1783

Variae observationes circa angulos in progressione geometrica progredientes

Leonhard Euler

igitur $m = \frac{1}{2}$; $n = \frac{5}{2}$, vt fit $l = \frac{3}{2}$, vnde quatuor numeri priores erunt

$$a = \tfrac{1}{2};\ b = \tfrac{5}{2};\ c = 6 \text{ et } d = 48;$$

vnde porro deducimus:

$$p = 57;\ q = 451\tfrac{1}{4};\ r = 931\tfrac{1}{2} \text{ et } s = 360;$$

ex his ergo deducitur:

$$z = \frac{4 \cdot 931\frac{1}{2} + 114 \cdot 361}{359^2} = \frac{44880}{359^2} = \frac{44880}{128881}$$

qui numeri multo funt minores quam praecedentes.

VARIAE

uatuor numeri
360;
80
881
edentes.
VARIAE

VARIAE OBSERVATIONES
CIRCA ANGVLOS
IN PROGRESSIONE GEOMETRICA PROGREDIENTES.

§. 1.

Cum pleraeque infignes proprietates, quae adhuc circa angulos, fiue arcus, eorumque finus, cofinus, tangentes, cotangentes, fecantes et cofecantes funt inueftigatae, ex confideratione arcuum in arithmetica progreffione crefcentium fint deriuatae: non minus notatu dignae videntur illae proprietates, quas ex confideratione arcuum in geometrica progreffione procedentium deducere licet; imprimis cum earum veritas plerumque multo magis abfcondita videatur, quocirca hoc loco plures eiusmodi proprietates euoluere conftitui.

§. 2. Primum fontem ad huiusmodi fpeculationes nobis aperit notiffima formula: $\text{fin.}\, 2\Phi = 2\, \text{fin.}\, \Phi . \text{cof.}\, \Phi$ vnde, fi s denotet arcum fiue angulum quemcunque, erit

$\text{fin.}\, s = 2\, \text{fin.}\, \frac{1}{2}s . \text{cof.}\, \frac{1}{2}s$; tum vero fimili modo erit
$\text{fin.}\, \frac{1}{2}s = 2\, \text{fin.}\, \frac{1}{4}s . \text{cof.}\, \frac{1}{4}s$, qui valor ibi fubftitutus praebet
$\text{fin.}\, s = 4\, \text{fin.}\, \frac{1}{4}s . \text{cof.}\, \frac{1}{2}s\, \text{cof.}\, \frac{1}{4}s$. Deinde quia porro eft
$\text{fin.}\, \frac{1}{4}s = 2\, \text{fin.}\, \frac{1}{8}s . \text{cof.}\, \frac{1}{8}s$, hoc valore fubftituto erit
$\text{fin.}\, s = 8\, \text{fin.}\, \frac{1}{8}s . \text{cof.}\, \frac{1}{2}s . \text{cof.}\, \frac{1}{4}s . \text{cof.}\, \frac{1}{8}s$. Pari modo progrediendo eft

$$\text{fin.}\, s = 16\, \text{fin.}\, \tfrac{1}{16}s . \text{cof.}\, \tfrac{1}{2}s . \text{cof.}\, \tfrac{1}{4}s . \text{cof.}\, \tfrac{1}{8}s . \text{cof.}\, \tfrac{1}{16}s$$

atque si hoc modo in infinitum progrediamur, denotante i numerum infinitum, seu potius infinitesimam potestatem ipsius 2, habebimus

$$\sin. s = i \sin. \tfrac{s}{i}.\ \cos. \tfrac{1}{2}s.\ \cos. \tfrac{1}{4}s.\ \cos. \tfrac{1}{8}s.\ \cos. \tfrac{1}{16}s.\ \text{etc.}$$

vbi quia arcus $\frac{s}{i}$ est infinite paruus, erit $\sin \frac{s}{i} = \frac{s}{i}$, ideoque $i \sin. \frac{s}{i} = s$, vnde adipiscimur hanc insignem proprietatem, vt sit

$$\sin. s = s \cos. \tfrac{1}{2}s.\ \cos. \tfrac{1}{4}s. \cos. \tfrac{1}{8}s. \cos. \tfrac{1}{16}s. \cos. \tfrac{1}{32}s.\ \text{etc. in infinitum.}$$

