1784

De ellipsi minima dato parallelogrammo rectangulo circumscribenda

Leonhard Euler

DE

ELLIPSI MINIMA

DATO PARALLELOGRAMMO RECTANGVLO CIRCVMSCRIBENDA.

Auctore
L. EVLERO.

Cum omni rectangulo infinitae ellipses circumscribi queant, quaestiones non parum curiosae videntur, quibus inter has ellipses vel ea quaeritur, cuius area futura sit minima, vel ea, quae minimam habitura sit perimetrum. Prior quidem nulla prorsus laborat difficultate; solutio tamen nihilominus attentione digna videtur; verum altera quaestio de perimetro maxime est ardua, ita vt eius vix solutionem perfectam exspectare liceat. Eo magis igitur vtile erit, conatus eam resoluendi in medium adferre.

Problema 1.

§. 1. *Circa datum rectangulum* M *m* N *n* *eam ellipsin* A *a* B *b* *describere, cuius area sit omnium minima.* Tab. I. Fig. 1.

Solutio.

Posito centro tam rectanguli quam ellipsis in puncto C vocentur semisses laterum rectanguli $CF = f$ et $CG = g$;

 semi-

semiaxes vero ellipsis sint $CA = a$ et $CB = b$; et quoniam punctum N in ipsa ellipsi est situm, ex elementis constat fore $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$. Cum nunc area ellipsis sit $= \pi . a b$; quantitates a et b ita definiri oportet, vt differentiale areae euanescat, sumtis scilicet semiaxibus a et b variabilibus; vnde nanciscimur $a\,db + b\,da = 0$, siue $da : db = a : -b$, quare ipsa illa aequatio $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$, differentietur et loco da et db scribantur proportionalia a et $-b$, ac prodibit $\frac{g^2}{b^2} = \frac{f^2}{a^2}$, hincque deducimus $\frac{2f^2}{a^2} = 1$, ideoque $a = f\sqrt{2}$ et $b = g . \sqrt{2}$. Definitis autem semiaxibus ipsa ellipsis facillime describitur.

Corollarium.

§. 2. Cum hinc semiaxes lateribus rectanguli prodierint proportionales, sitque $a : b = f : g$; euidens est, si ducantur rectae AB et FG, eas inter se fore parallelas; qua conditione ellipsis iam determinatur.

Scholion.

§. 3. Expedita igitur priore quaestione, alteram ne suscipere quidem licet, nisi ante in genere cuiusque ellipsis perimeter per seriem infinitam ita commode exprimatur, quae pro omnibus speciebus quam maxime conuergat. Etsi autem iam plures huiusmodi series inueniri queant; tamen vix vlla reperietur, quae sequenti, quam sum daturus, palmam praeripiat.

Lemma.

§. 4. Inuenire seriem maxime conuergentem, quae datae ellipsis perimetrum exhibeat.

Sit-

Sit quadrans ellipsis propositae A M B, cuius centrum in C et semiaxes $CA = a$ et $CB = b$. Ex centro C semiaxe maiore C A superstruatur quadrans circuli A L D, cuius ergo radius $AC = a$, ducaturque radius quicunque C L; tum vero ducatur adplicata L P, ellipsin in puncto M intersecans, pro quo vocentur coordinatae $CP = x$ et $PM = y$; et posito angulo $ACL = \phi$ colligimus $CP = x = a.\,\text{cos}.\,\phi$ et $PL = a.\,\text{sin}.\,\phi$. Quia vero est $CD : CB = PL : PM = a : b$, habebimus $y = b\,\text{sin}.\,\phi$, vnde fit $dx = -a\,d\phi.\,\text{sin}.\,\phi$ et $dy = b\,d\phi.\,\text{cos}.\,\phi$, hincque elementum ellipticum Tab. I. Fig. 2.

