1783

De seriebus, in quibus producta ex minis terminis contiguis datam constituunt progressionem

Leonhard Euler

DE SERIEBVS
IN QVIBVS
PRODVCTA EX BINIS TERMINIS CONTIGVIS DATAM CONSTITVVNT PROGRESSIONEM.

Propofita progreffione numerorum quacunque:

A, B, C, D, E, F, etc.

quaeftio, quam hic tractare ftatui, in hoc confiftit, vt inveniatur eiusmodi feries:

$a, b, c, d, e, f,$ etc.

in qua fit:

$ab=A$; $bc=B$; $cd=C$; $de=D$; $ef=E$; $fg=F$; etc.

vbi, etfi numeri A, B, C, D, etc. fint rationales, fatisque fimplici lege procedant, plerumque fieri folet, vt numeri a, b, c, d, etc. euadant adeo maxime tranfcendentes. Euidens autem eft totum negotium ad vnicum terminum huius feriei reuocari; quippe quo cognito reliqui omnes facillime definientur: inuento enim primo a reliqui ita fe habebunt:

$b=\frac{A}{a}$; $c=\frac{B}{b}$; $d=\frac{C}{c}$; $e=\frac{D}{d}$; etc.

 Dupli-

Duplicem autem ad ſolutionem huius quaeſtionis patere viam obſervavi, quarum altera interpolatione certae cuiusdam ſeriei abſolvatur, altera autem, quae magis directa videatur, ad fractiones continuas perducatur, quae duae methodi cum diuerſo plane modo negotium conficiant, earum collatio haud contemnendas proprietates patefaciet. Vtramque igitur methodum ſeorſim exponam, deinceps, quae fuerint eruta, inter ſe comparaturus.

Methodus prior,
interpolatione innixa.

§. 1. Conſideretur ſeries, ex quaeſita hoc modo formanda:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ a, & ab, & abc, & abcd, & abcde, & abcdef, & abcdefg, & \text{etc.}\end{matrix}$$

quae ob

$$ab = A,\; bc = B,\; cd = C,\; de = D,\; ef = E,\; \text{etc.}$$

in hanc abibit formam:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ a; & A; & aB; & AC; & aBD; & ACE; & aBDF; & \text{etc.}\end{matrix}$$

cuius ergo termini, locis paribus conſtituti, ob progreſſionem A, B, C, D, etc. datam, per ſe innoteſcunt.

§. 2. Cum ergo progreſſio terminorum alternorum

$$A;\; AC;\; ACE;\; ACEG;\; \text{etc.}$$

ſit cognita, eius interpolatio ad verum valorem termini quaeſiti

quaeſiti a manuducet. At iſta progreſſio ſemper ita eſt comparata, vt in infinitum continuata cum eiusmodi progreſſione ſimplici confundatur, cuius interpolatio nulli amplius difficultati ſit obnoxia. Plerumque autem illa progreſſio in infinitum producta in geometricam abire ſolet, ita vt interpolandi ſint medii proportionales inter binos contiguos.

§. 3. Si ergo ſeriem

$$a,\; A,\; aB,\; AC,\; aBD,\; ACE,\; \text{etc.}$$

totam vt geometricam ſpectemus, indeque terminos medios definiamus, ab initio multum fortaſſe a veritate aberrabimus; ſed quo longius progrediamur, eo propius ad veritatem accedemus, quam tandem in infinito plane aſſequemur. Hinc ſequentes determinationes ad verum continuo magis appropinquabunt:

$$aa = \frac{AA}{B} \qquad aa = \frac{AAC}{BB}$$

$$aa = \frac{AACC}{BBD} \qquad aa = \frac{AACCE}{BBDD}$$

$$aa = \frac{AACCEE}{BBDDF} \qquad aa = \frac{AACCEEG}{BBDDFF}$$

etc. etc.

Sicque revera in infinitum progrediendo erit

$$aa = A \cdot \frac{AC}{BB} \cdot \frac{CE}{DD} \cdot \frac{EG}{FF} \cdot \frac{GI}{HH} \cdot \frac{IL}{KK} \cdot \text{etc.}$$

§. 4. Expreſſio haec infinita verum valorem ipſius a exhibet, quoties progreſſio numerorum A, B, C, D, etc. ita eſt comparata, vt termini infiniteſimi inter ſe rationem aequalitatis teneant, illiusque expreſſionis factores tandem in vnitatem abeant. Veluti ſi pro A, B, C, D, etc. ſeries numerorum naturalium accipiatur, vt ſit

$$ab = 1,\ bc = 2,\ cd = 3,\ de = 4,\ ef = 5,\ fg = 6, \text{ etc.}$$

erit

$$aa = \frac{1.3}{2.2} \cdot \frac{3.5}{4.4} \cdot \frac{5.7}{6.6} \cdot \frac{7.9}{8.8} \cdot \frac{9.11}{10.10} \cdot \frac{11.13}{12.12} \cdot \text{ etc.}$$

Constat autem hoc productum infinitum, posita ratione diametri ad peripheriam $= 1 : \pi$, esse $= \frac{2}{\pi}$, ita vt sit $a = \sqrt{\frac{2}{\pi}}$, hincque $b \sqrt{\frac{\pi}{2}}$; $c = \frac{2\sqrt{2}}{\sqrt{\pi}}$; $d = \frac{3\sqrt{\pi}}{2\sqrt{2}}$; etc.

§. 5. Haec ergo series numerorum transcendentalium lege quadam vniformitatis procedit, quos numeros per approximationem euoluisse eorumque differentias notasse iuuabit.

	diff. 1.	diff. 2.	diff. 3.
$a = 0,7978845$		—	—
$b = 1,2533140$	4554295		
$c = 1,5957690$	3424550	1129745	
$d = 1,8799710$	2842020	582530	547215
$e = 2,1276920$	2477210	364810	217720
$f = 2,3499637$	2222717	254493	110317
$g = 2,5532304$	2032667	190050	64443
$h = 2,7416243$	1883929	148738	41312

Si enim pro a alius quicunque numerus assumeretur, ex eoque sequentes definirentur, in differentiis ingentes saltus essent appariturі.

§. 6. Eodem modo negotium procedit, si pro numeris A, B, C, D, etc. quaecunque capiatur progressio arithmetica. Quaerenda enim sit series $a, b, c, d, e,$ etc. ita vt sit

a b

$$ab = p;\ bc = p+q;\ cd = p+2q;\ de = p+3q; \text{ etc.}$$

& quia termini infinitesimi ad rationem aequalitatis accedunt, erit

$$aa = p \cdot \frac{p(p+2q)}{(p+q)(p+q)} \cdot \frac{(p+2q)(p+4q)}{(p+3q)(p+3q)} \cdot \frac{(p+4q)(p+6q)}{(p+5q)(p+5q)} \cdot \text{ etc.}$$

cuius expressionis valor ita per formulas integrales exhiberi potest, vt sit

$$aa = p \cdot \frac{\int z^{p+q-1}\, dz : \sqrt{(1 - z^{2q})}}{\int z^{p-1}\, dz : \sqrt{(1 - z^{2q})}}$$

posito post vtramque integrationem $z = 1$.

§. 7. Si pro A, B, C, D, etc. sumatur progressio mixta ex arithmetica & harmonica, vt talis series numerorum $a, b, c, d, e,$ etc. sit inuestiganda:

$$ab = \frac{p}{r};\ bc = \frac{p+q}{r+s};\ cd = \frac{p+2q}{r+2s};\ de = \frac{p+3q}{r+3s}; \text{ etc.}$$

quia & hic numeri A, B, C, D, etc. ad rationem aequalitatis conuergunt, erit

$$aa = \frac{p}{r} \cdot \frac{p(r+s)(p+2q)(r+s)}{r(p+q)(r+2s)(p+q)} \cdot \frac{(p+2q)(r+3s)(p+4q)(r+3s)}{(r+2s)(p+3q)(r+4s)(p+3q)} \text{ etc.}$$

cuius valor vt supra colligitur:

$$aa = \frac{p}{r} \cdot \frac{\int z^{p+q-1}\, dz : \sqrt{(1 - z^{2q})} \cdot \int z^{r-1}\, dz : \sqrt{(1 - z^{2s})}}{\int z^{p-1}\, dz : \sqrt{(1 - z^{2q})} \cdot \int z^{r+s-1}\, dz : \sqrt{(1 - z^{2s})}}$$

vbi iterum post integrationem poni oportet $z = 1$.

§. 8. Si fuerit $s = q$, numeri A, B, C, D, etc. continuo propius ad vnitatem accedunt, eique tandem fient aequales. Vnde cum seriei

$$a,\ A,\ aB,\ AC,\ aBD,\ ACE, \text{ etc.}$$

termini

termini infinitesimi inter se aequales sint consendi, inde concludetur

$$a = \frac{p}{r} \cdot \frac{(p+2q)(r+q)}{(p+q)(r+2q)} \cdot \frac{(p+4q)(r+3q)}{(p+3q)(r+4q)} \cdot \frac{(p+6q)(r+5q)}{(p+5q)(r+6q)} \text{ etc.}$$

quae expressio etiam ita referri potest:

$$a = \frac{q(r+q)}{r(p+q)} \cdot \frac{(p+2q)(r+3q)}{(r+2q)(p+3q)} \cdot \frac{(p+4q)(r+5q)}{(r+4q)(p+5q)} \text{ etc.}$$

cuius valor per formulas integrales est:

$$a = \frac{\int z^{r-1}\,dz : \sqrt{(1 - z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1 - z^{2q})}}.$$

§. 9. Hinc etiam casus, quo s & q sunt inaequales, facilius expediri potest. Sit enim $s = nnq$, ac ponatur $r = nnt$; tum vero statuatur:

$$a = \frac{\alpha}{n};\; b = \frac{\beta}{n};\; c = \frac{\gamma}{n};\; d = \frac{\delta}{n};\; e = \frac{\varepsilon}{n};\; \text{etc.}$$

eritque per conditionem praescriptam:

$$\alpha\beta = \frac{p}{t};\; \beta\gamma = \frac{p+q}{t+q};\; \gamma\delta = \frac{p+2q}{t+2q};\; \delta\varepsilon = \frac{p+3q}{t+3q};\; \text{etc.}$$

ex cuius convenientia cum praecedenti est

$$\alpha = \frac{p(t+q)}{t(p+q)} \cdot \frac{(p+2q)(t+3q)}{(t+2q)(p+3q)} \cdot \frac{(p+4q)(t+5q)}{(t+4q)(p+5q)} \text{ etc.}$$

ideoque

$$\alpha = \frac{\int z^{t-1}\,dz : \sqrt{(1 - z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1 - z^{2q})}}.$$

§. 10. Cum igitur sit

$$n = \sqrt{\frac{s}{q}};\; t = \frac{qr}{s} \text{ et } a = \frac{\alpha\sqrt{q}}{\sqrt{s}},$$

erit pro casu in §. 7. exposito:

$$a = \frac{\sqrt{q}}{\sqrt{s}} \cdot \frac{p(r+s)}{r(p+q)} \cdot \frac{(p+2q)(r+3s)}{(r+2s)(p+3q)} \cdot \frac{(p+4q)(r+5s)}{(r+4s)(p+5q)} \text{ etc.}$$

ac

ac per formulas integrales:

$$a = \frac{\sqrt{q}}{\sqrt{s}} \cdot \frac{\int z^{\frac{qr}{s}-1}\,dz : \sqrt{(1 - z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1 - z^{2q})}},$$

vbi si in numeratore pro z^q scribatur z^s, fiet

$$a = \frac{\sqrt{s}}{\sqrt{q}} \cdot \frac{\int z^{r-1}\,dz : \sqrt{(1 - z^{2s})}}{\int z^{p-1}\,dz : \sqrt{(1 - z^{2q})}},$$

cuius ergo quadratum aequetur necesse est formulae supra inuentae, ita vt sit

$$\frac{s}{q} \frac{\int z^{r-1}\,dz : \sqrt{(1 - z^{2s})}}{\int z^{p-1}\,dz : \sqrt{(1 - z^{2q})}} = \frac{p}{r} \cdot \frac{\int z^{p+q-1}\,dz : \sqrt{(1 - z^{2q})}}{\int z^{r+s-1}\,dz : \sqrt{(1 - z^{2s})}}.$$

§. 11. Harum ergo formularum consensus casu, quo post integrationem statuitur $z = 1$, sequens nobis suppeditat Theorema:

$$pq \int \frac{z^{p-1}\,dz}{\sqrt{(1 - z^{2q})}} \cdot \int \frac{z^{p+q-1}\,dz}{\sqrt{(1 - z^{2q})}} = rs \int \frac{z^{r-1}\,dz}{\sqrt{(1 - z^{2s})}} \cdot \int \frac{z^{r+s-1}\,dz}{\sqrt{(1 - z^{2s})}}.$$

cuius veritatem quidem iam alibi ex aliis principiis demonstratam dedi. Hinc ergo sequitur, sumendo $r = s = 1$ fore

$$pq \int \frac{z^{p-1}\,dz}{\sqrt{(1 - z^{2q})}} \cdot \int \frac{z^{p+q-1}\,dz}{\sqrt{(1 - z^{2q})}} = \frac{\pi}{2}, \text{ ob}$$

$$\int \frac{dz}{\sqrt{(1 - zz)}} = \frac{\pi}{2} \text{ et } \int \frac{z\,dz}{\sqrt{(1 - zz)}} = 1.$$

§. 12. Contemplemur igitur aliquot exempla.