§. 3. Hinc igitur ipse arcus s per eius sinum et cosinus arcuum continuo in ratione dupla decrescentium ita pulcherrime definitur, vt sit

$$s = \frac{\sin. s.}{\cos. \frac{1}{2}s. \cos. \frac{1}{4}s. \cos. \frac{1}{8}s. \cos. \frac{1}{16}s. \cos. \frac{1}{32}s.\ \text{etc.}}$$

at quia $\frac{1}{\cos. \Phi} = \sec. \Phi$, erit per expressionem integram

$$s = \sin. s. \sec. \tfrac{1}{2}s. \sec. \tfrac{1}{4}s. \sec. \tfrac{1}{8}s. \sec. \tfrac{1}{16}.. \sec. \tfrac{1}{32}s. \sec. \tfrac{1}{64}s.\ \text{etc.}$$

quae expressio satis commode geometrice repraesentari potest, quemadmodum iam alio loco ostendi.

§. 4. Quia hic arcus s per productum exprimitur, sumendis logarithmis habebimus.

$$l s = l \sin. s + l \sec. \tfrac{1}{2}s + l \sec. \tfrac{1}{4}s + l \sec. \tfrac{1}{8}s + l \sec. \tfrac{1}{16}s + l \sec. \tfrac{1}{32}s + \text{etc.}$$

vnde si accipiamus $s = \frac{\pi}{2} = 90°$, fiet

$$l \tfrac{\pi}{2} = 0 + l \sec. 45° + l \sec. 22°,30' + l \sec. 11°,15' + l \sec. 5°,37\tfrac{1}{2}.\ \text{etc.}$$

vnde calculo instituto erit

l sec.

, denotante i
potestatem
s. etc.
$\frac{s}{i}$, ideoque
roprietatem,
$\frac{1}{32}s.$ etc. in
us finum et
ecrescentium
. etc.
tegram
. sec. $\frac{1}{64}s.$ etc.
aesentari po-
exprimitur,
sec. $\frac{1}{32}s$ + etc,
c. 5°,37½. etc.
l sec.

$$\begin{aligned}
l \sec 45° &= 0,1505150 \\
l \sec. 22°,30' &= 0,0343847 \\
l \sec. 11°,15' &= 0,0084261 \\
l \sec. 5°,37\tfrac{1}{2} &= 0,0020963 \\
l \sec. 2°,48\tfrac{3}{4} &= 0,0005235 \\
l \sec. 1°,24\tfrac{3}{8} &= 0,0001309 \\
l \sec. 0°,42\tfrac{3}{16} &= 0,0000327 \\
l \sec. 0°,21\tfrac{3}{32} &= 0,0000082 \\
\text{reliqui omnes} &= 0,0000027 \\
l \tfrac{\pi}{2} &= 0,1961201 \\
l\, 2 &= 0,3010300 \\
l\, \pi &= 0,4971501
\end{aligned}$$

hincque $\pi = 3,1415941$ satis exacte, vti constat.

§. 5 Quo autem hinc nouas relationes deducamus, differentiemus postremam aequationem logarithmicam, et cum sit $d.\, l \sec. \Phi = \frac{d\Phi \sin. \Phi}{\cos. \Phi} = d\Phi \tan. \Phi$, orietur per ds diuidendo sequens aequatio:

$$\tfrac{1}{s} = \cot. s + \tfrac{1}{2} \tan \tfrac{1}{2}s + \tfrac{1}{4} \tan. \tfrac{1}{4}s + \tfrac{1}{8} \tan. \tfrac{1}{8}s + \tfrac{1}{16} \tan. \tfrac{1}{16}s + \text{etc.}$$

quae series quam citissime conuergit, id quod sequenti exemplo clarius patebit, in quo sumamus $s = 90° = \frac{\pi}{2}$, vnde fiet

$$\tfrac{2}{\pi} = \tfrac{1}{2} \tan. 45° + \tfrac{1}{4} \tan. 22°,30' + \tfrac{1}{8} \tan. 11°,15' + \tfrac{1}{16} \tan. 5°,37' + \text{etc.}$$

qui valores ex tabulis desumti dabunt.

