1780

De motibus maxime irregularibus, qui in systemate mundano locum habere possent, una cum methodo hujusmodi motus per temporis spatium quantumvis magnum prosequendi.

Leonhard Euler

lationibus delectantur, in eorum demonstrationem inquirant, cum nullum sit dubium, quin inde Theoria numerorum insignia incrementa sit adeptura.

Conclusio.

§. 39. Quatuor haec Theoremata postrema, quorum demonstratio adhuc desideratur, sequenti modo concinnius exhiberi possunt:

Existente s *numero quocunque primo, diuidantur tantum quadrata imparia* 1, 9, 25, 49, *etc. per diuisorem* 48, *notenturque residua, quae omnia erunt formae* 4q + 1, *quorum quoduis littera* α *indicetur, reliquorum autem numerorum, formae* 4q + 1, *qui inter residua non occurrunt, quilibet littera* $\mathfrak{A}$ *indicetur, quo facto si fuerit*

diuisor numerus primus formae	tum est
$4ns+\alpha$	$+s$ residuum et $-s$ residuum
$4ns-\alpha$	$+s$ residuum et $-s$ non-residuum
$4ns+\mathfrak{A}$	$+s$ non-residuum et $-s$ non-residuum
$4ns-\mathfrak{A}$	$+s$ non-residuum et $-s$ residuum.

OBSER-

onem inqui-
heoria nume-
strema, quo-
i modo con-
ntur tantum
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OBSER-

OBSERVATIONES
ANALYTICAE.

§. 1.

Inter alia, quae passim de fractionibus continuis sum commentatus, notatu digna videtur haec forma:

$$1+\cfrac{n}{2+\cfrac{n+1}{3+\cfrac{n+2}{4+\cfrac{n+3}{5+\cfrac{n+4}{6+\text{etc.}}}}}}$$

cuius valor, quoties n est numerus integer, sequenti modo exhiberi potest, denotante e numerum, cuius logarithmus est vnitas, vt sit $e = 2{,}718281828459045$

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\text{etc.}}}}} = \frac{1}{e-2};$$

$$1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\text{etc.}}}}} = e-1;$$

1 +

$$1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{5+\text{etc.}}}}}=2;$$

$$1+\cfrac{4}{2+\cfrac{5}{3+\cfrac{6}{4+\cfrac{7}{5+\text{etc.}}}}}=\frac{9}{4};$$

$$1+\cfrac{5}{2+\cfrac{6}{3+\cfrac{7}{4+\cfrac{8}{5+\text{etc.}}}}}=\frac{58}{31};$$

$$1+\cfrac{6}{2+\cfrac{7}{3+\cfrac{8}{4+\cfrac{9}{5+\text{etc.}}}}}=\frac{365}{156};$$

$$1+\cfrac{7}{2+\cfrac{8}{3+\cfrac{9}{4+\cfrac{10}{5+\text{etc.}}}}}=\frac{3006}{1057};$$

$$1+\cfrac{8}{2+\cfrac{9}{3+\cfrac{10}{4+\cfrac{11}{5+\text{etc.}}}}}=\frac{28357}{9178};$$

vbi

vbi modo prorſus ſingulari vſu venit, vt binae priores numerum tranſcendentem e implicent, dum ſequentes omnes numeris rationalibus exprimuntur.

§. 2. Hoc eo magis mirum videtur, quod etiam caſus praecedentes, vbi pro n vel cyphra vel numeri negatiui ponuntur, valoribus rationalibus continentur, quibus quidem caſibus ipſa fractionis continuae forma abrumpitur. Erit enim

$$1+\cfrac{0}{2+\cfrac{1}{3+\cfrac{2}{4+\text{etc.}}}}=1;$$

$$1-\cfrac{1}{2+\cfrac{0}{3+\cfrac{1}{4+\text{etc.}}}}=\frac{1}{2};$$

$$1-\cfrac{2}{2-\cfrac{1}{3+\cfrac{0}{4+\text{etc.}}}}=-\frac{1}{5};$$

$$1-\cfrac{3}{2-\cfrac{2}{3-\cfrac{1}{4+0}}}=-\frac{19}{14};$$

$$1-\cfrac{4}{2-\cfrac{3}{3-\cfrac{2}{4-\cfrac{1}{5+\text{etc.}}}}}=-\frac{151}{37};$$

1 — 5

$$1-\cfrac{5}{2-\cfrac{4}{3-\cfrac{3}{4-\cfrac{2}{5-\cfrac{1}{6+\text{etc.}}}}}}=-\tfrac{1091}{34};$$

Qua igitur lege tam hi valores, quam praecedentes inter se cohaereant, haud abs re fore arbitror, ostendisse. Imprimis autem iuuabit methodum exposuisse, qua isti valores inuestigari queant.

§. 3. Primum igitur obseruo, si pro numero quocunque n valor fractionis continuae ita indicetur:

$$f(n)=1+\cfrac{n}{2+n+\cfrac{1}{3+n+\cfrac{2}{4+n+\cfrac{3}{5+\text{etc.}}}}}$$

fore $f(n+1)=\frac{n(f(n)+1)}{f(n)+n-1}$; cuius veritas in valoribus indicatis perspicitur, cum sit:

$$f(1)=\tfrac{1}{e-2};\ f(2)=e-1;\ f(3)=2;\ f(4)=\tfrac{9}{4};$$
$$f(5)=\tfrac{52}{21};\ f(6)=\tfrac{365}{136};\ f(7)=\tfrac{3006}{1045};\ f(8)=\tfrac{28357}{9276};$$

et pro praecedentibus:

$$f(0)=1;\ f(-1)=\tfrac{1}{2};\ f(-2)=-\tfrac{1}{5};\ f(-3)=-\tfrac{19}{14};$$
$$f(-4)=-\tfrac{151}{37};\ f(-5)=-\tfrac{1091}{34};\ f(-6)=+\tfrac{7841}{887}.$$

Haec relatio inter binos valores contiguos intercedens non impedit, quominus casibus $n=1$ et $n=2$ sint transcendentes.

dentes inter
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qua isti va-
umero quo-
ur:
valoribus in-
$f(4)=\frac{9}{4}$;
8) $=\frac{28357}{9276}$;
) $=-\frac{19}{14}$;
$+\frac{7841}{887}$.
cedens non
t transcen-
dentes.

dentes. Posito enim $n=0$ fit $f(1)=\frac{0(1+1)}{1+0-1}=\frac{0}{0}$, quae expressio valori $\frac{1}{e-2}$ non aduersatur, etiamsi hic inde elici nequeat. Deinde posito $n=2$ prodit $f(3)=\frac{2(f(2)+1)}{f(2)+1}=2$, ita vt ipse valor $f(2)=e-1$ hic non in computum veniat.

§. 4. Inuestigatio autem horum valorum haud parum ardua videtur; quare, quemadmodum ad eos peruenerim, dilucide exponam, quandoquidem methodus, qua sum vsus, multo latius patet, ac fortasse ad alias praeclusas speculationes deducere potest. Sumsi igitur binos numeros indefinitos m et n, eorumque certam quandam functionem, quae sit p, sum contemplatus, vnde similes functiones eorundem numerorum, vna pluribusue vnitatibus auctorum, formaui, quas, sumta littera Φ pro signo huius functionis, ita repraesento:

$$p=\Phi(m \text{ et } n);\ p'=\Phi(m \text{ et } n+1);\ p''=\Phi(m \text{ et } n+2);$$
$$q=\Phi(m+1 \text{ et } n);\ q'=\Phi(m+1 \text{ et } n+1);\ q''=\Phi(m+1 \text{ et } n+2);$$
$$r=\Phi(m+2 \text{ et } n);\ r'=\Phi(m+2 \text{ et } n+1);\ r''=\Phi(m+2 \text{ et } n+2);$$
$$s=\Phi(m+3 \text{ et } n);\ s'=\Phi(m+3 \text{ et } n+1);\ s''=\Phi(m+3 \text{ et } n+2);$$
etc. etc.

Functionem autem Φ eius indolis esse statuo, vt sit

$$p=Am+Bn+C+\frac{Dnn+En+F}{p'},\ \text{erit}$$
$$q=A(m+1)+Bn+C+\frac{Dnn+En+F}{q'},$$
$$r=A(m+2)+Bn+C+\frac{Dnn+En+F}{r'},$$
$$s=A(m+3)+Bn+C+\frac{Dnn+En+F}{s'},$$
etc.

 §. 5.