I. Si esse debeat

$$ab = 1;\; bc = 2;\; cd = 3;\; de = 4;\; ef = 5;\; \text{etc.}$$

erit

erit

$$aa = 1 \cdot \frac{1.3}{2.2} \cdot \frac{3.5}{4.4} \cdot \frac{5.7}{6.6} \cdot \frac{7.9}{8.8} \cdot \text{ etc. et}$$
$$aa \frac{\int z\,dz : \sqrt{(1-zz)}}{\int dz : \sqrt{(1-zz)}} = \frac{2}{\pi}.$$

II. Si esse debeat

$$ab = 1,\ bc = 3,\ cd = 5,\ de = 7,\ ef = 9, \text{ etc.}$$

erit

$$aa = \frac{1.5}{3.3} \cdot \frac{5.9}{7.7} \cdot \frac{9.13}{11.11} \cdot \frac{13.17}{15.15} \cdot \text{ etc.}$$

seu

$$aa = \frac{\int zz\,dz : \sqrt{(1-z^4)}}{\int dz : \sqrt{(1-z^4)}}.$$

Cum vero sit ex theoremate modo exposito

$$\frac{\pi}{4} = \int \frac{dz}{\sqrt{(1-z^4)}} \cdot \int \frac{zz\,dz}{\sqrt{(1-z^4)}}, \text{ colligitur}$$
$$a = \frac{2}{\sqrt{\pi}} \int \frac{zz\,dz}{\sqrt{(1-z^4)}}.$$

III. Si esse debeat

$$ab = 1,\ bc = 4,\ cd = 7,\ de = 10,\ ef = 13, \text{ etc.}$$

erit

$$aa = \frac{1.7}{4.4} \cdot \frac{7.13}{10.10} \cdot \frac{13.19}{16.16} \cdot \text{ etc.}$$

seu

$$aa = \frac{\int z^4\,dz : \sqrt{(1-z^6)}}{\int dz : \sqrt{(1-z^6)}}, \text{ hincque}$$
$$a = \frac{\sqrt{6}}{\sqrt{\pi}} \int \frac{z^3\,dz}{\sqrt{(1-z^6)}}.$$

IV. Si generalius esse debeat

$$ab = p;\ bc = p+q;\ cd = p+2q;\ de = p+3q;$$
$$ef = p+4q; \text{ etc.}$$

per reductionem, ope Theorematis superioris instituendam, colli-

colligemus.

$$a = p \frac{\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p-1}\,dz}{\sqrt{(1-z^{2q})}}.$$

§. 13. Haec exempla ex progressione arithmetica sunt desumta, quibus adiungamus aliquot, in quibus numerorum A, B, C, D, etc. progressio est mixta ex arithmetica et harmonica.

I. Si esse debeat

$$ab = \tfrac{1}{2};\ bc = \tfrac{2}{3};\ cd = \tfrac{3}{4};\ de = \tfrac{4}{5};\ ef = \tfrac{5}{6}; \text{ etc.}$$

ob

$$p = 1,\ q = 1,\ r = 2,\ s = 1, \text{ erit}$$
$$a = \frac{1.3}{2.2} \cdot \frac{3.5}{4.4} \cdot \frac{5.7}{6.6} \cdot \frac{7.9}{8.8} \text{ etc.}$$

seu

$$a = \frac{\int z\,dz : \sqrt{(1-zz)}}{\int dz : \sqrt{(1-zz)}} = \frac{2}{\pi}.$$

II. Si esse debeat

$$ab = \tfrac{1}{2};\ bc = \tfrac{3}{4};\ cd = \tfrac{5}{6};\ de = \tfrac{7}{8};\ ef = \tfrac{9}{10}; \text{ etc.}$$

ob

$$p = 1;\ q = 2;\ r = 2;\ s = 2; \text{ erit}$$
$$a = \frac{1.4}{3.3} \cdot \frac{5.8}{6.7} \cdot \frac{9.12}{10.11} \cdot \frac{13.16}{14.15} \cdot \frac{17.20}{18.19} \cdot \text{ etc.}$$

seu

$$a = \frac{\int zz\,dz : \sqrt{(1-z^4)}}{\int dz : \sqrt{(1-z^4)}} = \frac{\pi}{4} : \int \frac{dz}{\sqrt{(1-z^4)}} = \int \frac{zz\,dz}{\sqrt{(1-z^4)}}.$$

III. Si esse debeat

$$ab = \tfrac{1}{1};\ bc = \tfrac{2}{3};\ cd = \tfrac{3}{5};\ de = \tfrac{4}{7};\ ef = \tfrac{5}{9}; \text{ etc.}$$

ob

$$p = 1;\ q = 1;\ r = 1;\ s = 2; \text{ erit}$$

$$a = \frac{1}{\sqrt{2}} \cdot \frac{1.3}{1.2} \cdot \frac{3.7}{5.4} \cdot \frac{5.11}{9.6} \cdot \frac{7.15}{13.8} \cdot \frac{9.19}{17.10} \cdot \text{ etc.}$$

vel

$$a = \frac{\sqrt{2}}{1} \cdot \frac{\int dz : \sqrt{(1-z^4)}}{\int dz : \sqrt{(1-zz)}} = \frac{2\sqrt{2}}{\pi} \cdot \int \frac{dz}{\sqrt{(1-z^4)}} = \frac{1}{\sqrt{2}} : \int \frac{zz\,dz}{\sqrt{(1-z^4)}}.$$

Productum autem ex hoc valore et praecedente manifesto est $= \frac{1}{\sqrt{2}}$.

Methodus altera, per fractiones continuas.

§. 14. Seriem inueniendam ita cum indicibus repraesentemus

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & & & & n & n+1 & \cdot \\ a, & b, & c, & d, & e, & \text{etc.} & - & - & - & x, & y, \end{matrix}$$

ac primo inuestigemus eam seriem, in qua sit

$$ab = p;\; bc = p+q;\; cd = p+2q;\; de = p+3q; \text{ etc.}$$

vt sit per methodum praecedentem

$$aa = p \cdot \frac{p(p+2q)}{(p+q)(p+q)} \cdot \frac{(p+2q)(p+4q)}{(p+3q)(p+3q)} \cdot \frac{(p+4q)(p+6q)}{(p+5q)(p+5q)} \cdot \text{ etc.}$$

et

$$aa = p \cdot \frac{\int z^{p+q-1}\,dz : \sqrt{(1-z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}},$$

seu

$$a = p\frac{\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p-1}\,dz}{\sqrt{(1-z^{2q})}}, \text{ hinc}$$

$$b = \frac{(p+q)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+2q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+q-1}\,dz}{\sqrt{(1-z^{2q})}} \text{ et}$$

$$c = \frac{(p+2q)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+3q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+2q-1}\,dz}{\sqrt{(1-z^{2q})}}$$

ideoque

$x =$

$$x = \frac{(p+nq)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+(n+1)q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+nq-1}\,dz}{\sqrt{(1-z^{2q})}}.$$

§. 15. Cum ergo pro hac serie in genere sit $xy = p + nq$, quantitas x eiusmodi functio indicis n esse debet, vt posito in ea $n+1$ loco n prodeat y, fiatque productum $xy = p + nq$: quod cum rationalitati aduersetur, quaeri conuenit valores quadratorum xx et yy, ex aequatione

$$xxyy = pp + 2npq + nnqq;$$

quandoquidem ratio illa functionum etiam ad quadrata patet. Hacc igitur inuestigatio commode latius extendetur ad resolutionem huius aequationis:

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma;$$

vnde valor ipsius xx pluribus modis ad fractiones continuas reduci potest, qui sequentibus lemmatibus innituntur.

Lemma I.

§. 16. Proposita hac aequatione:

$$(X + \lambda n + \mu)(Y + \lambda n + \nu) = \zeta\zeta nn + 2\zeta\eta n + \theta,$$

in qua Y perinde ex $n+1$ atque X ex n definitur: ponatur

$$X + \lambda n + \mu = \zeta n + f + \frac{h}{X'} \text{ et}$$
$$Y + \lambda n + \nu = \zeta n + g + \frac{h}{Y'},$$

vt sit

$$X = (\zeta - \lambda) n + f - \mu + \frac{h}{X'} \text{ et}$$
$$Y = (\zeta - \lambda) n + g - \nu + \frac{h}{Y'},$$

 vbi

ubi jam X' et Y' sint novae functiones similes ipsarum n et $n+1$; atque necesse est sit

$$g-\nu=\zeta-\lambda+f-\mu, \text{ seu } g=\zeta-\lambda-\mu+\nu+f.$$

§. 17. Hoc posito aequatio praescripta abit in hanc:

$$\zeta\zeta nn+\zeta(f+g)n+fg+\tfrac{k(\zeta n+f)}{Y'}+\tfrac{k(\zeta n+g)}{X'}+\tfrac{kk}{X'Y'}$$
$$=\zeta\zeta nn+2\zeta\eta n+\theta.$$

Statuatur $f+g=2\eta$ et $k=fg-\theta$, ut prodeat

$$X'Y'+(\zeta n+f)X'+(\zeta n+g)Y'+fg-\theta=0,$$

seu

$$(X'+\zeta n+g)(Y'+\zeta n+f)=\zeta\zeta nn+\zeta(f+g)n+\theta,$$

quae similis est formae propositae. At ob $f+g=2\eta$, habebitur

$$\zeta-\lambda-\mu+\nu+2f=2\eta;\ f=\eta+\tfrac{\lambda-\zeta+\mu-\nu}{2}, \text{ et}$$
$$g=\eta-\tfrac{\lambda+\zeta-\mu+\nu}{2},$$

hincque

$$k=fg-\theta=\eta\eta-\tfrac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\theta.$$

§. 18. Quocirca aequatio proposita

$$(X+\lambda n+\mu)(Y+\lambda n+\nu)=\zeta\zeta nn+2\zeta\eta n+\theta\ ,$$

ope huius substitutionis:

$$X=(\zeta-\lambda)n+\eta+\frac{\lambda-\zeta-\mu-\nu}{2}+\frac{\eta\eta-\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\theta}{X'}$$

$$Y=(\zeta-\lambda)n+\eta-\frac{\lambda+\zeta-\mu-\nu}{2}+\frac{\eta\eta-\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\theta}{Y'}$$

reducitur ad hanc aequationem ipsi propositae similem:

$$(X'+\zeta n+\eta-\tfrac{\lambda+\zeta-\mu+\nu}{2})(Y'+\zeta n+\eta+\tfrac{\lambda-\zeta+\mu-\nu}{2})$$
$$=\zeta\zeta nn+2\zeta\eta n+\theta.$$

§. 19.

ipsarum n
$-\nu+f.$
it in hanc:
$+g) + \tfrac{kk}{X'Y'}$
at
$g-\theta=0,$
$+g)n+\theta,$
$+g=2\eta,$
$-\mu-\nu}{2}$, et
$-\theta.$
$\zeta\eta n+\theta$,
$\zeta+\mu-\nu)^2-\theta$
$\zeta+\mu-\nu)^2-\theta$
similem:
$\lambda-\zeta+\mu-\nu$
§. 19.