$$\begin{aligned}
\tfrac{1}{2} \tan. 45° &= 0,5000000 \\
\tfrac{1}{4} \tan. 22°,30' &= 0,1035534 \\
\tfrac{1}{8} \tan. 11°,15' &= 0,0248640 \\
\tfrac{1}{16} \tan. 5°,37\tfrac{1}{2} &= 0,0061557 \\
\tfrac{1}{32} \tan. 2°,48\tfrac{3}{4} &= 0,0015352 \\
\tfrac{1}{64} \tan. 1°,24\tfrac{3}{8} &= 0,0003836 \\
\text{pro reliquis} &= 0,0001278 \\
\tfrac{2}{\pi} &= 0,6366197
\end{aligned}$$

hinc $\pi \frac{1}{0,6366191} = \frac{2}{0,3183097}$

§. 6.

§. 6. Si postremam aequationem denuo differentiemus, ad seriem perueniemus multo magis conuergentem; cum enim sit $d.\cot.\Phi = \frac{-d\Phi}{(\sin.\Phi)^2}$ et $d.\mathrm{tag}.\Phi = \frac{d\Phi}{(\cos.\Phi)^2} = d\Phi \sec.\Phi^2$, reperiemus

$$-\frac{1}{ss} = \frac{-1}{(\sin.s)^2} + \frac{1}{4}\sec.\frac{1}{2}s^2 + \frac{1}{16}\sec.\frac{1}{4}s^2 + \frac{1}{64}\sec.\frac{1}{8}s^2 + \text{etc.}$$

siue

$$\frac{1}{4}\sec.\frac{1}{2}s^2 + \frac{1}{16}\sec.\frac{1}{4}s^2 + \frac{1}{64}\sec.\frac{1}{8}s^2 + \frac{1}{256}\sec.\frac{1}{16}s^2 = \frac{1}{(\sin.s)^2} - \frac{1}{ss}.$$

§. 7. Accommodemus eadem ratiocinia ad rationem triplam, secundum quam arcus decrescant; hunc in finem consideremus formulam

$$\sin.3\Phi = 4\sin.\Phi\cos.\Phi^2 = \sin.\Phi(4 - 4\sin.\Phi^2),$$

quae dat $\sin.3\Phi = 3\sin.\Phi(1 - \frac{4}{3}\sin.\Phi^2)$; vnde si s denotet arcum quemcunque, erit $\sin.s = 3\sin.\frac{1}{3}s(1 - \frac{4}{3}\sin\frac{s^2}{3})$; similique modo erit $\sin.\frac{1}{3}s = 3\sin.\frac{s}{9}(1 - \frac{4}{3}\sin.\frac{s^2}{9})$, ita vt nunc sit $\sin.s = 9\sin\frac{1}{9}s(1 - \frac{4}{3}\sin.\frac{s^2}{3})(1 - \frac{4}{3}\sin.\frac{s^2}{9})$

Si tales substitutiones in infinitum continuentur, peruenietur tandem vt ante ad hanc expressionem:

$$\sin.s = s(1 - \frac{4}{3}\sin.\frac{s^2}{3})(1 - \frac{4}{3}\sin\frac{s^2}{9})(1 - \frac{4}{3}\sin.\frac{s^2}{27}) \text{ etc.}$$