§. 5. Cum igitur p', q', r', s', etc. oriantur ex p, q, r, s, etc. si, seruato numero m, alter n vnitate augeatur, erit simili modo:

$$p' = A\,m + B\,(n+1) + C + \frac{D(n+1)^2 + E(n+1) + F}{p''}$$
$$q' = A\,(m+1) + B\,(n+1) + C + \frac{D(n+1)^2 + E(n+1) + F}{q''}$$
$$r' = A\,(m+2) + B\,(n+1) + C + \frac{D(n+1)^2 + E(n+1) + F}{r''}$$

tum vero ob eandem rationem:

$$p'' = A\,m + B\,(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{p'''},$$
$$q'' = A\,(m+1) + B\,(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{q'''},$$
$$r'' = A\,(m+2) + B\,(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{r'''},$$

etc.

atque

$$p''' = A\,m + B\,(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{p''''},$$
$$q''' = A\,(m+1) + B\,(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{q''''},$$
$$r''' = A\,(m+2) + B\,(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{r''''},$$

etc.

sicque porro vlterius progrediendo.

§. 6. Hinc functio p sequenti modo per fractionem continuam infinitam exprimetur:

$$p = Am + Bn + C + \cfrac{Dn^2 + En + F}{Am + B(n+1) + C + \cfrac{D(n+1)^2 + E(n+1) + F}{Am + B(n+2) + C + \cfrac{D(n+2)^2 + E(n+2) + F}{Am + B(n+3) + C + \text{etc.}}}}$$

vnde seruato n, si loco m successiue scribantur numeri $m+1$, $m+2$, $m+3$, etc. prodibunt valores functionum

oriantur ex vnitate au·
er fractio-
numeri functio-
num

num q, r, s, t, etc. per similes fractiones continuas expressae. Nunc igitur quaeritur, cuiusmodi relatio sit intercessura inter functiones p et q? Qua inuenta per superiorem analogiam simul relatio inter omnes functiones hic exhibitas constabit. Quod cum a priori determinatu nimis difficile videatur, coniectura vtendum censeo.

§. 7. Videamus ergo, num inter p et q huiusmodi relatio statui queat:

$$(p + (\alpha - A)m + (\beta - B)n + \gamma - C)(q + (\delta - A)m + (\varepsilon - B)n + \zeta - A - C)$$
$$= \lambda mm + \mu m + \nu,$$

vnde seruato m, si loco n scribatur $n+1$, erit

$$(p' + (\alpha - A)m + (\beta - B)(n+1) + \gamma - C) \times$$
$$\times (q' + (\delta - A)m + (\varepsilon - B)(n+1) + \zeta - A - C) = \lambda mm + \mu m + \nu.$$

At si ibi pro p et q superiores valores per p' et q' substituantur, prodibit:

$$\left(\alpha m + \beta n + \gamma + \frac{Dnn + En + F}{p'}\right)\left(\delta m + \varepsilon n + \zeta + \frac{Dnn + En + F}{q'}\right)$$
$$= \lambda mm + \mu m + \nu$$

quae euoluitur in hanc:

$$(\alpha m + \beta n + \gamma)(\delta m + \varepsilon n + \zeta)\, p'q' - (\lambda mm + \mu m + \nu)\, p'q'$$
$$+ (\alpha m + \beta n + \gamma)(Dnn + En + F)\, p'$$
$$+ (\delta m + \varepsilon n + \zeta)(Dnn + En + F)\, q'$$
$$+ (Dnn + En + F)^2 = 0,$$

quae cum illa congruere debet. Vnde perspicuum est esse oportere

$$(\alpha m + \beta n + \gamma)(\delta m + \varepsilon n + \zeta) - \lambda mm - \mu m - \nu = \theta\,(Dnn + En + F)$$

vt diuisione per $\theta\,(Dnn + En + F)$ instituta fiat

$$p'q' + \tfrac{1}{\theta}(\alpha m + \beta n + \gamma)\, p' + \tfrac{1}{\theta}(\delta m + \varepsilon n + \zeta)\, q' + \tfrac{1}{\theta}(Dnn + En + F) = 0,$$

 quae

quae per factores repraesentata ita exhibeatur:

$$\left(p' + \frac{\delta m + \varepsilon n + \zeta}{\theta}\right)\left(q' + \frac{\alpha m + \beta n + \gamma}{\theta}\right) = \frac{(\alpha m + \beta n + \gamma)(\delta m + \varepsilon n + \zeta)}{\theta\theta}$$
$$- \tfrac{1}{\theta}(D nn + E n + F),$$

seu

$$\left(p' + \frac{\delta m + \varepsilon n + \zeta}{\theta}\right)\left(q' + \frac{\alpha m + \beta n + \gamma}{\theta}\right) = \frac{\lambda mm + \mu m + \nu}{\theta\theta}.$$

§. 8. Comparetur haec forma cum priori:

$$(p' + (\alpha - A) m + (\beta - B) n + \beta + \gamma - B - C) \times$$
$$\times (q' + (\delta - A) m + (\varepsilon - B) n + \varepsilon + \zeta - A - B - C)$$
$$= \lambda mm + \mu m + \nu,$$

vnde statim colligitur $\theta\theta = 1$, ideoque vel $\theta = 1$ vel $\theta = -1$. Tum vero esse debet:

$$\delta = \theta(\alpha - A);\ \varepsilon = \theta(\beta - B);\ \zeta = \theta(\beta + \gamma - B - C),$$
$$\alpha = \theta(\delta - A);\ \beta = \theta(\varepsilon - B);\ \gamma = \theta(\varepsilon + \zeta - A - B - C).$$

Quia ergo valor $\theta = 1$ non conuenit, ponamus $\theta = -1$, vt habeamus:

$$\alpha + \delta = A;\ \beta + \varepsilon = B;\ \beta + \gamma + \zeta = B + C \text{ et}$$
$$\gamma + \varepsilon + \zeta = A + B + C$$

hincque $\varepsilon - \beta = A$; ergo:

$$\beta = \tfrac{1}{2}(B - A);\ \varepsilon = \tfrac{1}{2}(A + B) \text{ et } \gamma + \zeta = \tfrac{1}{2}(A + B) + C.$$

Praeterea vero haec conditio est adimplenda:

$$(\alpha m + \beta n + \gamma)(\delta m + \varepsilon n + \zeta) = \lambda mm + \mu m + \nu - D nn - E n - F$$
$$= \alpha\delta mm + \alpha\varepsilon mn + \alpha\zeta m + \beta\zeta n + \gamma\zeta$$
$$+ \beta\varepsilon nn + \beta\delta mn + \gamma\delta m + \gamma\varepsilon n.$$

Erit ergo:

$$\lambda = \alpha\delta;\ \mu = \alpha\zeta + \gamma\delta;\ D = -\beta\varepsilon;\ E = -\beta\zeta - \gamma\varepsilon$$
$$\nu - F = \gamma\zeta \text{ et } \alpha\varepsilon + \beta\delta = 0,$$

vnde

r: $+\gamma)(\delta m + \varepsilon n + \zeta)$ / $\theta\theta$
$mm + \mu m + \nu$ / $\theta\theta$.
priori:
$- B - C) \times$
$- A - B - C)$
el $\theta = 1$ vel
$- B - C),$
$- A - B - C).$
us $\theta = -1,$
$= B + C$ et
$+ B) + C.$
$nn - E n - F$
$+ \gamma\zeta,$
$\beta\zeta - \gamma\varepsilon$
vnde

vnde primo fit $D = -\beta\varepsilon = \tfrac{1}{4}(AA - BB)$; deinde

$$\tfrac{1}{2}\alpha(A + B) - \tfrac{1}{2}\delta(B - A) = 0, \text{ seu } \delta = \frac{A + B}{A - B}\alpha;$$

ideoque

$$\alpha = \tfrac{1}{2}(A - B) \text{ et } \delta = \tfrac{1}{2}(A + B).$$

Tum vero erit

$$E + \tfrac{1}{2}\zeta(B - A) + \tfrac{1}{2}\gamma(A + B) = 0, \text{ seu}$$
$$E + \tfrac{1}{4}B(A + B) + \tfrac{1}{2}BC + \tfrac{1}{2}A(\gamma - \zeta) = 0,$$

hincque

$$\zeta - \gamma = \frac{BC}{A} + \frac{B(A + B)}{2A} + \frac{2E}{A},$$

ergo

$$\zeta = \tfrac{1}{4}(A + B) + \tfrac{1}{2}C + \frac{BC}{2A} + \frac{B(A + B)}{4A} + \frac{E}{A},$$
$$\gamma = \tfrac{1}{4}(A + B) + \tfrac{1}{2}C - \frac{BC}{2A} - \frac{B(A + B)}{4A} - \frac{E}{A},$$

siue hoc modo:

$$\zeta = \tfrac{1}{4}(A + B + 2C)\left(1 + \frac{B}{A}\right) + \frac{E}{B} = \frac{(A + B)(A + B + 2C)}{4A} + \frac{E}{A}$$
$$\gamma = \tfrac{1}{4}(A + B + 2C)\left(1 - \frac{B}{A}\right) - \frac{E}{A} = \frac{(A - B)(A + B + 2C)}{4A} - \frac{E}{A}.$$

