



1783

Observationes analyticae

Leonhard Euler

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lationibus delectantur, in eorum demonstrationem inquirent, cum nullum sit dubium, quin inde Theoria numerorum insignia incrementa sit adeptura.

Conclusio.

§. 39. Quatuor haec Theoremata postrema, quorum demonstratio adhuc desideratur, sequenti modo concinnius exhiberi possunt:

Existente s numero quocunque primo, dividantur tantum quadrata imparia 1, 9, 25, 49, etc. per divisorem 4s, noventurque residua, quae omnia erunt formae 4q + 1, quorum quodvis littera a dividitur, reliquorum autem numerorum, formae 4q + 1, qui inter residua non occurrunt, quibus littera s indicetur, quo facto se fuerit

| | |
|---------------------------------------|--------------------------------------|
| divisor numerus primus formae 4ns + a | tum est |
| 4ns + a | + s, residuum et - s residuum |
| 4ns + a | + s residuum et - s non-residuum |
| 4ns + 2a | + s non-residuum et - s non-residuum |
| 4ns - 2a | + s non-residuum et - s residuum. |

OBSER-

onem inquisitionem numer-

stema, quomodo con-

itur tantum per divisorem erunt formae litterae, reliqua s indicetur, quo facto se fuerit

n
iduum
n-residuum
fiduum.

OBSER-

OBSERVATIONES ANALYTICAE.

§. I.

Inter alia, quae passim de fractionibus continuis sum commentatus, notatu digna videtur haec formula:

$$\frac{1+n}{2+n-1} = \frac{3+n+2}{4+n+3} = \frac{5+n+4}{6+etc.}$$

cuius valor, quoties n est numerus integer, sequenti modo exhiberi potest, denotante e numerum, cuius logarithmus est unitas, ut sit e = 2, 718281828459045

$$\frac{1+n}{2+n} = \frac{3+n}{4+n} = \frac{5+n}{6+n} = e-1;$$

$$\frac{1+n}{2+n} = \frac{3+n}{4+n} = \frac{5+n}{6+n} = e-1;$$

$$\frac{1+n}{2+n} = \frac{3+n}{4+n} = \frac{5+n}{6+n} = e-1;$$

1+

218) 86 (218

$$\begin{array}{r} 1+3 \\ 2+4 \\ 3+5 \\ 4+6 \\ 5+7 \\ \text{etc.} \end{array} = 2;$$

$$\begin{array}{r} 1+4 \\ 2+5 \\ 3+6 \\ 4+7 \\ 5+8 \\ \text{etc.} \end{array} = 2\frac{1}{2};$$

$$\begin{array}{r} 1+5 \\ 2+6 \\ 3+7 \\ 4+8 \\ 5+9 \\ \text{etc.} \end{array} = 3\frac{1}{3};$$

$$\begin{array}{r} 1+6 \\ 2+7 \\ 3+8 \\ 4+9 \\ 5+10 \\ \text{etc.} \end{array} = 4\frac{1}{4};$$

$$\begin{array}{r} 1+7 \\ 2+8 \\ 3+9 \\ 4+10 \\ 5+11 \\ \text{etc.} \end{array} = 5\frac{1}{5};$$

$$\begin{array}{r} 1+8 \\ 2+9 \\ 3+10 \\ 4+11 \\ 5+12 \\ \text{etc.} \end{array} = 6\frac{1}{6};$$

vbi

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vbi modo proptus singulari vlti venit, vt binae, priores numerum transcendentem e implicent, dum sequentes omnes numeris rationalibus exprimentur.

§. 2. Hoc eo magis mirum videtur, quod etiam casus praecedentes, vbi pro n vel cyphra vel numeri negativi ponuntur, valoribus rationalibus continentur, quibus quidem casibus ipsa fractionis continuatae forma inspicitur. Erit enim

$$\begin{array}{r} 1+0 \\ 2+1 \\ 3+2 \\ 4+3 \\ \text{etc.} \end{array} = 1\frac{1}{2};$$

$$\begin{array}{r} 1-1 \\ 2+0 \\ 3+1 \\ 4+2 \\ \text{etc.} \end{array} = 1\frac{1}{3};$$

$$\begin{array}{r} 1-2 \\ 2-1 \\ 3+0 \\ 4+1 \\ \text{etc.} \end{array} = 1\frac{1}{4};$$

$$\begin{array}{r} 1-3 \\ 2-2 \\ 3-1 \\ 4+0 \\ \text{etc.} \end{array} = 1\frac{1}{5};$$

$$\begin{array}{r} 1-4 \\ 2-3 \\ 3-2 \\ 4-1 \\ 5+0 \\ \text{etc.} \end{array} = 1\frac{1}{6};$$

vbi

$$x-5 = -\frac{101}{74};$$

$$\frac{2-4}{3-3} = \frac{4-2}{5-1} \text{ etc.}$$

Qua igitur lego: hactenus hi valores, quam praecedentes inter se cohaerent, haud abs re fore arbitrator, offendite. Imprimis autem iuvabit methodum exposuisse, qua illi valores inuestigari queant.

§. 3. Primum igitur obtineo, si pro numero quocunque n valor fractionis continuae ita indicetur:

$$f(n) = x + \frac{n}{2+n-1}$$

$$= \frac{3+n-1}{3+n-2}$$

$$= \frac{4+n+2}{4+n+3}$$

$$= \frac{5+\text{etc.}}{5+\text{etc.}}$$

fore $f(n+1) = \frac{n(f(n)-x)}{f(n)+n-1}$; cuius veritas in valoribus indicatis purpiscitur, cum sit:

$$f(1) = \frac{1}{1}; f(2) = e-x; f(3) = 2; f(4) = \frac{2}{3};$$

$$f(5) = \frac{5}{11}; f(6) = \frac{26}{55}; f(7) = \frac{100}{195}; f(8) = \frac{347}{557};$$

et pro praecedentibus:

$$f(0) = 1; f(-1) = \frac{1}{5}; f(-2) = -\frac{1}{5}; f(-3) = -\frac{19}{15};$$

$$f(-4) = -\frac{151}{17}; f(-5) = -\frac{1901}{22}; f(-6) = -\frac{244}{57}.$$

Haec relatio inter binos valores contiguos intercedens non impedit, quominus casibus $n = 1$ et $n = 2$ sint tranfcedentes.

dentem. Posito enim $n = 0$ fit $f(1) = \frac{2(0+1)}{1+0} = 2$, quae expressio valori $\frac{2}{1}$ non adhaeretur, etiam si hic inde elici nequeat. Deinde posito $n = 2$ prodit $f(3) = \frac{2(2+1)}{1+2} = 2$, ita ut ipse valor $f(2) = e-x$ hic non in computum veniat.

§. 4. Inuestigatio autem horum valorum haud parum ardua videtur; quare, quemadmodum ad eos pervenerim, dilucide exponam, quandoquidem methodus, quam vsum, multo latius patet, ac fortasse ad alias praecellitas speculationes deducere potest. Summi igitur binos numeros indefinitos m et n , eorumque certam quandam functionem, quae sit ϕ , sum contempnatus, vnde similes functiones eorundem numerorum, vna pluribusque variabilibus auctorum, formavi, quas, summa littera ϕ pro signo huius functionis, ita repraesentabo:

$$\phi = \phi(m \text{ et } n); \phi' = \phi(m \text{ et } n+1); \phi'' = \phi(m \text{ et } n+2);$$

$$q = \phi(m+1 \text{ et } n); q' = \phi(m+1 \text{ et } n+1); q'' = \phi(m+1 \text{ et } n+2);$$

$$r = \phi(m+2 \text{ et } n); r' = \phi(m+2 \text{ et } n+1); r'' = \phi(m+2 \text{ et } n+2);$$

$$s = \phi(m+3 \text{ et } n); s' = \phi(m+3 \text{ et } n+1); s'' = \phi(m+3 \text{ et } n+2);$$

etc.

Functionem autem ϕ eius indolis esse statuo, ut sit

$$\phi = A m + B n + C + \frac{D m + E n + F}{G}$$

$$q = A(m+1) + B n + C + \frac{D n + E n + F}{q}$$

$$r = A(m+2) + B n + C + \frac{D n + E n + F}{r}$$

$$s = A(m+3) + B n + C + \frac{D n + E n + F}{s}$$

etc.