$$d\phi.\,\surd(a^2\,\text{sin}.\,\phi^2 + b^2\,\text{cos}.\,\phi^2);$$

quocirca integrando arcus ellipticus A M erit

$$= \int.\,d\phi.\,\surd(a^2\,\text{sin}.\,\phi^2 + b^2.\,\text{cos}.\,\phi^2),$$

integrali ita sumto, vt euanescat posito $\phi = 0$. Hinc vero ipse quadrans ellipticus A M B reperietur, statuendo $\phi = 90$. Sicque totum negotium ad idoneam integrationem formulae

$$d\phi\,\surd(a^2\,\text{sin}.\,\phi^2 + b^2\,\text{cos}.\,\phi^2)$$

reducitur. Cum autem sit

$$\text{sin}.\,\phi^2 = \frac{1 - \text{cos}.\,2\phi}{2} \text{ et } \text{cos}.\,\phi^2 = \frac{1 + \text{cos}.\,2\phi}{2}$$

nostra formula integranda abibit in hanc:

$$\int d\phi.\,\surd.\left(\frac{a^2 + b^2}{2} - \frac{(a^2 - b^2)}{2}\,\text{cos}.\,2\phi\right)$$

quam concinniorem reddemus, ponendo

$$a^2 + b^2 = c^2 \text{ et } \frac{a^2 - b^2}{a^2 + b^2} = n;$$

tum enim erit arcus

$$AM = \frac{c}{\surd 2}\int.\,d\phi\,\surd(1 - n\,\text{cos}.\,2\phi)$$

a qua igitur simplici formula rectificatio ellipsis pendet. Conuertamus ergo hanc formulam irrationalem in seriem,

 quae

quae erit

$$\sqrt{(1 - n.\ \text{cof.}\ 2\Phi)} = 1 - \tfrac{1}{2}.\ n.\ \text{cof.}\ 2\Phi$$
$$- \tfrac{1.\ 1}{2.\ 4}\ n^2.\ \text{cof.}\ 2\Phi^2 - \tfrac{1.\ 1.\ 3}{2.\ 4.\ 6}\ n^3\ \text{cof.}\ 2\Phi^3$$
$$- \tfrac{1.\ 1.\ 3.\ 5}{2.\ 4.\ 6.\ 8}\ n^4\ \text{cof.}\ 2\Phi^4 - \text{etc.}$$

Ad has integrationes peragendas in fubfidium vocemus hanc reductionem:

$$\int d\Phi.\ \text{cof.}\ 2\Phi^{\lambda+2} = \tfrac{\lambda+1}{\lambda+2} \int d\Phi.\ \text{cof.}\ 2\Phi^{\lambda}$$
$$+ \tfrac{1}{2\lambda+4}\ \text{fin.}\ 2\Phi.\ \text{cof.}\ 2\Phi^{\lambda+1},$$

vbi terminus poftremus cafu $\Phi = 0$ fponte euanefcit, ita vt nulla conftante opus fit adiicienda. Extendamus autem pro inftituto noftro hanc integrationem vsque ad $\Phi = 90$, quo fit $2\Phi = 180°$, ac denuo euanefcet terminus ille abfolutus; quocirca reductio noftra erit

$$\int d\Phi.\ \text{cof.}\ 2\Phi^{\lambda+2} = \tfrac{\lambda+1}{\lambda+2} \int d\Phi.\ \text{cof.}\ 2\Phi^{\lambda}$$

cuius ope ex binis terminis primoribus noftrae feriei fequentes omnes facillime integrabuntur, a termino fcilicet $\Phi = 0$ vfque ad $\Phi = 90° = \frac{\pi}{2}$; quemadmodum fequens tabula declarat.

$$\int d\Phi = \Phi = \tfrac{\pi}{2}$$
$$\int d\Phi\ \text{cof.}\ 2\Phi = 0$$
$$\int d\Phi.\ \text{cof.}\ 2\Phi^2 = \tfrac{1}{2}.\tfrac{\pi}{2}$$
$$\int d\Phi.\ \text{cof.}\ 2\Phi^3 = 0$$
$$\int d\Phi.\ \text{cof.}\ 2\Phi^4 = \tfrac{1}{2}.\tfrac{3}{4}.\tfrac{\pi}{2}$$
$$\int d\Phi.\ \text{cof.}\ 2\Phi^5 = 0$$
$$\int d\Phi.\ \text{cof.}\ 2\Phi^6 = \tfrac{1.\ 3.\ 5}{2.\ 4.\ 6}.\tfrac{\pi}{2}$$
$$\int d\Phi.\ \text{cof.}\ 2\Phi^7 = 0$$
$$\int d\Phi.\ \text{cof.}\ 2\Phi^8 = \tfrac{1.\ 3.\ 5.\ 7}{2.\ 4.\ 6.\ 8}.\tfrac{\pi}{2}$$

etc.