§. 9. Relatio ergo inter p et q assumta subsistere nequit, nisi sit $D = \tfrac{1}{4}(AA - BB)$, qui valor si ipsi D tribuatur, sequentes litterae ita se habebunt:

$$\alpha = \tfrac{1}{2}(A - B);\ \delta = \tfrac{1}{2}(A + B);\ \gamma = \frac{(A - B)(A + B + 2C)}{4A} - \frac{E}{A},$$
$$\beta = -\tfrac{1}{2}(A - B);\ \varepsilon = \tfrac{1}{2}(A + B);\ \zeta = \frac{(A + B)(A + B + 2C)}{4A} + \frac{E}{A},$$
$$\lambda = \tfrac{1}{4}(AA - BB) = D,$$
$$\mu = \frac{(AA - BB)(A + B + 2C)}{4A} - \frac{BE}{A},$$
$$\nu = \frac{(AA - BB)(A + B + 2C)^2}{16AA} - \frac{BE(A + B + 2C)}{2AA} - \frac{EE}{AA} + F,$$

hincque porro

$$\alpha - A = -\tfrac{1}{2}(A + B);\ \beta - B = -\tfrac{1}{2}(A + B);$$
$$\gamma - C = \frac{AA - BB}{4A} - \frac{C(A + B)}{2A} - \frac{E}{A},$$

 δ − A

$$\delta - A = -\tfrac{1}{2}(A - B);\quad \varepsilon - B = \tfrac{1}{2}(A - B);$$
$$\zeta - A - C = -\tfrac{(A-B)(3A+B)}{4A} - \tfrac{C(A-B)}{2A} + \tfrac{E}{A},$$

vnde inter p et q haec resultat aequatio:

$$\left(p - \tfrac{1}{2}(A + B)(m + n) + \tfrac{AA-BB}{4A} - \tfrac{C(A+B)}{2A} - \tfrac{E}{A}\right) \times$$
$$\times\left(q - \tfrac{1}{2}(A - B)(m - n) - \tfrac{(A-B)(3A+B)}{4A} - \tfrac{C(A-B)}{2A} + \tfrac{E}{A}\right)$$
$$= \lambda m^2 + \mu m + \nu.$$

§. 10. Ponamus ad abbreuiandum

$$P = \tfrac{(A+B)(A-B)}{4A} - \tfrac{C(A+B)}{2A} - \tfrac{E}{A},$$
$$Q = \tfrac{(A-B)(3A+B)}{4A} + \tfrac{C(A-B)}{2A} - \tfrac{E}{A},$$

vt fit

$$\left(p - \tfrac{1}{2}(A+B)(m+n) + P\right)\left(q - \tfrac{1}{2}(A-B)(m-n) - Q\right)$$
$$= \lambda m^2 + \mu m + \nu,$$

erit

$$p = \tfrac{1}{2}(A+B)(m+n) - P + \frac{\lambda m^2 + \mu m + \nu}{q - \frac{1}{2}(A-B)(m-n) - Q}.$$

Simili vero modo est

$$q = \tfrac{1}{2}(A+B)(m+n) + \tfrac{1}{2}(A+B) - P + \frac{\lambda(m+1)^2 + \mu(m+1) + \nu}{r - \frac{1}{2}(A-B)(m-n) - \frac{1}{2}(A-B) - Q},$$

$$r = \tfrac{1}{2}(A+B)(m+n) + (A+B) - P + \frac{\lambda(m+2)^2 + \mu(m+2) + \nu}{s - \frac{1}{2}(A-B)(m-n) - (A-B) - Q},$$

vnde fit

$$q - \tfrac{1}{2}(A-B)(m-n) - Q = Bm + An + \tfrac{1}{2}(A+B) - P - Q$$
$$+ \frac{\lambda(m+1)^2 + \mu(m+1) + \nu}{r - \frac{1}{2}(A-B)(m-n) - \frac{1}{2}(A-B) - Q},$$

$$r - \tfrac{1}{2}(A-B)(m-n) - \tfrac{1}{2}(A-B) - Q = Bm + An + \tfrac{1}{2}(A+3B) - P - Q$$
$$+ \frac{\lambda(m+2)^2 + \mu(m+2) + \nu}{s - \frac{1}{2}(A-B)(m-n) - (A-B) - Q},$$

Est

Est vero

$$P + Q = \tfrac{(A-B)(3A+B)}{2A} - \tfrac{BC}{A} - \tfrac{2E}{A},$$

hincque

$$q - \tfrac{1}{2}(A-B)(m-n) - Q = Bm + An + B + \tfrac{BB-AA+2BC+4E}{2A} + \text{etc.}$$

Quare si breuitatis gratia ponatur

$$\tfrac{BB - AA + 2BC + 4E}{2A} = G,\ \text{erit}$$

$$p = \tfrac{1}{2}(A+B)(m+n) + \tfrac{BB-AA}{4A} + \tfrac{C(A+B)}{2A} + \tfrac{E}{A}$$
$$+ \cfrac{\lambda m^2 + \mu m + \nu}{B(m+1) + An + G + \cfrac{\lambda(m+1)^2 + \mu(m+1) + \nu}{B(m+2) + An + G + \cfrac{\lambda(m+2)^2 + \mu(m+2) + \nu}{B(m+3) + An + G + \text{etc.}}}}$$

vel valorem G introducendo:

$$p = \tfrac{1}{2}(A+B)(m+n) + \tfrac{1}{2}(C+G)$$
$$+ \cfrac{\lambda m^2 + \mu m + \nu}{B(m+1) + An + G + \cfrac{\lambda(m+1)^2 + \mu(m+1) + \nu}{B(m+2) + An + G + \cfrac{\lambda(m+2)^2 + \mu(m+2) + \nu}{B(m+3) + An + G + \text{etc.}}}}$$

Tum vero etiam erit

$$P = -\tfrac{1}{2}(C + G)\ \text{et}\ Q = \tfrac{1}{2}(A - B) + \tfrac{1}{2}(C - G)$$

ideoque

$$\left(p - \tfrac{1}{2}(A+B)(m+n) - \tfrac{1}{2}(C+G)\right)\left(q - \tfrac{1}{2}(A-B)(m+1-n) - \tfrac{1}{2}(C-G)\right)$$
$$= \lambda m^2 + \mu m + \nu.$$

§. 11. Quodsi ergo proposita fuerit huiusmodi fractio continua infinita:

$f =$

$$p = Am + Bn + C + \cfrac{Dn^2 + En + F}{Am + B(n+1) + C + D(n+1)^2 + E(n+1) + \cfrac{F}{Am + B(n+2) + C + D(n+2)^2 + E(n+2) + \cfrac{F}{Am + B(n+3) + C + \text{etc.}}}}$$

in qua sit $D = \frac{1}{4}(AA - BB)$; indeque haec altera formetur:

$$q = A(m+1) + Bn + C + \cfrac{Dn^2 + En + F}{A(m+1) + B(n+1) + C + D(n+1)^2 + E(n+1) + \cfrac{F}{A(m+1) + B(n+2) + C + D(n+2)^2 + E(n+2) + \cfrac{F}{A(m+1) + B(n+3) + C + \text{etc.}}}}$$

dum loco m vbique scribitur $m + 1$; primo relatio inter p et q assignari potest hoc modo: Ponatur breuitatis gratia:

$$\frac{BB - AA + 2BC + 4E}{2A} = G \text{ et}$$

$$\lambda = \tfrac{1}{4}(AA - BB),$$

$$\mu = \tfrac{1}{2}(AA - BB) + \tfrac{1}{2}(AC - BG),$$

$$\nu = \tfrac{1}{4}CC + \tfrac{1}{2}(C - G)(A + B) - \tfrac{1}{4}GG + F,$$

seu

$$\nu = F + \tfrac{1}{4}(C - G)(A + B + C + G),$$

eritque ista relatio:

$$\left(p - \tfrac{1}{2}(A+B)(m+n) - \tfrac{1}{2}(C+G)\right)\left(q - \tfrac{1}{2}(A-B)(m+1-n) - \tfrac{1}{2}(C-G)\right) = \lambda m^2 + \mu m + \nu;$$

tum vero praeterea functio p etiam huic alteri fractioni continuae aequatur:

$$p = \tfrac{1}{2}(A+B)(m+n) + \tfrac{1}{2}(C+G) + \cfrac{\lambda m^2 + \mu m + \nu}{B(m+1) + An + G + \cfrac{\lambda(m+1)^2 + \mu(m+1) + \nu}{B(m+2) + An + G + \cfrac{\lambda(m+2)^2 + \mu(m+2) + \nu}{B(m+3) + An + G \text{ etc.}}}}$$