§. 8. Factores hos nimis complicatos sequenti modo in simpliciores resoluere licet; namque forma generalis $1 - \frac{4}{3}\sin.\Phi^2$, ob $\sin\Phi^2 = \frac{1}{2} - \frac{1}{2}\cos.2\Phi$, reducitur ad hanc: $\frac{1}{3} + \frac{2}{3}\cos.2\Phi$, quam hoc modo referre licet: $\frac{2\cos.60^\circ + 2\cos.2\Phi}{3}$.
Cum iam sit

$$\cos.a + \cos.b = 2(\cos.\tfrac{(a+b)}{2})(\cos.\tfrac{(a-b)}{2}), \text{ erit}$$

$$\cos.60^\circ + \cos.2\Phi = 2\cos.(30^\circ + \Phi)\cos.(30^\circ - \Phi)$$

quae forma in $\frac{2}{3}$ ducta praebebit

$$1 - \frac{4}{3}\sin.\Phi^2 = \frac{4}{3}\cos.(30^\circ + \Phi)\cos.(30^\circ - \Phi).$$

Quare

Qu ... uo differen- app ... nergentem; ... $d\Phi$ sec. Φ^2, ... s^2 + etc. quo ... $\frac{1}{(\sin.s)^2} - \frac{1}{ss}$. ... a ad ratio- ... t; hunc in qui ... in. Φ^2), vnit ... si s deno- ... tern ... ita vt com ... mas ... ontinuentur, ... tem: ... etc. id ... os sequenti erit ... forma gene- ... educitur ad ... , erit ... $-\Phi$) tatu valu ries ...). Quare

Quare si haec reductio ad singulos factores supra inuentos applicetur, habebimus sequens productum infinitum:

$$\frac{\sin.s}{s} = \frac{4}{3}\cos.(30^\circ + \tfrac{s}{3})\cos.(30^\circ - \tfrac{s}{3}).$$
$$\frac{4}{3}\cos.(30^\circ + \tfrac{s}{9})\cos.(30^\circ - \tfrac{s}{9})$$
$$\frac{4}{3}\cos.(30^\circ + \tfrac{s}{27})\cos.(30^\circ - \tfrac{s}{27}) \text{ etc.}$$

quod per secantes ita exhibebitur:

$$\frac{s}{\sin.s} = \frac{3}{4}\sec(30^\circ + \tfrac{s}{3})\sec.(30^\circ - \tfrac{s}{3}).$$
$$\frac{3}{4}\sec.(30^\circ + \tfrac{s}{9})\sec.(30^\circ - \tfrac{s}{9})$$
$$\frac{3}{4}\sec.(30^\circ + \tfrac{s}{27})\sec.(30^\circ - \tfrac{s}{27}).$$

qui factores, quo magis arcus s diminuitur, eo propius ad vnitatem accedunt.

§. 9. Si nunc logarithmos sumamus et singulos terminos differentiemus, factores illi numerici $\frac{3}{4}$, vtpote constantes, penitus ex calculo excedent; et quia, vt supra vidimus est $d.l.\sec.\Phi = d\Phi\,\mathrm{tag}.\Phi$. obtinebitur sequens aequatio:

$$\frac{1}{s} = \mathrm{cotag}.s + \tfrac{1}{3}\mathrm{tag}.(30^\circ + \tfrac{s}{3}) + \tfrac{1}{9}\mathrm{tag}.(30^\circ + \tfrac{s}{9}) + \tfrac{1}{27}\mathrm{tag}.(30^\circ + \tfrac{s}{27})$$
$$- \tfrac{1}{3}\mathrm{tag}.(30^\circ - \tfrac{s}{3}) - \tfrac{1}{9}\mathrm{tag}.(30^\circ - \tfrac{s}{9}) - \tfrac{1}{27}\mathrm{tag}.(30^\circ - \tfrac{s}{27})$$

id quod exemplo comprobasse iuuabit. Sit igitur $s = \frac{\pi}{2}$ eritque

$$\frac{2}{\pi} = \tfrac{1}{3}\mathrm{tag}.60^\circ + \tfrac{1}{9}\mathrm{tag}.40^\circ + \tfrac{1}{27}\mathrm{tag}.(33^\circ,20') + \tfrac{1}{81}\mathrm{tag}.(31^\circ,6\tfrac{2}{3}')$$
$$- \tfrac{1}{3}\mathrm{tag}.0^\circ - \tfrac{1}{9}\mathrm{tag}.20^\circ - \tfrac{1}{27}\mathrm{tag}.(26^\circ,40') - \tfrac{1}{81}\mathrm{tag}.(28^\circ,53\tfrac{1}{3}')$$
$$+ \tfrac{1}{243}\mathrm{tag}.(30^\circ,22\tfrac{2}{9}') \quad \text{etc.}$$
$$- \tfrac{1}{243}\mathrm{tag}.(29^\circ,37\tfrac{7}{9}')$$