§. 19. Simili modo eadem aequatio proposita

$$(X+\lambda n+\mu)(Y+\lambda n+\nu)=\zeta\zeta nn+2\zeta\eta n+\theta$$

factis his substitutionibus:

$$X=(\zeta-\lambda)n+\eta+\frac{\lambda-\zeta-\mu-\nu}{2}+\frac{\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\eta\eta+\theta}{X'}$$

$$Y=(\zeta-\lambda)n+\eta-\frac{\lambda+\zeta-\mu-\nu}{2}+\frac{\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\eta\eta+\theta}{Y'}$$

reducitur ad hanc sui similem:

$$(X'-\zeta n-\eta+\tfrac{\lambda-\zeta+\mu-\nu}{2})(Y'-\zeta n-\eta-\tfrac{\lambda-\zeta-\mu+\nu}{2})$$
$$=\zeta\zeta nn+2\zeta\eta n+\theta.$$

Lemma II.

§. 20. Proposita hac aequatione:

$$(X+\lambda n+\mu)(Y+\lambda n+\nu)=\zeta\zeta nn+2\zeta\eta n+\theta$$

in qua Y perinde ab $n+1$ atque X ab n pendet, ponatur

$$X+\lambda n+\mu=\zeta n+f+\tfrac{hn+k}{X'}$$
$$Y-\lambda n+\nu=\zeta n+g+\tfrac{hn+h+k}{Y'},$$

ubi ob similitudinem functionum esse debet ut ante

$$g=\zeta-\lambda-\mu+\nu+f.$$

§. 21. Porro substitutione horum valorum facta habebimus:

$$\zeta\zeta nn+\zeta(f+g)n+fg+\tfrac{(\zeta n+f)(hn+h+k)}{Y'}+\tfrac{(\zeta n+g)(hn+k)}{X'}$$
$$+\tfrac{(hn+k)(hn+h+k)}{X'Y'}=\zeta\zeta nn+2\zeta\eta n+\theta,$$

unde fit:

ζζ

$$(\zeta(f+g-2\eta)n+fg-\theta)X'Y'+(\zeta n+f)(bn+b+k)X'$$
$$+(\zeta n+g)(bn+k)Y'+(bn+k)(bn+b+k)=0,$$

quae vt similis sit formae propositae, diuisibilis esse debet per $\zeta(f+g-2\eta)n+fg-\theta$; cui quantitati ergo vel $bn+k$, vel $bn+b+k$ aequale vel multiplum statui oportet.

§. 22. Sit primo

$$bn+k=\alpha\zeta(f+g-2\eta)n+\alpha(fg-\theta),$$

et $\zeta n+f$ submultiplum ipsius $\zeta(f+g-2\eta)n+fg-\theta$ esse oportet; vnde fit $f(f+g-2\eta)=fg-\theta$, seu $ff=2\eta f-\theta$, hincque

$$f=\eta+\sqrt{(\eta\eta-\theta)} \text{ et } g=\zeta-\lambda-\mu+\nu+\eta+\sqrt{(\eta\eta-\theta)}:$$

quare porro

$$b=\alpha\zeta(f+g-2\eta) \text{ et } k=\alpha(fg-\theta),$$

et aequatio resultans euadet:

$$X'Y'+\frac{\alpha\zeta(f+g-2\eta)n+\alpha\zeta(f+g-2\eta)+\alpha(fg-\theta)}{f+g-2\eta}X'$$
$$+\alpha(\zeta n+g)Y'+\alpha\alpha(\zeta(f+g-2\eta)n$$
$$+\zeta(f+g-2\eta)+fg-\theta)=0.$$

§. 23. Vt fractiones tollamus ponamus:

$$\alpha=f+g-2\eta=\zeta-\lambda-\mu+\nu+2\sqrt{(\eta\eta-\theta)}$$

sicque fiet

$$X'Y'+(\zeta(f+g-2\eta)n+\zeta(f+g-2\eta)+fg-\theta)X'$$
$$+(\zeta(f+g-2\eta)n+g(f+g-2\eta))Y'$$
$$+(f+g-2\eta)(\zeta(f+g-2\eta)n+\zeta(f+g-2\eta)+fg-\theta)=0.$$

Verum si fractiones non curemus, habebimus:

X'

n+b+k)X'
b+k)=o,
s esse debet
i ergo vel
m statui o-
−θ),
n+fg−θ
ff=2ηf−θ,
+√(ηη−θ):
),
g−θ) X'
−2η)n
.
us:
η−θ)
fg−θ)X'
η)) Y'
2η)+fg−θ)=o.
:
X'

$$X'Y'+\alpha(\zeta n+\zeta+\tfrac{fg-\theta}{f+g-2\eta})X'+\alpha(\zeta n+g)Y'$$
$$+\alpha\alpha(f+g-2\eta)(\zeta n+\zeta+\tfrac{fg-\theta}{f+g-2\eta})=0,$$

quae aequatio, posito breuitatis gratia $\frac{fg-\theta}{f+g-2\eta}=\varepsilon$, reducitur ad hanc propositae similem:

$$(X'+\alpha(\zeta n+g))(Y'+\alpha(\zeta n+\zeta+\varepsilon))$$
$$=\alpha\alpha(\zeta\zeta nn+\zeta(\zeta+\varepsilon-f+2\eta)+(\zeta+\varepsilon)(2\eta-f))$$
$$=\alpha\alpha(\zeta n+\zeta+\varepsilon)(\zeta n+2\eta-f)$$

§. 24. Proposita ergo aequatione

$$(X+\lambda n+\mu)(Y+\lambda n+\nu)=\zeta\zeta nn+2\zeta\eta n+\theta,$$

si breuitatis gratia ponatur

$$f=\eta+\sqrt{(\eta\eta-\theta)};\ g=\zeta-\lambda-\mu+\nu+\eta+\sqrt{(\eta\eta-\theta)}$$

atque

$$\frac{fg-\theta}{f+g-2\eta}=\varepsilon,$$

sequens substitutio:

$$X=(\zeta-\lambda)n+f-\mu+\frac{\zeta(f+g-2\eta)n+fg-\theta}{X'},$$
$$Y=(\zeta-\lambda)n+g-\nu+\frac{\zeta(f+g-2\eta)(n+1)+fg-\theta}{Y'},$$

suppeditabit sequentem aequationem propositae similem:

$$(X'+\zeta n+g)(Y'+\zeta(n+1)+\varepsilon)$$
$$=\zeta\zeta nn+\zeta(\zeta+\varepsilon-f+2\eta)n+(\zeta+\varepsilon)(2\eta-f).$$

§. 25. Quemadmodum hic sumsimus $\alpha=1$, ita positio $\alpha=-1$, manentibus iisdem abbreuiationibus, dabit hanc substitutionem:

$$X=(\zeta-\lambda)n+f-\mu+\frac{\zeta(2\eta-f-g)n-fg+\theta}{X'},$$
$$Y=(\zeta-\lambda)n+g-\nu+\frac{\zeta(2\eta-f-g)(n+1)-fg+\theta}{Y'},$$

vnde

vnde oritur haec aequatio similis propositae:

$$(X' - \zeta n - g)(Y' - \zeta(n+1) - \varepsilon)$$
$$= \zeta\zeta nn + \zeta(\zeta + \varepsilon - f + 2\eta)n + (\zeta + \varepsilon)(2\eta - f).$$

§. 26. Ponamus porro esse

$$hn + h + k = \zeta(f + g - 2\eta)n + fg - \theta,$$

vt sit

$$h = \zeta(f + g - 2\eta) \text{ et } k = fg - \theta - \zeta(f + g - 2\eta)$$

ac necesse est, vt fiat

$$\zeta(f + g - 2\eta)n + fg - \theta = (f + g - 2\eta)(\zeta n + g)$$

ideoque

$$g(f + g - 2\eta) = fg - \theta, \text{ seu } g = \eta + \sqrt{(\eta\eta - \theta)},$$

hincque

$$f = \lambda - \zeta + \mu - \nu + \eta + \sqrt{(\eta\eta - \theta)}.$$

Aequatio autem resultans erit

$$X'Y' + (\zeta n + f)X' + \left(\zeta n - \zeta + \frac{fg - \theta}{f + g - 2\eta}\right)Y'$$
$$+ (f + g - 2\eta)\left(\zeta(n-1) + \frac{fg - \theta}{f + g - 2\eta}\right) = 0,$$

quae, posito $\frac{fg - \theta}{f + g - 2\eta} = \varepsilon$, abit in hanc:

$$(X' + \zeta n - \zeta + \varepsilon)(Y' + \zeta n + f) = \zeta\zeta nn + \zeta n(2\eta - g - \zeta + \varepsilon)$$
$$+ (\varepsilon - \zeta)(2\eta - g) = (\zeta n - \zeta + \varepsilon)(\zeta n + 2\eta - g).$$

§. 27. Proposita ergo aequatione

$$(X + \lambda n + \mu)(Y + \lambda n + \nu) = \zeta\zeta nn + 2\zeta\eta n + \theta,$$

si ponatur breuitatis gratia,

f =

$$f = \lambda - \zeta + \mu - \nu + \eta + \sqrt{(\eta\eta - \theta)};$$
$$g = \eta + \sqrt{(\eta\eta - \theta)} \text{ atque } \varepsilon = \frac{fg - \theta}{f + g - 2\eta},$$

sequens substitutio:

$$X = (\zeta - \lambda)n + f - \mu + \frac{\zeta(f + g - 2\eta)(n-1) + fg - \theta}{X'},$$
$$Y = (\zeta - \lambda)n + g - \nu + \frac{\zeta(f + g - 2\eta)n + fg - \theta}{Y'},$$

praebebit hanc aequationem propositae similem:

$$(X' + \zeta n - \zeta + \varepsilon)(Y' + \zeta n + f)$$
$$= \zeta\zeta nn + \zeta(2\eta - g - \zeta + \varepsilon)n + (\varepsilon - \zeta)(2\eta - g).$$

§. 28. Simili modo, manentibus iisdem abbreuiaturis, eadem aequatio proposita ope harum substitutionum:

$$X = (\zeta - \lambda)n + f - \mu + \frac{\zeta(2\eta - f - g)(n-1) - fg + \theta}{X'},$$
$$Y = (\zeta - \lambda)n + g - \nu + \frac{\zeta(2\eta - f - g)n - fg + \theta}{Y'},$$

reducetur ad hanc aequationem propositae similem:

$$(X' - \zeta n + \zeta - \varepsilon)(Y' - \zeta n - f)$$
$$= \zeta\zeta nn + \zeta(2\eta - g - \zeta + \varepsilon)n + (\varepsilon - \zeta)(2\eta - g).$$

Ope ergo harum senarum reductionum in §§. 18, 19, 24, 25, 27, 28, traditarum omnes huiusmodi aequationes infinitis modis per fractiones continuas resolui poterunt.

Resolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

per §. 18.