§. 12. Exempla supra proposita hinc nascuntur, si statuatur:

$$D = \tfrac{1}{4}(AA - BB) = 0.$$

Sit ergo $B = A$, vt habeantur istae fractiones continuae:

$$p = A(m+n) + C + \cfrac{En + F}{A(m+n+1) + C + \cfrac{E(n+1) + F}{A(m+n+2) + C + \cfrac{E(n+2) + F}{A(m+n+3) + C + \text{etc.}}}}$$

$$q = A(m+n+1) + C + \cfrac{En + F}{A(m+n+2) + C + \cfrac{E(n+1) + F}{A(m+n+3) + C + \cfrac{E(n+2) + F}{A(m+n+4) + C + \text{etc.}}}}$$

quarum relatio, posito

$$G = C + \tfrac{2E}{A};\ \mu = \tfrac{1}{2}A(C - G) \text{ et}$$

$$\nu = F + \tfrac{1}{4}(C - G)(2A + C + G), \text{ seu}$$

$$\mu = -E;\ \nu = F - \tfrac{E}{2A}(2A + C + G), \text{ erit}$$

$$\left(p - A(m+n) - \tfrac{1}{2}(C + G)\right)\left(q - \tfrac{1}{2}(C - G)\right) = \mu m + \nu,$$

et quantitas p etiam alio modo per fractionem continuam infinitam exprimetur.

$$p=A(m+n)+\tfrac{1}{2}(C+G)+\cfrac{\mu m+\nu}{A(m+n+1)+G+\cfrac{\mu(m+1)+\nu}{A(m+n+2)+G+\cfrac{\mu(m+2)+\nu}{A(m+n+3)+G+\text{etc.}}}}$$

§. 13. Cum igitur hic tam numeratores quam denominatores progressionem arithmeticam constituant, eorum formam simpliciorem reddamus, sintque binae fractiones continuae infinitae:

$$p=a+\cfrac{f}{a+b+\cfrac{f+g}{a+2b+\cfrac{f+2g}{a+3b+\cfrac{f+3g}{a+b+\text{etc.}}}}}$$

$$q=a+b+\cfrac{f}{a+2b+\cfrac{f+g}{a+3b+\cfrac{f+2g}{a+4b+\cfrac{f+3g}{a+5b+\text{etc.}}}}}$$

ita vt ex fractione p nascatur q, si loco a scribatur $a+b$. Erit ergo:

$A(m+n)+C=a$; $En+F=f$; $A=b$; $E=g$;

hincque $C=a-b(m+n)$; et $F=f-gn$; vnde conficitur:

$G=a-b(m+n)+\frac{2g}{b}$; $C-G=\frac{-2g}{b}$;

$C+G=2a-2b(m+n)+\frac{2g}{b}$; tum

$\mu=-g$; $\nu=f-gn-g-\frac{g}{b}(a-b(m+n)+\frac{g}{b})$

seu

-G+ etc.
tores quam
constituant,
binae frac-
f+3g / a+b+etc.
f+3g / +5b+etc.
itur a+b.
b; E=g;
nde confi-
n) + g/b)
seu

seu $\nu=f-\frac{g(a+b)}{b}+gm-\frac{gg}{bb}$, ergo

$\mu m+\nu=f-\frac{g(a+b)}{b}-\frac{gg}{bb}$.

$A(m+n)+\frac{1}{2}(C+G)=a+\frac{g}{b}$ et

$\frac{1}{2}(C-G)=-\frac{g}{b}$.

Quare inter p et q haec habetur relatio:

$$(p-a-\tfrac{g}{b})(q+\tfrac{g}{b})=f-\tfrac{g(a+b)}{b}-\tfrac{gg}{bb},\text{ seu}$$

$$pq+\tfrac{g}{b}p-(a+\tfrac{g}{b})q=f-g$$

vnde pro p etiam haec habetur fractio continua:

$$p=a+\frac{g}{b}+\cfrac{f-\frac{g(a+b)}{b}-\frac{gg}{bb}}{a+b+\frac{2g}{b}+\cfrac{f-\frac{g(a+2b)}{b}-\frac{gg}{bb}}{a+2b+\frac{2g}{b}+\cfrac{f-\frac{g(a+3b)}{b}-\frac{gg}{bb}}{a+3b+\frac{2g}{b}\text{ etc.}}}}$$

§. 14. Vt fractiones tollamus ponamus $g=bb$, vt habeamus has fractiones continuas:

$$p=a+\cfrac{f}{a+b+\cfrac{f+bb}{a+2b+\cfrac{f+2bb}{a+3b+\cfrac{f+3bb}{a+4b\text{ etc.}}}}}$$

$$q=a+b+\cfrac{f}{a+2b+\cfrac{f+bb}{a+3b+\cfrac{f+2bb}{a+4b+\cfrac{f+3bb}{a+5b\text{ etc.}}}}}$$

quarum relatio ita se habet, vt sit

$$(p-a-b)(q+b)=f-(a+b)b-bb, \text{ seu}$$
$$pq+bp-(a+b)q=f-bb$$

vnde pro p elicitur haec quoque fractio continua:

$$p=a+b+\cfrac{f-(a+b)b-bb}{a+b+2b+\cfrac{f-(a+2b)b-bb}{a+2b+2b+\cfrac{f-(a+3b)b-bb}{a+3b+2b+\text{etc.}}}}$$

Cum igitur haec fractio continua primae sit aequalis, haec autem abrumpatur, quoties fuerit $f=(a+ib)b+bb$, denotante i numerum integrum positiuum, toties valor primae rationaliter assignari potest.

§. 15. Ex relatione inter p et q inuenta per p quoque q ita exprimitur:

$$q=-b+\frac{f-(a+b)b-bb}{-a-b+p}$$

et cum p oriatur ex q, si loco a scribatur $a-b$; si seriei $p, q, r,$ etc. termini praecedentes sint $o, n, m,$ etc. erit

$$p=-b+\frac{f-ab-bb}{-a+b-b+o};$$
$$o=-b+\frac{f-(a-b)b-bb}{-a+2b-b+n};$$
$$n=-b+\frac{f-(a-2b)b-bb}{-a+3b-b+m};$$
etc.

vnde pro p etiam haec fractio continua obtinetur:

$$p=-b+\cfrac{f-ab-bb}{b-a-2b+\cfrac{f+(b-a)b-bb}{2b-a-2b+\cfrac{f+(2b-a)b-bb}{3b-a-2b+\text{etc.}}}}$$

quam eandem ex nostris formulis generalibus inuenissemus, si

si supra §. 12. posuissemus $B=-A$. Quare etiam valor rationaliter exprimi poterit, quoties fuerit $bb=(ib-a)b+f$, nisi forte his casibus denominator, isti numeratori euanescenti subiectus, quoque euanescat.

§. 16. Ex ipsa autem fractione continua proposita:

$$p=a+\cfrac{f}{a+b+\cfrac{f+bb}{a+2b+\cfrac{f+2bb}{a+3b+\text{etc.}}}}$$

alia immediate hoc modo deduci potest ipsi aequalis. Cum enim sit:

$p=a+\frac{f}{p'}$; $p'=a+b+\frac{f+bb}{p''}$; $p''=a+2b+\frac{f+2bb}{p'''}$; etc.

erit regrediendo:

$p_{\prime}=a-b+\frac{f-bb}{p}$; $p_{\prime\prime}=a-2b+\frac{f-2bb}{p_{\prime}}$; $p_{\prime\prime\prime}=a-3b+\frac{f-3bb}{p_{\prime\prime}}$;

hincque

$p=\frac{f-bb}{b-a+p_{\prime}}$; $p_{\prime}=\frac{f-2bb}{2b-a+p_{\prime\prime}}$; $p_{\prime\prime}=\frac{f-3bb}{3b-a+p_{\prime\prime\prime}}$; etc.

vnde concludimus:

$$p=\cfrac{f-bb}{b-a+\cfrac{f-2bb}{2b-a+\cfrac{f-3bb}{3b-a+\cfrac{f-4bb}{4b-a+\text{etc.}}}}}$$

ita vt etiam casibus $f=ibb$ valor rationaliter exhiberi queat.

