1783

Varia artificia in serierum indolem inquirendi

Leonhard Euler

VARIA ARTIFICIA
IN SERIERVM INDOLEM INQVIRENDI.

Eiusmodi saepe occurrunt series, quarum origo etsi satis est perspicua, earum tamen lex progressionis et natura maxime est abscondita, et nonnisi insignibus adhibitis artificiis analyticis inuestigari potest. In genere quidem huiusmodi artificia vix ita proponere licet, vt eorum vsus luculenter perspiciatur; sed potius eorum vis in exemplis commodissime ostenditur, vnde simul ratio ac necessitas ea excogitandi multo clarius intelligitur. Seriem igitur seu numerorum progressionem omnino singularem hic contemplabor, quae oritur, si potestates trinomii $1+x+xx$ euoluantur, atque ex singulis termini tantum medii, qui maximis numeris afficiuntur, in ordinem disponantur: ita enim enascitur numerorum series eo magis notata digna, quo minus lex progressionis perspicitur. Ea autem explorata pulcherrimae affectiones agnoscuntur, in quo negotio maxima vis artificiorum analyticorum potissimum cernitur. Imprimis autem haec series memorabile documentum exhibet, quanta circumspectione in inductione, cui plerumque in huiusmodi inuestigationibus non parum tribui solet, versari debeamus, cum hic eiusmodi inductio occurrat. quae, etiamsi maxime confirmata videatur, tamen in errorem inducat.

Euo-

Euolutio potestatum trinomii.

$$1+x+x^2$$
$$1+2x+3x^2+2x^3+x^4$$
$$1+3x+6x^2+7x^3+6x^4+3x^5+x^6$$
$$1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$$
$$1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$$
$$1+6x+21x^2+50x^3+90x^4+126x^5+141x^6+126x^7+90x^8+50x^9+21x^{10}+6x^{11}+x^{12}$$
etc.

Ex singulis his formis terminos tantum medios excerpo, qui hanc suppeditant progressionem:

$$x+3x^2+7x^3+19x^4+51x^5+141x^6+\text{ etc.}$$

cuius naturam hic inuestigare constitui, vbi quidem, omissis potestatibus ipsius x, totum negotium ad hanc progressionem numericam reducitur:

$$1,\ 3,\ 7,\ 19,\ 51,\ 141,\ 393,\ \text{etc.}$$

Consideratio I.

Seriem hanc perpendenti mox in mentem venit, quemlibet terminum cum triplo praecedentis haud incongrue comparari posse, quia hanc seriem in infinitum continuatam cum progressione geometrica tripla confundi debere ex eius origine est manifestum. Illi ergo, ad duos terminos vltra continuatae, terminos praecedentes triplicatos subscribo, indices vero superius noto, hoc modo:

Ind.

tiones continuas exprimi possit, videamus.

§. 62.

Ind.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
A...	1,	1,	3,	7,	19,	51,	141,	393,	1107,	3139
B...		3,	3,	9,	21,	57,	153,	423,	1179,	3321
C...		2,	0,	2,	2,	6,	12,	30,	72,	182
D...		1,	0,	1,	1,	3,	6,	15,	36,	91
E...		1,	0,	1,	1,	2,	3,	5,	8,	13

vbi series A. est ipsa proposita, quae a serie B, illius triplicata, ablata, relinquit seriem C; huius vero terminis bisectis prodit series D, cuius singuli termini sunt numeri trigonales, quibus suas subscripsi radices, vnde series nata est E.

§. 2. In hac serie E terminorum ordo ita comparatus videtur, vt quilibet aequetur summae binorum praecedentium, atque haec conclusio, ex inspectione nata, quoniam per decem seriei terminos confirmatur, ita certa videtur, vt neque dubitare liceat, quin omnes termini seriei D sint numeri trigonales; neque quin eorum radices seriem recurrentem illam simplicem, qua quilibet terminus est aggregatum binorum antecedentium, constituant. Saepe certe in huius generis inuestigationibus eiusmodi inductionibus confidere solemus, quae minus firmo fundamento innituntur.

