



1783

Evolutio producti infiniti

$(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. in
seriem simplicem

Leonhard Euler

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Record Created:

2018-09-25

Recommended Citation

Euler, Leonhard, "Evolutio producti infiniti $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. in seriem simplicem" (1783). *Euler Archive - All Works*. 541.

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EVOLVTIO
 PRODVCTI INFINITI

$$(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$$

IN SERIEM SIMPLICEM.

Auctore
 L. E V L E R O.

§. 1.

Posito $s = (1-x)(1-xx)(1-x^3)(1-x^4)$ etc. facile patet fore:

$s = 1 - x - xx(1-x) - x^3(1-x)(1-xx) - x^4(1-x)(1-xx)(1-x^3) -$ etc.
 quae series cum iam sit infinita, quaeritur, si singuli eius termini euoluantur, qualis series secundum simplices potestates ipsius x sit proditura. Cum igitur duo primi termini $1-x$ iam sint euoluti, loco reliquorum omnium scribatur littera A , ita vt sit $s = 1 - x - A$, ideoque

$$A = xx(1-x) + x^3(1-x)(1-xx) + x^4(1-x)(1-xx)(1-x^3) \text{ etc.}$$

§. 2. Quoniam hi termini omnes factorem habent communem $1-x$, eo euoluto singuli termini discerpentur in binas partes quas ita repraesentemus:

$$A = xx + x^3(1-xx) + x^4(1-xx)(1-x^3) + x^5(1-xx)(1-x^3)(1-x^4) - x^3 - x^4(1-xx) - x^5(1-xx)(1-x^3) - x^6(1-xx)(1-x^3)(1-x^4)$$

Hinc

Hinc iam binæ partes eadem potestate ipsius x affectæ in vnam contrahantur, ac resultabit pro A sequens forma:

$$A = xx - x^3 - x^7(1 - xx) - x^9(1 - xx)(1 - x^2) - x^{11}(1 - x^2)(1 - x^2)(1 - x^4) - \text{etc.}$$

vbi duo termini primi $xx - x^3$ iam sunt euoluti; sequentes autem procedunt per has potestates: $x^7, x^9, x^{11}, x^{13}, x^{15}$ quarum exponentes binario crescunt.

§. 3. Ponamus nunc simili modo vt ante

$$A = xx - x^3 - B, \text{ ita vt fit}$$

$$B = +x^7(1 - xx) + x^9(1 - xx)(1 - x^2) + x^{11}(1 - xx)(1 - x^2)(1 - x^4) + \text{etc.}$$

cuius omnes termini habent factorem communem $1 - xx$, quo euoluto singuli termini in binas partes discerpantur, vti sequitur:

$$B = x^7 + x^9(1 - x^2) + x^{11}(1 - x^2)(1 - x^4) + x^{13}(1 - x^2)(1 - x^4)(1 - x^5) \text{ etc.} \\ - x^9 - x^{11}(1 - x^2) - x^{13}(1 - x^2)(1 - x^4) - x^{15}(1 - x^2)(1 - x^4)(1 - x^5) \text{ etc.}$$

Hic iterum bini termini, qui eandem potestatem ipsius x habent præfixam, in vnam colligantur et prodibit:

$$B = x^7 - x^{12} - x^{15}(1 - x^2) - x^{18}(1 - x^2)(1 - x^4) - x^{21}(1 - x^2)(1 - x^4)(1 - x^5) - \text{etc.}$$

vbi iam potestates ipsius x crescunt ternario.

§. 4. Statuatur nunc porro $B = x^7 - x^{12} - C$, ita

vt fit

$$C = x^{15}(1 - x^2) + x^{18}(1 - x^2)(1 - x^4) + x^{21}(1 - x^2)(1 - x^4)(1 - x^5) + \text{etc.}$$

et iam singuli termini per euolutionem factoris $1 - x^2$ in binas partes resoluantur, fietque:

C =

$$C = x^{15} + x^{15}(1-x^4) + x^{21}(1-x^4)(1-x^5) + x^{24}(1-x^4)(1-x^5)(1-x^6) - x^{18} - x^{27}(1-x^4) - x^{24}(1-x^4)(1-x^5) - x^{27}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

vbi denuo membra, quibus eadem potestas ipsius x praefixa, in vnum contracta praebent

$$C = x^{15} - x^{12} - x^{26}(1-x^4) - x^{30}(1-x^4)(1-x^5) - x^{24}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

vbi potestates praefixae quaternario crescunt.