§. 29. Cum hic sit

$X = xx$; $Y = yy$; $\lambda = 0$; $\mu = 0$; $\nu = 0$; $\zeta = \alpha$; $\eta = \beta$ et $\theta = \gamma$, prodibit haec substitutio:

xx

$$xx = \alpha n + \beta - \tfrac{1}{2}\alpha + \frac{\beta\beta - \tfrac{1}{4}\alpha\alpha - \gamma}{X'},$$

$$yy = \alpha n + \beta + \tfrac{1}{2}\alpha + \frac{\beta\beta - \tfrac{1}{4}\alpha\alpha - \gamma}{Y'},$$

quae deducit ad hanc secundam aequationem:

$$(X' + \alpha n + \beta + \tfrac{1}{2}\alpha)(Y' + \alpha n + \beta - \tfrac{1}{2}\alpha) = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

§. 30. Ad hanc simili modo resoluendam, ob

$$\lambda = \alpha,\ \mu = \beta + \tfrac{1}{2}\alpha;\ \nu = \beta - \tfrac{1}{2}\alpha;\ \zeta = \alpha,\ \eta = \beta,\ \theta = \gamma,$$

consequemur hanc substitutionem:

$$X' = 0 + \frac{\beta\beta - \tfrac{1}{4}\alpha\alpha - \gamma}{X''};\quad Y' = 0 + \frac{\beta\beta - \tfrac{1}{4}\alpha\alpha - \gamma}{Y''},$$

quae deducit ad hanc tertiam aequationem:

$$(X'' + \alpha n + \beta - \tfrac{1}{2}\alpha)(Y'' + \alpha n + \beta + \tfrac{1}{2}\alpha) = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

Haec autem porro istas substitutiones praebet:

$$X'' = 0 + \frac{\beta\beta - \tfrac{1}{4}\alpha\alpha - \gamma}{X'''},\ \text{et}\ Y'' = 0 + \frac{\beta\beta - \tfrac{1}{4}\alpha\alpha - \gamma}{Y'''},$$

vnde ob $X''' = X'$ et $Y''' = Y'$ nihil vltra concludi potest.

Resolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

per §. 19.

§. 31. Factis his substitutionibus:

$$xx = \alpha n - \tfrac{1}{2}\alpha + \beta + \frac{\tfrac{1}{4}\alpha\alpha - \beta\beta + \gamma}{X},$$

$$yy = \alpha n + \tfrac{1}{2}\alpha + \beta + \frac{\tfrac{1}{4}\alpha\alpha - \beta\beta + \gamma}{Y},$$

perue-

peruenitur ad hanc aequationem:

$$(X - \alpha n - \tfrac{1}{2}\alpha - \beta)(Y - \alpha n + \tfrac{1}{2}\alpha - \beta) = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

quae secundum §. 19. reducta, ob

$$\lambda = -\alpha;\ \mu = -\tfrac{1}{2}\alpha - \beta;\ \nu = \tfrac{1}{2}\alpha - \beta;\ \zeta = \alpha;$$
$$\eta = \beta;\ \theta = \gamma;$$

dat has substitutiones:

$$X = 2\alpha n - \alpha + 2\beta + \frac{\tfrac{1}{4}\alpha\alpha - \beta\beta + \gamma}{X'},$$

$$Y = 2\alpha n + \alpha + 2\beta + \frac{\tfrac{1}{4}\alpha\alpha - \beta\beta + \gamma}{Y'},$$

vnde nascitur haec noua aequatio:

$$(X' - \alpha n - \tfrac{1}{2}\alpha - \beta)(Y' - \alpha n + \tfrac{1}{2}\alpha - \beta) = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

§ 34. Haec aequatio vlterius reducatur, et ob

$$\lambda = -\alpha;\ \mu = -\tfrac{1}{2}\alpha - \beta;\ \nu = \tfrac{1}{2}\alpha - \beta;\ \zeta = \alpha,\ \eta = \beta,\ \theta = \gamma,$$

habebimus has substitutiones:

$$X' = 2\alpha n - \alpha + 2\beta + \frac{\tfrac{1}{4}\alpha\alpha - \beta\beta + \gamma}{X''},$$

$$Y' = 2\alpha n + \alpha + 2\beta + \frac{\tfrac{1}{4}\alpha\alpha - \beta\beta + \gamma}{Y''},$$

hincque hanc aequationem nouam:

$$(X'' - \alpha n - \tfrac{1}{2}\alpha - \beta)(Y'' - \alpha n + \tfrac{1}{2}\alpha - \beta) = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

vnde sequentes substitutiones facile colliguntur.

§. 33. Quodsi ergo ad abbreviandum ponatur:

$$\alpha n - \tfrac{1}{2}\alpha + \beta = N\ \text{et}\ \beta\beta - \gamma = B,$$

valor ipsius xx sequenti fractione continua exprimetur:

$xx =$

$$xx = N + \cfrac{\frac{1}{4}\alpha\alpha - B}{2N + \cfrac{\frac{9}{4}\alpha\alpha - B}{2N + \cfrac{\frac{25}{4}\alpha\alpha - B}{2N + \cfrac{\frac{49}{4}\alpha\alpha - B}{2N + \cfrac{\frac{81}{4}\alpha\alpha - B}{2N - \text{etc.}}}}}}$$

qui conuenit aequationi propositae

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

Resolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

per §. 24.

§. 34. Cum hic sit

$$\lambda = 0;\ \mu = 0;\ \nu = 0;\ \zeta = \alpha;\ \eta = \beta;\ \theta = \gamma; \text{ erit}$$

$$f = \beta + \surd(\beta\beta - \gamma),\ g = \alpha + \beta + \surd(\beta\beta - \gamma)$$

hinc

$$fg - \theta = \alpha\beta + 2\beta\beta - 2\gamma + (\alpha + 2\beta)\surd(\beta\beta - \gamma) \text{ et}$$

$$f + g - 2\eta = \alpha + 2\surd(\beta\beta - \gamma).$$

Ponatur ergo

$$\frac{fg - \theta}{f + g - 2\eta} = \beta + \surd(\beta\beta - \gamma) = \varepsilon, \text{ ita vt fit}$$

$$\varepsilon = f \text{ et } g = \alpha + f,$$

vnde oriuntur hae substitutiones:

$$xx = \alpha n + f + \frac{(f + g - 2\eta)(\alpha n + f)}{X} = (\alpha n + f)\left(1 + \frac{\alpha + 2\surd(\beta\beta - \gamma)}{X}\right)$$

$$yy = \alpha n + g + \frac{(f + g - 2\eta)(\alpha n + g)}{Y} = (\alpha n + g)\left(1 + \frac{\alpha + 2\surd(\beta\beta - \gamma)}{Y}\right),$$

indeque haec aequatio noua:

$$(X + \alpha n + g)(Y + \alpha n + \alpha + f) = \alpha\alpha nn + \alpha(\alpha + 2\beta)n$$
$$+ (\alpha + f)(2\beta - f).$$

§. 35.

γ; erit
−γ)
−γ) et
fit
(ββ−γ) / X
β−γ)),
+ 2β) n
§. 35.

§. 35. Ponamus $\beta + \surd(\beta\beta - \gamma) = \delta$, vt fit $f = \delta$;

$$g = \alpha + \delta \text{ et } f + g - 2\eta = \alpha - 2\beta + 2\delta,$$

sicque substitutiones

$$xx = \alpha n + \delta + \frac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{X}$$

$$yy = \alpha(n + 1) + \delta + \frac{(\alpha - 2\beta + 2\delta)(\alpha(n + 1) + \delta)}{Y}$$

dabunt hanc aequationem:

$$(X + \alpha n + \alpha + \delta)(Y + \alpha n + \alpha + \delta)$$
$$= \alpha\alpha nn + \alpha(\alpha + 2\beta)n + (\alpha + \delta)(2\beta - \delta).$$

§. 36. Pro huius aequationis reductione est

$$\lambda = \alpha;\ \mu = \alpha + \delta;\ \nu = \alpha + \delta;\ \zeta = \alpha;\ \eta = \beta + \tfrac{1}{2}\alpha;$$
$$\theta = (\alpha + \delta)(2\beta - \delta),$$

vnde ob

$$ff - (2\beta + \alpha)f + (\alpha + \delta)(2\beta - \delta) = 0,$$

erit vel $f = \alpha + \delta$, vel $f = 2\beta - \delta$; at prior positio non vlterius deducit, vnde posteriori vtendo erit

$$f = 2\beta - \delta;\ g = 2\beta - \delta \text{ et } \varepsilon = 2\beta - \delta,$$

sicque hae obtinentur substitutiones:

$$X = 2\beta - \alpha - 2\delta + \frac{(2\beta - \alpha - 2\delta)(\alpha n + 2\beta - \delta)}{X'},$$

$$Y = 2\beta - \alpha - 2\delta + \frac{(2\beta - \alpha - 2\delta)(\alpha(n + 1) + 2\beta - \delta)}{Y'},$$

quae hanc praebent aequationem:

$$(X' + \alpha n + 2\beta - \delta)(Y' + \alpha n + \alpha + 2\beta - \delta)$$
$$= \alpha\alpha nn + 2\alpha(\alpha + \beta)n + (\alpha + \delta)(\alpha + 2\beta - \delta).$$

§. 37. Nunc igitur est

$$\lambda = \alpha;\ \mu = 2\beta - \delta;\ \nu = \alpha + 2\beta - \delta,\ \zeta = \alpha;\ \eta = \alpha + \beta;$$
$$\theta = (\alpha + \delta)(\alpha + 2\beta - \delta),$$

vnde ob

ff –

$$ff - 2(\alpha+\beta)f + (\alpha+\delta)(\alpha+2\beta-\delta) = 0,$$

sumatur valor

$$f = \alpha + 2\beta - \delta,\ \text{erit}\ g = 2\alpha + 2\beta - \delta,$$

atque

$$\varepsilon = \frac{(\alpha+2\beta-\delta)(\alpha+\beta-2\delta)}{\alpha+2\beta-2\delta} = \alpha + 2\beta - \delta.$$

Quare haec substitutio:

$$X' = \alpha + \frac{(\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{X''}$$

$$Y' = \alpha + \frac{(\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{Y''}$$

dabit hanc aequationem:

$$(X''+\alpha n+2\alpha+2\beta-\delta)(Y''+\alpha n+2\alpha+2\beta-\delta)$$
$$=\alpha\alpha nn+\alpha(3\alpha+2\beta)n+(2\alpha+2\beta-\delta)(\alpha+\delta).$$

§. 38. Si vtamur altero valore $f = \alpha + \delta$, fit $g = 2\alpha + \delta$ et $\varepsilon = \alpha + \delta$, et facta substitutione

$$X' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{X''},$$

$$Y' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{Y''},$$

nanciscimur hanc aequationem:

$$(X''+\alpha n+2\alpha+\delta)(Y''+\alpha n+2\alpha+\delta)$$
$$=\alpha\alpha nn+(3\alpha+2\beta)n+(2\alpha+\delta)(\alpha+2\beta-\delta).$$

§. 39. Prosequamur hanc posteriorem aequationem, quia magis similis est secundae, cum ex ea nascatur ponendo $\delta+\alpha$ pro δ et $\beta+\alpha$ pro β, vnde prodit haec substitutio:

$$X'' = 2\beta - \alpha - 2\delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+\alpha-\delta)}{X'''},$$

$$Y'' = 2\beta - \alpha - 2\delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+2\alpha-\delta)}{Y'''},$$

quae ducit ad hanc aequationem:

$$(X'''+\alpha n+2\beta+\alpha-\delta)(Y'''+\alpha n+2\alpha+2\beta-\delta)$$
$$=\alpha\alpha nn+2\alpha(2\alpha+\beta)n+(2\alpha+\delta)(2\alpha+2\beta-\delta).$$

§. 40.

) -δ). ·δ, fit), -δ),) ·δ). juatio- nafca- prodit ·α−δ), 2α−δ), 3−δ). §. 40.