§. 17. En ergo quatuor fractiones continuas inter se aequales:

I. P=

I. $p = a + \cfrac{f}{a + b + \cfrac{f + bb}{a + 2b + \cfrac{f + 2bb}{a + 3b + \cfrac{f + 3bb}{a + 4b \text{ etc.}}}}}$

II. $p = \cfrac{f - bb}{b - a + \cfrac{f - 2bb}{2b - a + \cfrac{f - 3bb}{3b - a + \cfrac{f - 4bb}{4b - a \text{ etc.}}}}}$

III. $p = a + b + \cfrac{f - (a+b)b - bb}{a + b + 2b + \cfrac{f - (a+2b)b - bb}{a + 2b + 2b + \cfrac{f - (a+3b)b - bb}{a + 3b + 2b \text{ etc.}}}}$

IV. $p = -b + \cfrac{f - ab - bb}{b - a - 2b + \cfrac{f + (b-a)b - bb}{2b - a - 2b + \cfrac{f + (2b-a)b - bb}{3b - a - 2b + \text{ etc.}}}}$

§. 18. Quo ad formam initio propoſitam propius accedamus, ſit $a = m$; $f = n$; $b = 1$ et $b = 1$; atque habebimus:

I. $p = m + \cfrac{n}{m + 1 + \cfrac{n + 1}{m + 2 + \cfrac{n + 2}{m + 3 + \cfrac{n + 3}{m + 4 \text{ etc.}}}}}$

II. $p =$

etc.

$-bb$

etc.

propius

que ha-

II. $p =$

II. $p = \cfrac{n - 1}{-m + 1 + \cfrac{n - 2}{-m + 2 + \cfrac{n - 3}{-m + 3 + \cfrac{n - 4}{-m + 4 \text{ etc.}}}}}$

III. $p = m + 1 + \cfrac{n - m - 2}{m + 3 + \cfrac{n - m - 3}{m + 4 + \cfrac{n - m - 4}{m + 5 + \cfrac{n - m - 5}{m + 6 \text{ etc.}}}}}$

IV. $p = -1 + \cfrac{n - m - 1}{-m - 1 + \cfrac{n - m}{-m + \cfrac{n - m + 1}{-m + 1 + \cfrac{n - m + 2}{-m + 2 \text{ etc.}}}}}$

Quare denotante i numerum integrum poſitiuum, cyphra non excluſa, fractionis noſtrae continuae valor rationaliter exprimi poterit, his caſibus:

I. $n = i$; II. $n = m + 2 + i$; III. $n = m + 1 - i$;

niſi forte incommodum ſupra memoratum locum inueniat.

§. 19. Ex caſibus $n = i$ raro valor quaeſitus reperitur, ob memoratum incommodum, quo etiam denominator in nihilum abit. Si enim $n = 1$, quo fit:

$p = m + \cfrac{1}{m + 1 + \cfrac{2}{m + 2 + \cfrac{3}{m + 3 + \cfrac{4}{m + 4 \text{ etc.}}}}}$

certe

certe non est $p = 0$, etsi secunda forma id ostendere videtur, vnde affirmare possumus esse:

$$0 = 1 - m - \cfrac{1}{2 - m - \cfrac{2}{3 - m - \cfrac{3}{4 - m - \cfrac{4}{5 - m \text{ etc.}}}}}$$

Quodsi prima forma generalis ad hunc casum accommodetur, erit $a = 1 - m$; $b = 1$; $f = -1$, et $h = -1$, vnde secunda dat

$$p = 1 - \cfrac{1}{m + \cfrac{1}{m + 1 + \cfrac{2}{m + 1 \text{ etc.}}}}$$

tertia vero:

$$p = -m - \cfrac{m}{-m + \cfrac{1 - m}{1 - m + \cfrac{2 - m}{2 - m \text{ etc.}}}}$$

et quarta

$$p = +1 - \cfrac{1 - m}{m - 2 - \cfrac{2 - m}{m - 1 - \cfrac{3 - m}{m - \text{ etc.}}}}$$

quae ergo nihilo sunt aequales.

§. 20.

ndere vi-
c.
accomme-
- 1, vn-
§. 20.

§. 20. Cum ex forma secunda §. 18. sit

$$\frac{n-1}{p} = 1 - m + \cfrac{n - 2}{2 - m + \cfrac{n - 3}{3 - m + \cfrac{n - 4}{4 - m \text{ etc.}}}}$$

si haec cum prima generali comparetur, erit

$a = 1 - m$; $b = 1$; $f = n - 2$; et $h = + 1$;

vnde forma tertia praebet:

$$\frac{n-1}{p} = -m + \cfrac{n - m - 1}{-m + \cfrac{n - m}{1 - m + \cfrac{n - m + 1}{2 - m + \cfrac{n - m + 2}{3 - m \text{ etc.}}}}}$$

et quarta:

$$\frac{n-1}{p} = 1 + \cfrac{n - m - 2}{m + 2 + \cfrac{n - m - 3}{m + 3 + \cfrac{n - m - 4}{m + 4 \text{ etc.}}}}$$

sicque duae nouae expressiones pro p habentur; similique modo plures aliae exhiberi possunt.

§. 21. Inuento autem valore p facile definitur haec fractio continua:

$$x = m + \cfrac{n + 1}{m + 1 + \cfrac{n + 2}{m + 2 + \cfrac{n + 3}{m + 3 + \text{ etc.}}}}$$

Scri-

Scribatur enim ibi $m-1$ loco m, ponaturque

$$q = m-1+\cfrac{n}{m+\cfrac{n+1}{m+1+\cfrac{n+2}{m+2 \text{ etc.}}}} = m-1+\frac{n}{x}$$

at ex tertia erit

$$q = m+\cfrac{n-m-1}{m+2+\cfrac{n-m-2}{m+3+\cfrac{n-m-3}{m+4 \text{ etc.}}}} = m+\frac{n-m-1}{p+1}$$

quibus binis valoribus ipsius q aequatis fit

$$-1+\frac{n}{x} = \frac{n-m-1}{p+1}, \text{ et } \frac{n}{x} = \frac{n-m+p}{p+1},$$

sicque $x = \frac{n(p+1)}{p-m+n}$.

§. 22. Cum igitur, posito $n = m+2$, fit $p = m+1$, erit

$$m+1 = p = m+\cfrac{m+2}{m+1+\cfrac{m+3}{m+2+\cfrac{m+4}{m+3 \text{ etc.}}}}$$

similique modo, ponendo $n = m+3$; $n = m+4$; etc.

$$m+1+\frac{1}{m+3} = m+\cfrac{m+3}{m+1+\cfrac{m+4}{m+2+\cfrac{m+5}{m+3 \text{ etc.}}}} = q$$

$m+2$,

$$m+1+\cfrac{2}{m+3+\cfrac{1}{m+4}} = m+\cfrac{m+4}{m+1+\cfrac{m+5}{m+2+\cfrac{m+6}{m+3 \text{ etc.}}}} = r$$

$$m+1+\cfrac{3}{m+3+\cfrac{2}{m+4+\cfrac{1}{m+5}}} = m+\cfrac{m+5}{m+1+\cfrac{m+6}{m+2+\cfrac{m+7}{m+3 \text{ etc.}}}} = s$$

$$m+1+\cfrac{4}{m+3+\cfrac{3}{m+4+\cfrac{2}{m+5+\cfrac{1}{m+6}}}} = m+\cfrac{m+6}{m+1+\cfrac{m+7}{m+3+\cfrac{m+8}{m+4 \text{ etc.}}}} = t$$

vnde, posito $m = 1$, casus rationales supra obseruati consequuntur. Hi autem valores ita progrediuntur, vt sit:

$$p = m+1;\quad q = \frac{(m+2)(p+1)}{p+2};$$
$$r = \frac{(m+3)(q+1)}{q+3};\quad s = \frac{(m+4)(r+1)}{r+4}; \text{ seu}$$
$$q = (m+2)\left(1-\frac{1}{p+2}\right);$$
$$r = (m+3)\left(1-\frac{2}{q+3}\right);$$
$$s = (m+4)\left(1-\frac{3}{r+4}\right);$$

etc.

quae expressiones etiam ita exhiberi possunt:

$$q = m+2-\frac{(m+1)}{m+3};$$
$$r = m+3-\cfrac{2(m+3)}{m+5-\cfrac{(m+1)}{m+3}};$$
$$s = m+4-\cfrac{3(m+4)}{m+7-\cfrac{2(m+3)}{m+5-\cfrac{(m+1)}{m+3}}}.$$

O 2

Facta

$$O = m+1-\cfrac{1}{m+3-\cfrac{2}{m+4-\cfrac{3}{m+5-\cfrac{4}{m+6-\cfrac{5}{m+7 \text{ etc.}}}}}}$$

Hinc autem finitum valorem ipſius O expectare non licet, cum caſu $m = 1$ certe ſit tranſcendens, qui quemadmodum ſit inueſtigandus, exponamus.

