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Quomodo sinus et cosinus angulorum multiplorum per producta exprimi queant

Leonhard Euler

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ex qua deducimus ipsam aequationem nostram demonstrandam

$$\frac{1}{s} = \cot. s + \tfrac{1}{3}\,\mathrm{tag.}(30^\circ + \tfrac{s}{3}) + \tfrac{1}{9}\,\mathrm{tag.}(30^\circ + \tfrac{s}{9}) + \tfrac{1}{27}\,\mathrm{tag.}\,30^\circ + \tfrac{s}{27} + \text{etc.}$$
$$- \tfrac{1}{3}\,\mathrm{tag.}(30^\circ - \tfrac{s}{3}) - \tfrac{1}{9}\,\mathrm{tag.}(30^\circ - \tfrac{s}{9}) - \tfrac{1}{27}\,\mathrm{tag.}\,30^\circ - \tfrac{s}{27} - \text{etc.}$$

§. 13. Quin etiam simili modo huiusmodi series pro maioribus rationibus, quibus arcus s continuo diminuitur, exhibere licet. Cum enim sit

$$\sin 4\Phi = 8 \sin. \Phi \cos. (45^\circ + \Phi) \cos. (45^\circ - \Phi) \cos. \Phi$$

pro ratione quadrupla erit

$$\frac{1}{s} = \cot. s + \tfrac{1}{4}\,\mathrm{tag.}\,\tfrac{s}{4} + \tfrac{1}{16}\,\mathrm{tag.}\,\tfrac{s}{16} + \tfrac{1}{64}\,\mathrm{tag.}\,\tfrac{s}{64} + \text{etc.}$$
$$+ \tfrac{1}{4}\,\mathrm{tag.}(45^\circ + \tfrac{s}{4}) + \tfrac{1}{16}\,\mathrm{tag.}(45^\circ + \tfrac{s}{16}) + \tfrac{1}{64}\,\mathrm{tag}(45^\circ + \tfrac{s}{64}) + \text{etc.}$$
$$- \tfrac{1}{4}\,\mathrm{tag.}(45^\circ - \tfrac{s}{4}) - \tfrac{1}{16}\,\mathrm{tag.}(45^\circ - \tfrac{s}{16}) - \tfrac{1}{64}\,\mathrm{tag.}(45^\circ - \tfrac{s}{64}) - \text{etc.}$$

Porro cum sit

$$\sin. 5\Phi = 16 \sin. \Phi \cos (18^\circ + \Phi) \cos. (18^\circ - \Phi) \cos. (54^\circ + \Phi) \cos. (54^\circ - \Phi)$$

reperiemus pro ratione quintupla

$$\frac{1}{s} = \cot. s + \tfrac{1}{5}\,\mathrm{tag.}(18^\circ + \tfrac{s}{5}) + \tfrac{1}{25}\,\mathrm{tag.}(18^\circ + \tfrac{s}{25})$$
$$- \tfrac{1}{5}\,\mathrm{tag.}(18^\circ - \tfrac{s}{5}) - \tfrac{1}{25}\,\mathrm{tag.}(18^\circ - \tfrac{s}{25})$$
$$+ \tfrac{1}{5}\,\mathrm{tag.}(54^\circ + \tfrac{s}{5}) + \tfrac{1}{25}\,\mathrm{tag.}(54^\circ + \tfrac{s}{25}) \quad \text{etc.}$$
$$- \tfrac{1}{5}\,\mathrm{tag.}(54^\circ - \tfrac{s}{5}) - \tfrac{1}{25}\,\mathrm{tag.}(54^\circ - \tfrac{s}{25}).$$

Pari modo vlterius progredi liceret, verum series resultarent nimis perplexae quam vt attentione dignae viderentur.

QVO-

m nostram de-
tag. 30°+s/27 + etc.
tag. 30°−s/27 − etc.
huiusmodi series
s continuo dimi-
(45° − Φ) cos. Φ
tc.
tag (45°+s/64) + etc.
tag. (45°−s/64) − etc.
54°+Φ)cos.(54°−Φ)
(18° + s/25)
(18° − s/25)
(54° + s/25) etc.
(54° − s/25).
verum series re-
tione dignae vide-
QVO-

QVOMODO SINVS ET COSINVS
ANGVLORVM MVLTIPLORVM
PER PRODVCTA EXPRIMI QVEANT.

§. 1.

Proposito angulo quocunque Φ ponatur breuitatis gratia:

$$\cos. \Phi + \sqrt{-1} \sin. \Phi = p \quad \text{et} \quad \cos. \Phi - \sqrt{-1} \sin. \Phi = q,$$

erit $pq = 1$; tum vero

$$p^n = \cos. n\Phi + \sqrt{-1} \sin. n\Phi \quad \text{et} \quad q^n = \cos. n\Phi - \sqrt{-1} \sin. n\Phi,$$

vnde fit

$$p^n + q^n = 2 \cos. n\Phi \quad \text{et} \quad p^n - q^n = 2\sqrt{-1} \sin. n\Phi;$$

Res igitur eo redit, vt formulae $p^n + q^n$ et $p^n - q^n$ in factores resoluantur.