§. 40. Haec aequatio porro vti in §. 38. tractata ope harum substitutionum:

$$X''' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{X''''}$$

$$Y''' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+3\alpha+\delta)}{Y''''},$$

reducitur ad hanc:

$$(X''''+\alpha n+3\alpha+\delta)(Y''''+\alpha n+3\alpha+\delta)$$
$$=\alpha\alpha nn+\alpha(5\alpha+2\beta)n+(3\alpha+\delta)(2\alpha+2\beta-\delta)$$

haecque vlterius per has substitutiones:

$$X'''' = 2\beta - \alpha - 2\delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+2\alpha-\delta)}{X'''''}$$

$$Y'''' = 2\beta - \alpha - 2\delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+3\alpha-\delta)}{Y'''''},$$

ad istam reducitur:

$$(X'''''+\alpha n+2\beta+2\alpha-\delta)(Y'''''+\alpha n+3\alpha+2\beta-\delta)$$
$$=\alpha\alpha nn+2\alpha(3\alpha+\beta)n+(3\alpha+\delta)(3\alpha+2\beta-\delta).$$

§. 41. Hinc ergo valor ipsius xx ex hac aequatione:

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

posito breuitatis causa

$$\beta + \sqrt{(\beta\beta-\gamma)} = \delta\ \text{et}\ \alpha - 2\beta + 2\delta = A,\ \text{erit}$$

$$xx = \alpha n+\delta+\cfrac{A(\alpha n+\delta)}{-A-\cfrac{A(\alpha n+2\beta-\delta)}{A+\cfrac{A(\alpha n+\alpha+\delta)}{-A-\cfrac{A(\alpha n+\alpha+2\beta-\delta)}{A+\cfrac{A(\alpha n+2\alpha+\delta)}{-A-\cfrac{A(\alpha n+2\alpha+2\beta-\delta)}{A+\cfrac{A(\alpha n+3\alpha+\delta)}{-A-\cfrac{A(\alpha n+3\alpha+2\beta-\delta)}{A+\text{etc.}}}}}}}}}$$

Haec

Haec autem expreſſio euoluta praebet pro xx ipſum illud productum ex infinitis factoribus conſtans, quod per methodum priorem elicitur.

§. 42. Iſta fractio continua ſimplicius hoc modo exprimi poteſt:

$$xx = \alpha n + \delta - \cfrac{(\alpha n+\delta)}{1+\alpha n+2\beta-\delta + \cfrac{A-(\alpha n+\alpha+\delta)}{1+\alpha n+\alpha+2\beta-\delta + \cfrac{A-(\alpha n+2\alpha+\delta)}{1+\alpha n+2\alpha+2\beta-\delta + \cfrac{A-(\alpha n+3\alpha+\delta)}{1+\alpha n+3\alpha+2\beta-\delta + \cfrac{A-(\alpha n+4\alpha+\delta)}{1+\text{etc.}}}}}}$$

Sin autem formulae §. 37 hoc modo vlterius reducentur, inuenitur haec expreſſio ab initio irregularis:

$$xx = \alpha n + \delta - \cfrac{(\alpha n+\delta)}{1+\alpha n+2\beta-\delta + \cfrac{\alpha-(\alpha n+\alpha+2\beta-\delta)}{1+\alpha n+\alpha+\delta + \cfrac{A-(\alpha n+2\alpha+2\beta-\delta)}{1+\alpha n+2\alpha+\delta + \cfrac{A-(\alpha n+3\alpha+2\beta-\delta)}{1+\alpha n+3\alpha+\delta + \cfrac{}{A-\text{etc.}}}}}}.$$

§. 43.

§. 43. Si vtraque expreſſio capite communi truncetur, pro 2β valor aſſumtus $\alpha + 2\delta - A$ ſubſtituatur, inſuperque pro $\alpha n + \alpha + \delta$ ſcribatur N, habebitur haec aequalitas:

$$\cfrac{A-N}{1+N+\alpha-A + \cfrac{A-N-\alpha}{1+N+2\alpha-A + \cfrac{A-N-2\alpha}{1+N+3\alpha-A + \cfrac{}{A-\text{etc.}}}}}$$

$$= \alpha - \cfrac{N-\alpha+A}{1+N + \cfrac{A-N-2\alpha+A}{1+N+\alpha + \cfrac{A-N-3\alpha+A}{1+N+2\alpha + \cfrac{}{A-\text{etc.}}}}}$$

vbi pro A, α et N numeri quicunque aſſumi poſſunt.

Reſolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

ope §. 25.

§. 44. Prima ſubſtitutio, ex reſolutione praecedente, ſumendis X et Y negatiuis, petita

$$xx = \alpha n + \delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+\delta)}{X}$$

$$yy = \alpha(n+1) + \delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+\alpha+\delta)}{Y};$$

D 2 poſito

posito $\delta = \beta + \sqrt{(\beta\beta - \gamma)}$ deducit ad hanc aequationem:

$$(X - \alpha n - \alpha - \delta)(Y - \alpha n - \alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(\alpha + 2\beta)n + (\alpha + \delta)(2\beta - \delta),$$

quae cum §. 25. comparata praebet

$$\lambda = -\alpha;\ \mu = -\alpha - \delta;\ \nu = -\alpha - \delta;\ \zeta = \alpha;$$
$$\eta = \tfrac{1}{2}\alpha + \beta;\ \theta = (\alpha + \delta)(2\beta - \delta),$$

vnde colligitur

$$ff - (\alpha + 2\beta)f + (\alpha + \delta)(2\beta - \delta) = 0.$$

Sit $f = \alpha + \delta$, erit $g = 3\alpha + \delta$ et $\varepsilon = \alpha + \delta$; hincque nafcitur haec fubftitutio:

$$X = 2\alpha n + 2\alpha + 2\delta - \frac{(3\alpha - 2\beta + 2\delta)(\alpha n + \alpha + \delta)}{X'}$$
$$Y = 2\alpha n + 4\alpha + 2\delta - \frac{(3\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{Y'}$$

quae ducit ad fequentem aequationem:

$$(X' - \alpha n - 3\alpha - \delta)(Y' - \alpha n - 2\alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(2\alpha + 2\beta)n + (2\alpha + \delta)(2\beta - \delta).$$

§. 45. Tractetur haec aequatio fimili modo fecundum §. 25. et ob valores

$$\lambda = -\alpha;\ \mu = -3\alpha - \delta;\ \nu = -2\alpha - \delta;\ \zeta = \alpha;$$
$$\eta = \alpha + \beta;\ \theta = (2\alpha + \delta)(2\beta - \delta).$$

erit

$$ff - (2\alpha + 2\beta)f + (2\alpha + \delta)(2\beta - \delta) = 0,$$

vnde fumatur $f = 2\alpha + \delta$, fietque $g = 5\alpha + \delta$ et $\varepsilon = 2\alpha + \delta$. Habebitur ergo ifta fubftitutio:

$$X' = 2\alpha n + 5\alpha + 2\delta - \frac{(5\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{X''}$$
$$Y' = 2\alpha n + 7\alpha + 2\delta - \frac{(5\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{Y''},$$

quae

quae praebet hanc aequationem:

$$(X'' - \alpha n - 5\alpha - \delta)(Y'' - \alpha n - 3\alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(3\alpha + 2\beta)n + (3\alpha + \delta)(2\beta - \delta).$$

§. 46. Nunc igitur eodem modo erit

$$\lambda = -\alpha;\ \mu = -5\alpha - \delta;\ \nu = -3\alpha - \delta;$$
$$\zeta = \alpha;\ \eta = \tfrac{3}{2}\alpha + \beta;\ \theta = (3\alpha + \delta)(2\beta - \delta);$$

vnde ob $f = 3\alpha + \delta$ colligitur $g = 7\alpha + \delta$ et $\varepsilon = 3\alpha + \delta$. Subftitutio ergo

$$X'' = 2\alpha n + 8\alpha + 2\delta - \frac{(7\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{X'''},$$
$$Y'' = 2\alpha n + 10\alpha + 2\delta - \frac{(7\alpha - 2\beta + 2\delta)(\alpha n + 4\alpha + \delta)}{Y'''},$$

iftam dabit aequationem:

$$(X''' - \alpha n - 7\alpha - \delta)(Y''' - \alpha n - 4\alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(4\alpha + 2\beta) + (4\alpha + \delta)(2\beta - \delta).$$

§. 47. Cum lex progreffionis hic fit fatis manifefta, facile concluditur fore:

$$xx = \alpha n + \delta - \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + 2\alpha + 2\delta - \cfrac{(3\alpha - 2\beta + 2\delta)(\alpha n + \alpha + \delta)}{2\alpha n + 5\alpha + 2\delta - \cfrac{(5\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{2\alpha n + 8\alpha + 2\delta - \cfrac{(7\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{2\alpha n + 11\alpha + 2\delta - \text{etc.}}}}}$$

vbi notandum eft, ex aequatione propofita

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

duplici modo dari δ, cum fit $\delta = \beta \pm \sqrt{(\beta\beta - \gamma)}$, ficque binae eiusmodi feries obtinentur, quarum altera proditura fuiffet, fi vbique pro f alteros valores affumfiffemus.

Alia

Alia resolutio,

binos valores ipsius f alternando.

§. 48. Sumamus in resolutione §. 44. $f=2\beta-\delta$, vt sit $g=2\alpha+2\beta-\delta$ et $\varepsilon=2\beta-\delta$, erit substitutio:

$$X=2\alpha n+\alpha+2\beta-\frac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{X'},$$

$$Y=2\alpha n+3\alpha+2\beta-\frac{(\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{Y'}$$

vnde resultat haec aequatio:

$$(X'-\alpha n-2\alpha-2\beta+\delta)(Y'-\alpha n-\alpha-2\beta+\delta)$$
$$=\alpha\alpha nn+\alpha(2\alpha+2\beta)n+(\alpha+\delta)(\alpha+2\beta-\delta),$$

quae ex superiori oritur, si ibi pro δ scribatur $-\alpha+2\beta-\delta$, quo valore in sequentibus retento fiet:

$$xx=\alpha n+\delta\cdot\cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+2\beta-\cfrac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{2\alpha n+3\alpha+4\beta-2\delta-\cfrac{(3\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{2\alpha n+6\alpha+4\beta-2\delta-\cfrac{(5\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{2\alpha n+9\alpha+4\beta-2\delta-\text{etc.}}}}}$$

§ 49. At aequatio modo eruta cum §. 25. collata dat

$$\lambda=-\alpha;\ \mu=-2\alpha-2\beta+\delta;\ \nu=-\alpha-2\beta+\delta;$$
$$\zeta=\alpha;\ \eta=\alpha+\beta;\ \theta=(\alpha+\delta)(\alpha+2\beta-\delta)\ \text{et}$$
$$ff-2(\alpha+\beta)f+(\alpha+\delta)(\alpha+2\beta-\delta)=0.$$

Si hic sumeremus $f=\alpha+2\beta-\delta$, haberemus formulam modo inuentam. Sit ergo $f=\alpha+\delta$, erit $g=4\alpha+\delta$, et $\varepsilon=\alpha+\delta$, ideoque:

$$X'=2\alpha n+3\alpha+2\beta-\frac{(2\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{X''},$$

$$Y'=2\alpha n+5\alpha+2\beta-\frac{(2\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{Y''},$$

hinc-

hincque ista nascitur aequatio:

$$(X''-\alpha n-4\alpha-\delta)(Y''-\alpha n-2\alpha-\delta)$$
$$=\alpha\alpha nn+\alpha(3\alpha+2\beta)n+(2\alpha+\delta)(\alpha+2\beta-\delta),$$

quae ex praecedente oritur, si ibi pro δ scribatur $-\alpha+\delta$, sicque erit:

$$xx=\alpha n+\delta\cdot\cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+2\beta-\cfrac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{2\alpha n+3\alpha+2\beta-\cfrac{(3\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{2\alpha n+6\alpha+2\beta-\text{etc.}}}}$$