Euleri Opusc. Anal. Tom. I. M §. 5.

dentem interndiffe. Imqua illi valore numero quorur:

$$f(4) = \frac{2}{3};$$

$$8) = \frac{2417}{3572};$$

$$) = -\frac{12}{11};$$

cedens non t tranfcedentes.

valoribus in-

§. 5. Cum igitur $p', q', r', s', t', etc.$ orientantur ex $p, q, r, s, etc.$ si, servato numero m , alter n unitate au-geatur, erit simili modo:

$$p' = A m + B (n + 1) + C + \frac{D(a+1)^2 + E(a+1) + F}{p'}$$

$$q' = A (m + 1) + B (n + 1) + C + \frac{D(a+1)^2 + E(a+1) + F}{q'}$$

$$r' = A (m + 2) + B (n + 1) + C + \frac{D(a+1)^2 + E(a+1) + F}{r'}$$

tum vero ob eandem rationem:

$$p'' = A m + B (n + 2) + C + \frac{D(a+2)^2 + E(a+2) + F}{p''}$$

$$q'' = A (m + 1) + B (n + 2) + C + \frac{D(a+2)^2 + E(a+2) + F}{q''}$$

$$r'' = A (m + 2) + B (n + 2) + C + \frac{D(a+2)^2 + E(a+2) + F}{r''}$$

atque

$$p''' = A m + B (n + 3) + C + \frac{D(a+3)^2 + E(a+3) + F}{p'''}$$

$$q''' = A (m + 1) + B (n + 3) + C + \frac{D(a+3)^2 + E(a+3) + F}{q'''}$$

$$r''' = A (m + 2) + B (n + 3) + C + \frac{D(a+3)^2 + E(a+3) + F}{r'''}$$

sique porro ulterius progrediendo.

§. 6. Hinc factio p sequenti modo per fractionem continuam infinitam exprimitur:

$$p = \frac{A m + B n + C + D n^2 + E n + F}{A m + B (n + 1) + C + D (n + 1)^2 + E (n + 1) + F}$$

$$\frac{A m + B (n + 2) + C + D (n + 2)^2 + E (n + 2) + F}{A m + B (n + 3) + C + etc.}$$

Vnde servato n , si loco m succedant scribantur numeri $m + 1, m + 2, m + 3, etc.$ prodibunt valores functionum

orientantur ex unitate au.

$$\frac{D(a+1) + F}{E(a+1) + F}$$

$$\frac{E(a+1) + F}{E(a+1) + F}$$

$$\frac{(a+1) + F}{(a+1) + F}$$

$$\frac{(a+1) + F}{(a+1) + F}$$

$$\frac{(n+1) + F}{(n+1) + F}$$

$$\frac{(n+1) + F}{(n+1) + F}$$

er factio-

$$\frac{E(n+2) + F}{(n+2) + F} + C + etc.$$

numeri functionum

num $q, r, s, t, etc.$ per similes fractiones continuas expressae. Nunc igitur quaeritur, cuiusmodi relatio sit intercedura inter functiones p et q ? Qua inventa per superiores analogiam simul relatio inter omnes functiones hic exhibitas constabit. Quod cum a priori determinari nimis difficile videatur, consuetura utendum censet.

§. 7. Videamus ergo, num inter p et q huiusmodi relatio statui queat:

$$(p + (a-A)m + (\beta-B)n + \gamma - C)(q + (\delta-A)m + (\epsilon-B)n + \zeta - A - C)$$

$$= \lambda m m + \mu m + \nu,$$

vnde servato m , si loco n scribatur $n + 1$, erit

$$(p' + (a-A)m + (\beta-B)(n+1) + \gamma - C) \times$$

$$\times (q' + (\delta-A)m + (\epsilon-B)(n+1) + \zeta - A - C) = \lambda m m + \mu m + \nu.$$

At si ibi pro p et q superiores valores per p' et q' substituamur, probabit:

$$(a m + \beta n + \gamma + \frac{D n^2 + E n + F}{p'}) (\delta m + \epsilon n + \zeta + \frac{D n^2 + E n + F}{q'})$$

$$= \lambda m m + \mu m + \nu$$

quae evolvitur in hanc:

$$(a m + \beta n + \gamma) (\delta m + \epsilon n + \zeta) p' q' - (\lambda m m + \mu m + \nu) p' q'$$

$$+ (a m + \beta n + \gamma) (D n^2 + E n + F) p'$$

$$+ (\delta m + \epsilon n + \zeta) (D n^2 + E n + F) q'$$

$$+ (D a n + E n + F)^2 = 0,$$

quae cum illa congruere debet. Vnde perspicuum est esse oportere

$$(a m + \beta n + \gamma) (\delta m + \epsilon n + \zeta) - \lambda m m - \mu m - \nu = \theta (D n^2 + E n + F)$$

ve divisione per $\theta (D n^2 + E n + F)$ insituta fiat

$$p' q' + \frac{(a m + \beta n + \gamma) (\delta m + \epsilon n + \zeta) q' + \theta (D n^2 + E n + F)}{\theta (D n^2 + E n + F)} = 0,$$

quae

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quae per factores representata ita exhibentur:

$$(p' + \delta m + \epsilon + \zeta)(p' + \alpha m + \beta n + \gamma) = \frac{(\alpha m + \beta n + \gamma)(\delta m + \epsilon + \zeta)}{\beta}$$
$$- \frac{1}{2}(D m n + E n + F)$$

$$(p' + \delta m + \epsilon + \zeta)(p' + \alpha m + \beta n + \gamma) = \frac{\lambda m m + \mu m + \nu}{\beta}$$

§. 8. Comparetur haec forma cum priori:

$$(p' + (\alpha - A)m + (\beta - B)n + \beta + \gamma - B - C) \times$$
$$\times (p' + (\delta - A)m + (\epsilon - B)n + \epsilon + \zeta - A - B - C)$$
$$= \lambda m m + \mu m + \nu$$

unde statim colligitur $\theta = 1$, ideoque vel $\theta = 1$ vel $\theta = -1$. Tum vero esse debet:

$$\delta = \theta(\alpha - A); \epsilon = \theta(\beta - B); \zeta = \theta(\beta + \gamma - B - C);$$
$$\alpha = \theta(\delta - A); \beta = \theta(\epsilon - B); \gamma = \theta(\epsilon + \zeta - A - B - C).$$

Quia ergo valor $\theta = 1$ non convenit, ponamus $\theta = -1$, ut habeamus:

$$\alpha + \delta = A; \beta + \epsilon = B; \beta + \gamma + \zeta = B + C \text{ et}$$
$$\gamma + \epsilon + \zeta = A + B + C$$

hincque $\epsilon - \beta = A$; ergo:

$$\beta = \frac{1}{2}(B - A); \epsilon = \frac{1}{2}(A + B) \text{ et } \gamma + \zeta = \frac{1}{2}(A + B) + C.$$

Praeterea vero haec conditio est adimplenda:

$$(\alpha m + \beta n + \gamma)(\delta m + \epsilon n + \zeta) = \lambda m m + \mu m + \nu - D m n - E n - F$$
$$= \alpha \delta m m + \alpha \epsilon m n + \alpha \zeta m + \beta \zeta n + \gamma \zeta,$$
$$+ \beta \epsilon n n + \beta \delta m n + \gamma \delta m + \gamma \epsilon n.$$

Erit ergo:

$$\lambda = \alpha \delta; \mu = \alpha \zeta + \gamma \delta; D = -\beta \zeta; E = -\beta \zeta - \gamma \epsilon$$
$$\nu - F = \gamma \zeta \text{ et } \alpha \epsilon + \beta \delta = 0,$$

unde

$$\frac{m + n + \dots}{\beta}$$

$$\frac{m + n + \dots}{\beta}$$

priori:

$$-B - C) \times$$
$$-A - B - C)$$

$$\theta = 1 \text{ vel}$$

$$-B - C);$$

$$-A - B - C).$$

$$\text{us } \theta = -1,$$

$$= B + C \text{ et}$$

$$+ B) + C.$$

$$n n - E n - F$$
$$+ \gamma \zeta,$$

$$- \beta \zeta - \gamma \epsilon$$

unde

unde primo sic $D = -\beta \zeta = \frac{1}{2}(A - B - C)$; deinde $\frac{1}{2}\alpha(A + B) - \frac{1}{2}\delta(B - A) = 0$, seu $\delta = \frac{A + B}{A - B}\alpha$;

ideoque

$$\alpha = \frac{1}{2}(A - B) \text{ et } \delta = \frac{1}{2}(A + B).$$

Tum vero erit:

$$E + \frac{1}{2}\zeta(B - A) + \frac{1}{2}\gamma(A + B) = 0, \text{ seu}$$

$$E + \frac{1}{2}B(A + B) + \frac{1}{2}BC + \frac{1}{2}A(\gamma - \zeta) = 0,$$

hincque

$$\zeta - \gamma = \frac{BC}{A} + \frac{B(A + B)}{A} + \frac{2E}{A}.$$

ergo

$$\zeta = \frac{1}{2}(A + B) + \frac{1}{2}C + \frac{BC}{A} + \frac{B(A + B)}{A} + \frac{E}{A},$$

$$\gamma = \frac{1}{2}(A + B) + \frac{1}{2}C - \frac{BC}{A} - \frac{B(A + B)}{A} - \frac{E}{A}.$$

sive hoc modo:

$$\zeta = \frac{1}{2}(A + B + 2C) \left(\frac{1 + \frac{B}{A}}{1 - \frac{B}{A}} \right) - \frac{E}{A} = \frac{(A + B)(A + B + 2C) + 2E}{A}$$
$$\gamma = \frac{1}{2}(A + B + 2C) \left(\frac{1 - \frac{B}{A}}{1 + \frac{B}{A}} \right) - \frac{E}{A} = \frac{(A + B)(A + B + 2C) - 2E}{A}.$$