§. 25. Ex forma ergo prima formentur hae formulae:

$$O = m + \tfrac{m+1}{A};\; A = m+1+\tfrac{m+2}{A};\; B = m+2+\tfrac{m+3}{C};\; \text{etc.}$$

eritque

$$OA = mA + m + 1;\; AB = (m+1)B + m + 2;\; \text{etc.}$$

Statuatur

$$O = -1 + \tfrac{1}{\omega};\; A = -1 + \tfrac{1}{\alpha};\; B = -1 + \tfrac{1}{\beta};\; \text{etc.}$$

ac reperientur hae formulae:

$$\alpha + (m+1)\omega = 1;\; \beta + (m+2)\alpha = 1;\; \gamma + (m+3)\beta = 1;\; \text{etc.}$$

ſeu

$$\omega = \tfrac{1}{m+1} - \tfrac{\alpha}{m+1};\; \alpha = \tfrac{1}{m+2} - \tfrac{\beta}{m+2};\; \beta = \tfrac{1}{m+3} - \tfrac{\gamma}{m+3};$$

vnde per ſeriem conſuetam fit

$$\omega = \tfrac{1}{m+1} - \tfrac{1}{(m+1)(m+2)} + \tfrac{1}{(m+1)(m+2)(m+3)} - \tfrac{1}{(m+1)(m+2)(m+3)(m+4)} \text{ etc.}$$

cuius valor eſt $\omega = \frac{1}{e}\int e^z z^m dz$, integrali hoc ita ſumto, vt euaneſcat poſito $z = 0$, quo facto ſtatui debet $z = 1$. Hinc caſibus quibus m eſt numerus integer erit

ſi

-5

m+7 etc.

e non li-
qui quem-

hae for-

+3/C; etc.

+2; etc.

; etc.

)β=1; etc.

-γ/(m+3);

+3)(m+4) etc.

ſumto, vt
et z = 1.

ſi

ſi $m = 0$; $\omega = \frac{e-1}{e}$; et $O = \frac{1}{e-1}$

ſi $m = 1$; $\omega = \frac{1}{e}$; et $O = e - 1$

ſi $m = 2$; $\omega = \frac{e-2}{e}$; $O = \frac{2}{e-2}$

ſi $m = 3$; $\omega = \frac{-2e+6}{e}$, $O = \frac{3e-6}{6-2e} = \frac{3(e-2)}{2(3-e)}$

ſi $m = 4$; $\omega = \frac{9e-24}{e}$; $O = \frac{24-8e}{9e-24} = \frac{4(6-2e)}{3(3e-8)} = \frac{8(3-e)}{3(3e-8)}$

ſi $m = 5$; $\omega = \frac{120-44e}{e}$; $O = \frac{45e-120}{120-44e} = \frac{5(9e-24)}{4(30-11e)} = \frac{15(3e-8)}{4(30-11e)}$

ſi $m = 6$; $\omega = \frac{265e-720}{e}$, $O = \frac{720-264e}{265e-720} = \frac{6(120-44e)}{5(53e-144)} = \frac{24(30-11e)}{5(53e-144)}$

ſi $m = 7$; $\omega = \frac{5040-1854e}{e}$; $O = \frac{1855e-5040}{5040-1854e} = \frac{7(265e-720)}{6(720-309e)} = \frac{35(53e-144)}{6(720-309e)}$

Niſi m eſt numerus integer, valor ipſius O per numerum e, cuius logarithmus $= 1$, exprimi nequit.

§. 26. Ponamus $m = 0$, quorſum omnes caſus, quibus m eſt numerus integer, referri poſſunt, eritque

$$O = \tfrac{1}{e-1};\; N = 0;\; M = -1;\; L = \tfrac{-2+2M}{M+2} = -4;$$
$$K = \tfrac{-3+3L}{L+3} = 15;\; I = \tfrac{-4+4K}{K+4} = \tfrac{56}{19};$$
$$H = \tfrac{-5+5I}{I+5} = \tfrac{185}{151};\; G = \tfrac{-6+6H}{H+6} = \tfrac{204}{1091};$$

vnde ſequentes oriuntur fractiones continuae:

$$1 = 0 + \cfrac{2}{1 + \cfrac{3}{2 + \cfrac{4}{3 + \text{etc.}}}}$$

$$\frac{1}{e-1} = 0 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \text{etc.}}}}$$

[illegible]

Facta autem euolutione inuenitur:

$$p = m + 1;\quad q = \frac{(m+2)(m+2)}{m+3};$$
$$r = \frac{(m+3)(mm+5m+7)}{mm+7m+13};$$
$$s = \frac{(m+4)(m^3+9m^2+29m+34)}{m^3+12mm+50m+73};$$
$$t = \frac{(m+5)(m^4+14m^3+77m^2+200m+209)}{m^4+18m^3+125m^2+400m+501}.$$

§. 23. Denotent $p, q, r, s, t,$ etc. fractiones continuas in §. praecedente exhibitas, sitque

$$p = (m+1)\tfrac{a}{\alpha};\quad q = (m+2)\tfrac{b}{\beta};\quad r = (m+3)\tfrac{c}{\gamma};\ \text{etc.}$$

erit $a = 1$, $\alpha = 1$; reliquae vero litterae ita a se inuicem pendent, vt sit:

$b = (m+1)a + \alpha;$	$\beta = (m+1)a + 2\alpha;$
$c = (m+2)b + \beta;$	$\gamma = (m+2)b + 3\beta;$
$d = (m+3)c + \gamma;$	$\delta = (m+3)c + 4\gamma;$
$e = (m+4)d + \delta;$	$\varepsilon = (m+4)d + 5\delta;$
etc.	etc.

vnde relatio inter litteras latinas et graecas seorsim colligitur:

$b = (m+2)a;$	$\beta = (m+3)\alpha;$
$c = (m+4)b - 1(m+1)a;$	$\gamma = (m+5)\beta - 1(m+2)\alpha;$
$d = (m+6)c - 2(m+2)b;$	$\delta = (m+7)\gamma - 2(m+3)\beta;$
$e = (m+8)d - 3(m+3)c;$	$\varepsilon = (m+9)\delta - 3(m+4)\gamma;$
etc.	etc.

Inuentis autem denominatoribus $\alpha, \beta, \gamma, \delta,$ etc. erunt numeratores:

$$b = \beta - \alpha,\ c = \gamma - 2\beta,\ d = \delta - 3\gamma,\ e = \varepsilon - 4\delta,\ \text{etc.}$$

At si numeratores $a, b, c, d, e,$ iam sint inuenti, erunt denominatores:

$$\alpha = a;\ \beta = 2b - (m+1)a;$$
$$\gamma = 3c - 2(m+2)b;\ \delta = 4d - 3(m+3)c;$$
$$\varepsilon = 5e - 4(m+4)d;\ \text{etc.}$$

Vidi-

es con- etc. inuicem colli- $-2)a;$ $-3)\beta;$ $4)\gamma;$ erunt etc. erunt $c;$ Vidi-

Vidimus autem esse:

$a = 1$	$\alpha = 1$
$b = m+2$	$\beta = m+3$
$c = mm+5m+7$	$\gamma = m^2+7m+13$
$d = m^3+9m^2+29m+34$	$\delta = m^3+12m^2+50m+73$
$e = m^4+14m^3+77m^2+200m+209$	$\varepsilon = m^4+18m^3+120m^2+400m+501.$

§. 24. Ex valore cuiusque fractionis continuae §. 22. definiri quoque potest valor praecedentis, hoc modo:

$$p = \frac{m+2-2q}{q-(m+2)};\quad q = \frac{m+3-3r}{r-(m+3)};\quad r = \frac{m+4-4s}{s-(m+4)};\ \text{etc.}$$

vnde si fractiones continuae ordine praecedentes designentur litteris O, N, M, etc. erit

$$O = \frac{m+1-p}{p-(m+1)};\ N = \frac{m}{O-m};\ M = \frac{m-1+N}{N-(m-1)};\ L = \frac{m-2+2M}{M-(m-2)};\ \text{etc.}$$