§. 50. Verum illa aequatio altero modo resoluta, ob valores

$$\lambda=-\alpha;\ \mu=-4\alpha-\delta;\ \nu=-2\alpha-\delta;$$
$$\zeta=\alpha;\ \eta=\tfrac{3}{2}\alpha+\beta;\ \theta=(2\alpha+\delta)(\alpha+2\beta-\delta);\ \text{dat}$$
$$ff-(3\alpha+2\beta)f+(2\alpha+\delta)(\alpha+2\beta-\delta)=0;$$

vnde nunc sumamus

$f=\alpha+2\beta-\delta$, vt sit $g=5\alpha+2\beta-\delta$ et $\varepsilon=\alpha+2\beta-\delta$

prodibitque haec substitutio:

$$X''=2\alpha n+5\alpha+2\beta-\frac{(2\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{X'''},$$

$$Y''=2\alpha n+7\alpha+2\beta-\frac{(2\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{Y'''},$$

quae ducit ad hanc aequationem:

$$(X'''-\alpha n-5\alpha-2\beta-\delta)(Y'''-\alpha n-2\alpha-2\beta+\delta)$$
$$=\alpha\alpha nn+\alpha(4\alpha+2\beta)n+(2\alpha+\delta)(2\alpha+2\beta-\delta).$$

§. 51. Cum lex progressionis hinc iam colligi possit, habebimus:

$$xx=\alpha n+\delta-\cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+2\beta-\cfrac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{2\alpha n+3\alpha+2\beta-\cfrac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{2\alpha n+5\alpha+2\beta\cdot\cfrac{(3\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{2\alpha n+7\alpha+2\beta-\text{etc.}}}}}$$

quae

quae fractio continua ob satis concinnam progressionis legem est notatu digna.

Resolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

Per §. 28.

§. 52. Posito $\delta = \beta + \surd(\beta\beta - \gamma)$, ob $\lambda = 0$, $\mu = 0$, $\nu = 0$, $\zeta = \alpha$, $\eta = \beta$, $\theta = \gamma$, erit $g = \delta$, $f = -\alpha + \delta$ et $\varepsilon = \delta$; vnde substitutio

$$xx = \alpha n - \alpha + \delta + \frac{(-\alpha - 2\beta + 2\delta)(\alpha n - \alpha + \delta)}{X},$$

$$yy = \alpha n + \delta + \frac{(-\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{Y},$$

dabit hanc aequationem ex §. 27.

$$(X + \alpha n - \alpha + \delta)(Y + \alpha n + \delta)$$
$$= \alpha\alpha nn + \alpha(2\beta - \alpha)n + (\delta - \alpha)(2\beta - \delta).$$

Sumtis autem X et Y negatiuis, vt fit ex §. 28.

$$xx = \alpha n - \alpha + \delta + \frac{(\alpha + 2\beta - 2\delta)(\alpha n - \alpha + \delta)}{X},$$

$$yy = \alpha n + \delta + \frac{(\alpha + 2\beta - 2\delta)(\alpha n + \delta)}{Y},$$

habebitur:

$$(X - \alpha n + \alpha - \delta)(Y - \alpha n - \delta)$$
$$= \alpha\alpha nn - \alpha(\alpha - 2\beta)n - (\alpha - \delta)(2\beta - \delta).$$

§. 53. Haec aequatio porro secundum easdem formulas tractata praebet:

$$\lambda = -\alpha;\ \mu = \alpha - \delta;\ \nu = -\delta;\ \zeta = \alpha;$$
$$\eta = -\tfrac{1}{2}\alpha + \beta;\ \theta = (\delta - \alpha)(2\beta - \delta),\ \text{vnde fit}$$
$$gg - (2\beta - \alpha)g + (\delta - \alpha)(2\beta - \delta) = 0,$$

ergo

ergo vel $g = \delta - \alpha$, vel $g = 2\beta - \delta$, et $f = -\alpha + g$ atque $\varepsilon = g$. Quare substitutio erit:

$$X = 2\alpha n - 2\alpha + g + \delta + \frac{(2\beta - 2g)(\alpha n - \alpha + g)}{X'},$$

$$Y = 2\alpha n + g + \delta + \frac{(2\beta - 2g)(\alpha n + g)}{Y'},$$

quae ducit ad hanc aequationem:

$$(X' - \alpha n + \alpha - g)(Y' - \alpha n + \alpha - g)$$
$$= \alpha\alpha nn + \alpha(2\beta - 2\alpha)n + (g - \alpha)(2\beta - \alpha - g).$$

§. 54. Retineamus hanc litteram g geminum valorem inuoluentem, et sequentes per g', g'' indicemus. Cum ergo hic sit $\lambda = -\alpha$; $\mu = \alpha - g$; $\nu = \alpha - g$; $\zeta = \alpha$; $\eta = -\alpha + \beta$; $\theta = (g - \alpha)(2\beta - \alpha - g)$; erit vel $g' = g - \alpha$, vel $g' = 2\beta - \alpha - g$; hincque $f = -2\alpha + g'$ et $\varepsilon = g'$, ideoque

$$X' = 2\alpha n - 3\alpha + g + g' + \frac{(2\beta - 2g')(\alpha n - \alpha + g')}{X''},$$

$$Y' = 2\alpha n - \alpha + g + g' + \frac{(2\beta - 2g')(\alpha n + g')}{Y''},$$

vnde prodit haec aequatio:

$$(X'' - \alpha n + \alpha - g')(Y'' - \alpha n + 2\alpha - g')$$
$$= \alpha\alpha nn + \alpha(2\beta - 3\alpha)n + (g' - \alpha)(2\beta - 2\alpha - g').$$

§. 55. Nunc igitur porro erit:

$$\lambda = -\alpha;\ \mu = \alpha - g';\ \nu = 2\alpha - g';\ \zeta = \alpha;$$
$$\eta = \beta - \tfrac{3}{2}\alpha;\ \theta = (g' - \alpha)(2\beta - 2\alpha - g');$$

hincque vel $g'' = g' - \alpha$, vel $g'' = 2\beta - 2\alpha - g'$ et

$$f = -3\alpha + g'' \ \text{atque}\ \varepsilon = g''.$$

Quare ſubſtitutio

$$X'' = 2\alpha n - 4\alpha + g' + g'' + \frac{(2\beta - 2g'')(\alpha n - \alpha + g'')}{X'''},$$

$$Y'' = 2\alpha n - 2\alpha + g' + g'' + \frac{(2\beta - 2g'')(\alpha n + g'')}{Y'''},$$

dabit hanc aequationem:

$$(X''' - \alpha n + \alpha - g'')(Y''' - \alpha n + 3\alpha - g'')$$
$$= \alpha\alpha nn + \alpha(2\beta - 4\alpha)n + (g'' - \alpha)(2\beta - 3\alpha - g'').$$

§. 56. Iam pro huius aequationis reſolutione eſt:

$$\lambda = -\alpha;\ \mu = \alpha - g'';\ \nu = 3\alpha - g'';\ \zeta = \alpha;$$
$$\eta = \beta - 2\alpha;\ \theta = (g'' - \alpha)(2\beta - 3\alpha - g'')$$

vnde vel

$$g''' = g'' - \alpha, \text{ vel } g''' = 2\beta - 3\alpha - g'',$$
$$f = -4\alpha + g''' \text{ atque } \varepsilon = g'''.$$

Quare ex ſubſtitutione

$$X''' = 2\alpha n - 5\alpha + g'' + g''' + \frac{(2\beta - 2g''')(\alpha n - \alpha + g''')}{X''''},$$

$$Y''' = 2\alpha n - 3\alpha + g'' + g''' + \frac{(2\beta - 2g''')(\alpha n + g''')}{Y''''},$$

orietur haec aequatio:

$$(X'''' - \alpha n + \alpha - g''')(Y'''' - \alpha n + 4\alpha - g''')$$
$$= \alpha\alpha nn + \alpha(2\beta - 5\alpha)n + (g''' - \alpha)(2\beta - 4\alpha - g''').$$

§. 57. His igitur colligendis ex aequatione propoſita

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

poſito $\delta = \beta + \sqrt{(\beta\beta - \gamma)}$, ſi pro litteris g, g', g'', g''' etc. ſequentes valores gemini aſſumantur:

$$g = \begin{cases}\delta - \alpha \\ 2\beta - \delta\end{cases};\ g' = \begin{cases} g - \alpha \\ 2\beta - \alpha - g\end{cases};\ g'' = \begin{cases} g' - \alpha \\ 2\beta - 2\alpha - g'\end{cases};\ g''' = \begin{cases} g'' - \alpha \\ 2\beta - 3\alpha - g''\end{cases}; \text{etc.}$$

colligetur pro xx ſequens valor:

$xx =$

$$xx = \alpha n - \alpha + \delta + \cfrac{(\alpha + 2\beta - 2\delta)(\alpha n - \alpha + \delta)}{2\alpha n - 2\alpha + \delta + g + \cfrac{(2\beta - 2g)(\alpha n - \alpha + g)}{2\alpha n - 3\alpha + g + g' + \cfrac{(2\beta - 2g')(\alpha n - \alpha + g')}{2\alpha n - 4\alpha + g' + g'' + \text{etc.}}}}$$

qui ergo ob geminos valores ſingularum δ, g, g', g'', g''' etc. in infinitum variari poteſt.

§. 58. Si eadem aequatio ſimili modo, retentis valoribus omnibus ambiguis, ſecundum §. 25. reſoluatur, ac ſumto $\delta = \beta \pm \sqrt{(\beta\beta - \gamma)}$ ponatur

$$f = \begin{cases}\alpha + \delta \\ 2\beta - \delta\end{cases};\ f' = \begin{cases}\alpha + f \\ \alpha + 2\beta - f\end{cases};\ f'' = \begin{cases}\alpha + f' \\ 2\alpha + 2\beta - f'\end{cases};\ f''' = \begin{cases}\alpha + f'' \\ 3\alpha + 2\beta - f''\end{cases}; \text{etc.}$$

erit

$$xx = \alpha n + \delta - \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + \alpha + \delta + f - \cfrac{(\alpha - 2\beta + 2f)(\alpha n + f)}{2\alpha n + 2\alpha + f + f' - \cfrac{(\alpha - 2\beta + 2f')(\alpha n + f')}{2\alpha n + 3\alpha + f' + f'' - \cfrac{(\alpha - 2\beta + 2f'')(\alpha n + f'')}{2\alpha n + 4\alpha + f'' + f''' - \text{etc.}}}}}$$

§. 59. Simili modo aequationem propoſitam ſecundum §. 24. tractando, ſi poſito $\delta = \beta \pm \sqrt{(\beta\beta - \gamma)}$ ſtatuatur vt ante:

$$f = \begin{cases}\alpha + \delta \\ 2\beta - \delta\end{cases};\ f' = \begin{cases}\alpha + f \\ \alpha + 2\beta - f\end{cases};\ f'' = \begin{cases}\alpha + f' \\ 2\alpha + 2\beta - f'\end{cases};\ f''' = \begin{cases}\alpha + f'' \\ 3\alpha + 2\beta - f''\end{cases}; \text{etc.}$$

erit

$$xx = \alpha n + \delta - \cfrac{(2\beta\alpha - \alpha - 2\delta)(\alpha n + \delta)}{-\alpha - \delta + f - \cfrac{(2\beta + \alpha - 2f)(\alpha n + f)}{-f + f' - \cfrac{(2\beta + \alpha - 2f')(\alpha n + f')}{-\alpha - f' + f'' - \cfrac{(2\beta + 3\alpha - 2f'')(\alpha n + f'')}{-f'' + f''' - \cfrac{(2\beta + 3\alpha - 2f''')(\alpha n + f''')}{-\alpha - f''' + f'''' - \cfrac{(2\beta + 5\alpha - 2f'''')(\alpha n + f'''')}{-f'''' + f''''' - \text{etc.}}}}}}}$$

§. 60.

