac{n}{0}\right] = 1$. Similique modo, cum vltima vnciarum quoque ſit vnitas, erit etiam $\left[\frac{n}{n}\right] = 1$, propterea quod erit

$$\left[\frac{n}{n}\right] = \frac{n(n-1)(n-2)\ \ldots\ldots\ 1}{1.\ 2.\ 3\ \ldots\ldots\ n}$$

vbi numerator manifeſto denominatori eſt aequalis.

Corollarium 2.

§. 20. Cum, quoties exponens n eſt numerus integer poſitiuus, tam omnes vnciae primam antecedentes, quam vltimam ſequentes, ſint nihilo aequales, iuxta nouum hunc exprimendi modum perpetuo erit

$$\left[\frac{n}{-1}\right] = 0;\ \left[\frac{n}{-2}\right] = 0;\ \left[\frac{n}{-3}\right] = 0;\ \text{etc.}$$

ita vt, denotante i numerum integrum poſitiuum quemcunque, ſemper ſit $\left[\frac{n}{-i}\right] = 0$. Simili modo pro vnciis vltimam ſequentibus ſemper erit

$$\left[\frac{n}{n+1}\right] = 0;\ \left[\frac{n}{n+2}\right] = 0;\ \left[\frac{n}{n+3}\right] = 0;\ \left[\frac{n}{n+4}\right] = 0;\ \text{etc.}$$

atque adeo in genere erit $\left[\frac{n}{n+i}\right] = 0$.

Lem-

Lemma 1.

§. 21. Recepto isto signandi modo semper erit $[\frac{p}{q}]=[\frac{p}{p-q}]$. Cum enim sit

$$[\frac{p}{q}]=\frac{p(p-1)(p-2)(p-3)\ldots\ldots(p-q+1)}{1.\ 2.\ 3.\ 4\ \ldots\ldots\ q};$$

similique modo

$$[\frac{p}{p-q}]=\frac{p(p-1)(p-2)(p-3)\ldots\ldots(q+1)}{1.\ 2.\ 3.\ 4\ \ldots\ldots\ (p-q)},$$

hae duae expressiones manifesto inter se sunt aequales; per crucem enim multiplicando, prior numerator in denominatorem posteriorem ductus praebet productum

$$1.\ 2.\ 3.\ \ldots\ (p-q)(p-q+1)(p-q+2)\ \ldots\ p$$

vbi factores sine vlla interruptione continuo vnitate crescunt, ita vt istud productum sit $1.\ 2.\ 3.\ 4\ \ldots\ldots\ p$. Simili modo denominator prior ductus in numeratorem posteriorem dat istud productum:

$$1.\ 2.\ 3.\ 4\ \ldots\ q.(q+1)(q+2)\ \ldots\ p$$

quod itidem est $1.\ 2.\ 3.\ 4\ \ldots\ldots\ p$, vt ante; sicque aequalitas harum duarum formularum $[\frac{p}{q}]$ et $[\frac{p}{p-q}]$ est demonstrata.

Corollarium.

§. 22. Hoc iam Lemma manifesto continet rationem, cur vnciae omnium ordinum, siue directe siue retro scriptae, eadem lege progrediantur.

Lemma 2.

§. 23. Introducta eadem vncias designandi ratione semper erit

$$[\frac{p}{q}]+[\frac{p}{q-1}]=[\frac{p+1}{q}].$$

Cum

Cum enim ſit

$$\left[\frac{p}{q}\right] = \frac{p(p-1)(p-2)(p-3)}{1.\ 2.\ 3.\ 4} \cdot \cdot \cdot \frac{(p-q+1)}{q} \text{ et}$$

$$\left[\frac{p}{q-1}\right] = \frac{p(p-1)(p-2)(p-3)}{1.\ 2.\ 3.\ 4} \cdot \cdot \cdot \frac{(p-q+2)}{q-1};$$

prior forma aequatur poſteriori ductae in $\frac{p-q+1}{q}$, ideoque erit

$$\left[\frac{p}{q}\right] + \left[\frac{p}{q-1}\right] = \left[\frac{p}{q-1}\right]\left(1 + \frac{p-q+1}{q}\right) = \left[\frac{p}{q-1}\right] \cdot \left(\frac{p+1}{q}\right),$$

quocirca habebimus

$$\left[\frac{p}{q}\right] + \left[\frac{p}{q-1}\right] = \left(\frac{p+1}{q}\right) \cdot \frac{p(p-1)(p-2)}{1.\ 2.\ 3} \cdot \cdot \cdot \cdot \frac{(p-q+2)}{q-1},$$

quae forma manifeſto conuenit cum hac:

$$\frac{(p+1)p(p-1)(p-2)}{1.\ 2.\ 3.\ 4} \cdot \cdot \cdot \cdot \cdot \frac{(p-q+2)}{q}$$

quae ergo, more ſignandi recepto, ita refertur: $\left[\frac{p+1}{q}\right]$, ita vt ſit $\left[\frac{p}{q}\right] + \left[\frac{p}{q-1}\right] = \left[\frac{p+1}{q}\right]$.

Corollarium 1.

§. 24. Si loco q ſcribamus $q+1$, formulis permutatis erit

$$\left[\frac{p}{q}\right] + \left[\frac{p}{q+1}\right] = \left[\frac{p+1}{q+1}\right]$$

ſimilique modo, numerum q continuo vnitate augendo, erit etiam vt ſequitur:

$$\left[\frac{p}{q+1}\right] + \left[\frac{p}{q+2}\right] = \left[\frac{p+1}{q+2}\right];$$

$$\left[\frac{p}{q+2}\right] + \left[\frac{p}{q+3}\right] = \left[\frac{p+1}{q+3}\right];$$

$$\left[\frac{p}{q+3}\right] + \left[\frac{p}{q+4}\right] = \left[\frac{p+1}{q+4}\right];$$

$$\left[\frac{p}{q+4}\right] + \left[\frac{p}{q+5}\right] = \left[\frac{p+1}{q+5}\right];$$

$$\left[\frac{p}{q+5}\right] + \left[\frac{p}{q+6}\right] = \left[\frac{p+1}{q+6}\right];$$

etc. etc.