§. 3. Si haec inductio veritati esset consentanea, pro inuento maximi momenti esset habenda, cum inde adeo terminus generalis seriei propositae A assignari posset: terminus scilicet indici n respondens foret

$$\tfrac{1}{10}3^n+\tfrac{1}{10}(-1)^n+\tfrac{1}{5}\left(\tfrac{1+\sqrt{5}}{2}\right)^n+\tfrac{1}{5}\left(\tfrac{1-\sqrt{5}}{2}\right)^n+\tfrac{1}{5}\left(\tfrac{3+\sqrt{5}}{2}\right)^n+\tfrac{1}{5}\left(\tfrac{3-\sqrt{5}}{2}\right)^n$$

et

§. 62.

et nostra progressio ex sequentibus tribus seriebus recurrentibus nasceretur:

											Scala relationis
A...	1,	1,	5,	13,	41,	121,	365,	1093,	3281	...	2, + 3
B...	2,	3,	7,	18,	47,	123,	322,	843,	2207	...	3, − 1
C...	2,	1,	3,	4,	7,	11,	18,	29,	47	...	1, + 1
D...	5,	5,	15,	35,	95,	255,	705,	1965,	5535,	et div. per 5	
E...	1,	1,	3,	7,	19,	51,	141,	393,	1107,	etc.	

Ex seriebus nempe recurrentibus A, B, C per singulorum terminorum additionem nascitur series D, cuius termini per 5 diuisi producunt ipsam nostram progressionem, saltem ad decem terminos.

§. 4. Quomodo expressionem huius termini generalis eruerim haud attinet docere, quandoquidem inductio superior, quantumuis fundata videatur, tamen veritati repugnat. Statim enim ac nostra progressio vlterius continuatur, et operationes vti §. 1. instituuntur, vt sequitur:

Ind.	5.	6.	7.	8.	9.	10.	11.
A...	51,	141,	393,	1107,	3139,	8953,	25653
B...	57,	153,	423,	1179,	3321,	9417,	26859
C...	6,	12,	30,	72,	182,	464,	1206
D...	3.	6,	15,	36,	91,	232,	603
E...	2,	3,	5,	8,	13,	...	...

in serie D termini 232 et 603 non amplius sunt trigonales, neque adeo lex seriei E vlterius valet. Hoc ergo exemplum inductionis illicitae eo magis est notatu dignum,

quod

E 3 ... etc. vel

quod mihi quidem eiusmodi casus nondum obtigerit, in quo tam speciosa inductio fefellerit.

Consideratio II.

§. 5. Repudiata ergo omni inductione progressionis nostrae indolem ex ipsa eius natura scrutari aggredior. Ac primo quidem euidens est, si in hac serie:

$$x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6,\ 393x^7,\ \text{etc.}$$

terminus indici n conueniens ponatur $= N x^n$, fore $N x^n$ ipsum terminum huius potestatis ipsius x, qui ex euolutione formulae $(1 + x + x^2)^n$ nascitur. Trinomium igitur $1 + x + xx$, binis prioribus partibus iunctis, tanquam binomium tracto, eritque:

$$(1+x+xx)^n = (1+x)^n + \tfrac{n}{1} xx(1+x)^{n-1} + \tfrac{n(n-1)}{1\cdot 2} x^4 (1+x)^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3} x^6 (1+x)^{n-3} + \text{etc.}$$

ex cuius singulis membris potestatem x^n elici oportet, indeque summa omnium collecta dabit nostrum terminum quaesitum $N x^n$.