§. 2. Consideremus primo formulam

$$p^n + q^n = 2 \cos. n\Phi,$$

quae, quoties n est numerus impar, factorem habet simplicem $p + q - 2\cos. \Phi$, ita vt his casibus $\cos. \Phi$ sit factor ipsius $\cos. n\Phi$: Pro reliquis factoribus autem ponamus factorem duplicem in genere esse, $pp - 2pq \cos. \omega + qq$, ita vt formula $p^n + q^n$ euanescat, posito ex qua deducimus ipſam aequationem noſtram demonſtrandam

$$\frac{1}{s} = \cot. s + \tfrac{1}{3}\,\text{tag.}\left(30^\circ + \tfrac{s}{3}\right) + \tfrac{1}{9}\,\text{tag.}\left(30^\circ + \tfrac{s}{9}\right) + \tfrac{1}{27}\,\text{tag.}\,30^\circ + \tfrac{s}{27} + \text{etc.}$$
$$- \tfrac{1}{3}\,\text{tag.}\left(30^\circ - \tfrac{s}{3}\right) - \tfrac{1}{9}\,\text{tag.}\left(30^\circ - \tfrac{s}{9}\right) - \tfrac{1}{27}\,\text{tag.}\,30^\circ - \tfrac{s}{27} - \text{etc.}$$

§. 13. Quin etiam ſimili modo huiusmodi ſeries pro maioribus rationibus, quibus arcus s continuo diminuitur, exhibere licet. Cum enim ſit

$$\text{ſin}\,4\Phi = 8\,\text{ſin.}\,\Phi\,\text{coſ.}\,(45^\circ + \Phi)\,\text{coſ.}\,(45^\circ - \Phi)\,\text{coſ.}\,\Phi$$

pro ratione quadrupla erit

$$\frac{1}{s} = \cot. s + \tfrac{1}{4}\,\text{tag.}\,\tfrac{s}{4} + \tfrac{1}{16}\,\text{tag.}\,\tfrac{s}{16} + \tfrac{1}{64}\,\text{tag.}\,\tfrac{s}{64} + \text{etc.}$$
$$+ \tfrac{1}{4}\,\text{tag.}\left(45^\circ + \tfrac{s}{4}\right) + \tfrac{1}{16}\,\text{tag.}\left(45^\circ + \tfrac{s}{16}\right) + \tfrac{1}{64}\,\text{tag}\left(45^\circ + \tfrac{s}{64}\right) + \text{etc.}$$
$$- \tfrac{1}{4}\,\text{tag.}\left(45^\circ - \tfrac{s}{4}\right) - \tfrac{1}{16}\,\text{tag.}\left(45^\circ - \tfrac{s}{16}\right) - \tfrac{1}{64}\,\text{tag.}\left(45^\circ - \tfrac{s}{64}\right) - \text{etc.}$$

Porro cum ſit

$$\text{ſin.}\,5\Phi = 16\,\text{ſin.}\,\Phi\,\text{coſ}\,(18^\circ + \Phi)\,\text{coſ.}\,(18^\circ - \Phi)\,\text{coſ.}\,(54^\circ + \Phi)\,\text{coſ.}\,(54^\circ - \Phi)$$

reperiemus pro ratione quintupla

$$\frac{1}{s} = \cot. s + \tfrac{1}{5}\,\text{tag.}\left(18^\circ + \tfrac{s}{5}\right) + \tfrac{1}{25}\,\text{tag.}\left(18^\circ + \tfrac{s}{25}\right)$$
$$- \tfrac{1}{5}\,\text{tag.}\left(18^\circ - \tfrac{s}{5}\right) - \tfrac{1}{25}\,\text{tag.}\left(18^\circ - \tfrac{s}{25}\right)$$
$$+ \tfrac{1}{5}\,\text{tag.}\left(54^\circ + \tfrac{s}{5}\right) + \tfrac{1}{25}\,\text{tag.}\left(54^\circ + \tfrac{s}{25}\right) \quad \text{etc.}$$
$$- \tfrac{1}{5}\,\text{tag.}\left(54^\circ - \tfrac{s}{5}\right) - \tfrac{1}{25}\,\text{tag.}\left(54^\circ - \tfrac{s}{25}\right).$$

Pari modo vlterius progredi liceret, verum ſeries reſultarent nimis perplexae quam vt attentione dignae viderentur.

QVO-

m noſtram de-
tag. 30° + s/27 + etc.
tag. 30° − s/27 − etc.
huiusmodi ſeries
s continuo dimi-
(45° − Φ) coſ. Φ
tc.
tag (45° + s/64) + etc.
ag. (45° − s/64) − etc.
54° + Φ) coſ. (54° − Φ)
(18° + s/25)
(18° − s/25)
(54° + s/25) etc.
(54° − s/25).
verum ſeries re-
tione dignae vide-
QVO-

QVOMODO SINVS ET COSINVS
ANGVLORVM MVLTIPLORVM
PER PRODVCTA EXPRIMI QVEANT.

§. 1.

Propoſito angulo quocunque Φ ponatur breuitatis gratia:

$$\text{coſ.}\,\Phi + \sqrt{-1}\,\text{ſin.}\,\Phi = p \quad \text{et} \quad \text{coſ.}\,\Phi - \sqrt{-1}\,\text{ſin.}\,\Phi = q,$$

erit $pq = 1$; tum vero

$$p^n = \text{coſ.}\,n\Phi + \sqrt{-1}\,\text{ſin.}\,n\Phi \quad \text{et} \quad q^n = \text{coſ.}\,n\Phi - \sqrt{-1}\,\text{ſin.}\,n\Phi,$$

vnde fit

$$p^n + q^n = 2\,\text{coſ.}\,n\Phi \quad \text{et} \quad p^n - q^n = 2\sqrt{-1}\,\text{ſin.}\,n\Phi;$$

Res igitur eo redit, vt formulae $p^n + q^n$ et $p^n - q^n$ in factores reſoluantur.