§. 60. Porro ex §. 27. si post
$\delta = \beta \pm \sqrt{(\beta\beta - \gamma)}$, statuatur

$$g = \begin{Bmatrix} \delta - \alpha \\ 2\beta - \delta \end{Bmatrix};\ g' = \begin{Bmatrix} g - \alpha \\ 2\beta - \alpha - g \end{Bmatrix};\ g'' = \begin{Bmatrix} g' - \alpha \\ 2\beta - 2\alpha - g' \end{Bmatrix};\ g''' = \begin{Bmatrix} g'' - \alpha \\ 2\beta - 3\alpha - g'' \end{Bmatrix};$$

erit

$$xx = an - \alpha + \delta - \cfrac{(2\beta - 2\delta + \alpha)(an - \alpha + \delta)}{g - \delta - \cfrac{(2\beta - 2g)(an - \alpha + g)}{g' - g + \alpha - \cfrac{(2\beta - 2g' - 2\alpha)(an - \alpha + g')}{g'' - g' - \cfrac{(2\beta - 2g'' - 2\alpha)(an - \alpha + g'')}{g''' - g'' + \alpha - \cfrac{(2\beta - 2g''' - 4\alpha)(an - \alpha + g''')}{g'''' - g''' - \cfrac{(2\beta - 2g'''' - 4\alpha)(an - \alpha + g'''')}{g''''' - g'''' + \alpha - \text{etc.}}}}}}}$$

§. 61. Possent autem permiscendis his reductionibus innumerabiles aliae fractiones continuae elici, quae omnes valorem ipsius xx exprimerent; verum his quaternis formis generalibus, quibus prima §. 33. exhibita addi potest, acquiescamus, easque ad casum quempiam determinatum accommodemus. Sit scilicet $xxyy = nn$, seu quaeratur eiusmodi series a, b, c, d, e, f, etc. vt sit $ab = 1;\ bc = 2;\ cd = 3;\ de = 4;\ ef = 5;$ etc. . . . $xy = n$, atque iam notauimus (12) fore

$$aa = \tfrac{2}{n};\ bb = \tfrac{n}{2};\ cc = \tfrac{2}{1}\cdot\tfrac{2}{n};\ dd = \tfrac{3}{2}\cdot\tfrac{n}{2};\ ee = \tfrac{2\cdot4}{1\cdot3}\cdot\tfrac{2}{n};\ \text{etc.}$$

Deinde vero ex §. 6. colligitur

$$xx = n.\frac{\int z^n\, dz : \sqrt{(1 - zz)}}{\int z^{n-1}\, dz : \sqrt{(1 - zz)}}$$

seu per productum infinitum:

$$xx = n.\frac{n(n+2)}{(n+1)^2}.\frac{(n+2)(n+4)}{(n+3)^2}.\frac{(n+4)(n+6)}{(n+5)^2}.\frac{(n+6)(n+8)}{(n+7)^2}\ \text{etc.}$$

Hunc igitur valorem ipsius xx quemadmodum per fractiones continuas exprimi possit, videamus.

§. 62.

§. 62. Cum igitur pro aequatione $xxyy = nn$ sit $\alpha = 1;\ \beta = 0;\ \gamma = 0;$ erit secundum §. 33. $N = n - \frac{1}{2}$, et $B = 0$, vnde fit:

$$xx = n - \tfrac{1}{2} + \cfrac{1:4}{2n - 1 + \cfrac{9:4}{2n - 1 + \cfrac{25:4}{2n - 1 + \cfrac{49:4}{2n - 1 + \text{etc.}}}}}$$

siue

$$2xx = 2n - 1 + \cfrac{1}{2(2n-1) + \cfrac{9}{2(2n-1) + \cfrac{25}{2(2n-1) + \cfrac{49}{2(2n-1) + \cfrac{81}{\text{etc.}}}}}}$$

§. 63. Porro ex §. 59. ob $\beta = 0;\ \gamma = 0$ et $\delta = 0;$ si sumatur

$$f = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix};\ f' = \begin{Bmatrix} 1 + f \\ 1 - f \end{Bmatrix};\ f'' = \begin{Bmatrix} 1 + f' \\ 2 - f' \end{Bmatrix};\ f''' = \begin{Bmatrix} 1 + f'' \\ 3 - f'' \end{Bmatrix};\ f'''' = \begin{Bmatrix} 1 + f''' \\ 4 - f''' \end{Bmatrix}$$

erit

$$xx = n + \cfrac{n}{f - 1 - \cfrac{(1 - 2f)(n + f)}{f' - f - \cfrac{(1 - 2f')(n + f')}{f'' - f' - 1 - \cfrac{(3 - 2f'')(n + f'')}{f''' - f'' - \cfrac{(3 - 2f''')(n + f''')}{f'''' - f''' - 1 - \cfrac{(5 - 2f'''')(n + f'''')}{f''''' - f'''' - \cfrac{(5 - 2f''''')(n + f''''')}{\text{etc.}}}}}}}}$$

vel

vel ex §. 58. sub iisdem denominationibus:

$$xx = n - \cfrac{n}{2n+1+f - \cfrac{(1+2f)(n+f)}{2n+2+f+f' - \cfrac{(1+2f')(n+f')}{2n+3+f'+f'' - \cfrac{(1+2f'')(n+f'')}{2n+4+f''+f''' - \cfrac{(1+2f''')(n+f''')}{2n+5+f'''+f'''' - \text{etc.}}}}}}$$

§. 64. Deinde posito

$$g = \begin{Bmatrix} -1 \\ 0 \end{Bmatrix};\ g' = \begin{Bmatrix} g-1 \\ -1-g \end{Bmatrix};\ g'' = \begin{Bmatrix} g'-1 \\ -2-g' \end{Bmatrix};\ g''' = \begin{Bmatrix} g''-1 \\ -3-g'' \end{Bmatrix};\ \text{etc.}$$

erit ex §. 60.

$$xx = n-1 - \cfrac{(n-1)}{g+2g(n-1+g) \over g'-g+1+2(1+g')(n-1+g') \over g''-g'+2(1+g'')(n-1+g'') \over g'''-g''+1+2(2+g''')(n-1+g''') \over g''''-g'''+\text{etc.}}$$

atque ex §. 57:

$$xx = n-1 + \cfrac{(n-1)}{2n-2+g - \cfrac{2g(n-1+g)}{2n-3+g+g' - \cfrac{2g'(n-1+g')}{2n-4+g'+g'' - \cfrac{2g''(n-1+g'')}{2n-5+g''+g''' - \text{etc.}}}}}$$

§. 65. Generaliter ergo pro serie a, b, c, d, etc. in qua sit

$$ab = p;\ bc = p+q;\ cd = p+2q;\ de = p+3q;\ \text{etc.}$$

$xy = p+nq$; ex superioribus constat esse.

$$xx = (p+nq) \cdot \frac{(p+nq)(p+(n+2)q)}{(p+(n+1)q)(p+(n+1)q)} \cdot \frac{(p+(n+2)q)(p+(n+4)q)}{(p+(n+3)q)(p+(n+3)q)}\ \text{etc.}$$

et

et per formulas integrales:

$$xx = (p+nq) \cdot \frac{\int z^{p+(n+1)q-1}\, dz : \sqrt{(1-z^{2q})}}{\int z^{p+nq-1}\, dz : \sqrt{(1-z^{2q})}}$$

posito $z = 1$. Iam ob $xxyy = qqnn + 2pqn + pp$ habebimus $\alpha = q$; $\beta = p$ et $\gamma = pp$; hinc $\delta = p$. Quare ex §. 33. erit $N = nq - \frac{q}{2} + p$ et $B = 0$; ideoque

$$xx = p + q(n-\tfrac{1}{2}) + \cfrac{\frac{1}{4}qq}{2p+q(2n-1) + \cfrac{\frac{9}{4}qq}{2p+q(2n-1) + \cfrac{\frac{25}{4}qq}{2p+q(2n-1) + \cfrac{\frac{49}{4}qq}{\text{etc.}}}}}$$

§. 66. At per reliquas formulas, si ponamus primo:

$$f = \begin{Bmatrix} q+p \\ p \end{Bmatrix};\ f' = \begin{Bmatrix} q+f \\ q+2p-f \end{Bmatrix};\ f'' = \begin{Bmatrix} q+f' \\ 2q+2p-f' \end{Bmatrix};\ f''' = \begin{Bmatrix} q+f'' \\ 3q+2p-f'' \end{Bmatrix};$$

etc.

habebimus ex §. 59.

$$xx = qn+p + \cfrac{q(qn+p)}{f-p-q - \cfrac{(q+2p-2f)(qn+f)}{f'-f - \cfrac{(q+2p-2f')(qn+f')}{f''-f'-q - \cfrac{(3q+2p-2f'')(qn+f'')}{f'''-f'' - \cfrac{(3q+2p-2f''')(qn+f''')}{\text{etc.}}}}}}$$

et ex §. 58.

$$xx = qn+p - \cfrac{q(qn+p)}{2qn+q+p+f - \cfrac{(q-2p+2f)(qn+f)}{2qn+2q+f+f' - \cfrac{(q-2p+2f')(qn+f')}{2qn+3q+f'+f''\ \text{etc.}}}}$$

vbi

vbi ex tribus numeris datis p, q, n, bini quicunque negatiui assumi possunt, quandoquidem aequatio resoluenda hinc nullam mutationem patitur.

§. 67. Deinde si ponamus

$$g = \left\{ \begin{matrix} p-q \\ p \end{matrix} \right\}; \quad g' = \left\{ \begin{matrix} g-q \\ 2p-q-g \end{matrix} \right\}; \quad g'' = \left\{ \begin{matrix} g'-q \\ 2p-2q-g' \end{matrix} \right\}; \quad g''' = \left\{ \begin{matrix} g''-q \\ 2p-3q-g'' \end{matrix} \right\} \text{etc}$$

erit per §. 60.

$$xx = qn-q+p-\cfrac{q(qn-q+p)}{g-p-\cfrac{2(p-g)(qn-q+g)}{g'-g+q-\cfrac{2(p-g'-q)(qn-q+g')}{g''-g'-\cfrac{2(p-g''-q)(qn-q+g'')}{g'''-g''+q-\text{etc.}}}}}$$

per §. 57.

$$xx = qn-q+p+\cfrac{q(qn-q+p)}{2qn-2q+p+g+\cfrac{2(p-g)(qn-q+g)}{2qn-3q+g+g'+\cfrac{2(p-g')(qn-q+g')}{2qn-4q+g'+g''-\text{etc.}}}}$$

§. 68. Verum de his expressionibus in infinitum excurrentibus tenendum est, eas saepenumero seriebus diuergentibus aequiualere, ita vt quo vlterius earum valores colligamus, eo magis a veritate aberremus: quod incommodum tamen in expressione prima non vsu venit. Quamobrem his casibus de iis eadem locum habent, quae de serierum diuergentium natura iam annotaui, scilicet eas spectandas esse tanquam formulas infinitas, ex euolatione cuiuspiam formulae finitae natas, quae nihilominus pro earum summa sit habenda, etiamsi vbicunque in collectione partium substiterimus, verum nunquam attingamus.