His

His igitur valoribus introductis quadrans noster ellipticus prodibit

$$AMB = \frac{\pi c}{2\sqrt{2}}\left(1 - \frac{1.1}{2.4}.\frac{1}{2}n^2 - \frac{1.1.3.5}{2.4.6.8}.\frac{1.3}{2.4}n^4 - \frac{1.1.3.5.7.9}{2.4.6.8.10.12}.\frac{1.3.5}{2.4.6}n^6 - \text{etc.}\right).$$

Quia hic singuli coëfficientes numerici praecedentes in se complectuntur, pro hac serie sequentem formam scribamus:

$$AMB = \frac{\pi c}{2\sqrt{2}}(1 - \alpha.n^2 - \alpha\beta.n^4 - \alpha\beta\gamma.n^6 - \alpha\beta\gamma\delta.n^8 - \text{etc.})$$

quarum litterarum valores ita progredientur:

$$\alpha = \frac{1.1}{2.4}.\frac{1}{2} = \frac{1.1}{4.4}$$

$$\beta = \frac{3.5}{6.8}.\frac{3}{4} = \frac{3.5}{8.8}$$

$$\gamma = \frac{7.9}{10.12}.\frac{5}{6} = \frac{7.9}{12.12}$$

$$\varepsilon = \frac{11.13}{14.16}.\frac{7}{8} = \frac{11.13}{16.16}.$$

Pro ellipsi igitur, cuius semiaxes sunt a et b, ponendo breuitatis gratia

$$a^2 + b^2 = c^2 \text{ et } \frac{a^2 - b^2}{a^2 + b^2} = n,$$

quadrans perimetri exprimetur sequenti serie:

$$\frac{\pi c}{2\sqrt{2}}\left(1 - \frac{1.1}{4.4}.n^2 - \frac{1.1.3.5}{4.4.8.8}n^4 - \frac{1.1.3.5.7.9}{4.4.8.8.12.12}n^6 - \text{etc.}\right)$$

quae series semper admodum conuergit, quantumuis axes ellipsis fuerint inter se diuersi, quia semper n est vnitate minor, ac praeterea coëfficientes numerici vehementer decrescunt.

Corollarium.

§. 5. Casus, quo haec series minime conuergit, est, quo $n = 1$, quod euenit, vbi $b = 0$, seu vbi axis coniugatus euanescit; tum autem manifestum est, quadrantem ellipti-

ellipticum ipſi ſemiaxi A C $= a$ aequalem fore; vnde quia etiam $c = a$, habebimus pro hoc caſu

$$\frac{2\sqrt{2}}{\pi} = 1 - \frac{1.1}{4.4} - \frac{1.1.3.5}{4.4.8.8} - \frac{1.1.3.5.7.9}{4.4.8.8.12.12} \text{ etc.}$$

quae ſeries vtique attentione digna videtur, idque eo magis, quod terminis continuo colligendis praebeat ſequentem inſignem aequalitatem:

$$\frac{3.5}{4.4} \cdot \frac{7.9}{8.8} \cdot \frac{11.13}{12.12} \cdot \frac{15.17}{16.16} \text{ etc.} = \frac{2\sqrt{2}}{\pi}$$

ideoque

$$\frac{\pi}{2\sqrt{2}} = \frac{4.4}{3.5} \cdot \frac{8.8}{7.9} \cdot \frac{12.12}{11.13} \text{ etc.}$$

cuius veritas ex iis, quae olim de productis infinitis protuli, facile elucet.

Corollarium 2.