§. 2. Conſideremus primo formulam

$$p^n + q^n = 2\,\text{coſ.}\,n\,\Phi,$$

quae, quoties n eſt numerus impar, factorem habet ſimplicem $p + q - 2\,\text{coſ.}\,\Phi$, ita vt his caſibus coſ. Φ ſit factor ipſius coſ. $n\,\Phi$: Pro reliquis factoribus autem ponamus factorem duplicem in genere eſſe, $pp - 2pq\,\text{coſ.}\,\omega + qq$, ita vt formula $p^n + q^n$ euaneſcat, poſito

$$pp - 2pq\,\text{cof.}\,\omega + qq = 0,$$

tum autem erit vel

$$p = q(\text{cof.}\,\omega + \sqrt{-1}\,\text{fin.}\,\omega) \text{ vel } p = q(\text{cof.}\,\omega - \sqrt{-1}\,\text{fin.}\,\omega),$$

hinque

$$p^n = q^n(\text{cof.}\,n\omega \pm \sqrt{-1}\,\text{fin.}\,n\omega)$$

ficque debebit effe

$$q^n(\text{cof.}\,n\omega \pm \sqrt{-1}\,\text{fin.}\,n\omega) + q^n = 0, \text{ fiue}$$

$$\text{cof.}\,n\omega \pm \sqrt{-1}\,\text{fin.}\,n\omega + 1 = 0,$$

vnde fit $\text{fin.}\,n\omega = 0$ et $\text{cof.}\,n\omega = -1$, tum autem fponte fit $\text{fin.}\,n\omega = 0$.

§. 3. Quia igitur $\text{cof.}\,n\omega = -1$, angulus $n\omega$ erit vel π, vel 3π, vel 5π, vel 7π, vel etc. Sicque fi i denotet numerum imparem quemcunque, erit $n\omega = i\pi$, hincque $\omega = \frac{i\pi}{n}$, quocirca factor duplex in genere erit

$$pp - 2pq\,\text{cof.}\,\tfrac{i\pi}{n} + qq.$$

§. 4. Cum nunc fit $pp + qq = 2\,\text{cof.}\,2\Phi$, ob $pq = 1$ erit ifte factor $2\,\text{cof.}\,2\Phi - 2\,\text{cof.}\,\frac{i\pi}{n}$, qui fponte in duos factores refoluitur. Cum enim fit

$$\text{cof.}\,A - \text{cof.}\,B = 2\,\text{fin.}\,\tfrac{B+A}{2}.\ \text{fin.}\,\tfrac{B-A}{2}, \text{ erit}$$

$$\text{cof.}\,2\Phi - \text{cof.}\,\tfrac{i\pi}{n} = 2\,\text{fin.}\,(\tfrac{i\pi}{2n} + \Phi).\,\text{fin.}\,(\tfrac{i\pi}{2n} - \Phi)$$

ficque vnus factor in genere erit

$$4\,\text{fin.}\,(\tfrac{i\pi}{2n} + \Phi)\,\text{fin.}\,(\tfrac{i\pi}{2n} - \Phi).$$

Hinc pro i fucceffiue numeros 1, 2, 3, 4, etc. fcribendo, erit:

$$2\,\text{cof.}\,n\Phi = 4\,\text{fin.}\,(\tfrac{\pi}{2n} + \Phi)\,\text{fin.}\,(\tfrac{\pi}{2n} - \Phi).$$

$$4\,\text{fin.}\,(\tfrac{3\pi}{2n} + \Phi).\,\text{fin.}\,(\tfrac{3\pi}{2n} - \Phi).\,4\,\text{fin.}\,(\tfrac{5\pi}{2n} + \Phi).\,\text{fin.}\,(\tfrac{5\pi}{2n} - \Phi),$$

donec omnino habeantur n factores.

§. 5.

§. 5. Percurramus igitur hanc expreffionem fecundum fingulos factores numeri n, eritque

fi $n = 1$	$2\,\text{cof.}\,\Phi = \text{fin.}\,(\frac{\pi}{2} - \Phi)$
fi $n = 2$	$2\,\text{cof}\,2\Phi = \text{fin}\,2^2(\frac{\pi}{4} - \Phi)\,\text{fin.}\,(\frac{\pi}{4} + \Phi)$
fi $n = 3$	$2\,\text{cof.}\,3\Phi = 2^3\,\text{fin.}\,(\frac{\pi}{6} - \Phi)\,\text{fin}\,(\frac{\pi}{6} + \Phi)\,\text{fin.}\,(\frac{3\pi}{6} - \Phi)$
fi $n = 4$	$2\,\text{cof.}\,4\Phi = 2^4\,\text{fin.}\,(\frac{\pi}{8} - \Phi)\,\text{fin.}\,(\frac{\pi}{8} + \Phi)\,\text{fin.}\,(\frac{3\pi}{8} - \Phi).\,\text{fin}(\frac{3\pi}{8} + \Phi)$
fi $n = 5$	$2\,\text{cof.}\,5\Phi = 2^5\,\text{fin.}(\frac{\pi}{10} - \Phi)\,\text{fin.}(\frac{\pi}{10} + \Phi)\,\text{fin}\,(\frac{3\pi}{10} - \Phi)\,\text{fin.}(\frac{3\pi}{10} + \Phi)\,\text{fin.}(\frac{5\pi}{10} - \Phi)$
fi $n = 6$	$2\,\text{cof.}\,6\Phi = 2^6\,\text{fin.}(\frac{\pi}{12} - \Phi)\,\text{fin.}(\frac{\pi}{12} + \Phi)\,\text{fin.}(\frac{3\pi}{12} - \Phi)\,\text{fin.}(\frac{3\pi}{12} + \Phi)\,\text{fin.}(\frac{5\pi}{12} - \Phi)\,\text{fin}\,(\frac{5\pi}{12} + \Phi)$