§. 9. Relatio ergo inter ϕ et q adsumpta subditur nequit, nisi sit: $D = \frac{1}{2}(A - B)$, qui valor ϕ ipsi D tribuatur, sequentes litterae ita se habebunt:

$$\alpha = \frac{1}{2}(A - B); \delta = \frac{1}{2}(A + B); \gamma = \frac{(A - B)(A + B + 2C) - 2E}{A};$$
$$\beta = -\frac{1}{2}(A - B); \epsilon = \frac{1}{2}(A + B); \zeta = \frac{(A + B)(A + B + 2C) + 2E}{A};$$
$$\lambda = \frac{1}{2}(A - B); \mu = D; \nu = \frac{(A - B)(A + B + 2C) - 2E}{A};$$

$$\mu = \frac{(A - B)(A + B + 2C) - 2E}{A};$$
$$\nu = \frac{(A - B)(A + B + 2C) - 2E}{A};$$

hincque porro

$$\alpha - A = -\frac{1}{2}(A + B); \beta - B = -\frac{1}{2}(A + B);$$

$$\gamma - C = \frac{A - B}{A}; \delta - D = \frac{A + B}{A};$$

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$\delta - A$

$$\delta - A = -\frac{1}{2}(A-B); \quad \epsilon - B = \frac{1}{2}(A-B);$$

$$\zeta - A - C = -\frac{(A-B)(A+B) - C(A-B)}{2A} + \frac{E}{A},$$

vnde inter p et q haec resultat aequatio:

$$\left(p - \frac{1}{2}(A+B)(m+n) + \frac{AA-BB - C(A+B) - E}{2A} \right) \times$$

$$\times \left(q - \frac{1}{2}(A-B)(m-n) - \frac{(A-B)(A+B) - C(A-B) - E}{2A} + \frac{E}{A} \right)$$

$$= \lambda m^2 + \mu m + \nu.$$

§. 10. Ponamus ad abbreviandum

$$P = \frac{(A+B)(A-B)}{2A} - \frac{C(A+B)}{2A} - \frac{E}{A},$$

$$Q = \frac{(A-B)(A+B)}{2A} + \frac{C(A-B)}{2A} - \frac{E}{A},$$

vt fit

$$(p - \frac{1}{2}(A+B)(m+n) + P) (q - \frac{1}{2}(A-B)(m-n) - Q)$$

$$= \lambda m^2 + \mu m + \nu,$$

erit

$$p = \frac{1}{2}(A+B)(m+n) - P + \frac{\lambda m^2 + \mu m + \nu}{q - \frac{1}{2}(A-B)(m-n) - Q}.$$

Simili vero modo erit

$$q = \frac{1}{2}(A+B)(m+n) + \frac{1}{2}(A+B) - P + \frac{\lambda(m+x)^2 + \mu(m+x) + \nu}{r - \frac{1}{2}(A-B)(m-n) - \frac{1}{2}(A-B) - Q},$$

$$r = \frac{1}{2}(A+B)(m+n) + (A+B) - P + \frac{\lambda(m+2)^2 + \mu(m+2) + \nu}{s - \frac{1}{2}(A-B)(m-n) - (A-B) - Q},$$

vnde fit

$$q - \frac{1}{2}(A-B)(m-n) - Q = Bm + An + \frac{1}{2}(A+B) - P - Q$$

$$+ \frac{\lambda(m+x)^2 + \mu(m+x) + \nu}{r - \frac{1}{2}(A-B)(m-n) - \frac{1}{2}(A-B) - Q},$$

$$r - \frac{1}{2}(A-B)(m-n) - \frac{1}{2}(A-B) - Q = Bm + An + \frac{1}{2}(A+3B) - P - Q$$

$$+ \frac{\lambda(m+2)^2 + \mu(m+2) + \nu}{s - \frac{1}{2}(A-B)(m-n) - (A-B) - Q},$$

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ER vero

$$P + Q = \frac{(A-B)(A+B)}{2A} - \frac{BC - E}{A},$$

hincque

$$q - \frac{1}{2}(A-B)(m-n) - Q = Bm + An + B + \frac{BB - AA + BC + E}{2A} + \text{etc.}$$

Quare si breuitatis gratia ponatur

$$\frac{BB - AA + BC + E}{2A} = G, \text{ erit}$$

$$p = \frac{1}{2}(A+B)(m+n) + \frac{BB - AA + C(A+B) + E}{2A} + \frac{\lambda m^2 + \mu m + \nu}{B(m+x) + An + G + \lambda \frac{(m+x)^2 + \mu(m+x) + \nu}{B(m+2) + An + G + \lambda \frac{(m+2)^2 + \mu(m+2) + \nu}{B(m+3) + An + G + \text{etc.}}}}$$

vel valorem G introducendo:

$$p = \frac{1}{2}(A+B)(m+n) + (C+G) + \frac{\lambda m^2 + \mu m + \nu}{B(m+x) + An + G + \lambda \frac{(m+x)^2 + \mu(m+x) + \nu}{B(m+2) + An + G + \lambda \frac{(m+2)^2 + \mu(m+2) + \nu}{B(m+3) + An + G + \text{etc.}}}}$$

Tam vero etiam erit

$$P = -\frac{1}{2}(C+G) \text{ et } Q = \frac{1}{2}(A-B) + \frac{1}{2}(C-G)$$

ideoque

$$(p - \frac{1}{2}(A+B)(m+n) - \frac{1}{2}(C+G)) (q - \frac{1}{2}(A-B)(m+n) - \frac{1}{2}(C-G))$$

$$= \lambda m^2 + \mu m + \nu.$$

§. 11. Quod si ergo proposita fuerit huiusmodi fractio constantia infinita:

f =

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$$p = Am + Bn + C$$

$$\frac{Dn^2 + En + F}{Am^2 + B(n+1) + C + D(n+1)^2 + E(n+1) + F}$$

$$\frac{Am + B(n+2) + C + D(n+2)^2 + E(n+2) + F}{Am + B(n+3) + C + etc.}$$

in qua sit $D = \frac{1}{2}(A^2 - B^2)$, indeque haec altera formetur:

$$q = A(m+x) + Bn + C$$

$$\frac{Dn^2 + En + F}{A(m+x) + B(n+1) + C + D(n+1)^2 + E(n+1) + F}$$

$$\frac{A(m+x) + B(n+2) + C + D(n+2)^2 + E(n+2) + F}{A(m+x) + B(n+3) + C + etc.}$$

dum loco m vbiq; scribitur $m+x$, primo relatio inter p et q assignari potest hoc modo: Ponatur breuitatis gratia:

$$\frac{B^2 - A^2 + 2BC + 4E}{2A} = G \text{ et}$$

$$\lambda = \frac{1}{2}(A^2 - B^2),$$

$$\mu = \frac{1}{2}(A^2 - B^2) + \frac{1}{2}(A^2 - B^2)G,$$

$$\nu = \frac{1}{2}CC + \frac{1}{2}(C-G)(A+B) - \frac{1}{2}GG + F,$$

seu

$$\nu = F + \frac{1}{2}(C-G)(A+B+C+G),$$

erique ista relatio:

$$(p - \frac{1}{2}(A+B)(m+n) - \frac{1}{2}(C+G)(q - \frac{1}{2}(A-B)(m+x-n) - \frac{1}{2}(C-G)) = \lambda m^2 + \mu m + \nu,$$

tum vero praetera functio p etiam huic alteri fractioni continuac acquatur:

$$p =$$

$$p = \frac{1}{2}(A+B)(m^2+n) + \frac{1}{2}(C+G)$$

$$\frac{\lambda m^2 + \mu m + \nu}{B(m+x) + A + G + \lambda(m+x)^2 + \mu(m+x) + \nu}$$

$$\frac{B(m+2) + A + G + \lambda(m+2)^2 + \mu(m+2) + \nu}{B(m+3) + A + G \text{ etc.}}$$