Corol-

Corollarium 2.

§. 25. Quodſi harum aequalitatum binas ſe inſequentes addamus, prodibunt iſtae nouae aequationes:

$$\left[\tfrac{p}{q}\right]+2\left[\tfrac{p}{q+1}\right]+\left[\tfrac{p}{q+2}\right]=\left[\tfrac{p+1}{q+1}\right]+\left[\tfrac{p+1}{q+2}\right]=\left[\tfrac{p+2}{q+2}\right]$$
$$\left[\tfrac{p}{q+1}\right]+2\left[\tfrac{p}{q+2}\right]+\left[\tfrac{p}{q+3}\right]=\left[\tfrac{p+1}{q+2}\right]+\left[\tfrac{p+1}{q+3}\right]=\left[\tfrac{p+2}{q+3}\right]$$
$$\left[\tfrac{p}{q+2}\right]+2\left[\tfrac{p}{q+3}\right]+\left[\tfrac{p}{q+4}\right]=\left[\tfrac{p+1}{q+3}\right]+\left[\tfrac{p+1}{q+4}\right]=\left[\tfrac{p+2}{q+4}\right]$$
$$\left[\tfrac{p}{q+3}\right]+2\left[\tfrac{p}{q+4}\right]+\left[\tfrac{p}{q+5}\right]=\left[\tfrac{p+1}{q+4}\right]+\left[\tfrac{p+1}{q+5}\right]=\left[\tfrac{p+2}{q+5}\right]$$
$$\left[\tfrac{p}{q+4}\right]+2\left[\tfrac{p}{q+5}\right]+\left[\tfrac{p}{q+6}\right]=\left[\tfrac{p+1}{q+5}\right]+\left[\tfrac{p+1}{q+6}\right]=\left[\tfrac{p+2}{q+6}\right]$$
$$\left[\tfrac{p}{q+5}\right]+2\left[\tfrac{p}{q+6}\right]+\left[\tfrac{p}{q+7}\right]=\left[\tfrac{p+1}{q+6}\right]+\left[\tfrac{p+1}{q+7}\right]=\left[\tfrac{p+2}{q+7}\right]$$
$$\left[\tfrac{p}{q+6}\right]+2\left[\tfrac{p}{q+7}\right]+\left[\tfrac{p}{q+8}\right]=\left[\tfrac{p+1}{q+7}\right]+\left[\tfrac{p+1}{q+8}\right]=\left[\tfrac{p+2}{q+8}\right]$$

etc. etc.

Corollarium 3.

§. 26. Quodſi denuo binas harum aequalitatum ſe inſequentes addamus, reperiemus primo:

$$\left[\tfrac{p}{q}\right]+3\left[\tfrac{p}{q+1}\right]+3\left[\tfrac{p}{q+2}\right]+\left[\tfrac{p}{q+3}\right]=\left[\tfrac{p+2}{q+2}\right]+\left[\tfrac{p+2}{q+3}\right]=\left[\tfrac{p+3}{q+3}\right].$$

Simili modo prodibunt ſequentes aequationes:

$$\left[\tfrac{p}{q+1}\right]+3\left[\tfrac{p}{q+2}\right]+3\left[\tfrac{p}{q+3}\right]+\left[\tfrac{p}{q+4}\right]=\left[\tfrac{p+3}{q+4}\right].$$

Eodemque modo porro:

$$\left[\tfrac{p}{q+2}\right]+3\left[\tfrac{p}{q+3}\right]+3\left[\tfrac{p}{q+4}\right]+\left[\tfrac{p}{q+5}\right]=\left[\tfrac{p+3}{q+5}\right];$$

parique modo vlterius progredi licebit, quousque libuerit; atque hinc ſequens problema reſoluere poterimus.

Problema.

§. 27. Sumtis pro p et q numeris quibuscunque integris poſitiuis, ſi praeterea littera n etiam huiusmodi numerum

 quem-

quemcunque denotet, inuestigare summam istius seriei:

$$\left[\frac{n}{0}\right] \cdot \left[\frac{p}{q}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{p}{q+1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{p}{q+2}\right] + \left[\frac{n}{3}\right] \cdot \left[\frac{p}{q+3}\right] + \text{etc.}$$

Solutio.

Cum sit $\left[\frac{n}{0}\right] = 1$, et istae formulae:

$$\left[\frac{n}{0}\right];\ \left[\frac{n}{1}\right];\ \left[\frac{n}{2}\right];\ \left[\frac{n}{3}\right];\ \text{etc.}$$

exhibeant vncias pro potestate exponentis n, in Corollariis praecedentibus vidimus, fore pro casu $n = 1$

$$\text{I. } \left[\frac{p}{q}\right] + \left[\frac{p}{q+1}\right] = \left[\frac{p+1}{q+1}\right];$$

tum vero pro casu $n = 2$ Corollarium secundum dedit:

$$\text{II. } \left[\frac{p}{q}\right] + 2\left[\frac{p}{q+1}\right] + \left[\frac{p}{q+2}\right] = \left[\frac{p+2}{q+2}\right];$$

deinde pro casu $n = 3$ in Corollario III. inuenimus:

$$\text{III. } \left[\frac{p}{q}\right] + 3\left[\frac{p}{q+1}\right] + 3\left[\frac{p}{q+2}\right] + \left[\frac{p}{q+3}\right] = \left[\frac{p+3}{q+3}\right];$$

atque si in eodem Corollario binas priores aequationes addamus, prodibit pro casu $n = 4$ ista aequatio:

$$\text{IV. } \left[\frac{p}{q}\right] + 4\left[\frac{p}{q+1}\right] + 6\left[\frac{p}{q+2}\right] + 4\left[\frac{p}{q+3}\right] + \left[\frac{p}{q+4}\right] = \left[\frac{p+4}{q+4}\right],$$

vnde iam satis luculenter perspicitur fore pro casu $n = 5$:

$$\text{V. } \left[\frac{p}{q}\right] + 5\left[\frac{p}{q+1}\right] + 10\left[\frac{p}{q+2}\right] + 10\left[\frac{p}{q+3}\right] + 5\left[\frac{p}{q+4}\right] + \left[\frac{p}{q+5}\right] = \left[\frac{p+5}{q+5}\right],$$

atque adeo iam in genere pronunciare licet, seriei in problemate propositae:

$$\left[\frac{n}{0}\right] \cdot \left[\frac{p}{q}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{p}{q+1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{p}{q+2}\right] + \left[\frac{n}{3}\right] \cdot \left[\frac{p}{q+3}\right] + \text{etc.}$$

summam esse $= \left[\frac{p+n}{q+n}\right]$, qua formula indicatur istud productum:

$$\frac{p+n}{1} \cdot \frac{p+n-1}{2} \cdot \frac{p+n-2}{3} \cdot \frac{p+n-3}{4} \cdots \frac{p-q+1}{q+n}.$$

Corol-

Corollarium 1.

§. 28. Cum per Lemma I. in genere sit
$[\frac{p}{q}] = [\frac{p}{p-q}]$,

erit nostrae seriei propositae summa etiam $= [\frac{p+n}{p-q}]$, qua forma exprimitur istud productum:

$$\frac{p+n}{1} \cdot \frac{p+n-1}{2} \cdot \frac{p+n-2}{3} \cdot \frac{p+n-3}{4} \cdot \; - \; - \; - \; - \; \frac{q+n+1}{p-q}.$$

Corollarium 2.

§. 29. Quodsi sumamus $q = 0$, istae formulae:
$[\frac{p}{0}] + [\frac{p}{1}] + [\frac{p}{2}] + [\frac{p}{3}] +$ etc.

exhibebunt vncias pro potestate exponentis p; quae ergo si singulatim ducantur in vncias pro potestate exponentis n, resultabit ista series:

$$[\tfrac{n}{0}] \cdot [\tfrac{p}{0}] + [\tfrac{n}{1}] \cdot [\tfrac{p}{1}] + [\tfrac{n}{2}] \cdot [\tfrac{p}{2}] + [\tfrac{n}{3}] \cdot [\tfrac{p}{3}] + \text{etc.}$$

cuius ergo summa erit $= [\frac{p+n}{n}]$, vel etiam $[\frac{p+n}{p}]$, quarum formularum illa dat istud productum:

$$\frac{p+n}{1} \cdot \frac{p+n-1}{2} + \frac{p+n-2}{3} \cdot \frac{p+n-3}{4} \; - \; - \; - \; - \; \frac{p+1}{n},$$

altera vero formula euoluta aequatur hunc producto:

$$\frac{p+n}{1} \cdot \frac{p+n-1}{2} \cdot \frac{p+n-2}{3} \cdot \frac{p+n-3}{4} \; - \; - \; - \; - \; \frac{n+1}{p},$$

sicque veritas secundi theorematis supra allati est demonstrata, quoniam litterae α, β, γ, etc. ibi denotabant vncias pro exponente n, alterae vero α', β', γ', etc. pro exponente n', cuius loco hic habemus p.

Corollarium 3.

§. 30. Si praeterea capiamus $p = n$, nostra series abibit in eam ipsam, quam in theoremate I sumus contemplatemplati, ſcilicet:

$$1 + [\tfrac{n}{1}]^2 + [\tfrac{n}{2}]^2 + [\tfrac{n}{3}]^2 + \text{etc.}$$

cuius loco ibi habuimus

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \text{etc.},$$

eius ergo ſumma ex hic allatis erit

$$[\tfrac{2n}{n}] = \tfrac{2n}{1} \cdot \tfrac{2n-1}{2} \cdot \tfrac{2n-2}{3} \cdot \tfrac{2n-3}{4} \cdots \tfrac{n+1}{n},$$

ſicque etiam theorema I. rigoroſe eſt demonſtratum.

Scholion.

§. 31. In ipſo quidem theoremate primo ſummam ſeriei per aliud productum, ſcilicet:

$$\tfrac{2}{1} \cdot \tfrac{6}{2} \cdot \tfrac{10}{3} \cdot \tfrac{14}{4} \cdot \tfrac{18}{5} \cdots \tfrac{4n-2}{n},$$

expreſſimus; verum hanc formam cum hic exhibita penitus conuenire facile oſtendi poteſt. Cum enim vtrinque denominatores ſint iidem, demonſtrandum eſt, hoc productum:

$$2n(2n-1)(2n-2) \cdots n+1,$$

ſemper aequale eſſe huic producto:

$$2.\ 6.\ 10.\ 14. \cdots (4n-2).$$

Hunc in finem ponamus prius productum $= P$, ſequens vero, quod oritur, ſi loco n ſcribamus $n+1$, deſignemus littera Q, ita vt ſit

$$Q = (2n+2)(2n+1)\,2n(2n-1)(2n-2) \cdots (n+2),$$

vnde patet fore

$$\tfrac{Q}{P} = \tfrac{(2n+2)(2n+1)}{n+1} = (4n+2), \text{ ideoque}$$

$$Q = (4n+2)P;$$

vnde patet, quomodo ex quouis valore pro n definiatur ſequens valor pro $n+1$. Quare cum pro caſu $n=1$ illud

illud productum P fit $= 2$; sequens productum Q erit $= 2.6$, quod ergo valet pro $n = 2$, quod si iam denuo vocetur $= P$, sequens productum Q erit $= 2.6.10$, pro casu $n = 3$; si hoc denuo designetur per P, erit sequens productum $Q = 2.6.10.14$, pro casu $n = 4$; vnde veritas huius identitatis manifesto elucet. Caeterum problema, quod modo tractauimus, multo latius patet, quam theoremata initio allata, quare operae pretium erit theorema inde natum hic ob oculos exponere.

Theorema generale.