Generaliter autem erit

$$\text{cof.}\,n\Phi = 2^{n-1}\,\text{fin.}(\tfrac{\pi}{2n} - \Phi)\,\text{fin}(\tfrac{\pi}{2n} + \Phi)\,\text{fin.}(\tfrac{3\pi}{2n} - \Phi)\,\text{fin.}(\tfrac{3\pi}{2n} + \Phi) \text{ etc.}$$

donec habeantur n factores

§. 6. Sumendis igitur logarithmis erit

$$l\,\text{cof.}\,n\Phi = l\,2^{n-1} + l\,\text{fin.}(\tfrac{\pi}{2n} - \Phi) + l\,\text{fin.}(\tfrac{\pi}{2n} + \Phi) + l\,\text{fin.}(\tfrac{3\pi}{2n} - \Phi) + l\,\text{fin.}(\tfrac{3\pi}{2n} + \Phi) + \text{etc.}$$

quae aequatio differentiata praebet

$$\frac{n\,d\Phi\,\text{fin.}\,n\Phi}{\text{cof.}\,n\Phi} = \frac{d\Phi\,\text{cof.}(\frac{\pi}{2n} - \Phi)}{\text{fin.}\,(\frac{\pi}{2n} - \Phi)} - \frac{d\Phi\,\text{cof.}(\frac{\pi}{2n} + \Phi)}{\text{fin.}\,(\frac{\pi}{2n} + \Phi)} + \frac{d\Phi\,\text{cof.}(\frac{3\pi}{2n} - \Phi)}{\text{fin.}\,(\frac{3\pi}{2n} - \Phi)} - \frac{d\Phi\,\text{cof.}\,\frac{3\pi}{2n} + \Phi)}{\text{fin.}\,\frac{3\pi}{2n} + \Phi)} + \text{etc.}$$

hoc eft

$$n\,\text{tag.}\,n\Phi = \text{cot.}(\tfrac{\pi}{2n} - \Phi) - \text{cot.}(\tfrac{\pi}{2n} + \Phi) + \text{cot.}(\tfrac{3\pi}{2n} - \Phi) - \text{cot.}(\tfrac{3\pi}{2n} + \Phi) + \text{etc.}$$

vnde deducuntur fequentes aequalitates memoratu dignae

I° $\text{tag.}\,\Phi = \text{cot.}\,(\frac{\pi}{2} - \Phi)$

II° $2\,\text{tag.}\,2\Phi = \text{cot}\,(\frac{\pi}{4} - \Phi) - \text{cot.}(\frac{\pi}{4} + \Phi) = \text{tg.}(\frac{\pi}{4} + \Phi) - \text{tg.}\,(\frac{\pi}{4} - \Phi)$

III° $\text{tag.}\,3\Phi = \text{cot.}\,(\frac{\pi}{6} - \Phi) - \text{cot.}\,(\frac{\pi}{6} + \Phi) + \text{cot.}\,(\frac{3\pi}{6} - \Phi)$, fiue

$\text{tag.}\,3\Phi = \text{tag.}\,(\frac{\pi}{3} + \Phi) - \text{tag.}\,(\frac{\pi}{3} - \Phi) + \text{tag.}\,\Phi$

IV° $\text{tag.}\,4\Phi = \text{cot.}(\frac{\pi}{8} - \Phi) - \text{cot.}(\frac{\pi}{8} + \Phi) + \text{cot.}(\frac{3\pi}{8} - \Phi) - \text{cot.}\,(\frac{3\pi}{8} + \Phi)$

fiue $4\,\text{tg.}\,4\Phi = \text{tag.}\,(\frac{3\pi}{8} + \Phi) - \text{tag.}\,(\frac{3\pi}{8} - \Phi) + \text{tg.}\,(\frac{\pi}{8} + \Phi) - \text{tag.}(\frac{\pi}{8} - \Phi)$

§. 7.

§. 7. Eodem modo tractemus formulam

$$p^n - q^n = 2\sqrt{-1}\,\sin n\phi$$

cujus factorem duplicem statuamus

$$pp - 2pq\cos\omega + qq$$

quo posito $= 0$ fit ut ante

$$p = q(\cos.\omega \pm \sqrt{-1}\sin.\omega)$$

hincque porro

$$p^n = q^n(\cos.n\omega \pm \sqrt{-1}\sin.n\omega)$$

sicque debebit esse

$$q^n(\cos.n\omega \pm \sqrt{-1}\sin.n\omega) - q^n = 0,\ \text{sive}\ \cos.n\omega \pm \sqrt{-1}\sin.n\omega - 1 = 0$$

Vnde fieri debet

$$\sin.n\omega = 0 \text{ ac } \cos.n\omega = 1$$

quam ob rem angulus $n\omega$ erit vel 0, vel 2π, vel 4π, vel 6π, vel in genere $2i\pi$, ideoque $\omega = \frac{2i\pi}{n}$, denotante i numeros omnes 1, 2, 3, 4, etc. Hinc igitur factor duplex in genere erit

$$pp - 2pq\cos.\tfrac{2i\pi}{n} + qq = 2\cos.2\phi - 2\cos.\tfrac{2i\pi}{n}$$

qui resolvitur in hos factores:

$$2\sin.(\tfrac{i\pi}{n} - \phi)\ 2\sin.(\tfrac{i\pi}{n} + \phi);$$

praeterea autem formula $p^n - q^n$ habet factorem simplicem

$$p - q = 2\sqrt{-1}\sin.\phi$$

consequenter habebimus

$$\sin n\phi = \sin.\phi.\ 2\sin.(\tfrac{i\pi}{n} - \phi).\ 2\sin.(\tfrac{i\pi}{n} + \phi) \text{ etc.}$$