§. 10. Haec postrema series eo magis videtur notatu digna, quod vix quisquam eius veritatem demonstrare valuerit, nisi eadem methodo fuerit vsus. Haec autem series sine dubio multo altioris est indaginis quam ea, ad

 quam

quam per euolutionem praecedentis casu sumus deducti, quae erat

$$\frac{1}{s} = \cot. s + \tfrac{1}{2}\,\mathrm{tag.}\tfrac{1}{2}s + \tfrac{1}{4}\,\mathrm{tag.}\tfrac{1}{4}s + \tfrac{1}{8}\,\mathrm{tag.}\tfrac{1}{8}s + \tfrac{1}{16}\,\mathrm{tag.}\tfrac{1}{16}s + \text{etc.}$$

cuius veritas ex notissima formula $2\cot. 2\Phi = \cot.\Phi - \mathrm{tag.}\Phi$ sequitur, vnde habemus $\mathrm{tag.}\Phi = \cot.\Phi - 2\cot. 2\Phi$. Hinc si loco singularum tangentium valores debiti substituantur, operatio sequenti modo instruetur:

$$\frac{1}{s} = \begin{cases} \cot. s + \tfrac{1}{2}\cot.\tfrac{1}{2}s + \tfrac{1}{4}\cot.\tfrac{1}{4}s + \tfrac{1}{8}\cot.\tfrac{1}{8}s \;\ldots\ldots\; \tfrac{1}{i}\cot.\tfrac{s}{i} \\ \cot. s - \tfrac{1}{2}\cot.\tfrac{1}{2}s - \tfrac{1}{4}\cot.\tfrac{1}{4}s - \tfrac{1}{8}\cot.\tfrac{1}{8}s \;\ldots\ldots \end{cases}$$

vbi omnes termini manifesto se mutuo destruunt, vsque ad vltimum $\tfrac{1}{i}\cot.\tfrac{s}{i}$, qui ad hanc formam redigatur: $\frac{\cos.\frac{s}{i}}{i\,\sin.\frac{s}{i}}$, denotante i numerum infinitum. Iam quia arcus $\frac{s}{i}$ est infinite paruus, erit $\cos.\frac{s}{i} = 1$, sinus vero ipsi arcui $\frac{s}{i}$ aequalis, ex quo vltimus iste terminus fit $= \frac{1}{s}$, qui est ipse valor huic seriei aequalis inuentus.

§. 11. Interim tamen etiam pro casu praesenti demonstratio directa simili modo exhibetur, ex formula, qua tangens anguli tripli exprimitur, repetenda. Si enim ponatur $\mathrm{tag.}\Phi = t$, quia habetur $\mathrm{tag.}\,3\Phi = \frac{3t - t^3}{1 - 3tt}$, erit $\cot.3\Phi = \frac{1 - 3tt}{3t - t^3}$ et $3\cot.3\Phi = \frac{3 - 9tt}{t(3 - tt)}$. Hinc subtrahatur $\cot.\Phi = \frac{1}{t}$, fietque $3\cot.3\Phi - \cot.\Phi = \frac{-8tt}{t(3 - tt)} = \frac{-8t}{3 - tt}$; hinc loco t substituamus eius valorem $\frac{\sin.\Phi}{\cos.\Phi}$ et habebimus,

$$3\cot.3\Phi - \cot.\Phi = \frac{-8\sin.\Phi.\cos.\Phi}{3\cos.\Phi^2 - \sin.\Phi^2}$$

cuius fractionis numeratorem et denominatorem sequenti modo tractemus. Cum sit $\cos.\Phi^2 = \frac{1}{2} + \frac{1}{2}\cos.2\Phi$ et $\sin.\Phi^2 = \frac{1}{2} - \frac{1}{2}\cos.2\Phi$, induet denominator hanc formam: $1 + 2\cos.2\Phi$, quae propterea