§. 32. Si litterae p, n et q denotent numeros integros quoscunque, huius seriei inde formatae:

$$[\tfrac{n}{0}].[\tfrac{p}{q}] + [\tfrac{n}{1}].[\tfrac{p}{q+1}] + [\tfrac{n}{2}].[\tfrac{p}{q+2}] + \text{etc.}$$

summa semper aequalis est huic formulae: $[\frac{p+n}{q+n}]$, vel etiam huic: $[\frac{p+n}{q-n}]$, quarum illa praebet istud productum:

$$\frac{p+n}{1}.\frac{p+n-1}{2}.\frac{p+n-2}{3} \ldots \frac{p-q+1}{q+n},$$

haec vero istud:

$$\frac{p+n}{1}.\frac{p+n-1}{2}.\frac{p+n-2}{3} \ldots \frac{q+n+1}{p-q}.$$

Corollarium 1.

§. 33. Quodsi ergo istam seriem littera S designemus, ita vt sit

$$S = [\tfrac{n}{0}].[\tfrac{p}{q}] + [\tfrac{n}{1}].[\tfrac{p}{q+1}] + \text{etc.}$$

erit $S = [\frac{p+n}{q+n}]$. Iam loco p scribamus $p+1$, seriemque inde natam per S' referamus, ita vt sit

$$S' = [\tfrac{n}{0}].[\tfrac{p+1}{q}] + [\tfrac{n}{1}].[\tfrac{p+1}{q+1}] + [\tfrac{n}{2}].[\tfrac{p+1}{q+2}] + \text{etc.}$$

 erit

erit $S' = [\frac{p+n+1}{q+n}]$, factaque euolutione erit

$$S' = \frac{p+n+1}{1} \cdot \frac{p+n}{2} \cdot \frac{p+n-1}{3} \cdot - - - - \frac{p-q+2}{q+n},$$

vnde colligitur fore

$$\frac{S'}{S} = \frac{p+n+1}{p-q+1}, \text{ ita vt fit}$$

$$S' = \frac{p+n+1}{p-q+1} \cdot S.$$

Corollarium 2.

§. 34. Simili modo ſi in noſtra ſerie S loco p ſcribamus $p + 2$, ac ſtatuamus:

$$S'' = [\tfrac{n}{0}] \cdot [\tfrac{p+2}{q}] + [\tfrac{n}{1}] \cdot [\tfrac{p+2}{q+1}] + [\tfrac{n}{2}] \cdot [\tfrac{p+2}{q+2}] + [\tfrac{n}{3}] \cdot [\tfrac{p+2}{q+3}] + \text{etc.}$$

erit $S'' = \frac{p+n+2}{p-q+2} S'$, quocirca habebimus nunc

$$S'' = \frac{p+n+1}{p-q+1} \cdot \frac{p+n+2}{p-q+2} \cdot S.$$

Simili modo ſi denuo litteram p vnitate augeamus, ac ſtatuamus:

$$S''' = [\tfrac{n}{0}] \cdot [\tfrac{p+3}{q}] + [\tfrac{n}{1}] \cdot [\tfrac{p+3}{q+1}] + [\tfrac{n}{2}] \cdot [\tfrac{p+3}{q+2}] + \text{etc.}$$

erit

$$S''' = \frac{p+n+3}{p-q+3} \cdot S'',$$

ideoque habebimus nunc

$$S''' = \frac{p+n+1}{p-q+1} \cdot \frac{p+n+2}{p-q+2} \cdot \frac{p+n+3}{p-q+3} \cdot S.$$

Sicque vlterius progredi licet, quousque libuerit, ita vt per valorem S omnes ſequentes: S', S'', S''', S'''', etc. facile exhiberi queant, id quod in ſequentibus inſignem vſum praeſtabit.

Pro-

Problema.

§. 35. Si quaepiam litterarum p, q et n, vel duae, vel adeo omnes fuerint fractiones, in verum valorem seriei propositae:

$$S = [\tfrac{n}{0}] \cdot [\tfrac{p}{q}] + [\tfrac{n}{1}] \cdot [\tfrac{p}{q+1}] + [\tfrac{n}{2}] \cdot [\tfrac{p}{q+2}] + \text{etc.}$$

per formulam integralem inquirere.

Solutio.

Supra in Theoremate II. obseruauimus, valorem istius producti:

$$\frac{n+n'}{1} \cdot \frac{n+n'-1}{2} \cdot \frac{n+n'-2}{3} - - - - \frac{n'+1}{n},$$

aequari vel huic formulae:

$$\text{I.} \quad \frac{1}{n \int x^{n'} dx (1-x)^{n-1}},$$

$$\text{II.} \quad \frac{n+n'}{n n' \int x^{n'-1} dx (1-x)^{n-1}}, \text{ vel huic:}$$

$$\text{III.} \quad \frac{1}{(n+n'+1) \int x^{n'} dx (1-x)^{n}}, \text{ vel etiam huic:}$$

Cum igitur nostro casu sit

$$S = \frac{p+n}{1} \cdot \frac{p+n-1}{2} \cdot \frac{p+n-2}{3} \cdot - - - - \frac{p-q+1}{q+n},$$

comparatione rite instituta patet, quod ibi fuerat n hic esse $q+n$, et quod ibi fuerat $n+n'$ hic esse $p+n$, ideoque quod ibi fuerat n', hic nobis erit $p-q$; quibus notatis triplici modo valorem ipsius S per formulas integrales exprimere poterimus, quae sunt:

$$\text{I.} \quad S = \frac{1}{(q+n) \int x^{p-q} dx (1-x)^{q+n-1}},$$

II.

$$\text{II. } S = \frac{p+n}{(p-q)(q+n)\int x^{p-q-1}\,dx\,(1-x)^{q+n-1}}$$

$$\text{III. } S = \frac{1}{(p+n+1)\int x^{p-q}\,dx\,(1-x)^{q+n}},$$

ſi quidem ſingula haec integralia a termino $x = 0$ vsque ad terminum $x = 1$ extendantur. Perpetuo autem perinde erit, quanam harum trium formularum vti velimus, quandoquidem inter ſe perfecte conueniunt, quemadmodum ex reductione integralium ſatis nota intelligitur. Manifeſtum autem eſt, quaecunque fractiones litteris p, q et n deſignentur, valorem ſummae S ad certam formulam integralem, ſiue quadraturam reuocari.