§. 6. Si ſeriem ante inuentam ſtatuamus

$$1 - \frac{1.1}{4.4} n^2 - \frac{1.1.3.5}{4.4.8.8} n^4 - \text{etc.} = s$$

per ea, quae olim de ſummatione huiusmodi ſerierum oſtendi, reperitur iſta aequatio differentialis ſecundi gradus:

$$\frac{4 n\, dds}{dn^2} + \frac{4\, ds}{dn} + \frac{ns}{1 - nn} = 0$$

vbi dn pro conſtante eſt ſumtum. Cum enim hinc ſit

$$\frac{4 n\, dds}{dn^2}(1 - nn) + \frac{4\, ds}{dn}(1 - nn) + ns = 0,$$

ſi fingamus

$$s = 1 + A nn + B n^4 + C n^6 + \text{etc.}$$

erit vt ſequitur:

$$\frac{4 n\, dds}{dn^2} = 4.2.1.A.n + 4.4.3.B.n^3$$
$$- 4.2.1.A.n^3$$
$$+ 4.6.5.C.n^5 + 4.8.7.D.n^7 \text{ etc.}$$
$$- 4.4.3.B.n^5 - 4.6.5.C.n^7 \text{ etc.}$$

4. ds

$$\frac{4\,ds}{dn}(1-nn) = 4.2.An + 4.4.B.n^3 + 4.6.Cn^5$$
$$- 4.2A.n^2 - 4.4.B.n^5$$
$$+ 4.8.D.n^7 \text{ etc.}$$
$$- 4.6.C.n^7 \text{ etc.}$$
$$ns = n + A.n^3 + Bn^5 + C.n^7 \text{ etc.}$$

vnde sequentes determinationes oriuntur:

$o = 4.4.A + 1$; ergo $A = \frac{-1}{4.4}$;

$o = 4.4.4.B - 3.5.A$; ergo $B = \frac{5.5}{8.8}A$;

$o = 4.6.6.C - 7.9.B$; ergo $C = \frac{7.9}{12.12}.B$;

$o = 4.8.8.D - 11.13.C$; ergo $D = \frac{11.13}{16.16}.C$;

etc.

vnde eadem series resultat, quam supra inuenimus.

Problema 2.

§. 7. Circa datum rectangulum $MmNn$ eam ellipsin describere, cuius perimeter sit minima.

Solutio.

Positis vt ante semilateribus rectanguli dati $CF = f$, $CG = g$, et semiaxibus ellipsis quaesitae $CA = a$ et $CB = b$ habemus primo $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$. Tum vero si ponamus Tab. I. Fig. 1.

$$a^2 + b^2 = c^2 \text{ et } \frac{a^2 - b^2}{a^2 + b^2} = n,$$

quadrantem perimetri modo ante inuenimus

$$= \frac{\pi.c}{2\sqrt{2}}(1 - \alpha n^2 - \alpha\beta n^4 - \alpha\beta\gamma n^6 - \alpha\beta\gamma\delta n^8 - \text{etc.}),$$

existente

$$\alpha = \frac{1.1}{4.4};\ \beta = \frac{5.5}{8.8};\ \gamma = \frac{7.9}{12.12};\ \delta = \frac{11.13}{16.16};\ \text{etc.}$$

quae

quae quantitas vt fiat minima, eius differentiale nihilo aequari debet, dum fcilicet litterae c et n tanquam variabiles tractantur, vnde diuidendo per $\frac{\pi}{2\sqrt{2}}$ fequens nafcetur aequatio:

$$\left.\begin{array}{l} dc(1 - \alpha nn - \alpha\beta n^4 - \alpha\beta\gamma n^6 - \alpha\beta\gamma\delta n^8 - \text{etc.}) \\ -cdn(2\alpha n + 4\alpha\beta n^3 + 6\alpha\beta\gamma n^5 + 8\alpha\beta\gamma\delta n^7 + \text{etc.}) \end{array}\right\} = 0$$

quae hoc modo repraefentetur:

$$\frac{dc}{c}(1 - \alpha n^2 - \alpha\beta n^4 - \alpha\beta\gamma.\, n^6 - \alpha\beta\gamma\delta n^8 - \text{etc.})$$
$$= \frac{dn}{n}(2\alpha n^2 + 4\alpha\beta n^4 + 6.\alpha\beta\gamma n^6 + 8.\alpha\beta\gamma\delta.\, n^8 - \text{etc.})$$