§. 12. Exempla supra proposita hinc nascuntur, si statuatur:

$$D = \frac{1}{2}(A^2 - B^2) = 0.$$

Sit ergo $B = A$, ut habeantur istae fractiones continuac:

$$p = A(m+n) + C$$

$$\frac{En + F}{A(m+n+1) + C + \frac{(n+1) + F}{A(m+n+2) + C + E(n+2) + F}}$$

$$\frac{A(m+n+3) + C + etc.}{A(m+n+4) + C + etc.}$$

$$q = A(m+n+1) + C$$

$$\frac{F n + F}{A(m+n+2) + C + E(n+1) + F}$$

$$\frac{A(m+n+3) + C + E(n+2) + F}{A(m+n+4) + C + etc.}$$

quarum relatio, posito

$$G = C + \frac{2E}{A}, \mu = \frac{1}{2}A(C-G) \text{ et}$$

$$\nu = F + \frac{1}{2}(C-G)(2A+C+G), \text{ seu}$$

$$\mu = -E; \nu = F - \frac{E}{A}(2A+C+G), \text{ erit}$$

$$(p - A(m+n) - \frac{1}{2}(C+G))(q - \frac{1}{2}(C-G)) = \mu m + \nu,$$

et quantitas p etiam alio modo per fractionem continuam infinitam exprimitur.

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N

$$p =$$

$\frac{1}{2}F$
+ etc.
for-

$\frac{n+2}{3} + F$
+ etc.
ulatis

-G))
tioni

$$p =$$

$$p = A(m+n) + K(C+G)$$

$$+ \frac{\mu m + \nu}{A(m+n+1) + G + \mu(m+1) + \nu} \\ A(m+n+2) + G + \mu(m+2) + \nu \\ A(m+n+3) + G + \dots \text{ etc.}$$

§. 13. Cum igitur hic tam numeratores quam denominatores progressionem arithmeticam constituent, eorum formam simpliciter reddamus, atque binas fractiones continuas infinitas:

$$p = a + \frac{f}{a + b + \frac{f + g}{a + 2b + \frac{f + 2g}{a + 3b + \frac{f + 3g}{a + b + \text{etc.}}}}$$

ita ut ex fractione p nascatur q , si loco a scribatur $a + b$.
Erit ergo:
 $A(m+n) + C = a$; $E n + F = f$; $A = b$; $E = g$;
hincque $C = a - b(m+n)$; et $F = f - g n$; unde con-

clinetur:

$$G = a - b(m+n) + \frac{2f}{b}; \quad C - G = \frac{2a - b(m+n)}{b};$$

$$C + G = 2a - 2b(m+n) + \frac{2f}{b}; \quad \text{tum}$$

$$\mu = -g; \quad \nu = f - g n - g - \frac{f}{b}(a - b(m+n) + \frac{f}{b})$$

feu

feu $\nu = f - \frac{E(a+b)}{b} + g m - \frac{Eg}{b}$, ergo

$$\mu m + \nu = f - \frac{E(a+b)}{b} - \frac{Eg}{b} \\ A(m+n) + \frac{1}{2}(C+G) = a + \frac{f}{b} \text{ et} \\ \frac{1}{2}(C-G) = -\frac{f}{b}.$$

Quare inter p et q haec habetur relatio:

$$(p - a - \frac{f}{b})(q + \frac{f}{b}) = f - \frac{E(a+b)}{b} - \frac{Eg}{b}; \text{ seu}$$

$$p q + \frac{f}{b} p - (a + \frac{f}{b}) q = f - g$$

unde pro p etiam haec habetur fractio continua:

$$p = a + \frac{f}{b} + \frac{f - \frac{E(a+b)}{b} - \frac{Eg}{b}}{a + b + \frac{2f}{b} + \frac{f - \frac{E(a+b)}{b} - \frac{Eg}{b}}{a + 2b + \frac{2f}{b} + \frac{f - \frac{E(a+b)}{b} - \frac{Eg}{b}}{a + 3b + \frac{2f}{b} + \frac{f - \frac{E(a+b)}{b} - \frac{Eg}{b}}{a + b + \text{etc.}}}}$$

§. 14. Ut fractiones tollamus ponamus $g = b h$, ut habeamus has fractiones continuas:

$$p = a + \frac{f}{a + b + \frac{f + b h}{a + 2b + \frac{f + 2 b h}{a + 3b + \frac{f + 3 b h}{a + 4 b \text{ etc.}}}}$$

$$q = a + b + \frac{f}{a + 2b + \frac{f + b h}{a + 3b + \frac{f + 2 b h}{a + 4b + \frac{f + 3 b h}{a + 5b \text{ etc.}}}}$$

N 2

quasi

1) + $\frac{f}{b}$
feu

$\frac{f + 3g}{a + b + \text{etc.}}$
utur $a + b$.
 b ; $E = g$;
unde conf-

quarum relatio ita se habet; ut sit

$$(p - a - b)(q + b) = f - (a + b)b - b^2; \text{ seu}$$

$$p q + b p - (a + b)q = f - b b.$$

Vnde pro p elicitur haec quoque fractio continua:

$$p = a + b + f - (a + b)b - b^2$$

$$a + b + 2b + f - (a + 2b)b - b^2$$

$$a + 3b + 2b + \text{etc.}$$

Cum igitur haec fractio continua primae sit equalis, haec autem abruptatur, quoties fuerit $f = (a + i)b + b^2$, deoatante i numerum integrum posituum, toties valor primae rationally assignari potest.

§. 15: Ex relatione inter p et q iuuenta per p quoque q ita exprimitur:

$$q = -b + \frac{f - (a + b)b - b^2}{a + b + p}$$

et cum p oriatur ex q , si loco a scribatur $a - b$; si seriei p , q , r , etc. termini praecedentes sint o , n , m , etc. erit.

$$p = -b + \frac{f - ab - b^2}{a + b + p}$$

$$o = -b + \frac{f - (a - b)b - b^2}{a + b + p}$$

$$n = -b + \frac{f - (a - n)b - b^2}{a + b + p}$$

etc.

Vnde pro p etiam haec fractio continua obinetur:

$$p = -b + f - ab - b^2$$

$$b - a - 2b + f + (b - a)b - b^2$$

$$2b - a - 2b + f + (2b - a)b - b^2$$

$$3b - a - 2b + \text{etc.}$$

quam eandem ex nostris formulis generalibus inueniſſemus, si

seri

haec
- b b;
valor

per p

seriei
cir

si supra §. 12. posuissimus $B = -A$. Quare etiam valor rationally exprimi poterit, quoties fuerit $b^2 = i(b - a)b + f$, nisi forte his casibus denominator, isti. numeratori euanescenti subiectus, quoque euanescat.

§. 16: Ex ipsa autem fractione continua proposita:

$$p = a + \frac{f}{a + b + \frac{f + b b}{a + 2b + \frac{f + 2 b b}{a + 3b + \text{etc.}}}}$$

alia immediate hoc modo deduci potest ipsa equalis. Cum enim sit:

$$p = a + \frac{f}{p}; p' = a + b + \frac{f + b b}{p'}$$

$$p'' = a + 2b + \frac{f + 2 b b}{p''}; \text{ etc.}$$

erit regressendo:

$$p_1 = a - b + \frac{f - b b}{p_1}; p_{II} = a - 2b + \frac{f - 2 b b}{p_{II}}; p_{III} = a - 3b + \frac{f - 3 b b}{p_{III}};$$

hincque

$$p = \frac{f - 3 b b}{a + p}; p_1 = \frac{f - 2 b b}{2b - a + p_1}; p_{II} = \frac{f - b b}{3b - a + p_{II}}; \text{ etc.}$$

vnde concludimus:

$$p = \frac{f - b b}{b - a + \frac{f - 2 b b}{2b - a + \frac{f - 3 b b}{3b - a + \text{etc.}}}}$$

ita ut etiam casibus $f = i b b$ valor rationally exhiberi queat.