Corollarium 1.

§. 36. Hinc igitur patet, innumerabiles iſtiusmodi ſeries communem ſummam habere poſſe. Veluti ſi alia quaecunque huius formae ſeries habeatur:

$$\left[\tfrac{N}{0}\right]\cdot\left[\tfrac{P}{Q}\right] + \left[\tfrac{N}{1}\right]\cdot\left[\tfrac{P}{Q+1}\right] + \left[\tfrac{N}{2}\right]\cdot\left[\tfrac{P}{Q+2}\right] + \text{etc.}$$

vt eius ſumma praecedenti euadat aequalis, requiritur primo vt ſit $Q + N = q + n$, ſecundo vt $P - Q = p - q$, ideoque $Q = q + n - N$ et $P = p + n - N$, vbi ergo N arbitrio noſtro relinquitur; ac dummodo litteris P et Q hi valores aſſignentur, ſeries inde reſultans ſemper aequalis erit ſeriei hic ſummatae.

Corollarium 2.

§. 37. Si, vt ante fecimus, loco p ſucceſſiue ſcribamus $p + 1$; $p + 2$; $p + 3$; etc. ac ſummas ſerierum inde de natarum per S', S'', S''', etc. designemus, hae per formulas integrales sequenti modo exprimentur:

$$\text{I. } S' = \frac{p+n+1}{(p-q+1)(q+n)\int x^{p-q}\,dx\,(1-x)^{q+n-1}}$$

$$\text{II. } S'' = \frac{(p+n+1)(p+n)}{(p-q+1)(p-q)(q+n)\int x^{p-q-1}\,dx\,(1-x)^{q+n-1}}$$

$$\text{III. } S''' = \frac{1}{(p-q+1)\int x^{p-q}\,dx\,(1-x)^{q+n}}.$$

Vnde satis liquet, quomodo etiam sequentes valores S'', S''', etc. exprimi debeant.

Scholion.

§. 38. Quodsi ergo litterae p, q et n denotent numeros integros, euidens est, singulas has formulas actu euadere integrabiles, indeque eadem producta enasci, quae supra pro summa S inuenimus; sin autem inter has litteras fractiones occurrant, summatio ad quadraturas eo altiores reducetur, quo magis fractiones fuerint complicatae; inter quas ii casus imprimis notatu digni occurrunt, quos ad arcus circulares reuocare licet, id quod vsu venit in ista formula:

$$\int \frac{x^{\lambda}}{(1-x)^{\lambda}} \cdot \frac{dx}{x}, \text{ siue } \int x^{\lambda-1}\,dx\,(1-x)^{-\lambda},$$

ita vt, comparatione cum prima nostrarum formularum facta, sit

$$p - q = \lambda - 1 \text{ et } q + n - 1 = -\lambda,$$

ideoque

$$p = -n \text{ et } q = 1 - n - \lambda.$$

Pro

Pro formula autem $\int \frac{x^\lambda}{(1-x)^\lambda} \cdot \frac{dx}{x}$ integranda statuamus $\frac{x}{1-x} = y$, fietque hinc $x = \frac{y}{1+y}$ et $\frac{dx}{x} = \frac{dy}{y(1+y)}$, ideoque formula nostra euadet $\int \frac{y^{\lambda-1}\, dy}{1+y}$; vbi notetur, casu $x = 0$ fore $y = 0$, at casu $x = 1$ fore $y = \infty$, ita vt hoc integrale a termino $y = 0$ vsque ad terminum $y = \infty$ capi oporteat. Quia nunc exponentem λ vt fractum spectamus, ponamus $\lambda = \frac{\mu}{\nu}$, et formula integralis erit $\int \frac{y^{\frac{\mu-\nu}{\nu}}\, dy}{1+y}$. Hic statuamus porro $y = z^\nu$, vt fit

$$y^{\frac{\mu-\nu}{\nu}} = z^{\mu-\nu} \text{ et } dy = \nu z^{\nu-1}\, dz,$$

vnde formula integralis erit $\nu \int \frac{z^{\mu-1}\, dz}{1+z^\nu}$. De hac autem formula notum est, eius integrale a termino $z = 0$ vsque ad $z = \infty$ esse $= \frac{\pi}{\sin. \frac{\mu\pi}{\nu}}$; quocirca, quoties fuerit

$$p = -n \text{ et } q = 1 - n - \frac{\mu}{\nu},$$

tum valor nostrae formulae integralis erit

$$\int x^{p-1}\, dx\, (1-x)^{q+n-1} = \frac{\pi}{\sin. \frac{\mu\pi}{\nu}}$$

vnde sequens problema resoluere poterimus.

Problema.

§. 39. Proposita hac serie:

$$S = \left[\tfrac{n}{0}\right]\left[\tfrac{p}{q}\right] + \left[\tfrac{n}{1}\right]\left[\tfrac{p}{q+1}\right] + \left[\tfrac{n}{2}\right]\left[\tfrac{p}{q+2}\right] + \text{etc.}$$

inue-

inuenire relationem inter numeros p, q et n, vt eius summa S per quadraturam circuli exprimi possit.

Solutio.

Inter ternas formulas integrales pro summa huius seriei S supra datas prima erat:

$$S = \frac{1}{(q+n)\int x^{p-q}\,dx\,(1-x)^{q+n-1}};$$

modo autem vidimus, quoties fuerit

$$p = -n \text{ et } q = 1 - n - \frac{\mu}{\nu},$$

toties fore

$$\int x^{p-q}\,dx\,(1-x)^{q+n-1} = \frac{\pi}{\text{sin.}\frac{\mu\pi}{\nu}},$$

quo valore substituto erit

$$S = \frac{\text{sin.}\frac{\mu\pi}{\nu}}{(q+n)\pi} = \frac{\nu\,\text{sin.}\frac{\mu\pi}{\nu}}{(\nu-\mu)\pi}.$$

Quo igitur iste valor locum habeat, duae conditiones requiruntur, quarum prima postulat, vt sit $p = -n$, siue $p + n = 0$, secunda vero, vt sit

$$q = 1 - n - \frac{\mu}{\nu}, \text{ siue } p - q = \frac{\mu}{\nu} - 1.$$

Corollarium.