Hic autem differentialia dc et dn certam inter fe tenere debent relationem, quam ex aequatione fundamentali $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$, peti oportet. Hic igitur ante omnia loco a et b litteras c et n introduci conuenit. Cum enim fit

$$a^2 + b^2 = c^2 \text{ et } a^2 - b^2 = c^2 n, \text{ erit}$$
$$a^2 = \tfrac{1}{2}c^2(1 + n) \text{ et } b^2 = \tfrac{1}{2}c^2(1 - n);$$

vnde noftra aequatio abibit in hanc:

$$\frac{2f^2}{1+n} + \frac{2g^2}{1-n} = c^2.$$

Quia f et g dantur, ponamus breuitatis gratia

$$f^2 + g^2 = b^2 \text{ et } \frac{f^2 - g^2}{f^2 + g^2} = i, \text{ fiue } f^2 - g^3 = b^2.\, i;$$

quo facto noftra aequatio erit $\frac{2b^2(1 - in)}{1 - nn} = c^2$, cuius fumamus logarithmos, eritque

$$l\, 2b^2 + l(1 - in) - l(1 - nn) = 2.\, lc.$$

quae aequatio differentiata dat

$$\frac{-idn}{1 - in} + \frac{2ndn}{1 - nn} = \frac{2dc}{c},$$

vnde fit

$$\frac{dc}{c} = \frac{ndn}{1 - nn} - \frac{idn}{2(1 - in)} = \frac{2n - i - inn}{2(1 - in)(1 - nn)}.\, dn$$

quo

quo valore ſubſtituto noſtra aequatio erit

$$(2n^2 - in - in^3)(1 - \alpha n^2 - \alpha\beta n^4 - \alpha\beta\gamma . n^6 - \alpha\beta\gamma\delta . n^8 \text{ etc.})$$
$$= 2(1 - in)(1 - nn)(2\alpha n^2 + 4\alpha\beta n^4 + 6\alpha\beta\gamma n^6 \text{ etc.})$$

in qua aequatione tantum duae inſunt quantitates i et n, quarum illa ex rectangulo datur, haec autem n ex illa debet definiri; quod ergo non aliter niſi reſolutione aequationis infinitae fieri poteſt. Conueniet igitur quantitatem n tanquam cognitam ſpectare indeque viciſſim i definire, quod ſi pro pluribus caſibus inſtituatur, facile iudicare licebit, quinam valor ipſius n cuiuis valori dato i reſpondeat.

Spectemus primo quantitatem n vt minimam, quippe cui etiam valor minimus ipſius i reſpondebit; reiectis ergo terminis β, γ, δ, etc. continentibus, habebimus

$$(2n^2 - in - in^3)(1 - \alpha n^2) = 2(1 - in)(1 - nn) . 2\alpha n^2$$

quae aequatio, reiectis poteſtatibus ipſius n tertio altioribus, abit in hanc:

$$2n^2 - 4\alpha n^3 - in - in^3 + 5\alpha i n^3 = 0,$$

vnde prodit

$$i = \frac{2n(1 - 2\alpha)}{1 + n^2 - 5\alpha . n^2}.$$

Quare ſi n fuerit fractio valde parua, erit proxime

$$i = 2n(1 - 2\alpha) = \tfrac{7}{4} . n$$

vnde viciſſim concludere licet, ſi i fuerit fractio quam minima, fore $n = \tfrac{4}{7} i$.