ideoque

$$\sin.n\phi = \sin.\phi.2\sin.(\tfrac{\pi}{n}-\phi)\,2\sin(\tfrac{\pi}{n}+\phi).2\sin(\tfrac{2\pi}{n}-\phi)2\sin(\tfrac{2\pi}{n}+\phi) \text{ etc.}$$

donec omnino prodeunt n factores. Erit ergo

$$\sin.n\phi = 2^{n-1}\sin.\phi.\sin.(\tfrac{\pi}{n}-\phi)\sin.(\tfrac{\pi}{n}+\phi)\sin.(\tfrac{2\pi}{n}-\phi)\sin.(\tfrac{2\pi}{n}+\phi) \text{ etc.}$$

§. 8.

m
eamus
si $n = 1$ | sir
si $n = 2$ | sir
si $n = 3$ | sir
si $n = 4$ | sir
si $n = 5$ | si
si $n = 6$ | si
$-\sin.n\omega-1=0$
eritque
l (
el 6π, vel in-
quae a
s 1, 2, 3, 4, etc.
$n\cos.n$
$\sin.n\phi$
$\frac{2i\pi}{n}$
sive
n c
donec
1 simplicem
special
ϕ) etc.
si $n = 1$
si $n = 2$
$(\frac{2\pi}{n}+\phi)$ etc.
si $n = 3$
si $n = 4$
$(\frac{2\pi}{n}+\phi)$ etc.
§. 8.

§. 8. Iam ex hac forma generali sequentes deducamus formas speciales:

si $n = 1$	$\sin.\phi = 2^0 \sin\phi$
si $n = 2$	$\sin.2\phi = 2\sin.\phi\sin.(\frac{\pi}{2}-\phi)$
si $n = 3$	$\sin.3\phi = 4\sin.\phi\sin.(\frac{\pi}{3}-\phi).\sin.(\frac{\pi}{3}+\phi)$
si $n = 4$	$\sin.4\phi = 8\sin.\phi\sin.(\frac{\pi}{4}-\phi)\sin.(\frac{\pi}{4}+\phi)\sin.(\frac{2\pi}{4}-\phi)$
si $n = 5$	$\sin.5\phi = 16\sin.\phi\sin.(\frac{\pi}{5}-\phi)\sin.(\frac{\pi}{5}+\phi)\sin.(\frac{2\pi}{5}-\phi)\sin.(\frac{2\pi}{5}+\phi)$
si $n = 6$	$\sin.6\phi = 32\sin.\phi\sin.(\frac{\pi}{6}-\phi)\sin.(\frac{\pi}{6}+\phi)\sin.(\frac{2\pi}{6}-\phi)\sin.(\frac{2\pi}{6}+\phi)\sin.(\frac{3\pi}{6}-\phi)$

§. 9. Sumamus hic etiam ut ante logarithmos, eritque

$$l\sin.n\phi = l\,2^{n-1} + l\sin.\phi + l\sin(\tfrac{\pi}{n}-\phi) + l\sin.(\tfrac{\pi}{n}+\phi) + \text{etc.}$$

quae aequatio differentiata et per $d\phi$ divisa praebet

$$\frac{n\cos.n\phi}{\sin.n\phi} = \frac{\cos.\phi}{\sin.\phi} - \frac{\cos.(\frac{\pi}{n}-\phi)}{\sin.(\frac{\pi}{n}-\phi)} + \frac{\cos.(\frac{\pi}{n}+\phi)}{\sin.(\frac{\pi}{n}+\phi)} - \frac{\cos.(\frac{2\pi}{n}-\phi)}{\sin.(\frac{2\pi}{n}-\phi)} + \text{etc}$$

sive

$$n\cot.n.\phi = \cot.\phi. - \cot.(\tfrac{\pi}{n}-\phi) + \cot.(\tfrac{\pi}{n}+\phi) - \cot.(\tfrac{2\pi}{n}-\phi) \text{ etc.}$$

donec habeantur n termini

§. 10. Hinc igitur sequentes obtinebimus formas speciales:

si $n = 1$	$\cot.\phi = \cot.\phi$
si $n = 2$	$2\cot.2\phi = \cot.\phi - \cot.(\frac{\pi}{2}-\phi)$
si $n = 3$	$3\cot.3\phi = \cot.\phi - \cot.(\frac{\pi}{3}-\phi) + \cot.(\frac{\pi}{3}+\phi)$
si $n = 4$	$4\cot.4\phi = \cot.\phi - \cot.(\frac{\pi}{4}-\phi) + \cot.\frac{\pi}{4}+\phi) - \cot.(\frac{2\pi}{n}-\phi)$

§. 11.