At est

$$O = m + \cfrac{m+1}{m+1+\cfrac{m+2}{m+2+\cfrac{m+3}{m+3+\cfrac{m+4}{m+4+\text{etc.}}}}}$$

cuius reliqui valores aequiualentes sunt

$$O = \cfrac{m}{1-m+\cfrac{m-1}{2-m+\cfrac{m-2}{3-m+\cfrac{m-3}{4-m+\text{etc.}}}}}$$

O 3 O =

merum inte-

§. 7. Continuatis autem quadratis ad $(\frac{p-1}{2})^2$ re-

$$\frac{6}{7} = 0 + \cfrac{3}{1 + \cfrac{4}{2 + \cfrac{5}{3 + \text{etc.}}}}$$

$$0 = 0 + \cfrac{0}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \text{etc.}}}}$$

$$\frac{81}{77} = 0 + \cfrac{4}{2 + \cfrac{5}{2 + \cfrac{6}{3 + \text{etc.}}}}$$

$$-1 = 0 - \cfrac{1}{1 + \cfrac{0}{2 + \cfrac{1}{3 + \text{etc.}}}}$$

$$\frac{216}{77} = 0 + \cfrac{5}{1 + \cfrac{6}{2 + \cfrac{7}{3 + \text{etc.}}}}$$

$$-4 = 0 - \cfrac{2}{1 - \cfrac{1}{2 + \cfrac{0}{3 + \text{etc.}}}}$$

$$\frac{2045}{351} = 0 + \cfrac{6}{1 + \cfrac{7}{2 + \cfrac{8}{3 + \text{etc.}}}}$$

$$15 = 0 - \cfrac{3}{1 - \cfrac{2}{2 - \cfrac{1}{3 - \text{etc.}}}}$$

qui-

quibus adiungi debent istae:

$$\frac{1}{e-1} = 0 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \text{etc.}}}}$$

$$\frac{1}{e-2} = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \text{etc.}}}}$$

$$\frac{1}{2e-5} = 2 + \cfrac{1}{3 + \cfrac{2}{4 + \cfrac{3}{5 + \text{etc.}}}}$$

$$\frac{1}{6e-16} = 3 + \cfrac{1}{4 + \cfrac{2}{5 + \cfrac{3}{6 + \text{etc.}}}}$$

$$\frac{1}{24e-65} = 4 + \cfrac{1}{5 + \cfrac{2}{6 + \cfrac{3}{7 + \text{etc.}}}}$$

$$\frac{1}{120e-326} = 5 + \cfrac{1}{6 + \cfrac{2}{7 + \cfrac{3}{8 + \text{etc.}}}}$$

Si enim fit

qui-

$$x = m + \cfrac{1}{m + 1 + \cfrac{2}{m + 2 + \cfrac{3}{m + 3 + \text{etc.}}}}$$

$$= m + 1 - \cfrac{m - 1}{m + 3 - \cfrac{m - 2}{m + 4 - \cfrac{m - 3}{m + 5 - \text{etc.}}}}$$

$$y = m + 1 + \cfrac{1}{m + 2 + \cfrac{2}{m + 3 + \cfrac{3}{m + 4 + \text{etc.}}}}$$

$$= m + 2 - \cfrac{m - 2}{m + 4 - \cfrac{m - 3}{m + 5 - \cfrac{m - 4}{m + 6 \text{ etc.}}}}$$

erit $y = \frac{n}{m + 1 - x}$.

§. 37. Verum nostrae inuestigationes multo latius patent, quas vt accuratius euoluamus, ad formulas §. 11. reuertamur, quae nihil de sua amplitudine amittunt, etiamsi numeros m et n nihilo aequales statuamus. Quare fractiones continuae considerandae erunt hae:

$$p = C + \cfrac{F}{C + B + F + E + \cfrac{D}{C + 2B + F + 2E + \cfrac{4D}{C + 3B + F + 3E + \cfrac{9D}{C + 4B + \text{etc.}}}}}$$

$q = C$

— etc.
etc.
4 + etc.
i + 6 etc.
lto latius
las §. 11.
tunt, et-
Quare
q = C

$$q = C + A + \cfrac{F}{C + A + B + F + E + \cfrac{D}{C + A + 2B + F + 2F + \cfrac{4D}{C + A + 3B + F + 3E + \cfrac{9D}{C + A + 4B + \text{etc.}}}}}$$

$$r = C + 2A + \cfrac{F}{C + 2A + B + F + E + \cfrac{D}{C + 2A + 2B + F + 2E + \cfrac{4D}{C + 2A + 3B + F + 3E + \cfrac{9D}{C + 2A + 4B \text{ etc.}}}}}$$

quae formae continuo vlterius continuantur, scribendo $C + A$ loco C. In singulis ergo denominatores progressionem arithmeticam, numeratores vero progressionem secundi ordinis constituunt, cuius differentiae secundae sunt constantes. Hic autem assumimus esse $D = \frac{1}{4}(AA - BB)$. Quodsi iam breuitatis gratia ponamus:

$$G = \frac{BB - AA + 2BC + 4E}{2A} = \frac{BC - 2D + 2E}{A} \text{ et}$$

$$\lambda = \tfrac{1}{4}(AA - BB) = D$$

$$\mu = \tfrac{1}{4}(AA - BB) + \tfrac{1}{2}(AC - BG) = \frac{(A + B + 2C)D - BE}{A}$$

$$\nu = F + \tfrac{1}{4}(C - G)(A + B + C + G), \text{ siue}$$

$$\nu = F + \frac{CD(A + C) + BD - EE}{AA} + \frac{B(A + B + 2C)(D - E)}{2AA}$$

erit

$$\left(p - \frac{(A + E)(C + 2D - 2E)}{2A}\right)\left(q - \frac{(A - B)(A + C) - 2CD - E)}{2A}\right) = \nu$$

hincque per aliam fractionem continuam

 $p =$

$$p=\tfrac{1}{2}(C+G)+\cfrac{\nu}{G+B+\nu+\mu+\lambda\cfrac{}{G+2B+\nu+2\mu+4\lambda\cfrac{}{G+3B+\text{etc.}}}}$$

§. 28. Propofita ergo fractione continua quacunque huius formae:

$$p=a+\cfrac{f}{a+b+f+g\cfrac{}{a+2b+f+2g+b\cfrac{}{a+3b+f+3g+3b\cfrac{}{a+4b+f+4g+6b\cfrac{}{a+5b+\text{etc.}}}}}}$$

ea in aliam fibi aequalem transmutari poteft. Comparatione enim facta eft.

$$C=a;\ B=b;\ F=f;\ E=g-\tfrac{1}{2}b;\ D=\tfrac{1}{2}b.$$

Capiatur ergo $A=\surd(bb+2b)$, tum vero

$$G=\tfrac{ab+2g-2b}{A};\ \lambda=\tfrac{1}{2}b;$$
$$\mu=\tfrac{1}{2}b+\tfrac{1}{2}(Aa-Gb)=\tfrac{1}{2}b+\tfrac{ab-b(g-b)}{A}$$
$$\nu=f+\tfrac{ab-b(g-b)}{2A}+\tfrac{aab-bb(g-b)-2ab(g-b)-2g(g-b)}{2AA}$$

hincque fiet

$$p=\tfrac{1}{2}(a+G)+\cfrac{\nu}{G+b+\nu+\mu+\lambda\cfrac{}{G+2b+\nu+2\mu+4\lambda\cfrac{}{G+3b+\nu+3\mu+9\lambda\cfrac{}{G+4b+\text{etc.}}}}}$$