Nunc igitur aliquanto propius ad veritatem accedamus, reiiciendo poteſtates ipſius n quinta ſuperiores, eritque

$$2n^2 - 4\alpha n^2 + 2\alpha n^4 - 8 . \alpha\beta n^4 - in - in^3 + 5\alpha i n^3 = 0,$$

vnde

vnde elicitur

$$i = \frac{2n(1-2\alpha)+2\alpha n^3(1-4\beta)}{1+n^2-5\alpha n^2},$$

et diuidendo

$$i = 2n(1-2\alpha) + 2\alpha n^3(1-4\beta) \\ - 2n^3(1-2\alpha)(1-5\alpha),$$

qui valor, ob $\alpha = \frac{1}{16}$ et $\beta = \frac{15}{64}$, praebet

$$i = \tfrac{7}{4}.\,n - \tfrac{153}{128}.\,n^3,$$

vnde viciſſim pro dato i concludimus

$$n = \tfrac{4}{7}\,i + \tfrac{306}{2401}.\,i^3$$

ſicque vlterius ad veritatem accedere licebit.

Euoluamus adhuc caſum, quo n maximum obtinet valorem, qui vnitati eſt aequalis et ponamus breuitatis gratia

$$1-\alpha-\alpha\beta-\alpha\beta\gamma\,.\,.\,.\,.=s \text{ et}$$
$$2\alpha+4\alpha\beta+6\alpha\beta\gamma\,.\,.\,.\,.=t$$

vt habeamus hanc aequationem:

$$2(1-i)s = 2(1-i)(1-nn)t, \text{ ſiue}$$
$$2(1-i)(s-(1-nn)t) = 0$$

vnde prior factor manifeſto dat $i=1$, id quod rei natura poſtulat; ſi enim latitudo ellipſis euaneſcat, hoc eſt ſi $b=0$, tum etiam latitudo rectanguli debet euaneſcere. Cum igitur euoluerimus tam caſus, quibus litterae i et n ſunt quam minimae, quam eum, vbi maximum ſortiuntur valorem $=1$, pro reliquis caſibus iudicium haud erit difficile. Cum enim pro exiguis valoribus habeamus $n=\frac{4}{7}i$, pro maximo autem $n=i$; in genere non multum a veritate aberrabimus, ſi ſtatuamus

$n =$

$$n = \frac{\frac{4}{7} i}{1 - \frac{3}{7} i i} = \frac{4 i}{7 - 3 . i i}.$$

Quamdiu enim i eſt fractio valde parua, erit $n = \frac{4}{7} i$; et ſi aliquanto maius fuerit, erit $n = \frac{4}{7} i + \frac{12}{49} i^3$, qui valor illo $n = \frac{4}{7} i + \frac{306}{2401} i^3$ aliquantillum maior eſt; verum vbi fit $i = 1$, iterum prodit $n = 1$.

Corollarium.

§. 8. Quo naturam huius ſolutionis penitius perſpiciamus, ponamus in genere:

$$1 - \alpha n^2 - \alpha \beta n^4 - \alpha \beta \gamma n^6 \ldots\ldots = s,$$

$$2 \alpha n^2 + 4 \alpha \beta n^4 + 6 \alpha \beta \gamma n^6 \ldots\ldots = t$$

vt noſtra aequatio fiat

$$(2 n^2 - i n - i n^3) s = 2 (1 - i n - n^2 + i n^3) t,$$

vnde ſequitur

$$i = \frac{2 n^2 s - 2 (1 - n^2) t}{n (1 + n^2) s - 2 n (1 - n^2) t},$$

vnde caſu $n = 1$ manifeſto fit $i = 1$. Pro alio vero caſu quocunque pro n aſſumto, quia ambas ſeries s et t ſatis expedite ſummare licet, verus valor ipſius i ſatis exacte definiri poterit.

Corollarium 2.

§. 9. Ambae ſeries, quae in noſtram ſolutionem ingrediuntur, manifeſto ita a ſe inuicem pendent, vt ſit $t = -\frac{n d s}{d n}$. Nunc autem ſtatuamus $t = s z$, vt fiat

$$i = \frac{2 n^2 - 2 (1 - n^2) z}{n (1 + n^2) - 2 n (1 - n^2) z},$$

ita vt hic vnicum valorem z quaeri oporteat, id quod ſe-

ſequenti modo fieri poterit. Cum ſit $t = -\frac{n\,ds}{dn}$, erit nunc $sz = -\frac{n\,ds}{dn}$, hincque $\frac{ds}{s} = -\frac{z\,dn}{n}$, et differentiando

$$\frac{dds}{s} - \frac{ds^2}{s^2} = + \frac{z\,dn^2}{n^2} - \frac{dz.dn}{n}.$$ Addatur

$$\frac{ds^2}{s^2} = \frac{z^2\,dn^2}{n^2},$$ fietque

$$\frac{dds}{s} = \frac{z\,dn^2}{n^2} + \frac{z^2\,dn^2}{n^2} - \frac{dz\,dn}{n}.$$