§. 6. Ex primo autem membro $(1+x)^n$, seu $(x+1)^n$, oritur facta euolutione, x^n; pro secundo autem membro, ex euolutione formulae $(x+1)^{n-1}$ terminus secundus $\frac{n-1}{1} x^{n-2}$ capi debet, qui in $\frac{n}{1} x^2$ ductus, dat $\frac{n(n-1)}{1\cdot 1} x^n$. Pro tertio porro membro, ex formula $(x+1)^{n-2}$ tertius terminus $\frac{(n-2)(n-3)}{1\cdot 2} x^{n-4}$, in factorem $\frac{n(n-1)}{1\cdot 2} x^4$ ductus praebet $\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} x^n$, sicque de ceteris membris; vnde nanciscimur

$$N = 1 + \tfrac{n(n-1)}{1\cdot 1} + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} + \tfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3} + \text{etc.}$$

quarum partium addendarum numerus pro quouis numero

erit, in
ogressio-
ggredior.
, etc.
re $N x^n$
euolutio-
n igitur
uam bi-
$1+x)^{n-2}$
rtet, in-
rminum
$)^n$, seu
o autem
terminus
$\frac{n-1)}{1} x^n$.
tertius
ductus
membris;
$\frac{-5)}{3}$ + etc.
nume-
ro

ro integro n fit finitus; sicque valor termini N facile assignari poterit. Facilius eadem expressio reperitur, si potestas trinomii ita euoluatur:

$$(x(1+x)+1)^n = x^n(1+x)^n + \tfrac{n}{1} x^{n-1}(1+x)^{n-1} + \tfrac{n(n-1)}{1\cdot 2} x^{n-2}(1+x)^{n-2} + \text{etc.}$$

vbi potestatis x^n coëfficiens ex primo membro fit 1, ex secundo $\frac{n}{1}\cdot\frac{n-1}{1}$, ex tertio $\frac{n(n-1)}{1\cdot 2}\cdot\frac{(n-2)(n-3)}{1\cdot 2}$ etc. vt supra.

Consideratio III.

§. 7. Inuenta expressione, qua in genere coefficiens potestatis x^n in nostra progressione definitur, primum obseruo, eam nullo modo ita simpliciorem reddi posse, vt ad formulam finitam reducatur. Etsi enim inuentio numeri N ad aequationem differentio-differentialem reuocari possit, ea tamen ita est comparata, vt nullo modo resolutionem admittat. Cum igitur omnis labor, in expressione pro N inuenta commodius exhibenda, inutiliter consumeretur, in id hic incumbam, vt legem eruam, qua in nostra progressione terminus quilibet ex aliquot praecedentibus definiri possit.

§. 8. Hunc in finem progressionem nostram ita repraesento:

$$x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5 \ - - - \ p x^{n-2},\ q x^{n-1},\ r x^n,$$

inuestigaturus, quomodo numerus r per praecedentes q et p determinari possit. Valores autem p, q, r ex superiori serie pro N inuenta habentur, quos quo analyticas operationes recipiant ita exprimo:

$$p = 1 + \tfrac{(n-2)(n-3)}{1\cdot 1} z^2 + \tfrac{(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2} z^4 + \text{etc.}$$

$$q = 1 + \tfrac{(n-1)(n-2)}{1\cdot 1} z^2 + \tfrac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2} z^4 + \text{etc.}$$

$$r = 1 + \tfrac{n(n-1)}{1\cdot 1} z^2 + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} z^4 + \text{etc.}$$

vnde

vnde, quemlibet a sequente subtrahendo, primo colligimus:

$$\frac{q-p}{z}=\frac{n-2}{1}z^2+\frac{(n-2)(n-3)(n-4)}{1.\ 1.\ 2}z^4+\frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1.\ 1.\ 2.\ 2.\ 3}z^6 \text{ etc.}$$

$$\frac{r-q}{z}=\frac{n-1}{1}z^2+\frac{(n-1)(n-2)(n-3)}{1.\ 1.\ 2}z^4+\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 1.\ 2.\ 2.\ 3}z^6 \text{ etc.}$$