§. 40. Quodsi ergo istae conditiones locum habeant, si successiue loco p scribamus $p+1$; $p+2$; $p+3$; etc., summae autem nostrae seriei hinc natae per S', S'', S''' etc. designentur, quoniam inuenimus

$$S = \frac{\nu\,\text{sin.}\frac{\mu\pi}{\nu}}{(\nu-\mu)\pi}, \text{ erit}$$

$S' =$

$$S' = \frac{p+n+1}{p-q+s} \cdot \frac{\nu \,\text{fin.}\, \frac{\mu\pi}{\nu}}{(\nu-\mu)\pi} = \frac{\nu^2}{\mu(\nu-\mu)\pi} \cdot \text{fin.}\, \frac{\mu\pi}{\nu};$$

$$S'' = \frac{p+n+2}{p-q+2}\, S' = \frac{2\nu^3}{\mu(\mu+\nu)(\nu-\mu)\pi} \cdot \text{fin.}\, \frac{\mu\pi}{\nu};$$

$$S''' = \frac{p+n+3}{p-q+3}\, S'' = \frac{1.2.3.\nu^4}{\mu(\mu+\nu)(\mu+2\nu)(\nu-\mu)\pi} \cdot \text{fin.}\, \frac{\mu\pi}{\nu};$$

$$S'''' = \frac{p+n+4}{p-q+4}\, S''' = \frac{1.2.3.4.\nu^5}{\mu(\mu+\nu)(\mu+2\nu)(\mu+3\nu)(\nu-\mu)\pi} \cdot \text{fin.}\, \frac{\mu\pi}{\nu}.$$

etc. etc.

Exemplum.

§. 41. Accommodemus haec ad casum theorematis nostri secundi, pro quo statui debet $q = 0$, vt nanciscamur hanc seriem:

$$S = [\tfrac{n}{0}] \cdot [\tfrac{p}{0}] + [\tfrac{n}{1}] \cdot [\tfrac{p}{1}] + [\tfrac{n}{2}] \cdot [\tfrac{p}{2}] + [\tfrac{n}{3}] \cdot [\tfrac{p}{3}] + \text{etc.}$$

Quia autem erat $q = 1 - n - \frac{\mu}{\nu}$, hic erit

$$n = \frac{\nu-\mu}{\nu}, \text{ ideoque } p = \frac{\mu-\nu}{\nu}.$$

Hic autem commode ipsum numerum n in computo retinere poterimus, ita vt sit $p = -n$, tum autem erit

$$\frac{\mu}{\nu} = 1 - n, \text{ siue } \mu = \nu(1-n),$$

ex quo nostra summa erit $S = \frac{\text{fin.}\,(1-n)\pi}{n\pi}$, quae ergo est summa huius seriei:

$$S = 1 + [\tfrac{n}{1}] \cdot [-\tfrac{n}{1}] + [\tfrac{n}{2}] \cdot [-\tfrac{n}{2}] + [\tfrac{n}{3}] \cdot [-\tfrac{n}{3}] + \text{etc.}$$

§. 42. Quodsi nunc numerum p vnitate augeamus, ob $p + 1 = 1 - n$ series nostra erit:

$$S' = 1 + [\tfrac{n}{1}]\, [\tfrac{1-n}{1}] + [\tfrac{n}{2}]\, [\tfrac{1-n}{2}] + [\tfrac{n}{3}]\, [\tfrac{1-n}{3}] + \text{etc.}$$

quamobrem propter $\mu = \nu(1-n)$ erit ista summa:

$S' = \frac{\text{fin.}\,(1-n)\pi}{n(1-n)\pi}$; vel quia $\text{fin.}\,(1-n)\pi = \text{fin.}\, n\pi$

com-

commodius habebimus

$$S = \frac{\sin. n\pi}{n\pi} \quad \text{et} \quad S' = \frac{\sin. n\pi}{n(1-n)\pi}.$$

§. 43. Quodsi iam porro statuamus

$$S'' = 1 + \left[\tfrac{n}{1}\right]\left[\tfrac{2-n}{1}\right] + \left[\tfrac{n}{2}\right]\left[\tfrac{2-n}{2}\right] + \left[\tfrac{n}{3}\right]\left[\tfrac{2-n}{3}\right] + \text{etc.}$$

reperietur ista summa $S'' = \frac{1 \cdot 2 \sin. n\pi}{n(1-n)(2-n)\pi}$. Simili modo si porro statuamus:

$$S''' = 1 + \left[\tfrac{n}{1}\right] \cdot \left[\tfrac{3-n}{1}\right] + \left[\tfrac{n}{2}\right] \cdot \left[\tfrac{3-n}{2}\right] + \left[\tfrac{n}{3}\right] \cdot \left[\tfrac{3-n}{3}\right] + \text{etc.}$$

prodibit ista summa $S''' = \frac{1 \cdot 2 \cdot 3 \sin. n\pi}{n(1-n)(2-n)(3-n)\pi}$. Hocque modo has series, quousque lubuerit, continuare licet.