§. 17. Ex ergo quatuor fractionibus continuas inter se aequales:

N 3

$$I, P =$$

I. $p = a + \frac{f}{a+b+f+bb}$

$\frac{a+2b+f+2bb}{a+3b+f+3bb}$

$\frac{a+4b}{a+4b}$ etc.

II. $p = \frac{f-b}{b-a+f-2bb}$

$\frac{3b-a+f-3bb}{4b-a}$ etc.

$\frac{4b-a}{4b-a}$ etc.

III. $p = a+b+f-(a+b)b-bb$

$\frac{a+b+2b+f-(a+2b)b-bb}{a+3b+2b+f-(a+3b)b-bb}$

$\frac{a+3b+2b}{a+3b+2b}$ etc.

IV. $p = -b+f-ab-bb$

$\frac{b-a-2b+f+(b-a)b-bb}{2b-a-2b+f+(2b-a)b-bb}$

$\frac{3b-a-2b}{3b-a-2b}$ etc.

§. 18. Quo ad formam initio propositam propius accedamus, sit $a = m$; $f = n$; $b = x$ et $b = 1$; atque habebimus:

I. $p = m + \frac{n}{m+1+n+1}$
 $\frac{m+2+n+1}{m+3+n+3}$
 $\frac{m+4}{m+4}$ etc.

II. $p =$

II. $p = \frac{n-x}{-m+1+n-x}$

$\frac{-m+2+n-2}{-m+3+n-3}$

$\frac{-m+4}{-m+4}$ etc.

III. $p = m+1+n-m-x$

$\frac{m+3+n-m-2}{m+4+n-m-3}$

$\frac{m+5+n-m-4}{m+6}$ etc.

IV. $p = -x+n-m-x$

$\frac{-m-1+n-m}{-m+n-m+1}$

$\frac{-m+2}{-m+2}$ etc.

Quare denotante i numerum integrum positivum; cyphra non excluda, fractionis nostrae continuae valor rationaliter exprimi poterit, His casibus:

I. $n = i$; II. $n = m+2+i$; III. $n = m+1-x-i$;

nisi forte incommodum supra memorarum locum inveniat.

§. 19. Ex casibus $n = i$ raro valor quaesitus reperitur, ob memoratum incommodum, quo etiam denominator in nihilum abit. Si enim $n = x$, quo fit:

$p = m + \frac{x}{m+1+x}$
 $\frac{m+2+x}{m+3+x}$
 $\frac{m+4}{m+4}$ etc.

certe

certe non est $p = 0$, est secunda forma id ostendere videtur, unde affirmare possumus esse:

$$0 = 1 - m - 1 \\ \frac{2 - m - 2}{3 - m - 3} \quad \frac{4 - m - 4}{5 - m \text{ etc.}}$$

Quodsi prima forma generalis ad hunc casum accommodetur, erit $a = 1 - m$; $b = 1$; $f = -1$, et $h = -1$, unde secunda dat

$$p = 1 - 1 \\ \frac{m + 1}{m + 1 + 2} \quad \frac{m + 1}{m + 1 \text{ etc.}}$$

tertia vero:

$$p = -m - m \\ \frac{-m + 1 - m}{1 - m + 2 - m} \quad \frac{2 - m \text{ etc.}}$$

et quarta

$$p = 1 - 1 - 1 - m \\ \frac{m - 2 - 2 - m}{m - 1 - 3 - m} \quad \frac{m - \text{etc.}}$$

quae ergo nihilo sunt aequales.

§. 20.

hinc videtur

accommodatur, unde

§. 20. Cum ex forma secunda §. 18. sit

$$\frac{1}{2} = 1 - m + n - 2 \\ \frac{2 - m + n - 3}{3 - m + n - 4} \quad \frac{4 - m \text{ etc.}}$$

si haec cum prima generali comparatur, erit

$d = 1 - m$; $b = 1$; $f = n - 2$; et $h = -1$;

unde forma tertia praebet:

$$\frac{1}{2} = -m + n - 1 - 1 \\ \frac{-m + n - 1}{2 - m + n - 2} \quad \frac{3 - m \text{ etc.}}$$

et quarta:

$$\frac{1}{2} = 1 + n - m - 2 \\ \frac{m + 2 + n - m - 3}{m + 3 + n - m - 4} \quad \frac{m + 4 \text{ etc.}}$$

haecque duae novae expressiones pro p habentur; similique modo plures aliae exhiberi possunt.

§. 21. Invenio autem valore p facile determinari haec

$$x = m + n + 1 \\ \frac{m + 1 + n + 2}{m + 2 + n + 3} \quad \frac{m + 3 + \text{etc.}}$$

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O

Scri-

Scribatur enim hi $m-1$ loco m , ponaturque.

$$q = m - 1 + m = 2m - 1 + \frac{m}{2}$$

$$m + n + 1$$

$$m + 1 + n + 2$$

$$m + 2 \text{ etc.}$$

at ex tertia erit

$$q = m + n - m - 1 = \frac{m + n - 1}{p + 1}$$

$$m + 2 + n - m - 2$$

$$m + 3 + n - m - 3$$

$$m + 4 \text{ etc.}$$

quibus binis valoribus ipsius q acquiritur sit

$$-1 + \frac{2}{p} = \frac{m + n - 1}{p + 1}, \text{ et } \frac{2}{p} = \frac{m + n - 1}{p + 1} + 1,$$

si que $x = \frac{m + n - 1}{p}$.

§. 22. Cum igitur, posito $n = m + 3$, sit $p = m + 1$,

$$m + 1 = p = m + 1 + 2$$

$$m + 1 + m + 3$$

$$m + 2 + m + 4$$

$$m + 3 \text{ etc.}$$

similique modo, ponendo, $n = m + 3$; $n = m + 4$; etc.

$$m + 1 + 1 = m + m + 3$$

$$m + 3 \quad m + 1 + m + 4$$

$$m + 2 + m + 5$$

$$m + 3 \text{ etc.}$$

$$m + 1 + 2 = 2m + m + 4 = m + 1 + m + 5 = m + 2 + m + 6 = m + 3 \text{ etc.}$$

$$m + 3 + 1$$

$$m + 4$$

$$m + 2 + m + 6$$

$$m + 1 + 3$$

$$m + 3 + 2$$

$$m + 4 + 1$$

$$m + 5$$

$$m + 1 + m + 6$$

$$m + 4 + 1$$

$$m + 1 + m + 7$$

$$m + 3 \text{ etc.}$$

$$m + 1 + 4$$

$$m + 3 + 3$$

$$m + 4 + 2$$

$$m + 5 + 1$$

$$m + 6$$

$$m + 3 + m + 8$$

$$m + 4 \text{ etc.}$$

unde, posito $m = 1$, casus rationales supra observari consue-
quuntur. Hi autem valores ita progrediuntur, ut sit:

$$p = m + 1; \quad q = \frac{(m+2)(p+1)}{p+1};$$

$$r = \frac{(m+3)(q+1)}{q+1}; \quad s = \frac{(m+4)(r+1)}{r+1}; \text{ seu}$$

$$q = (m+2) \left(1 - \frac{1}{p+2} \right);$$

$$r = (m+3) \left(1 - \frac{1}{q+3} \right);$$

$$s = (m+4) \left(1 - \frac{1}{r+4} \right);$$

etc.

quae expressiones etiam ita exhiberi possunt:

$$q = m + 2 - \frac{(m+1)}{m+1};$$

$$r = m + 3 - 2 \frac{(m+3)}{m+3};$$

$$s = m + 4 - 3 \frac{(m+4)}{m+4};$$

$$m + 7 - 2 \frac{(m+3)}{m+3}.$$

$$m + 5 - \frac{(m+1)}{m+1}.$$

Q 2

Extra

$$O = m + x - 1$$

$$\frac{m+3-2}{m+4-3} \quad \frac{m+5-4}{m+6-5} \quad \frac{m+7}{m+7} \text{ etc.}$$

Hinc autem finitum Valorem ipsius O expectare non licet, cum casu $m = x$ certe sit transcendens, qui quemadmodum sit investigandus, exponamus.