§. 9. Valoribus autem q et r differentiatis nanciscimur:

$$\frac{dq}{2dz}=\frac{(n-1)(n-2)}{1}z+\frac{(n-1)(n-2)(n-3)(n-4)}{1.\ 1.\ 2}z^3+\text{ etc.}$$

$$\frac{dr}{2dz}=\frac{n(n-1)}{1}z+\frac{n(n-1)(n-2)(n-3)}{1.\ 1.\ 2}z^3+\text{ etc.}$$

quae series cum praecedentibus facile comparantur, cum manifesto sit:

$$\frac{(n-1)(q-p)}{z}=\frac{z\,dq}{2\,dz} \text{ et } \frac{n(r-q)}{z}=\frac{z\,dr}{2\,dz},$$

vnde concludimus fore

$$dq=(n-1)(q-p).\frac{dz}{z} \text{ et } dr=n(r-q).\frac{dz}{z}.$$

§. 10. Deinde vero formae posteriores §. praec. differentiatae praebent:

$$\frac{dq-dp}{4dz}=(n-2)z+\frac{(n-2)(n-3)(n-4)}{1.\ 1}z^3+\frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1.\ 1.\ 2.\ 2}z^5 \text{ etc.}$$

$$\frac{dr-dq}{4dz}=(n-1)z+\frac{(n-1)(n-2)(n-3)}{1.\ 1}z^3+\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 1.\ 2.\ 2}z^5 \text{ etc.}$$

quae a primis hoc tantum differunt, quod hic coefficientes vno factore abundant; ibi autem per differentiationem iidem factores facile adiici possunt, hoc modo:

$$\frac{d.pz^{2-n}}{dz}=-(n-2)z^{1-n}-\frac{(n-2)(n-3)(n-4)}{1.\ 1}z^{3-n}$$
$$-\frac{(n-2)(n-3)(n-4)(n-5)(n-6)z^{5-n}}{1.\ 1.\ 2.\ 2} \text{ etc.}$$

$d.pz$

$$\frac{d.qz^{1-n}}{dz}=-(n-1)z^{-n}-\frac{(n-1)(n-2)(n-3)}{1.\ 1}z^{2-n}$$
$$-\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 1\ 2.\ 2}z^{4-n} \text{ etc.}$$

vnde manifestum est fieri

$$\frac{dq-dp}{4dz}+\frac{z^n d.pz^{2-n}}{dz}=0 \text{ et } \frac{dr-dq}{4dz}+\frac{z^{n+1}d.qz^{1-n}}{dz}=0,$$

et facta euolutione:

$$dq-dp+4zzdp-4(n-2)pzdz=0$$
$$dr-dq+4zzdq-4(n-1)qzdz=0,$$

§. 11. Cum igitur supra inuenerimus differentialia dq et dr per dz expressa, si hos valores in postrema aequatione substituamus, impetrabimus:

$$\frac{n(r-q)}{z}-\frac{(n-1)(q-p)}{z}+4(n-1)(q-p)z-4(n-1)qz=0,$$

ita vt differentialibus sublatis hic relatio finita inter p, q et r sit eruta, quae ita se habet:

$$n(r-q)=(n-1)(q-p)(1-4zz)+4(n-1)qzz$$

seu

$$n(r-q)=(n-1)(q+p(4zz-1)).$$

§. 12. Inuenimus ergo inter ternos valores continuos p, q, r, eiusmodi relationem, cuius ope ex binis datis tertius facile definitur, hocque multo generalius, quam pro nostro casu opus est, cum ista relatio pro quocunque numero z valeat. Quoniam igitur nostro casu est $z=1$, erit

$$n(r-q)=(n-1)(q+3p), \text{ seu } r=q+\frac{n-1}{n}(q+3p),$$

cuius

cuius formulae beneficio nostra progressio facile quousque libuerit continuari potest, in hunc modum:

A. .	1,	3,	7,	19,	51,	141,	393,	1107,	3139
B. .		3,	9,	21,	57,	153,	423,	1179	
C. .		6,	16,	40,	108,	294,	816,	2286,	
D. .		3,	4,	5,	6,	7,	8,	9,	
E. .		2,	4,	8,	18,	42,	102,	254	
F. .		4,	12,	32,	90,	252,	714,	2032	

Seriei scilicet A, quousque iam fuerit continuata, subscribantur iidem termini triplicati, eos vno loco promouendo, quae est series B; tum summa A + B dabit seriem C, cui subscripta progressione arithmetica D, diuisio C : D, praebet seriem E, vnde C − E suppeditat seriem F, cuius quiuis terminus, ad terminum supremum seriei A additus, eius sequentem suggerit.