§. 11. Si formulam pro cot. $n\Phi$ inuentam denuo differentiemus, ob $d\,\text{cot.}\,\theta = \frac{-d\theta}{\text{fin.}\,\theta^2}$, per $-\theta\Phi$ diuidendo habebimus,

$$\frac{nn}{\text{fin.}\,n\Phi^2} = \frac{1}{\text{fin.}\,\Phi^2} + \frac{1}{\text{fin.}(\frac{\pi}{n}-\Phi)^2} + \frac{1}{\text{fin.}(\frac{\pi}{n}+\Phi)^2} + \frac{1}{\text{fin.}(\frac{2\pi}{n}-\Phi)^2}\text{etc.}$$

donec habeantur n termini, vnde fequentes cafus notentur:

$$\text{fi}\ n=1 \Big| \frac{1}{\text{fin.}\,\Phi^2} = \frac{1}{\text{fin.}\,\Phi^2}$$

$$\text{fi}\ n=2 \Big| \frac{4}{\text{fin.}\,2\Phi^2} = \frac{1}{\text{fin.}\,\Phi^2} + \frac{1}{\text{fin.}\,(\frac{\pi}{2}-\Phi)^2}$$

$$\text{fi}\ n=3 \Big| \frac{9}{\text{fin}\,3\,\Phi^2} = \frac{1}{\text{fin}\,\Phi^2} + \frac{1}{\text{fin.}\,(\frac{\pi}{3}-\Phi)^2} + \frac{1}{\text{fin.}\,(\frac{\pi}{3}+\Phi)^2}$$

$$\text{fi}\ n=4 \Big| \frac{16}{\text{fin.}\,4\Phi^2} = \frac{1}{\text{fin.}\,\Phi^2} + \frac{1}{\text{fin.}(\frac{\pi}{4}-\Phi)^2} + \frac{1}{\text{fin.}(\frac{\pi}{4}+\Phi)^2} + \frac{1}{\text{fin.}(\frac{2\pi}{4}-\Phi)^2}\ \text{etc.}$$

Euolutio formulae

$$p^{2n} - 2p^n q^n\,\text{cof.}\,\theta + q^n.$$

§. 12. Sumamus hic ut ante

$$p = \text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi \ \text{et}\ q = \text{cof.}\,\Phi - \sqrt{-1}\,\text{fin}\,\Phi$$

ita vt ifta formula inuoluat hunc valorem:

$$2\,\text{cof.}\,2n\Phi - 2\,\text{cof.}\,\theta = 4\,\text{fin.}\,(n\Phi + \tfrac{1}{2}\theta)\,\text{fin.}\,(\tfrac{1}{2}\theta - n\Phi);$$

Iam fit $pp - 2pq\,\text{cof.}\,\omega + qq$ factor duplex huius formulae, quae ergo euanefcere debet pofito $p = q(\text{cof.}\,\omega \pm \sqrt{-1}\,\text{fin.}\,\omega)$, vnde facta fubftitutione prodibit

$$q^{2n}(\text{cof.}\,2n\omega \pm \sqrt{-1}\,\text{fin.}\,2n\omega) - 2q^{2n}\text{cof.}\,\theta(\text{cof.}\,\omega\omega \pm \sqrt{-}\text{fin.}\,n\omega) + q^{m} = 0$$

hoc

inuentam denuo
diuidendo habe-
$+\frac{1}{\text{fin.}(\frac{2\pi}{n}-\Phi)^2}$etc.
tentes cafus no-
$-\Phi)^2$
$\frac{1}{\text{fin.}(\frac{2\pi}{4}-\Phi)^2}$ etc.
$\sqrt{-1}\,\text{fin}\,\Phi$
:
$(\tfrac{1}{2}\theta - n\Phi)$;
ex huius formu-
$\omega \pm \sqrt{-1}\,\text{fin.}\,\omega)$,
$\sqrt{-}\text{fin.}\,n\omega) + q^{m} = 0$
hoc

hoc eft

$$\text{cof.}\,2n\omega \quad - 2\,\text{cof.}\,\theta\,\text{cof.}\,n\omega + 1 = 0$$
$$\pm \sqrt{-1}\,\text{fin.}\,2n\omega + 2\,\text{cof.}\,\theta\sqrt{-1}\,\text{fin.}\,n\omega$$

vnde nafcuntur hae duae aequationes:

$$\text{cof.}\,2n\omega - 2\,\text{cof.}\,\theta\,\text{cof.}\,n\omega + 1 = 0 \ \text{et}$$
$$\text{fin.}\,2n\omega - 2\,\text{cof.}\,\theta\,\text{fin.}\,n\omega = 0$$

Cum nunc fit

$$\text{cof.}\,2n\omega = 2\,\text{cof.}\,n\omega^2 - 1 \ \text{et fin.}\,2n\omega = 2\,\text{fin.}\,n\omega\,\text{cof.}\,n\omega$$

hae duae aequationes erunt

$$2\,\text{cof.}\,n\omega^2 - 2\,\text{cof.}\,\theta\,\text{cof.}\,n\omega = 0 \ \text{et}\ 2\,\text{fin.}\,n\omega\,\text{cof.}\,n\omega - \text{cof.}\,\theta\,\text{fin.}\,n\omega = 0,$$

fiue

$$\text{cof.}\,n.\omega - \text{cof.}\,\theta = 0 \ \text{et cof.}\,n\omega - \text{cof.}\,\theta = 0$$

vnde fequitur cof. $n\omega = \text{cof.}\,\theta$. Erit ergo vel $n\omega = \theta$, vel $n\omega = 2\pi + \theta$, vel $4\pi + \theta$, vel $6\pi + \theta$, vel in genere $n\omega = 2i\pi + \theta$, vnde fit in genere $\omega = \frac{2i\pi \pm \theta}{n}$, ita vt i denotet numeros 0, 1, 2, 3, 4, etc.