§. 25. Ex forma ergo prima formatur hae formulae:

mutae: $O = m + \frac{m}{A}; A = m + 1 + \frac{m+2}{A}; B = m + 2 + \frac{m+3}{B}; \text{ etc.}$

erique

Statuatur $O A = m A + m + 1; A B = (m+1) B + m + 2; \text{ etc.}$

ac reperientur hae formulae:

$O = -1 + \frac{1}{A}; A = -1 + \frac{1}{A}; B = -1 + \frac{1}{B}; \text{ etc.}$

feu

$\alpha + (m+1)\omega = 1; \beta + (m+2)\alpha = 1; \gamma + (m+3)\beta = 1; \text{ etc.}$

$\omega = \frac{1}{m+1} - \frac{\alpha}{m+1}; \alpha = \frac{1}{m+2} - \frac{\beta}{m+2}; \beta = \frac{1}{m+3} - \frac{\gamma}{m+3}; \text{ etc.}$

vnde per feriem confuetam fit

$\omega = \frac{1}{m+1} - \frac{1}{(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)} - \frac{1}{(m+1)(m+2)(m+3)(m+4)} + \text{ etc.}$

cuius valor est $\omega = \frac{1}{e^x} x^m a^x$, integrali hoc ita sumto, vt eueniat posito $x = 0$, quo facto statui debet $x = 1$. Hinc casibus quibus m est numerus integer erit

fi

$\frac{-5}{m+7} \text{ etc.}$

e non liqui quem-

hae for-

$\frac{+3}{e}; \text{ etc.}$

$+2; \text{ etc.}$

etc.

$\beta = 1; \text{ etc.}$

$\frac{-2}{m+2};$

$\frac{+3(e+1)}{e} \text{ etc.}$

sumto, vt ut $x = 1$.

fi

fi $m = 0; \omega = \frac{e-1}{e}; \text{ et } O = \frac{1}{e-1}$
 fi $m = 1; \omega = \frac{1}{e}; \text{ et } O = e - 1$
 fi $m = 2; \omega = \frac{e-2}{e^2}; O = \frac{e-1}{e-2}$
 fi $m = 3; \omega = \frac{e-3}{e^3}; O = \frac{e^2-2e+1}{e^2-3e+3}$
 fi $m = 4; \omega = \frac{9e-4}{e^4}; O = \frac{e^3-3e^2+2e-1}{e^3-4e^2+6e-4}$
 fi $m = 5; \omega = \frac{120-41e}{e^5}; O = \frac{45e^4-120e^3+100e^2-41e}{45e^4-120e^3+100e^2-41e}$
 fi $m = 6; \omega = \frac{252e-720}{e^6}; O = \frac{720e^5-252e^4-720e^3+513e^2-144}{720e^5-252e^4-720e^3+513e^2-144}$
 fi $m = 7; \omega = \frac{5040-1352e}{e^7}; O = \frac{1352e^6-5040e^5-1352e^4+9720e^3-2002e^2-52(1352-144)}{5040-1352e}$
 Nisi m est numerus integer, valor ipsius O per numerum e , cuius logarithmus = 1, exprimi nequit.

§. 26. Ponamus $m = 0$, quorundam omnes casus, quibus m est numerus integer, referri possunt, erique

$O = \frac{1}{e-1}; N = 0; M = -1; L = \frac{1}{e-1} + \frac{1}{e} = \frac{e+1}{e(e-1)}$

$K = \frac{1+e}{1+e} = 1; F = \frac{1+e}{1+e} = 1; R = \frac{1+e}{1+e} = 1$

$H = \frac{1+e+e^2}{1+e} = \frac{1+e+e^2}{1+e}; G = \frac{1+e+e^2}{1+e} = \frac{1+e+e^2}{1+e}$

vnde sequentes oriuntur fractiones continuae:

$1 = 0 + \frac{1}{1 + \frac{1}{3}}$

$\frac{1}{3} = 0 + \frac{1}{1 + \frac{1}{4}}$

$\frac{1}{4} = 0 + \frac{1}{1 + \frac{1}{2}}$

$\frac{1}{2} = 0 + \frac{1}{2 + \frac{1}{3}}$

$f = 0$

Facta autem evolutione invenitur:

$$p = m + 1; q = \frac{(m+1)(m+2)}{m+1};$$

$$r = \frac{(m+2)(m+3)(m+4)}{m+1};$$

$$s = \frac{(m+3)(m+4)(m+5)(m+6)}{m+1};$$

$$t = \frac{(m+4)(m+5)(m+6)(m+7)(m+8)}{m+1};$$

§. 23. Denotent p, q, r, s, t , etc. fractiones con-

tinnas in §. precedente exhibitas, sicut

$$p = (m+1) \frac{a}{b}; q = (m+2) \frac{b}{c}; r = (m+3) \frac{c}{d}; \text{ etc.}$$

erit $a = 1, \alpha = 1$; reliquae vero litterae ita a se invicem pendent, ut sit:

$$b = (m+1)a + \alpha;$$

$$c = (m+2)b + \beta;$$

$$d = (m+3)c + \gamma;$$

$$e = (m+4)d + \delta;$$

$$\text{etc.}$$

vnde ratio inter litteras latinas et graecae formam colligatur:

$$b = (m+2)a; \beta = (m+3)\alpha;$$

$$c = (m+4)b - 1(m+1)a; \gamma = (m+5)\beta - 2(m+2)\alpha;$$

$$d = (m+0)c - 2(m+2)b; \delta = (m+7)\gamma - 2(m+3)\beta;$$

$$e = (m+8)d - 3(m+3)c; \epsilon = (m+9)\delta - 5(m+4)\gamma;$$

$$\text{etc.}$$

Invenis autem denominatoribus $\alpha, \beta, \gamma, \delta, \epsilon$, etc. erunt numeratores:

$$b = \beta - \alpha, e = \gamma - 2\beta, d = \delta - 3\gamma, e = \epsilon - 4\delta, \text{ etc.}$$

At si numeratores a, b, c, d, e , iam sint inventi, erunt denominatores:

$$\alpha = a; \beta = 2b - (m+1)a;$$

$$\gamma = 3c - 2(m+2)b; \delta = 4d - 3(m+3)c;$$

$$\epsilon = 5e - 4(m+4)d \text{ etc.}$$

Vide

Vidimus autem esse:

$$a = 1; \beta = m + 3$$

$$b = m + 2; \gamma = m^2 + 7m + 13$$

$$c = m + 5m + 7; \delta = m^2 + 12m^2 + 50m + 73$$

$$d = m^2 + 9m^2 + 29m + 34; \epsilon = m^2 + 14m^2 + 77m^2 + 200m + 209$$

$$e = m^2 + 14m^2 + 77m^2 + 200m + 209; \zeta = m^2 + 18m^2 + 120m + 503$$

et con-

tinuas

etc.

nuicem

collig-

$$O = m + m + 1$$

$$N = \frac{m+1}{2(m+1)}; M = \frac{m-1+N}{N-(m-1)}; L = \frac{m+1+M}{N-(m-1)}$$

$$\text{etc.}$$

collig-

etc.

erunt

erunt

erunt

erunt

erunt

erunt

erunt

erunt

erunt

erunt

Vide

§. 24. Ex valore cuiusque fractionis continuae §. 22. definiti quoque potest valor praecedentis, hoc modo:

$$p = \frac{m+1-2q}{q-(m+2)}; q = \frac{m+1-3r}{r-(m+3)}; r = \frac{m+1-4s}{s-(m+4)}; \text{ etc.}$$

vnde si fractiones continuae ordine praecedentes designentur litteris O, N, M, etc. erit

$$O = \frac{m+1-2N}{2-(m+1)}; N = \frac{m}{O-M}; M = \frac{m-1+N}{N-(m-1)}; L = \frac{m+1+M}{N-(m-1)}$$

$$\text{etc.}$$

At est

$$O = \frac{m}{1-m+m-1}$$

$$N = \frac{m-1}{2-m+m-2}$$

$$M = \frac{m-2}{m-2+m-3}$$

$$L = \frac{m-3}{m-3+m-4}$$

$$\text{etc.}$$

cuius reliqui valores aequivalentes sunt

$$O = \frac{m}{1-m+m-1}$$

$$N = \frac{m-1}{2-m+m-2}$$

$$M = \frac{m-2}{m-2+m-3}$$

$$L = \frac{m-3}{m-3+m-4}$$

$$\text{etc.}$$

Vide

127) 127 (213

$$\begin{aligned}
 1 &= 0 + \frac{3}{1} \\
 &= 0 + \frac{3}{1} + \frac{4}{2} \\
 &= 0 + \frac{3}{1} + \frac{4}{2} + \frac{5}{3} + \text{etc.} \\
 0 &= 0 + \frac{0}{1} + \frac{1}{2} + \frac{2}{3} + \text{etc.} \\
 \frac{1}{2} &= 0 + \frac{4}{2} + \frac{5}{3} + \frac{6}{4} + \text{etc.} \\
 -1 &= 0 - \frac{1}{1} + \frac{0}{2} + \frac{1}{3} + \frac{2}{4} + \text{etc.} \\
 \frac{1}{3} &= 0 + \frac{5}{3} + \frac{6}{4} + \frac{7}{5} + \text{etc.} \\
 -4 &= 0 - \frac{2}{1} + \frac{1}{2} + \frac{0}{3} + \frac{1}{4} + \text{etc.} \\
 \frac{1045}{301} &= 0 + \frac{6}{1} + \frac{7}{2} + \frac{8}{3} + \text{etc.} \\
 15 &= 0 - \frac{3}{1} + \frac{2}{2} - \frac{1}{3} + \text{etc.}
 \end{aligned}$$