§. 13. Hoc ergo modo nostram progressionem vlterius continuemus:

A. . 1107	3139	8953	25653	73789	212941	616227
B. . 1179	3321	9417	26859	76959	221367	
C. . 2286	6460	18370	52512	150748	434308	
D. . 9	10	11	12	13	14	
E. . 254	646	1670	4376	11596	31022	
F. . 2032	5814	16700	48136	139152	403286	

vnde adiunctis potestatibus ipsius x, cum terminus ipsi x^n conueniens certo sit 1, vti etiam ex lege progressionis inuenta liquet, nostra progressio ita se habebit:

1, 1x,

facile quousque
, 1107, 3139
, 1179
, 2286,
, 9,
, 254
, 2032
inuata, subscri-
o promouendo,
abit seriem C,
diuisio C : D,
seriem F, cu-
m seriei A ad-
progressionem
12941 | 616227
21367
34308
14
31022
03286
terminus ipsi
ege progressio-
habebit:
1, 1x,

$1,\ 1x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6,\ 393x^7,$
$1107x^8,\ 3139x^9,\ 8953x^{10},\ 25653x^{11},\ 73789x^{12},$
$212941x^{13},\ 616227x^{14}$, etc.

$$- - - - \ p x^{n-2},\ q x^{n-1},\ r x^n,$$

et lex progressionis ita est comparata, vt sit:

$$r = q + \tfrac{n-1}{n}(q + 3p) = 2q + 3p - \tfrac{1}{n}(q + 3p).$$

§. 14. Imprimis autem hic notetur artificium, quo per differentialia ad istam relationem inter ternos terminos sequentes pertigimus, cum reuera hic nullius variabilitatis ratio habeatur. Iam quidem haud difficulter animaduertimus, eandem relationem sine differentiatione erui posse, si in ternis seriebus §. 8. haec multiplicatio adhibeatur, vt fiat $(\mathfrak{A} + a x x) p + \mathfrak{B} q + \mathfrak{C} r = 0$. Facile enim patebit, litteris $\mathfrak{A}$, a, $\mathfrak{B}$ et $\mathfrak{C}$, eiusmodi valores tribui posse, vt omnes ipsius x potestates in nihilum abeant, quod efficiendo ipsa superior relatio obtinetur. Verum initio rem consideranti hoc certe minus obuium videbatur.

Consideratio IV.

§. 15. Inuenta hac progressionis lege quaestio non minus curiosa occurrit, qua eiusdem progressionis in infinitum continuatae summa inuestigatur. Ponamus ergo

$$s = 1 + x + 3x^2 + 7x^3 \ldots p x^{n-2} + q x^{n-1} + r x^n + \text{etc.}$$

et cum inuenerimus $n(r - 2q - 3p) + q + 3p = 0$, hanc aequalitatem differentiando introducemus sequenti modo:

$$\frac{ds}{dx}=1+6x+21x^2+\ldots(n-2)px^{n-3}+(n-1)qx^{n-2}+nrx^{n-1}$$
$$-\frac{2d.xs}{dx}=-2-4x-18x^2 \qquad -2(n-1)px^{n-2}-2nqx^{n-1}$$
$$-\frac{3d.x^2s}{dx}=\ -6x-9x^2 \qquad -3npx^{n-1}$$
$$s=1+x+3x^2 \qquad +qx^{n-1}$$
$$3xs=\ 3x+3x^2 \qquad +3px^{n-1}$$

vnde consequimur:

$$\frac{ds-2d.xs-3d.x^2s}{dx}+s+3xs=0,$$

seu

$$(1-2x-3xx)ds-sdx-3xsdx=0.$$

Ex hac aequatione colligitur

$$\frac{ds}{s}=\frac{dx+3xdx}{1-2x-3xx}, \text{ hincque integrando}$$

$$s=\frac{1}{\sqrt{(1-2x-3xx)}}=\frac{1}{\sqrt{(1+x)(1-3x)}}.$$

§. 16. En ergo novam originem nostrae seriei, quippe quae oritur ex euolutione huius formae:

$(1-2x-3xx)^{-\frac{1}{2}}$, vnde calculo instituto haec ipsa series resultare deprehenditur:

$$1+x+3x^2+7x^3+19x^4+51x^5+141x^6+ \text{ etc.}$$

Simul vero hinc apparet, quanta futura sit summa huius seriei in infinitum continuatae pro quouis valore ipsius x; vbi quidem notandum est, si sit vel $x=-1$, vel $x=\frac{1}{3}$, summam fore infinitam; at si $x>\frac{1}{3}$, summa est imaginaria. Finita autem erit summa, si x contineatur intra limites $\frac{1}{3}$ et -1: et extra hos limites prodit semper summa imaginaria. Ita sumto $x=\frac{1}{4}$ erit

$$1+\frac{1}{4}+\frac{3}{4^2}+\frac{7}{4^3}+\frac{19}{4^4}+\frac{51}{4^5}=\frac{4}{\sqrt{5}}$$

Consi-

Consideratio V.

§. 17. Haec inuestigatio ad seriem terminorum mediorum, ex euolutione potestatum trinomii latius accepti: $a+bx+cx^2$, extendi potest. Posito enim in genere Nx^n pro termino medio potestatis $(a+bx+cxx)^n$, valor coefficientis N ita determinari poterit: Cum sit

$$(x(b+cx)+a)^n=x^n(b+cx)^n+\frac{n}{1}ax^{n-1}(b+cx)^{n-1}$$
$$+\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}(b+cx)^{n-2}+ \text{ etc.}$$

ex singulis membris colligantur termini potestate x^n affecti, ac reperietur:

$$N=b^n+\frac{n(n-1)}{1\cdot 1}ab^{n-2}c+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}a^2b^{n-4}c^2+ \text{ etc.}$$

seu, posito breuitatis gratia $\frac{ac}{bb}=g$, erit

$$N=b^n\left(1+\frac{n(n-1)}{1\cdot 1}g+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}g^2+ \text{ etc.}\right)$$

Vnde cum sumto $n=0$ fiat $N=1$, si hanc progressionem ita repraesentemus:

$$1+Ax+Bx^2+Cx^3+Dx^4+\ldots+Nx^n+ \text{ etc.}$$

hi coefficientes ita se habebunt:

$A=b$; $\qquad D=b^4(1+12g+6gg)$
$B=b^2(1+2g)$; $\qquad E=b^5(1+20g+30gg)$
$C=b^3(1+6g)$; $\qquad F=b^6(1+30g+90gg+20g^3)$.

§. 18. Vt inuestigemus quomodo quisque terminus per binos praecedentes determinetur, seriem ita exponamus:

$$1,\ bx,\ (1+2g)b^2x^2,\ (1+6g)b^3x^3 \ldots pb^{n-2}x^{n-2},\ qb^{n-1}x^{n-1},\ rb^nx^n$$

et pro g scribendo zz habebimus:

$$p = 1 + \frac{(n-2)(n-3)}{1\cdot 2} z^2 + \frac{(n-3)(n-4)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 4} z^4 + \text{etc.}$$

$$q = 1 + \frac{(n-1)(n-2)}{1\cdot 2} z^2 + \frac{(n-2)(n-3)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4} z^4 + \text{etc.}$$

$$r = 1 + \frac{n(n-1)}{1\cdot 2} z^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} z^4 \text{ etc.}$$

quae series sunt eaedem, quas supra iam tractaui; eritque ergo

$$n(r-q) = (n-1)(q + p(4zz-1)).$$

Loco zz ergo restituendo g, in serie nostra terminus r ita per ambos praecedentes determinatur, vt sit