§. 44. Quodsi iam characteres hic breuitatis gratia introductos more solito euoluamus, prima series hanc induet formam:

$$S = 1 - \frac{nn}{1} + \frac{nn(nn-1)}{1 \cdot 4} - \frac{nn(nn-1)(nn-4)}{1 \cdot 4 \cdot 9} + \frac{nn(nn-1)(nn-4)(nn-9)}{1 \cdot 4 \cdot 9 \cdot 16} - \text{etc.} = \frac{\sin. n\pi}{n\pi},$$

cuius summationis ratio aliunde ita ostendi potest. Diuidatur vtrinque per $nn-1$, fietque

$$\frac{S}{nn-1} = -1 + \frac{nn}{4} - \frac{nn(nn-4)}{1 \cdot 4 \cdot 9} + \frac{nn(nn-4)(nn-9)}{1 \cdot 4 \cdot 9 \cdot 16} - \frac{nn(nn-4)(nn-9)(nn-16)}{1 \cdot 4 \cdot 9 \cdot 16 \cdot 25} + \text{etc.}$$

Diuidamus porro vtrinque per $\frac{nn-4}{4}$, fietque

$$\frac{4S}{(nn-1)(nn-4)} = 1 - \frac{nn}{9} + \frac{nn(nn-9)}{9 \cdot 16} - \frac{nn(nn-9)(nn-16)}{9 \cdot 16 \cdot 25} + \text{etc.}$$

Diuidamus porro per $\frac{nn-9}{9}$, prodibitque

$$\frac{4 \cdot 9 \cdot S}{(nn-1)(nn-4)(nn-9)} = -1 + \frac{nn}{16} - \frac{nn(nn-16)}{16 \cdot 25} + \frac{nn(nn-16)(nn-25)}{16 \cdot 25 \cdot 36} - \text{etc.}$$

Diuidatur porro per $\frac{nn-16}{16}$, ac prodibit

$$\frac{4.9.16.S}{(nn-1)(nn-4)(nn-9)(nn-16)} = 1 - \frac{nn}{25} + \text{etc.}$$

Quare si istae operationes in infinitum continuentur, orietur tandem ista aequatio:

$$\frac{1.4.9.16.25\ldots S}{(nn-1)(nn-4)(nn-9)\,\text{etc.}} = \pm 1;$$

quae ambiguitas signi vt tollatur, in singulis factoribus denominatoris signa immutemus, habebimusque

$$\frac{1.4.9.16\ldots S}{(1-nn)(4-nn)(9-nn)(16-nn)\,\text{etc.}} = +1;$$

quocirca hinc sequitur fore

$$S = \frac{1-nn}{1}\cdot\frac{4-nn}{4}\cdot\frac{9-nn}{9}\cdot\frac{16-nn}{16}\ \text{etc. in infinitum,}$$

cuius producti infiniti valorem esse $= \frac{\sin. n\pi}{n\pi}$ iam satis constat. Si enim capiamus hic $n = \frac{1}{2}$, hinc fiet

$$\frac{2}{\pi} = \frac{1.3}{2.2}\cdot\frac{3.5}{4.4}\cdot\frac{5.7}{6.6}\cdot\frac{7.9}{8.8}\cdot\ \text{etc.}$$

quae est expressio notissima Wallisiana.

Scholion.

§. 45. Hactenus igitur veritatem nostrorum theorematum, quae primo aspectu maxime ardua merito sunt visa, ex planissimis Analyseos principiis solidissime demonstrauimus. Datur autem adhuc alia via ad eundem scopum perducens ex doctrina combinationum deducta, quam quamquam ab instituto non parum aliena videatur, hic clarius exponamus.

Problema.

Si habeatur manipulus schedularum vel chartarum, quarum numerus sit $= s$, inter quas reperiantur n chartae

certis

certis signis notatae, atque ex hoc manipulo forte extrahantur k chartae, inuestigare numeros casuum, quibus vel nulla illarum chartarum notatarum inter istas k chartas extractas reperiatur, vel vnica, vel duae tantum, vel tres, vel quatuor etc. vel adeo omnes n, si quidem numerus n non excedat numerum k.

Solutio.

§. 46. Cum numerus omnium chartarum sit $= s$, si inde vnica charta extraheretur, multitudo varietatum foret $= s$; sin autem duae tantum extraherentur, numerus varietatum foret $\frac{s(s-1)}{1 \cdot 2}$, qui numerus per nostros characteres expressus erit $[\frac{s}{2}]$; sin autem tres chartae extrahantur, numerus varietatum colligitur esse $= \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} = [\frac{s}{3}]$, atque hinc concludimus, si numerus chartarum extractarum fuerit $= k$, numerum omnium varietatum possibilium fore $= [\frac{s}{k}]$.

§. 47. Cum nunc numerus chartarum notarum sit $= n$, quaeramus primo, quot modis euenire queat, vt earum nulla inter k chartas extractas occurrat; ad hoc inueniendum excludamus omnes chartas signatas ex nostro manipulo integro, et numerus reliquarum chartarum erit $= s - n$; vnde si k chartae extrahantur, numerus omnium varietatum erit $= [\frac{s-n}{k}]$, qui numerus omnes continet casus, quibus nulla chartarum notatarum inter extractas reperietur.

§. 48. Inuestigemus nunc, quot modis euenire possit, vt vnica charta notata inter extractas reperiatur; ne-

necesse igitur est, vt reliquae extractae, quarum numerus est $k-1$, sint non notatae, quarum numerus cum sit $s-n$, si inde tantum $k-1$ chartae extrahantur, numerus omnium varietatum erit $=[\frac{s-n}{k-1}]$. Quare si his singulis casibus vnam chartam notatam adiungamus, id quod n variis modis fieri potest, numerus omnium horum casuum erit

$$=n[\tfrac{s-n}{k-1}]=[\tfrac{n}{1}].[\tfrac{s-n}{k-1}].$$

§. 49. Inuestigemus simili modo omnes casus, quibus duae chartae notatae inter k extractas reperientur; reliquae ergo harum chartarum, quarum numerus est $=k-2$, debent esse non notatae, ideoque ex numero chartarum $s-n$ desumtae, vnde numerus omnium varietatum erit $=[\frac{s-n}{k-2}]$; quibus ergo singulis insuper duas chartas notatas adiungi oportebit, id quod casibus $\frac{n(n-1)}{1.\ 2.}$ $=[\frac{n}{2}]$ fieri potest; vnde numerus omnium casuum, quibus duae tantum chartae notatae inter extractas reperientur, erit $[\frac{n}{2}].[\frac{s-n}{k-2}]$. Eodem modo facile patebit, vt tantum tres chartae notatae inter extractas occurrant, numerum omnium casuum possibilium fore $[\frac{n}{3}].[\frac{s-n}{k-3}]$. Porro igitur vt quatuor chartae notatae inter extractas reperiantur, numerus omnium varietatum possibilium erit