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quibus adiungi debent istae:

$$\begin{aligned}
 \frac{1}{2} &= 0 + \frac{1}{1} + \frac{2}{2} + \frac{3}{3} + \text{etc.} \\
 \frac{1}{3} &= 1 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \text{etc.} \\
 \frac{1}{4} &= 2 + \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \text{etc.} \\
 \frac{1}{5} &= 3 + \frac{1}{4} + \frac{2}{5} + \frac{3}{6} + \text{etc.} \\
 \frac{1}{6} &= 4 + \frac{1}{5} + \frac{2}{6} + \frac{3}{7} + \text{etc.} \\
 \frac{1}{7} &= 5 + \frac{1}{6} + \frac{2}{7} + \frac{3}{8} + \text{etc.}
 \end{aligned}$$

Si enim fit

Euleri Opus, Anal. Tom. I.

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$x = m + 1$

$\frac{m+1+2}{m+1+2}$

$\frac{m+2+3}{m+2+3}$

$\frac{m+3+4}{m+3+4}$ etc.

$= m + 1 - m - 1$

$\frac{m+3-m-2}{m+3-m-2}$

$\frac{m+4-m-3}{m+4-m-3}$

$\frac{m+5-m-4}{m+5-m-4}$ etc.

$y = m + 1 + 1$

$\frac{m+2+2}{m+2+2}$

$\frac{m+3+3}{m+3+3}$

$\frac{m+4+4}{m+4+4}$ etc.

$= m + 3 - m - 2$

$\frac{m+4-m-3}{m+4-m-3}$

$\frac{m+5-m-4}{m+5-m-4}$

$\frac{m+6-m-5}{m+6-m-5}$ etc.

erit $y = \frac{m}{m+1} - \frac{m}{m+2}$

§. 37. Verum nostrae investigationes multo latius patent, quas vt accuratius euoluamus, ad formulas §. 11. reuertamur, quae nihil de sua amplitudine amittunt, etiam numeros m et n nihilo aequales statuamus. Quare fractiones continuas considerandae erunt haec:

$p = C + F$
 $\frac{C+B+F+K+D}{C+2B+F+2E+4D}$
 $\frac{C+3B+F+3E+9D}{C+4B+etc.}$

$q = C$

$q = C + A + \frac{F}{C+A+B+F+E+D}$

$\frac{C+A+2B+F+2E+4D}{C+A+3B+F+3E+9D}$

$\frac{C+A+4B+etc.}{C+A+4B+etc.}$

$r = C + 2A + \frac{F}{C+2A+B+F+E+D}$

$\frac{C+2A+2B+F+2E+4D}{C+2A+3B+F+3E+9D}$

$\frac{C+2A+4B+etc.}{C+2A+4B+etc.}$

quae formae continuo vltius continuantur, scribendo C+A loco C. In singulis ergo denominatores progressionem arithmeticam, numeratores vero progressionem secundae ordinis constituent, cuius differentiae secundae sunt constantes. Hic autem affirmari esse $D = \frac{1}{2}(AA - BB)$. Quodsi iam breuitatis gratia ponamus:

$G = \frac{BB - AA + 2BC + 4E}{2A} = \frac{BC - 2D + 2E}{A}$ etc

$\lambda = \frac{1}{2}(AA - BB) = D$

$\mu = \frac{1}{2}(AA - BB) + \frac{1}{2}(AC - BG) = \frac{A+B+3CD-2E}{A}$

$\nu = F + \frac{1}{2}(C - G)(A + B + C + G)$, siue

$\nu = F + \frac{CD(A+C) + BD - EE}{A} + \frac{BA + E + C(D - E)}{2A}$

erit $(p - \frac{(A+B)(C+D-E)}{2A})(q - \frac{(A-B)(A+C-2CD-E)}{2A}) = \nu$

hincque per aliam fractionem continuam

$p = 2$

$q =$

$\frac{4+etc.}{1+6 etc.}$

Ita latius has §. 11. sunt, etc. Quare

$q = C$

$$p = \frac{1}{2}(G+G) + \nu$$

$$G+B+\nu+\mu+\lambda$$

$$G+2B+\nu+2\mu+4\lambda$$

$$G+3B+\text{etc.}$$

que huius formae:

$$p = a + \frac{f}{a+b+f+g}$$

$$\frac{a+2b+f+2g+b}{a+3b+f+3g+3b}$$

$$\frac{a+4b+f+4g+4b}{a+5b+\text{etc.}}$$

$$\frac{a+4b+f+4g+4b}{a+5b+\text{etc.}}$$

ea in aliam sibi aequalem transmutari potest. Comparatione enim facta est.

$$C = a; B = b; F = f; E = g - \frac{1}{2}b; D = \frac{1}{2}b.$$

Capiatur ergo $A = \nu (b + 2b)$, tum vero

$$G = \frac{ab+2g-ab}{2A}; \lambda = \frac{1}{2}b;$$

$$\mu = \frac{1}{2}b + \frac{1}{2}(Aa - Gb) = \frac{1}{2}b + \frac{ab-bg-b}{2A}$$

$$\nu = f + \frac{ab-\frac{1}{2}g-b}{2A} + \frac{aab-bb(g-b)-2ab(g-b)-2g(g-b)}{2AA}$$

hincque fiet

$$p = \frac{1}{2}(a+G) + \nu$$

$$G+b+\nu+\mu+\lambda$$

$$G+2b+\nu+2\mu+4\lambda$$

$$G+3b+\nu+3\mu+9\lambda$$

$$G+4b+\text{etc.}$$

§. 29. Si ergo fuerit $f = 0$, huius possemus fractionis continuae valor certe est $= a$, quicumque numeri

reliquis litteris tribuantur. Statuatur ergo $g - b = k$,

ut sit

$$A = \nu (bb + 2b); G = a^2 + 2b^2; \lambda = \frac{1}{2}b; \mu = \frac{1}{2}b + \frac{ab-b^2}{A}$$

$$\nu = \frac{ab-b^2}{2A} + \frac{aab-b^2k-2abk-2bk-2k^2}{2AA}$$

eritque:

$$\frac{1}{2}(a+G) = G + \nu$$

$$G+b+\nu+\mu+\lambda$$

$$G+2b+\nu+2\mu+4\lambda$$

$$G+3b+\nu+3\mu+9\lambda$$

$$G+4b+\text{etc.}$$

ut si litterae a, b, A et G pro datis habeantur, erit

$$\lambda = \frac{AA-b^2}{2A}; \mu = \frac{1}{2}A + \frac{Aa-Gb}{2A} = \frac{AA-2b^2}{2A} + \frac{Aa-Gb}{2A}$$

$$\nu = \frac{a^2+ab+aa+Gb-Aa-Ga}{2A} = \frac{1}{2}(a-G)(a+b+A+G)$$

Hinc:

$$\nu + \mu + \lambda = \frac{1}{2}(a-b+A-G)(a+2b+2A+G),$$

$$\nu + 2\mu + 4\lambda = \frac{1}{2}(a-2b+2A-G)(a+3b+3A+G),$$

$$\nu + 3\mu + 9\lambda = \frac{1}{2}(a-3b+3A-G)(a+4b+4A+G).$$

Ponatur ad formulam contrahendam:

$$a - G = 2a; A - b = 2\gamma; a + G = 2\beta; A + b = 2\delta;$$

$$\alpha = \frac{a(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+(\alpha+\gamma)(\beta+2\delta)}$$

$$\frac{\beta-\alpha+2(\delta-\gamma)+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+(\alpha+3\gamma)(\beta+4\delta)}$$

$$\frac{\beta-\alpha+4(\delta-\gamma)\text{etc.}}{\beta-\alpha+4(\delta-\gamma)\text{etc.}}$$

cuius veritas in pluribus exemplis sponte elucet.