$$r = q + \frac{n-1}{n}(q + (4g-1)p), \text{ seu}$$

$$r = 2q + (4g-1)p - \frac{1}{n}(q + (4g-1)p).$$

§. 19. Ponamus $4g - 1 = b$, vt sit $b = \frac{4ee-bb}{bb}$, et cum lex progressionis praebeat

$$r = 2q + bp - \frac{1}{n}(q + bp),$$

et omissis potestatibus $b^n x^n$ bini termini initiales sint 1 et 1, progressio nostra erit:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 1, & 1, & \frac{1+b}{2}, & \frac{1+2b}{3}, & \frac{11+10b+3bb}{8}, & \frac{11+13b+11bb}{6}, \end{array}$$

vnde sumto $b = 3$ series ante tractata resultat. Sin autem capiatur $b = -1$, seu $g = 0$, omnes termini in vnitatem abeunt, id quod ex relatione $n(r-q) = (n-1)(q-p)$ liquet; statim enim ac duo termini contigui p et q sunt aequales, omnes iisdem aequales fiant necesse est.

Consi-

etc.
etc.
eritque
minus r
$\frac{bb}{}$, et
sint 1
in au-
n vni-
$(q-p)$
sunt
Consi-

Consideratio VI.

§. 20. Inuestigationem summae huius progressionis multo generalius instituamus, sitque

$$s = A + Bx + Cx^2 \ldots\ldots + px^{n-2} + qx^{n-1} + rx^n + \text{etc.}$$

cuius seriei lex progressionis ita comparata concipiatur, vt sit:

$$n(ap + bq + cr) = fp + gq,$$

et calculum vt supra §. 15. instituendo habebimus:

$$\begin{aligned} \frac{a\,d.xxs}{dx} &= \quad 2aAx + 3aBxx \ldots\ldots\ldots + nap\,x^{n-1} \\ \frac{b\,d.xs}{dx} &= bA + 2bBx + 3bCxx \ldots\ldots\ldots + nbq\,x^{n-1} \\ \frac{c\,ds}{dx} &= cB + 2cCx + 3cDxx \ldots\ldots\ldots + ncr\,x^{n-1} \end{aligned}$$

$$\begin{aligned} gs &= Ag + gBx + gCxx \ldots\ldots\ldots + gq\,x^{n-1} \\ fxs &= \quad fAx + fBxx \ldots\ldots\ldots + fp\,x^{n-1}. \end{aligned}$$

Quocirca ex indole seriei fiat necesse est:

$$\frac{a\,d.xxs + b\,d.xs + c\,ds}{dx} - (fx+g)s = (b-g)A + cB$$

seu

$$\frac{ds(axx+bx+c)}{dx} + s((2a-f)x + (b-g)) = (b-g)A + cB.$$

§. 21. Cum igitur habeamus:

$$ds + \frac{s\,dx((2a-f)x + (b-g))}{axx+bx+c} = \frac{(b-g)A\,dx + cB\,dx}{axx+bx+c},$$

aequationis huius integratio ita institui debet, vt posito $x = 0$ fiat $s = A$, ex quo haec summatio nullam habet difficultatem. Accommodemus ergo haec ad seriem ante inuentam, quae erat

$$s = 1 + x + \frac{1+b}{2}x^2 + \frac{1+2b}{3}x^3 + \ldots\ldots + px^{n-2} + qx^{n-1} + rx^n + \text{etc.}$$

pro qua est:

$n(b,p$

$$n(bp + 2q - r) = bp + q \text{ et } A = 1,\ B = 1,$$

et facta applicatione fiet

$$a = b,\ b = 2,\ c = -1,\ f = b,\ g = 1,$$

vnde valorem summae s ex hac aequatione definiri oportet:

$$ds + \frac{s\,dx(bx+1)}{bxx+2x-1} - \frac{b\,dx - 2\,dx}{bxx+2x-1} = 0,$$

hincque colligitur

$$s\sqrt{(bxx + 2x - 1)} = \sqrt{-1}, \text{ seu}$$

$$s = \frac{1}{\sqrt{(1 - 2x - bxx)}}.$$

§. 22. Restituamus valorem §. 19. assumtum:

$$h = 4g - 1 = \frac{4ac - bb}{bb},$$

et loco x scribamus bx, vt haec series sit summanda:

$$s = 1 + bx + (bb + 2ac)x^2 + (b^3 + 6abc)x^3 + (b^4 + 12acbb + 6aacc)x^4 + \text{etc.}$$

eritque eius summa

$$s = \frac{1}{\sqrt{(1 - 2bx + (bb - 4ac)xx)}}, \text{ seu}$$

$$s = \frac{1}{\sqrt{((1 - bx)^2 - 4acxx)}}.$$

Ipsius autem seriei origo est, vt singuli eius termini sint medii ex potestatibus $(a + bx + cxx)^n$ excerpti. Tum vero lex progressionis ita est comparata, vt positis in ea ternis terminis se inuicem sequentibus: px^{n-2}, qx^{n-1}, rx^n, coefficiens r ita per binos praecedentes definiatur, vt sit:

$$r = bq + \frac{n-1}{n}(bq + (4ac - bb)p), \text{ seu}$$

$$r = \frac{2n-1}{n}bq + \frac{n-1}{n}(4ac - bb)p$$

§. 23.

B = 1,
i oportet:
mtum:
ianda:
c) x³
etc.
mini sint
i. Tum
tis in ea
qx^{n-1},
efiniatur,
§. 23.

§. 23. Si ponatur $bb = 4ac$, ita vt sit

$$a + bx + cxx = (\sqrt{a} + x\sqrt{c})^2,$$

quilibet terminus nostrae progressionis per solum praecedentem ita determinatur, vt sit $r = \frac{2n-1}{n} \cdot 2q\sqrt{ac}$. Ponatur hoc casu $a = 1$, $c = 1$ et $b = 2$, vt series nostra constet ex terminis mediis potestatum $(1 + 2x + xx)^n$, seu $(1 + x)^{2n}$, eritque $r = \frac{2(2n-1)}{n}q$, et ipsa series

$$s = 1 + 2x + \frac{2 \cdot 6}{1 \cdot 2}x^2 + \frac{2 \cdot 6 \cdot 10}{1 \cdot 2 \cdot 3}x^3 + \frac{2 \cdot 6 \cdot 10 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \text{etc.}$$

cuius summa fit $s = \frac{1}{\sqrt{(1 - 4x)}}$, vti quidem per se est manifestum.

Consideratio VII.

§. 24. Ex seriei praecedentis

$$s = 1 + bx + (bb + 2ac)x^2 + (b^3 + 6abc)x^3 + \text{etc.}$$

summa inuenta

$$s = ((1 - bx)^2 - 4acxx)^{-\frac{1}{2}},$$

vicissim eius terminus generalis, seu coefficiens potestatis x^n erui potest. Cum enim, euolutione more solito facta, sit:

$$s = \frac{1}{1 - bx} + \frac{2}{1} \cdot \frac{acxx}{(1 - bx)^3} + \frac{2 \cdot 6}{1 \cdot 2} \cdot \frac{a^2c^2x^4}{(1 - bx)^5} + \frac{2 \cdot 6 \cdot 10}{1 \cdot 2 \cdot 3} \cdot \frac{a^3c^3x^6}{(1 - bx)^7} \text{ etc.}$$

colligantur ex singulis membris potestates x^n;

ex primo quidem oritur . . $b^n x^n$,

ex secundo . . . . $\frac{2}{1} \cdot \frac{n(n-1)}{1 \cdot 2} acb^{n-2}x^n$,

ex tertio . . . . . . $\frac{2 \cdot 6}{1 \cdot 2} \cdot \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^2c^2b^{n-4}x^n$,

qui in vnam summam collecti dabunt:

$$b^n x^n \left(1 + \frac{n(n-1)}{1 \cdot 1}\frac{ac}{bb} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2}\frac{aacc}{b^4} + \text{etc.}\right)$$

omnino vti supra ex huius seriei origine elicuimus.

OBSER-