Supra autem dedimus hanc inter s et n aequationem:

$$\frac{4n.dds}{dn^2} + \frac{4n\,ds}{dn} + \frac{ns}{1-nn} = 0,$$

quae per s diuiſa praebet

$$\frac{4n}{dn^2}.\frac{dds}{s} + \frac{4}{dn}.\frac{ds}{s} + \frac{n}{1-nn} = 0;$$

hic autem valores modo inuenti ſubſtituti producunt

$$\frac{4.z}{n} + \frac{4.z^2}{n} - \frac{4\,dz}{dn} - \frac{4z}{n} + \frac{n}{1-nn} = 0,$$ vnde fit

$$dz = \frac{n\,dn}{4(1-nn)} + \frac{z^2\,dn}{n},$$

quae eſt aequatio differentialis tantum primi gradus, ſimilis formae, qua olim aequatio Riccatiana referri eſt ſolita. Semper autem reſolutionem ita inſtitui decet, vt littera n tanquam cognita ſpectetur, et pro ea debitus valor ipſius z eruatur; tum vero inde valor litterae i determinetur. Sic enim viciſſim affirmare poterimus, huic ipſi valori ipſius i aſſumtum valorem litterae n conuenire; quo inuento, cum ſit $c^2 = \frac{2b^2(1-in)}{1-nn}$, etiam innoteſcit quantitates c; at denique ex c et n ipſi ſemiaxes a et b definiuntur.

Corollarium 3.

§. 10. Quo omnes iſti calculi facilius expediri queant, valores coëfficientium α, β, γ, δ, in fractionibus decimalibus exhibeamus, vna cum earum logarithmis:

$l\alpha =$

$l\alpha = 8,7958800$; $\alpha = 0,0625000$.
$l\alpha\beta = 8,1657913$; $\alpha\beta = 0,0146484$.
$l\alpha\beta\gamma = 7,8067693$; $\alpha\beta\gamma = 0,0064087$.
$l\alpha\beta\gamma\delta = 7,5538653$; $\alpha\beta\gamma\delta = 0,0035799$.
$l\alpha\beta\gamma\delta\varepsilon = 7,3583455$; $\alpha\beta\gamma\delta\varepsilon = 0,0022821$.
$l\alpha\beta\gamma\delta\varepsilon\zeta = 7,1988959$; $\alpha\beta\gamma\delta\varepsilon\zeta = 0,0015808$.

Exemplum.

§. 11. Euoluamus casum, quo semiaxes ellipsis a et b sunt in ratione dupla, siue $a : b = 2 : 1$, vnde fit

$$n = \frac{a^2 - b^2}{a^2 + b^2} = \frac{3}{5}, \text{ hinc } n^2 = \frac{9}{25},$$

vnde valores singulorum terminorum ita se habebunt pro vtraque serie s et t:

$\alpha n^2 = 0,0225000$	$2\alpha n^2 = 0,0450000$
$\alpha\beta n^4 = 0,0018984$	$4\alpha\beta n^4 = 0,0073934$
$\alpha\beta\gamma n^6 = 0,0002990$	$6\alpha\beta\gamma n^6 = 0,0017940$
$\alpha\beta\gamma\delta n^8 = 0\ 0000601$	$8\alpha\beta\gamma\delta n^8 = 0,0004808$
$\alpha\beta\gamma\delta\varepsilon n^{10} = 0,0000138$	$10\alpha\beta\gamma\delta\varepsilon n^{10} = 0,0001380$
$\alpha\beta\gamma\delta\varepsilon\zeta n^{12} = 0,0000034$	$12\alpha\beta\gamma\delta\varepsilon\zeta n^{12} = 0,0000408$
- - - - 10	- - - - 215
0,0247757	$t = 0,0550685$
ergo $s = 0,9752242$	