§. 30. Si eadem positiones retineantur, numerus autem f non nihilo aequalis capiatur, habebitur haec fractio continua:

$$p = \alpha + \beta + \frac{f}{\alpha + \beta + (\delta - \gamma) + \frac{f + (\beta\gamma - \alpha\delta) + 2\gamma\delta}{\alpha + \beta + 2(\delta - \gamma) + \frac{f + 2(\beta\gamma - \alpha\delta) + 6\gamma\delta}{\alpha + \beta + 3(\delta - \gamma) + \frac{f + 3(\beta\gamma - \alpha\delta) + 12\gamma\delta}{\alpha + \beta + 4(\delta - \gamma) + \text{etc.}}}}$$

quae transformatur in hanc sibi aequalem:

$$p = \beta + \frac{f + \alpha(\beta + \delta)}{\beta - \alpha + (\delta - \gamma) + \frac{f + (\alpha + \gamma)(\beta + \alpha\delta)}{\beta - \alpha + 2(\delta - \gamma) + \frac{f + (\alpha + 2\gamma)(\beta + 3\delta)}{\beta - \alpha + 3(\delta - \gamma) + \frac{f + (\alpha + 3\gamma)(\beta + 4\delta)}{\beta - \alpha + 4(\delta - \gamma) + \text{etc.}}}}$$

unde si vel γ vel δ evanescens capiatur. casus ante tractatus exurgit. Haec autem binarum fractionum continuarum aequalitas omnia, quae haecenus sunt expolita, in se complectitur.

§. 31. Ex his oriuntur formae, quas littera q denotavimus, si loco α et β scribamus $\alpha + \gamma$ et $\beta + \delta$, ita ut sit

$$q = \alpha + \beta + \gamma + \delta + \frac{f}{\alpha + \beta + 2\delta + \frac{f + (\beta\gamma - \alpha\delta) + 2\gamma\delta}{\alpha + \beta + \gamma + 3\delta + \frac{f + 2(\beta\gamma - \alpha\delta) + 6\gamma\delta}{\alpha + \beta + 2\gamma + 4\delta + \frac{f + 3(\beta\gamma - \alpha\delta) + 12\gamma\delta}{\alpha + \beta + 3\gamma + 5\delta + \text{etc.}}}}$$

Itemque

$$q =$$

numerus
r haec

$$\frac{+ 12\gamma\delta}{+ \text{etc.}}$$

trac-
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forma q
+ δ ,

$$\frac{+ 12\gamma\delta}{+ \text{etc.}}$$

$$q =$$

$$q = \beta + \delta + \frac{f + (\alpha + \gamma)(\beta + \alpha\delta)}{\beta - \alpha + 2(\delta - \gamma) + \frac{f + (\alpha + 2\gamma)(\beta + 3\delta)}{\beta - \alpha + 3(\delta - \gamma) + \frac{f + (\alpha + 3\gamma)(\beta + 4\delta)}{\beta - \alpha + 4(\delta - \gamma) + \text{etc.}}}}$$

ita: ut inter has binas expressiones subsistat haec relatio:

$$(p - \beta)(q - \alpha - \gamma) = f + \alpha(\beta + \delta), \text{ seu}$$

$$pq - (\alpha + \gamma)p - \beta q + \beta\gamma - \alpha\delta = f$$

cuius ope aequalitas binarum superiorum formularum methode substitutionum, qua supra vidimus, demonstrari potest.

§. 32. Si ponamus $f + \alpha(\beta + \delta) = g$, ut sit $f = g - \alpha(\beta + \delta)$, prior forma ita se habebit:

$$p = \alpha + \beta + \frac{g - (\beta + \delta)}{\beta - \alpha + 3(\delta - \gamma) + \frac{g - (\alpha - \gamma)(\beta + \alpha\delta)}{\beta - \alpha + 2(\delta - \gamma) + \frac{g - (\alpha - 2\gamma)(\beta + 3\delta)}{\beta - \alpha + 3(\delta - \gamma) + \frac{g - (\alpha - 3\gamma)(\beta + 4\delta)}{\beta - \alpha + 4(\delta - \gamma) + \text{etc.}}}}$$

cui aequalis est ista:

$$q = \beta + \delta + \frac{f + \alpha(\beta + \delta)}{\beta - \alpha + (\delta - \gamma) + \frac{f + (\alpha + \gamma)(\beta + \alpha\delta)}{\beta - \alpha + 2(\delta - \gamma) + \frac{f + (\alpha + 2\gamma)(\beta + 3\delta)}{\beta - \alpha + 3(\delta - \gamma) + \frac{f + (\alpha + 3\gamma)(\beta + 4\delta)}{\beta - \alpha + 4(\delta - \gamma) + \text{etc.}}}}$$

Atque haec formae maxime idoneae videntur, quarum aequalitas methodo directa explorari ac demonstrari possit. Talis autem methodus etiam nunc desideratur. Nullum autem

tem

tem est dubium, quin ea patefacta multa praeclara incrementa Analyticos expectare liceat. Cum igitur prior forma finita evadat, si fuerit $g = (a - i\gamma)(\beta + (i + x)\delta)$, intelligimus etiam posterioris valorem rationaliter exprimi posse, quoties fuerit

$$f = (a - i\gamma)(\beta + (i + x)\delta) - a(\beta + \delta)$$

feu

$$f = i(a\delta - \beta\gamma - (i + x)\gamma\delta),$$

denotante i numerum integrum quemcumque.



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DISQUISITIO ACCURATIOR
CIRCA RESIDVA

DEX DIVISIONE QUADRATORVM ALTIORVMQVE
POTESTATVM PER NUMEROS PRIMOS
RELICTA.

§. IX.

Si numerus quadratus aa per numerum primum p dividatur, residuum relictum littera a indicetur; similique modo litterae β, γ, δ , etc. mihi denotabunt residua in divisione quadratorum bb, cc, dd , etc. relictā.

§. 2. Erit ergo $a = a - n p$, quia residuum a prodit, si a quadrato aa multipulum numeri p auferatur, idque maximum, ut residuum a ipso divitore p minus reddatur. Nihil autem impedit, quominus multipulum $n p$ minus accipiat quadrato aa , unde residuum a prodit agatur, sicque valor infra p deprimi potest.

§. 3. Idem igitur residuum a multis modis exhiberi potest, quoniam cunctae hae formae $a \pm n p$ eandem naturam continent. Perinde scilicet est, siue residuum ex divisione quadrati aa per numerum p ortum dicatur esse a , siue
Euleri Opusc. Anal. Tom. I. Q $a \pm p$

tem est dubium, quin ea patefacta multa praecleara incrementa Analyticos expectare liceat. Cum igitur prior forma finita euadat, si fuerit $g = (\alpha - i\gamma)(\beta + (i + 1)\delta)$, intellegimus etiam posterioris valorem rationaliter exprimi posse, quoties fuerit:

$$f = (\alpha - i\gamma)(\beta + (i + 1)\delta) - \alpha(\beta + \delta)$$

$$f = i(\alpha\delta - \beta\gamma - (i + 1)\gamma\delta),$$

denotante i numerum integrum quemcumque.

lata incrementa prior forma $(i)\delta$, intellegimus exprimi

DISQUISITIO ACCURATIOR CIRCA RESIDVA

DE DIVISIONE QUADRATORVM ALIORVMQVE
POTESTATVM PER NUMEROS PRIMOS
RELICTA.

§. I.

Si numerus quadratus aa , per numerum primum p dividatur, residuum relictum littera a indicetur; similiqve modo litterae β, γ, δ , etc. mihi denotabunt residua in divisione quadratorum bb, cc, dd , etc. relictia.

§. 2. Erit ergo $a = a^2 - n^2p$, quia residuum a prodit, si a quadrato aa multipulum numeri p auferatur, idque maximum, ut residuum a ipso divitore p minus reddatur. Nihil autem impedit, quominus multipulum n^2p maius accipiat quadrato aa , unde residuum a prodit auctum, sique eius valor infra p deprimi potest.

§. 3. Idem igitur residuum a multis modis exhiberi potest, quoniam cunctae hae formae $a + mp$ eandem naturam continent. Perinde scilicet est, siue residuum ex divisione quadrati aa per numerum p ortum dicatur esse a , siue
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