$$=[\tfrac{n}{4}].[\tfrac{s-n}{k-4}].$$

§. 50. Hinc igitur in genere concludimus, vt λ chartae notatae inter extractas inueniantur, numerum omnium varietatum possibilium fore $=[\frac{n}{\lambda}].[\frac{s-n}{k-\lambda}]$, qui numerus

merus duplici modo ad nihilum redigetur, primo scilicet, vti initio obseruauimus, si fuerit $\lambda > n$, tum vero etiam si fuerit $\lambda > k$; binis nimirum his casibus talis extractio, qualis desideratur, locum plane habere nequit. Quare si numerus chartarum notatarum n non fuerit maior quam numerus extractarum k, numerus omnium casuum possibilium, quibus omnes n chartae inter extractas occurrent, vbi $\lambda = n$, erit

$$[\tfrac{n}{n}] \cdot [\tfrac{s-n}{k-n}] = [\tfrac{s-n}{k-n}], \text{ ob } [\tfrac{n}{n}] = 1.$$

§. 51. Sin autem numerus chartarum notatarum n maior fuerit quam extractarum k, vltimus casus erit is, quo omnes k chartae extractae simul erunt notatae, quamobrem hic sumi debet $\lambda = k$, et numerus omnium horum casuum possibilium erit

$$[\tfrac{n}{k}] \cdot [\tfrac{s-n}{0}] = [\tfrac{n}{k}], \text{ ob } [\tfrac{s-n}{0}] = 1.$$

§. 52. Quo omnes hos diuersos casus clarius ante oculos exponamus, subiungamus sequentem tabellam, cuius columna prior indicet, quot chartae notatae inter extractas occurrere debeant, posterior vero columna indicat numerum omnium casuum, quibus hoc euenire potest

Chartarum notatarum inter extractas occurrentium numerus	Numerus omnium casuum possibilium, quibus hoc euenire potest
0	$\left[\frac{s-n}{k}\right]$
1	$\left[\frac{n}{1}\right]\cdot\left[\frac{s-n}{k-1}\right]$
2	$\left[\frac{n}{2}\right]\cdot\left[\frac{s-n}{k-2}\right]$
3	$\left[\frac{n}{3}\right]\cdot\left[\frac{s-n}{k-3}\right]$
4	$\left[\frac{n}{4}\right]\cdot\left[\frac{s-n}{k-4}\right]$
5	$\left[\frac{n}{5}\right]\cdot\left[\frac{s-n}{k-5}\right]$
.	.
.	.
.	.
in genere λ	$\left[\frac{n}{\lambda}\right]\cdot\left[\frac{s-n}{k-\lambda}\right]$

quas formulas eo vsque continuari oportet, donec euanescant; atque hinc sponte fluit alia demonstratio theorematum supra allatorum, ac praecipuae theorematis generalis §. 32 allati, quam hic euoluamus.

Demonstratio.

theorematis generalis §. 32. allati.

§. 53. Quodsi numeros casuum in tabula superioris paragraphi assignatos eo vsque continuemus, donec evanescant, eosque omnes in vnam summam colligamus, prodibit numerus omnium plane casuum, quibus vel nulla chartarum notatarum inter extractas reperietur, vel vnica tantum, vel duae, vel tres, vel quatuor, etc. vsque ad ad finem; quae ergo summa aequalis esse debebit numero omnium varietatum, quae in k chartis extractis locum habere possunt, quem numerum vidimus esse $=[\frac{s}{k}]$; quocirca si ponamus:

$$S=[\tfrac{n}{0}]\cdot[\tfrac{s-n}{k}]+[\tfrac{n}{1}]\cdot[\tfrac{s-n}{k-1}]+[\tfrac{n}{2}]\cdot[\tfrac{s-n}{k-2}]+[\tfrac{n}{3}]\cdot[\tfrac{s-n}{k-3}]+\text{etc.}$$

erit $S=[\frac{s}{k}]$.

§. 54. Haec quidem series ab illa, quae in theoremate tractatur adhuc dissidet, verum facile ad hanc formam reduci potest ope nostri lemmatis I., quo erat $[\frac{p}{q}]=[\frac{p}{p-q}]$. Hac enim reductione facta series superior sequentem induet formam:

$$S=[\tfrac{n}{0}]\cdot[\tfrac{s-n}{s-n-k}]+[\tfrac{n}{1}]\cdot[\tfrac{s-n}{s-n-k+1}]+[\tfrac{n}{2}]\cdot[\tfrac{s-n}{s-n-k+2}]$$
$$+[\tfrac{n}{3}]\cdot[\tfrac{s-n}{s-n-k+3}]+\text{etc.}$$

quae ergo summa erit $S=[\frac{s}{k}]$, vel etiam $S=[\frac{s}{s-k}]$.

§. 55. Nunc vero series in theoremate superiori summata erat haec:

$$S=[\tfrac{n}{0}]\cdot[\tfrac{p}{q}]+[\tfrac{n}{1}]\cdot[\tfrac{p}{q+1}]+[\tfrac{n}{2}]\cdot[\tfrac{p}{q+2}]+[\tfrac{n}{3}]\cdot[\tfrac{p}{q+3}]+\text{etc.}$$

ad quam formam seriem hic inuentam reuocabimus, si statuamus $s-n=p$ et $s-n-k=q$, vnde litterae s et k ita determinantur, vt fit $s=p+n$ et $k=p-q$; quocirca per ea, quae hic exposuimus, summa seriei propositae erit $S=[\frac{p+n}{p-q}]$, vel etiam $S=[\frac{p+n}{q+n}]$, quam eandem summam huic seriei in theoremate superiori assignauimus.

SOLV-