§. 69. Examinemus etiam seriem a, b, c, d, etc. in qua sit

$$ab = \frac{\beta}{\gamma}; \quad bc = \frac{\alpha+\beta}{\alpha+\gamma}; \quad cd = \frac{2\alpha+\beta}{2\alpha+\gamma}; \quad de = \frac{3\alpha+\beta}{3\alpha+\gamma}; \text{ etc.}$$

hincque in genere $xy = \frac{\alpha n+\beta}{\alpha n+\gamma}$; ac habebitur

$$x = \frac{\alpha n+\beta}{\alpha n+\gamma} \cdot \frac{\alpha(n+1)+\gamma}{\alpha(n+1)+\beta} \cdot \frac{\alpha(n+2)+\beta}{\alpha(n+2)+\gamma} \cdot \frac{\alpha(n+3)+\gamma}{\alpha(n+3)+\beta} \text{ etc.}$$

siue

$$x = \frac{\int z^{\alpha n+\gamma-1}\, dz : \sqrt{(1-z^{2\alpha})}}{\int z^{\alpha n+\beta-1}\, dz : \sqrt{(1-z^{2\alpha})}},$$

Cum nunc, si esset $n=\infty$, foret $x=y=1$, valores x et y continuo magis ad vnitatem accedent; quare ponatur $x = 1+\frac{A}{X}$ et $y = 1+\frac{A}{Y}$, fietque

$$(X+A)(Y+A) = \frac{\alpha n+\beta}{\alpha n+\gamma} X\,Y, \text{ seu}$$

$$(\beta-\gamma)XY - A(\alpha n+\gamma)X - A(\alpha n+\gamma)Y = AA(\alpha n+\gamma).$$

Sit $A = \beta-\gamma$, seu

$$x = 1+\frac{\beta-\gamma}{X} \text{ et } y = 1+\frac{\beta-\gamma}{Y},$$

habebiturque

$$XY - (\alpha n+\gamma)X - (\alpha n+\gamma)Y = +(\beta-\gamma)(\alpha n+\gamma)$$

siue

$$(X-\alpha n-\gamma)(Y-\alpha n-\gamma) = (\alpha n+\beta)(\alpha n+\gamma).$$

§. 70. Ex hac iam aequatione valores X et Y per formulas supra datas infinitis modis exhiberi possunt, ex quibus §. 19. maxime conuergentem suppeditat. Cum autem sit

$$\lambda = -\alpha;\ \mu = -\gamma;\ \nu = -\gamma;\ \zeta = \alpha;\ \eta = \tfrac{\beta+\gamma}{2} \text{ et } \theta = \beta\gamma,$$

fiet

$$X = 2\alpha n - \alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma + \frac{\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{X'}$$

$$Y = 2\alpha n + \alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma + \frac{\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{Y'}$$

hincque emergit ista noua aequatio:

$$(X' - \alpha n - \alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma)(Y' - \alpha n + \alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma) = (\alpha n + \beta)(\alpha n + \gamma).$$

§. 70. Quodsi haec aequatio denuo simili modo euoluatur, ob

$$\lambda = -\alpha;\ \mu = -\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma;\ \nu = \alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma;\ \zeta = \alpha;$$
$$\eta = \tfrac{\beta+\gamma}{2} \text{ et } \theta = \beta\gamma,$$

vt sit $\eta\eta - \theta = \frac{1}{4}(\beta-\gamma)^2$, orietur haec substitutio:

$$X' = 2\alpha n - \alpha + \beta + \gamma + \frac{4\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{X''},$$

$$Y' = 2\alpha n + \alpha + \beta + \gamma + \frac{4\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{Y''},$$

quae praebet istam aequationem:

$$(X'' - \alpha n - 2\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma)(Y'' - \alpha n + 2\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma) = (\alpha n + \beta)(\alpha n + \gamma).$$

§. 72. Iam cum hic sit

$$\lambda = -\alpha;\ \mu = -2\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma;\ \nu = 2\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma;\ \zeta = \alpha;$$
$$\eta = \tfrac{\beta+\gamma}{2};\ \theta = \beta\gamma;$$

orietur haec substitutio:

X =

βγ,
$\frac{1}{2}\gamma)$
modo
$= \alpha;$
$\frac{1}{2}\gamma)$
$= \alpha;$
X =

$$X'' = 2\alpha n - \alpha + \beta + \gamma + \frac{9\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{X'''},$$

$$Y'' = 2\alpha n + \alpha + \beta + \gamma + \frac{9\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{Y'''},$$

hincque ista aequalitas:

$$(X''' - \alpha n - 3\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma)(Y''' - \alpha n + 3\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma) = (\alpha n + \beta)(\alpha n + \gamma).$$

§. 73. Hoc modo progrediendo consequemur tandem aequationis propositae $xy = \frac{\alpha n + \beta}{\alpha n + \gamma}$ hanc resolutionem:

$$x = 1 + \cfrac{\beta - \gamma}{2\alpha n - \alpha + \frac{1}{2}\beta + \frac{3}{2}\gamma + \cfrac{\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{2\alpha n - \alpha + \beta + \gamma + \cfrac{4\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{2\alpha n - \alpha + \beta + \gamma + \cfrac{9\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{2\alpha n - \alpha + \beta + \gamma \text{ etc.}}}}}$$

ita vt sit

$$x = \frac{\int z^{\alpha n + \gamma - 1}\, dz : \sqrt{(1 - z^{2\alpha})}}{\int z^{\alpha n + \beta - 1}\, dz : \sqrt{(1 - z^{2\alpha})}},$$

§. 74. Percurramus quaedam exempla, sitque primo $\alpha = 1$, $\beta = 2$ et $\gamma = 0$, erit

$$x = \frac{\int z^{n-1}\, dz : \sqrt{(1 - zz)}}{\int z^{n+1}\, dz : \sqrt{(1 - zz)}} = \frac{n+1}{n},$$

ideoque

$$x = 1 + \cfrac{2}{2n + \cfrac{1 - 1}{2n + 1 \text{ etc.}}} = 1 + \tfrac{1}{n},$$

 vti

vti patet. Atque in genere si $\alpha = 1$, quoties $\beta - \gamma$ est numerus par, valor ipsius x rationaliter exprimitur.

§. 75. Manente $\alpha = 1$, sit $\beta = 1$ et $\gamma = 0$, erit

$$x = \frac{\int z^{n-1}\, dz : \sqrt{(1 - zz)}}{\int z^n\, dz : \sqrt{(1 - zz)}},$$

ex aequatione $xy = \frac{n+1}{n}$; at per fractionem continuam:

$$x = 1 + \cfrac{1}{2n - \frac{1}{2} + \cfrac{1 - \frac{1}{4}}{2n + \cfrac{4 - \frac{1}{4}}{2n + \cfrac{9 - \frac{1}{4}}{2n + \text{etc.}}}}}$$

$$= 1 + \cfrac{2}{4n - 1 + \cfrac{1.3}{4n + \cfrac{3.5}{4n + \cfrac{5.7}{4n + \text{etc.}}}}}$$

Sumto autem $\beta = 0$ et $\gamma = 1$, prodit valor reciprocus

$$\frac{1}{x} = 1 - \cfrac{1}{2n + \frac{1}{2} + \cfrac{1 - \frac{1}{4}}{2n + \cfrac{4 - \frac{1}{4}}{2n + \text{etc.}}}}$$

$$= 1 - \cfrac{2}{4n + 1 + \cfrac{1.3}{4n + \cfrac{3.5}{4n + \cfrac{5.7}{4n + \text{etc.}}}}}$$

cuius consensus cum praecedente facile perspicitur.

§. 76.

β − γ est
itur.
= o, erit
atinuam:
x.
c.
iprocus
§. 76.

§. 76. Quodsi iam pro n successiue numeri 1, 2, 3, 4, etc. substituantur, reperientur sequentes fractiones continuae:

$$\frac{\pi}{2} = 1 + \cfrac{2}{3 + \cfrac{1.3}{4 + \cfrac{3.5}{4 + \cfrac{5.7}{4 + \cfrac{7.9}{4 + \text{etc.}}}}}}$$

$$\frac{2}{1} \cdot \frac{2}{\pi} = 1 + \cfrac{2}{7 + \cfrac{1.3}{8 + \cfrac{3.5}{8 + \cfrac{5.7}{8 + \cfrac{7.9}{8 + \text{etc.}}}}}}$$

$$\frac{1.3}{2.2} \cdot \frac{\pi}{2} = 1 + \cfrac{2}{11 + \cfrac{1.3}{12 + \cfrac{3.5}{12 + \cfrac{5.7}{12 + \cfrac{7.9}{12 + \text{etc.}}}}}}$$

$$\frac{2.2.4}{1.3.3} \cdot \frac{2}{\pi} = 1 + \cfrac{2}{15 + \cfrac{1.3}{16 + \cfrac{3.5}{16 + \cfrac{5.7}{16 + \cfrac{7.9}{16 + \text{etc.}}}}}}$$

§. 77.

§. 77. Hinc etiam vicissim huiusmodi fractionum continuarum valores inuestigari poterunt. Sit enim proposita haec fractio:

$$s = \cfrac{\alpha\alpha - \delta\delta}{m + \cfrac{4\alpha\alpha - \delta\delta}{m + \cfrac{9\alpha\alpha - \delta\delta}{m + \cfrac{16\alpha\alpha - \delta\delta}{m + \text{etc.}}}}}$$

erit $\beta - \gamma = 2\delta$ et $2\alpha n + \beta + \gamma = m$; vnde $\beta = 2\delta + \gamma$ et $2\alpha n = m - 2\delta - 2\gamma$, sicque

$$x = 1 + \frac{2\delta}{m - \alpha - \delta + 1} = \frac{m - \alpha + \delta + 1}{m - \alpha - \delta + 1},$$

ergo

$$s = \frac{2\delta}{x - 1} - m + \alpha + \delta.$$

Verum est

$$x = \frac{\int z^{\frac{1}{2}m - \delta - 1} dz : \sqrt{(1 - z^{2\alpha})}}{\int z^{\frac{1}{2}m + \delta - 1} dz : \sqrt{(1 - z^{2\alpha})}},$$

vnde si ponatur

$$\int \frac{z^{\frac{1}{2}m - \delta - 1} dz}{\sqrt{(1 - z^{2\alpha})}} = P \text{ et } \int \frac{z^{\frac{1}{2}m + \delta - 1}}{\sqrt{(1 - z^{2\alpha})}} = Q,$$

erit

$$s = \frac{(m - \alpha + \delta) Q - (m - \alpha - \delta) P}{P - Q},$$

§. 78. Hinc si ponatur $\delta = \varepsilon\sqrt{-1}$, vt sit

$s =$

i fractionum
it enim pro-
$\beta = 2\delta + \gamma$
$= Q,$
vt fit

$$s = \cfrac{\alpha\alpha + \varepsilon\varepsilon}{m + \cfrac{4\alpha\alpha + \varepsilon\varepsilon}{m + \cfrac{9\alpha\alpha + \varepsilon\varepsilon}{m + \cfrac{16\alpha\alpha + \varepsilon\varepsilon}{m + \text{etc.}}}}}$$

ob $z^{-\delta} = z^{-\varepsilon\sqrt{-1}} = e^{-\varepsilon\sqrt{-1} \cdot lz} = \text{cos.}\, \varepsilon lz - \sqrt{-1}.\ \text{sin.}\, \varepsilon lz$ et $z^{\delta} = \text{cos.}\, \varepsilon lz + \sqrt{-1}.\ \text{sin.}\, \varepsilon lz$, statuatur

$$\int \frac{z^{\frac{1}{2}m - 1} dz\, \text{cos.}\, \varepsilon lz}{\sqrt{(1 - z^{2\alpha})}} = R \text{ et } \int \frac{z^{\frac{1}{2}m - 1} dz\, \text{sin.}\, \varepsilon lz}{\sqrt{(1 - z^{2\alpha})}} = S,$$

erit $P = R - S\sqrt{-1}$; et $Q = R + S\sqrt{-1}$; ideoque

$$s = \frac{2(m - \alpha) S\sqrt{-1} + 2\varepsilon R\sqrt{-1}}{-2S\sqrt{-1}} = \alpha - m - \varepsilon \frac{R}{S},$$

vbi notandum est integralia R et S ita sumi debere, vt posito $z = 0$ euanescant, tum vero poni $z = 1$.

VARIA