§. 29. Si ergo fuerit $f=0$, huius poftremae fractionis continuae valor certe eft $=a$, quicunque numeri reli-

rel
ve
eri tinua quacun-
$+4g+6b$
vt $+5b+\text{etc.}$
Compara-
hi $D=\tfrac{1}{2}b.$
P. $-b)-2g(g-b)$
a
$3\mu+9\lambda$
$4b+$ etc.
c poftremae fra-
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reliquis litteris tribuantur. Statuatur ergo $g-b=k$, vt fit

$$A=\surd(bb+2b);\ G=\tfrac{ab+2k}{A};\ \lambda=\tfrac{1}{2}b;\ \mu=\tfrac{1}{2}b+\tfrac{ab-bk}{A}\ \text{et}$$
$$\nu=\tfrac{ab-bk}{2A}+\tfrac{aab-bbk-2abk-2bk-2kk}{2AA},$$

eritque:

$$\tfrac{1}{2}(a+G)=G+\cfrac{\nu}{G+b+\nu+\mu+\lambda\cfrac{}{G+2b+\nu+2\mu+4\lambda\cfrac{}{G+3b+\nu+3\mu+9\lambda\cfrac{}{G+4b+\text{etc.}}}}}$$

vbi fi litterae a, b, A et G pro datis habeantur, erit

$$\lambda=\tfrac{AA-bb}{4};\ \mu=\lambda+\tfrac{Aa-Gb}{2}=\tfrac{AA-bb}{4}+\tfrac{Aa-Gb}{2},$$
$$\nu=\tfrac{aa+ab+Aa+Gb-AG-GG}{4}=\tfrac{1}{4}(a-G)(a+b+A+G)$$

hinc

$$\nu+\mu+\lambda=\tfrac{1}{4}(a-b+A-G)(a+2b+2A+G),$$
$$\nu+2\mu+4\lambda=\tfrac{1}{4}(a-2b+2A-G)(a+3b+3A+G),$$
$$\nu+3\mu+9\lambda=\tfrac{1}{4}(a-3b+3A-G)(a+4b+4A+G).$$

Ponatur ad formulam contrahendam:

$$a-G=2\alpha;\ A-b=2\gamma;\ a+G=2\beta;\ A+b=2\delta;\ \text{etc.}$$

$$a=\cfrac{\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+4(\delta-\gamma)\text{etc.}}}}}$$

cuius veritas in pluribus exemplis fponte elucet.

 §. 30.

§. 30. Si eaedem positiones retineantur, numerus autem f non nihilo aequalis capiatur, habebitur haec fractio continua:

$$p=\alpha+\beta+\cfrac{f}{\alpha+\beta+(\delta-\gamma)+\cfrac{f+(\beta\gamma-\alpha\delta)+2\gamma\delta}{\alpha+\beta+2(\delta-\gamma)+\cfrac{f+2(\beta\gamma-\alpha\delta)+6\gamma\delta}{\alpha+\beta+3(\delta-\gamma)+\cfrac{f+3(\beta\gamma-\alpha\delta)+12\gamma\delta}{\alpha+\beta+4(\delta-\gamma)+\text{etc.}}}}}$$

quae transformatur in hanc sibi aequalem:

$$p=\beta+\cfrac{f+\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+4(\delta-\gamma)+\text{etc.}}}}}$$

vnde si vel γ vel δ euanescens capiatur, casus ante tractatus exsurgit. Haec autem binarum fractionum continuarum aequalitas omnia, quae hactenus sunt exposita, in se complectitur.

§. 31. Ex his oriuntur formae, quas littera q denotauimus, si loco α et β scribamus $\alpha+\gamma$ et $\beta+\delta$, ita vt sit

$$q=\alpha+\beta+\gamma+\delta+\cfrac{f}{\alpha+\beta+2\delta+\cfrac{f+(\beta\gamma-\alpha\delta)+2\gamma\delta}{\alpha+\beta-\gamma+3\delta+\cfrac{f+2(\beta\gamma-\alpha\delta)+6\gamma\delta}{\alpha+\beta-2\gamma+4\delta+\cfrac{f+3(\beta\gamma-\alpha\delta)+12\gamma\delta}{\alpha+\beta-3\gamma+5\delta+\text{etc.}}}}}$$

itemque

$q=$

$$q=\beta+\delta+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+4(\delta-\gamma)+\text{etc.}}}}$$

ita vt inter has binas expressiones subsistat haec relatio:

$$(p-\beta)(q-\alpha-\gamma)=f+\alpha(\beta+\delta),\ \text{seu}$$
$$pq-(\alpha+\gamma)p-\beta q+\beta\gamma-\alpha\delta=f$$

cuius ope aequalitas binarum superiorum formularum methodo substitutionum, qua supra vsi sumus, demonstrari potest.

§. 32. Si ponamus $f+\alpha(\beta+\delta)=g$, vt sit $f=g-\alpha(\beta+\delta)$, prior forma ita se habebit:

$$p=\alpha+\beta+\cfrac{g-(\beta+\delta)}{\alpha+\beta+(\delta-\gamma)+\cfrac{g-(\alpha-\gamma)(\beta+2\delta)}{\alpha+\beta+2(\delta-\gamma)+\cfrac{g-(\alpha-2\gamma)(\beta+3\delta)}{\alpha+\beta+3(\delta-\gamma)+\cfrac{g-(\alpha-3\gamma)(\beta+4\delta)}{\alpha+\beta+4(\delta-\gamma)+\text{etc.}}}}}$$

cui aequalis est ista:

$$p=\beta+\cfrac{f+\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+4(\delta-\gamma)+\text{etc.}}}}}$$

Atque hae formae maxime idoneae videntur, quarum aequalitas methodo directa explorari ac demonstrari possit. Talis autem methodus etiamnunc desideratur. Nullum autem

tem eſt dubium, quin ea patefacta multa praeclara incrementa Analyſeos expectare liceat. Cum igitur prior forma finita euadat, ſi fuerit $g = (\alpha - i\gamma)(\beta + (i+1)\delta)$, intelligimus etiam poſterioris valorem rationaliter exprimi poſſe, quoties fuerit

$$f = (\alpha - i\gamma)(\beta + (i+1)\delta) - \alpha(\beta + \delta)$$

ſeu

$$f = i(\alpha\delta - \beta\gamma - (i+1)\gamma\delta),$$

denotante i numerum integrum quemcunque.

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DISQVISITIO ACCVRATIOR CIRCA RESIDVA EX DIVISIONE QVADRATORVM ALTIORVMQVE POTESTATVM PER NVMEROS PRIMOS RELICTA.

§. 1.

Si numerus quadratus aa per numerum primum p diuidatur, reſiduum relictum littera α indicetur; ſimilique modo litterae β, γ, δ, etc. mihi denotabunt reſidua in diuiſione quadratorum bb, cc, dd, etc. relicta.

§. 2. Erit ergo $\alpha = aa - np$, quia reſiduum α prodit, ſi a quadrato aa multiplum numeri p auferatur, idque maximum, vt reſiduum α ipſo diuiſore p minus reddatur. Nihil autem impedit, quominus multiplum np maius accipiatur quadrato aa, vnde reſiduum α prodit negatiuum, ſicque eius valor infra $\frac{1}{2}p$ deprimi poteſt.

§. 3. Idem igitur reſiduum α multis modis exhiberi poteſt, quoniam cunctae hae formae $\alpha \pm mp$ eandem naturam continent. Perinde ſcilicet eſt, ſiue reſiduum ex diuiſione quadrati aa per numerum p ortum dicatur eſſe α, ſiue tem est dubium, quin ea patefacta multa praeclara incrementa Analyseos expectare liceat. Cum igitur prior forma finita euadat, si fuerit $g = (\alpha - i\gamma)(\beta + (i+1)\delta)$, intelligimus etiam posterioris valorem rationaliter exprimi posse, quoties fuerit

$$f = (\alpha - i\gamma)(\beta + (i+1)\delta) - \alpha(\beta + \delta)$$

seu

$$f = i(\alpha\delta - \beta\gamma - (i+1)\gamma\delta),$$

denotante i numerum integrum quemcunque.

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DISQVISITIO ACCVRATIOR CIRCA RESIDVA EX DIVISIONE QVADRATORVM ALTIORVMQVE POTESTATVM PER NVMEROS PRIMOS RELICTA.

§. 1.

Si numerus quadratus aa per numerum primum p diuidatur, residuum relictum littera α indicetur; similique modo litterae β, γ, δ, etc. mihi denotabunt residua in diuisione quadratorum bb, cc, dd, etc. relicta.

§. 2. Erit ergo $\alpha = aa - np$, quia residuum α prodit, si a quadrato aa multiplum numeri p auferatur, idque maximum, vt residuum α ipso diuisore p minus reddatur. Nihil autem impedit, quominus multiplum np maius accipiatur quadrato aa, vnde residuum α prodit negatiuum, sicque eius valor infra $\frac{1}{2}p$ deprimi potest.

§. 3. Idem igitur residuum α multis modis exhiberi potest, quoniam cunctae hae formae $\alpha \pm mp$ eandem naturam continent. Perinde scilicet est, siue residuum ex diuisione quadrati aa per numerum p ortum dicatur esse α, siue

 $\alpha \pm p$