terea ita referri poterit

terea ita referri poterit: $2\cos.60^\circ + \cos.2\Phi$, quae porro ob $\cos. a + \cos. b = 2\cos.\frac{a+b}{2}.\cos.\frac{a-b}{2}$ reducitur ad hanc: $4\cos.(30^\circ + \Phi)\cos.(30^\circ - \Phi)$, numerator autem manifesto fit $-4\sin.2\Phi$, ita vt iam habeamus

$$3\cot.3\Phi - \cot\Phi = \frac{-\sin.2\Phi}{\cos.(30^\circ + \Phi).\cos.(30^\circ - \Phi)}.$$

Quia nunc in genere est

$$\sin.2\Phi = \sin.(a+\Phi)\cos.(a-\Phi) - \cos.(a+\Phi)\sin.(a-\Phi)$$

sumamus $a = 30^\circ$ et habemus sequentem aequationem:

$$3\cot.3\Phi - \cot\Phi = -\frac{\sin.(30^\circ+\Phi)\cos.(30^\circ-\Phi) + \cos.(30^\circ+\Phi)\sin.(30^\circ-\Phi)}{\cos.(30^\circ+\Phi)\cos.(30^\circ-\Phi)}$$

$$= -\mathrm{tag.}(30^\circ + \Phi) + \mathrm{tag.}(30^\circ - \Phi)$$

quocirca pertigimus ad hanc aequationem notatu dignam:

$$\cot.3\Phi = \tfrac{1}{3}\cot.\Phi - \tfrac{1}{3}\mathrm{tag.}(30^\circ + \Phi) + \tfrac{1}{3}\mathrm{tag.}(30^\circ - \Phi)$$

§. 12. Iam pro nostro casu loco 3Φ scribendo s statim nanciscimur

$$\cot. s = \tfrac{1}{3}\cot\tfrac{s}{3} - \tfrac{1}{3}\mathrm{tag.}(30^\circ + \tfrac{s}{3}) + \tfrac{1}{3}\mathrm{tag.}(30^\circ - \tfrac{s}{3}).$$

Simili vero modo vlterius erit

$$\tfrac{1}{3}\cot.\tfrac{s}{3} = \tfrac{1}{9}\cot.\tfrac{s}{9} - \tfrac{1}{9}\mathrm{tag.}(30^\circ + \tfrac{s}{9}) + \tfrac{1}{9}\mathrm{tag.}(30^\circ - \tfrac{s}{9}).$$

Eodem porro modo fit

$$\tfrac{1}{9}\cot.\tfrac{s}{9} = \tfrac{1}{27}\cot\tfrac{s}{27} - \tfrac{1}{27}\mathrm{tag.}(30^\circ + \tfrac{s}{27}) + \tfrac{1}{27}\mathrm{tag.}(30^\circ - \tfrac{s}{27}).$$

et si hoc modo in infinitum progrediamur perueniemus tandem ad huiusmodi cotangentem:

$\tfrac{1}{i}\cot.\tfrac{s}{i} = \frac{\cos.\frac{s}{i}}{i\sin.\frac{s}{i}}$; quamobrem nostra aequatio perducta erit ad hanc formam:

$$\cot. s = -\tfrac{1}{3}\mathrm{tag.}(30^\circ + \tfrac{s}{3}) - \tfrac{1}{9}\mathrm{tag.}(30^\circ + \tfrac{s}{9}) - \tfrac{1}{27}\mathrm{tag.}(30^\circ + \tfrac{s}{27}) \ldots\ldots + \tfrac{1}{s}$$
$$+ \tfrac{1}{3}\mathrm{tag.}(30^\circ - \tfrac{s}{3}) + \tfrac{1}{9}\mathrm{tag.}(30^\circ - \tfrac{s}{9}) + \tfrac{1}{27}\mathrm{tag.}(30^\circ - \tfrac{s}{27}) \ldots\ldots$$

ex