Hinc reperitur $\log. z = 8,7517967$. Cum nunc sit

$$n\,i = \frac{3}{5}\,i = \frac{2n^2 - 2(1 - n^2)z}{1 + n^2 - 2(1 - n^2)z} = \frac{18 - 32.z}{34 - 32.z},$$

reperiemus

$$\tfrac{3}{5}\,i = 0,50300, \text{ hincque } i = 0,838333.$$

Hinc ergo viciſſim, ſi pro dato rectangulo fuerit

$$i = \frac{f^2 - g^2}{f^2 + g^2} = 0,838333,$$

tum

tum pro ellipsi satisfaciente erit

$$n = \frac{3}{5}, \quad \text{siue } a : b = 2 : 1.$$

Problema 3.

§. 12. Si data fuerit species rectanguli, cui ellipsin minimae perimetri circumscribi oporteat, eius speciem per aequationem finitam definire.

Solutio.

Cum species rectanguli contineatur ratione inter eius latera, littera nostra i eius speciem declarat, cum sit $i = \frac{f^2 - g^2}{f^2 + g^2}$; deinde quia species ellipsis ratione inter eius axes indicatur, ea in nostra littera n comprehendetur, cum sit $n = \frac{a^2 - b^2}{a^2 + b^2}$. Requiritur ergo aequatio finitis terminis expressa, quae relationem inter has quantitates i et n exhibeat. Nunc autem vidimus esse

$$i\,n = \frac{2n^2 - 2(1 - n^2)z}{1 + n^2 - 2(1 - n^2)z},$$

vbi z per hanc aequationem differentialem determinatur:

$$dz = \frac{n\,dn}{4(1 - nn)} + \frac{z^2.\,dn}{n}.$$

Nihil aliud igitur superest, nisi vt hinc littera z eliminetur. Quod quo facilius fieri possit, ponamus $2(1 - n^2)z = x$, vt fiat $i\,n = \frac{2n^2 - x}{1 + n^2 - x}$; at ob

$$z = \frac{x}{2(1 - nn)} \text{ erit } dz = \frac{dx}{2(1 - nn)} + \frac{n\,x\,dn}{(1 - nn)^2};$$

vnde orietur

$$2(1 - nn)dx + 4nx\,dn = (1 - nn)n\,dn + \frac{x^2.\,dn}{n}.$$

Ex illa autem aequatione, ponendo v loco $i\,n$, nascitur

$$x = \frac{2n^2 - (1 + n^2)v}{1 - v}, \text{ hincque}$$

$dx =$

$$dx = -\frac{dv(1-nn)}{(1-v)^2} + \frac{4ndn - 2nvdn}{1-v},$$

quo ſubſtituto prodit

$$-\frac{2dv(1-nn)^2}{(1-v)^2} + 8ndn = (1-nn)ndn$$
$$+ \frac{4n^4 - 4n^2(1+n^2)v + (1+n^2)^2v^2}{(1-v)^2} \cdot \frac{dn}{n},$$

et fractionibus ſublatis hinc colligitur

$$-\frac{2dv(1-nn)^2}{dn} = -7n + 3n^3 + 10.nv + \frac{1}{n}v^2$$
$$- 2n^3.v - 5n.v^2.$$

Nunc denique loco v ſuum valorem in ſubſtituamus, ac prodit

$$-\frac{2n(1-nn)^2.di}{dn} = -7.n + 3n^3$$
$$+ 2i(1+3nn)$$
$$+ i^2.n(1-5.n^2).$$

Quodſi igitur conſtructio huius aequationis concedatur, non ſolum pro ſingulis valoribus ipſius n conuenientes valores pro i, ſed etiam viciſſim pro ſingulis valoribus ipſius i reſpondentes ipſius n elicere licebit. Haecque profecto aequatio multo ſimplicior euaſit, quam initio ſperare licuiſſet. Quodſi eam adeo integrare, vel ſaltem conſtruere liceret, analyſi inſigne incrementum acceſſiſſe foret cenſendum.