§. 13. Formulae igitur noftrae factor duplex in genere erit

$$pp + qq - 2pq\,\text{cof.}\,(\tfrac{2i\pi \pm \theta}{n}).$$

Eft vero

$$pp + qq = 2\,\text{cof.}\,2\Phi \ \text{et}\ pq = 1,$$

vnde hic factor erit

$$2\,(\text{cof.}\,2\Phi - \text{cof.}\,(\tfrac{2i\pi \pm \theta}{n}))$$

qui reducitur ad hos factores fimplices:

$$4\,\text{fin.}\,\frac{2i\pi + 2n\Phi \pm \theta}{2n}\,\text{fin.}\,\frac{2i\pi \pm \theta - 2n\Phi}{2n}.$$

vnde

vnde loco i ſcribendo numeros 1, 2, 3, 4, etc. factores noſtrae formulae erunt

$$4\,\text{ſin.}\tfrac{2n\Phi+\theta}{2n}\text{ſin.}\tfrac{\theta-2n\Phi}{2n}.\,4\,\text{ſin.}\tfrac{2\pi+2n\Phi+\theta}{2n}\text{ſin.}\tfrac{2\pi+\theta-2n\Phi}{2n}\,4\,\text{ſin.}\tfrac{4\pi+2n\Phi+\theta}{2n}.$$
$$\text{ſin.}\tfrac{4\pi+\theta-2n\Phi}{2n}.\,4\,\text{ſin.}\tfrac{6\pi+2n\Phi+\theta}{2n}\text{ſin.}\tfrac{6\pi+\theta-2n\Phi}{2n}$$

qui factores eouſque continuari debent, donec eorum numerus fiat $= n$

§. 14. Cum igitur hoc productum aequale ſit formulae $4\,\text{ſin}\,(n\Phi+\frac{1}{2}\theta)\,\text{ſin.}(\frac{1}{2}\theta-n\Phi)$, et in noſtro producto factor numericus ſit $4^n = 2^{2n}$, per 4 diuidendo habebimus hanc aequationem:

$$\text{ſin.}(n\Phi+\tfrac{1}{2}\theta)\,\text{ſin.}(\tfrac{1}{2}\theta-n\Phi) = 2^{2n-2}\,\text{ſin.}(\tfrac{2n\Phi+\theta}{2n})\,\text{ſin.}(\tfrac{\theta-2n\Phi}{2n}).$$
$$\text{ſin.}(\tfrac{2\pi+2n\Phi+\theta}{2n})\,\text{ſin.}(\tfrac{2\pi+\theta-2n\Phi}{2n}).$$
$$\text{ſin.}(\tfrac{4\pi+2n\Phi+\theta}{2n})\,\text{ſin.}(\tfrac{4\pi+\theta-2n\Phi}{2n})\text{ etc.}$$

quae aequatio quo concinnior reddatur ponamus $\theta = 2n\alpha$ et erit

$$\text{ſin.}\,n(\alpha+\Phi)\,\text{ſin.}\,n(\alpha-\Phi) = 2^{2n-2}\,\text{ſin.}(\alpha+\Phi)\,\text{ſin.}(\alpha-\Phi).$$
$$\text{ſin.}(\tfrac{\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\alpha-\Phi).$$
$$\text{ſin.}(\tfrac{2\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{2\pi}{n}+\alpha-\Phi)\text{ etc.}$$

§. 15. Haec autem expreſſio non eſt noua, ſed iam in praecedente continetur, quae erat.

$$\text{ſin.}\,n\Phi = 2^{n-1}\,\text{ſin.}\,\Phi.\,\text{ſin.}(\tfrac{\pi}{n}-\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\Phi)\,\text{ſin}(\tfrac{2\pi}{n}-\Phi)\,\text{ſin.}(\tfrac{2\pi}{n}+\Phi)\text{ etc.}$$

et cum ſit $\text{ſin}(o-\Phi) = \text{ſin.}(\pi - o + \Phi)$, erit,

$$\text{ſin}(\tfrac{\pi}{n}-\Phi) = \text{ſin.}(\tfrac{(n-1)\pi}{n}+\Phi);\ \text{ſin.}(\tfrac{2\pi}{n}-\Phi) = \text{ſin.}(\tfrac{(n-2)\pi}{n}+\Phi).$$

$\text{ſin.}(\frac{3\pi}{n}-\Phi) = \text{ſin.}(\frac{(n-3)\pi}{n}+\Phi)$, vnde illa expreſſio reducetur ad hanc formam:

ſin.

, etc. factores
Φ 4 ſin. $\frac{4\pi+2n\Phi+\theta}{2n}$.
vbi
Quo
ec eorum nuhinc
a aequale ſit
oſtro producto
do habebimus
quae
ſi
is $\theta = 2n\alpha$ et
Φ) ſin. $(\alpha-\Phi)$.
forn
nata
exiſt
eſt noua, ſed
n. $(\frac{2\pi}{n}+\Phi)$ etc.
ſi p
$(\frac{(n-2)\pi}{n}+\Phi)$.
tum
reſſio reduceſin.

$$\text{ſin.}\,n\Phi = 2^{n-1}\,\text{ſin.}\,\Phi.\,\text{ſin.}(\tfrac{\pi}{n}+\Phi)\,\text{ſin.}(\tfrac{2\pi}{n}+\Phi)$$
$$\text{ſin.}(\tfrac{3\pi}{n}+\Phi)\ldots\ldots\text{ſin.}(\tfrac{(n-1)\pi}{n}+\Phi)$$

vbi arcus in progreſſione arithmetica continua progrediuntur. Quod ſi iam hic loco Φ ſcribamus primo $\alpha+\Phi$, deinde $\alpha-\Phi$, hinc duae formulae ſequentes naſcentur:

$$\text{ſin.}\,n(\alpha+\Phi) = 2^{n-1}\,\text{ſin.}(\alpha+\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\alpha+\Phi)$$
$$\text{ſin.}(\tfrac{2\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{3\pi}{n}+\alpha+\Phi)\text{ etc.}$$
$$\text{ſin.}\,n(\alpha-\Phi) = 2^{n-1}\,\text{ſin}(\alpha-\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\alpha-\Phi)$$
$$\text{ſin.}(\tfrac{2\pi}{n}+\alpha-\Phi)\,\text{ſin.}(\tfrac{3\pi}{n}+\alpha-\Phi)\text{ etc.}$$

quae duae aequationes in ſe inuicem ductae praebent

$$\text{ſin.}\,n(\alpha+\Phi)\,\text{ſin.}\,n(\alpha-\Phi) = 2^{2n-2}\,\text{ſin.}(\alpha+\Phi)$$
$$\text{ſin.}(\alpha-\Phi).\,\text{ſin.}(\tfrac{\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\alpha-\Phi)$$
$$\text{ſin.}(\tfrac{2\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{2\pi}{n}+\alpha-\Phi)\,\text{ſin.}(\tfrac{3\pi}{n}+\alpha+\Phi)\text{ etc.}$$

§. 16. Si nunc attendamus ad originem harum formularum, quandoquidem ex noſtra formula

$$p^{2n} - 2p^n q^n \cos.\,2n\alpha + q^{2n}$$

nata eſt haec:

$$4\,\text{ſin.}\,n(\alpha+\Phi)\,\text{ſin.}\,n(\alpha-\Phi)$$

exiſtente

$$p = \cos.\Phi + \sqrt{-1}\,\text{ſin.}\,\Phi \text{ et}$$
$$q = \cos.\Phi - \sqrt{-1}\,\text{ſin.}\,\Phi,$$

ſi ponamus

$$f = \cos.(\alpha+\Phi) + \sqrt{-1}\,\text{ſin.}(\alpha+\Phi) \text{ et}$$
$$g = \cos.(\alpha+\Phi) - \sqrt{-1}\,\text{ſin.}(\alpha+\Phi),$$

tum erit $f^n - g^n = 2\sqrt{-1}\,\text{ſin.}\,n(\alpha+\Phi)$.

 Dein-

Deinde si ponamus

$$h = \text{cos.}(\alpha - \Phi) + \sqrt{-1}\,\text{sin}(\alpha - \Phi) \text{ et}$$
$$k = \text{cos.}(\alpha - \Phi) - \sqrt{-1}\,\text{sin.}(\alpha - \Phi)$$

erit simili modo

$$h^n - k^n = 2\sqrt{-1}\,\text{sin.}\,n(\alpha - \Phi),$$

unde erit

$$(f^n - g^n)(h^n - k^n) = -4\,\text{sin.}\,n(\alpha + \Phi)\,\text{sin.}\,n(\alpha - \Phi) = -p^{2n} + 2p^n q^n \,\text{cos.}\,2n\alpha - q^{2n}.$$

Ad hoc demonstrandum notetur esse

$$f = p(\text{cos.}\alpha + \sqrt{-1}\,\text{sin.}\alpha);$$
$$g = q(\text{cos.}\alpha - \sqrt{-1}\,\text{sin.}\alpha);$$
$$h = q(\text{cos.}\alpha + \sqrt{-1}\,\text{sin.}\alpha);$$
$$k = p(\text{cos.}\alpha - \sqrt{-1}\,\text{sin.}\alpha);$$

vnde fit

$$f^n = p^n(\text{cos.}n\alpha + \sqrt{-1}\,\text{sin.}n\alpha);$$
$$g^n = q^n(\text{cos.}n\alpha - \sqrt{-1}\,\text{sin.}n\alpha);$$
$$h^n = q^n(\text{cos.}n\alpha + \sqrt{-1}\,\text{sin.}n\alpha);$$
$$k^n = p^n(\text{cos.}n\alpha - \sqrt{-1}\,\text{sin.}n\alpha);$$

Ponamus breuitatis ergo

$$\text{cos.}n\alpha + \sqrt{-1}\,\text{sin.}n\alpha = A; \quad \text{cos.}n\alpha - \sqrt{-1}\,\text{sin.}n\alpha = B$$

vt fit

$$f^n = Ap^n;\ g^n = Bq^n;\ h^n = Aq^n \text{ et } k^n = Bp^n,$$

hincque porro

$$f^n - g^n = Ap^n - Bq^n \text{ et } h^n - k^n = Aq^n - Bp^n$$

quae duae formulae multiplicatae praebent

$$(f^n - g^n)(h^n - k^n) = (A^2 + B^2)p^n q^n - AB(p^{2n} + q^{2n});$$

vbi

) et
)
) sin. $n(\alpha - \Phi) =$
$\sqrt{-1}\,\text{sin.}n\alpha = B$
$k^n = Bp^n,$
$Aq^n - Bp^n$
bent
$- AB(p^{2n} + q^{2n});$
vbi

vbi cum sit

$$AB = 1 \text{ et } AA + BB = 2\,\text{cos.}\,2n\alpha$$

hoc productum erit

$$-p^{2n} + 2p^n q^n \,\text{cos.}\,2n\alpha - q^{2n}$$

quod est id ipsum quod inuenimus.

Corollarium.

Hinc igitur intelligimus, formulam

$$p^{2n} - 2p^n q^n \,\text{cos.}\,2n\alpha + q^{2n}$$

resolui in hos duos factores:

$$(Ap^n - Bq^n) \text{ et } (Bp^n - Aq^n)$$

existente

$$A = \text{cos.}\,n\alpha + \sqrt{-1}\,\text{sin.}\,n\alpha$$
$$B = \text{cos.}\,n\alpha - \sqrt{-1}\,\text{sin.}\,n\alpha.$$