§. 5. Statuatur $C = x^{15} - x^{22} - D$, vt fit

$$D = x^{26}(1-x^4) + x^{30}(1-x^4)(1-x^5) + x^{24}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

qui termini per euolutionem factoris $1-x^4$ in binos discerpantur hoc modo:

$$D = x^{26} + x^{30}(1-x^5) + x^{34}(1-x^5)(1-x^6) + x^{38}(1-x^5)(1-x^6)(1-x^7) - x^{20} - x^{24}(1-x^5) - x^{28}(1-x^5)(1-x^6) - x^{42}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

Nunc binis vt haecenus contrahendis colligitur

$$D = x^{26} - x^{35} - x^{40}(1-x^5) - x^{45}(1-x^5)(1-x^6) - x^{50}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

Hic igitur potestates ipsius x quinario crescunt.

§. 6. Statuatur $D = x^{26} - x^{55} - E$, ita vt fit

$$E = x^{40}(1-x^5) + x^{45}(1-x^5)(1-x^6) + x^{50}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

ac resolutione in binas partes vt haecenus instituta prodit

$$E = x^{40} + x^{45}(1-x^6) + x^{50}(1-x^6)(1-x^7) + x^{55}(1-x^6)(1-x^7)(1-x^8) - x^{45} - x^{50}(1-x^6) - x^{55}(1-x^6)(1-x^7) - x^{60}(1-x^6)(1-x^7)(1-x^8) \text{ etc.}$$

Contractis vero binis terminis in vnum prodibit

$$E = x^{10} - x^{51} - x^{57} (1 - x^6) - x^{63} (1 - x^6) (1 - x^7) - x^{69} (1 - x^6) (1 - x^7) (1 - x^8) \text{ etc.}$$

vbi potestates ipsius x senario crescunt.

§. 7. Cum lex, qua istae operationes ulterius sunt continuandae satis sit perspicua, si postremi valores pro singulis litteris A, B, C, D, inuenti ordine substituantur, pro serie quaesita reperiemus sequentem formam:

$$S = 1 - x, -xx + x^5, +x^7 - x^{12}, -x^{15} + x^{20}, +x^{26} - x^{35}, -x^{40} + x^{51}, \text{ etc.}$$

Hic igitur tota quaestio huc reducitur, vt ordo definiatur, quo exponentes potestatum ipsius x continuo ulterius au-
gentur, quandoquidem ex operationibus institutis iam satis est manifestum signa terminorum $+$ et $-$ ita alternatim se excipere, vt ambo gementur.

§. 8. Quo igitur in hanc legem inquiramus, videamus quomodo in singulis litteris isti numeri sint orti. Hunc in finem primos saltem cuiusque litterae terminos in eius forma prima exhibitos ordine disponamus

$A = xx(1-x)$	$7 = 3 + 4 = 3 + 1 + 3 = 3 + 1 + 1 + 2$
$B = x^7(1-xx)$	$15 = 4 + 11 = 4 + 2 + 9 = 4 + 2 + 2 + 7$
$C = x^{15}(1-x^5)$	$26 = 5 + 21 = 5 + 3 + 18 = 5 + 3 + 3 + 15$
$D = x^{26}(1-x^7)$	$40 = 6 + 34 = 6 + 4 + 30 = 6 + 4 + 4 + 26$
$E = x^{40}(1-x^8)$	$57 = 7 + 50 = 7 + 5 + 45 = 7 + 5 + 5 + 40$
etc.	etc.

Hic scilicet ex euolutione litterae A vidimus, numerum 7 oriri ex aggregato $3 + 4$, tum vero 4 oriri ex $1 + 3$, ac denique 3 ex $1 + 2$, quae ergo resolutio dabit

$$7 = 3 + 4 = 3 + 1 + 3 = 3 + 1 + 1 + 2.$$

At

Atque idem ordo in sequentibus litteris est observatus, vbi ultimi numeri procedunt ordine 2, 7, 15, 26, 40.

§. 9. Ex his iam manifestum est, numerorum 2, 7, 15, 26, 40, 57, etc. differentias progressionem arithmeticam constituere, vnde horum numerorum terminus generalis erit:

$$2 + 5(n-1) + \frac{2(n-1)(n-2)}{1,2} = \frac{3nn+n}{2}.$$

Exponentes autem, qui hos antecedunt, erant 1, 5, 12, 22, 35, 51 ab illis numeris 1, 2, 3, 4, 5, et in genere ipso numero n , ita vt exponents, qui formulam $\frac{3nn+n}{2}$ praecedit, futurus sit $\frac{3nn-n}{2}$.

§. 10. Nunc igitur seriem simplicem inuentam quae aequalis est producto infinito proposito

$$(1-x)(1-xx)(1-x^3)(1-x^4) \text{ etc.}$$

perfecte cognoscimus. Cum enim haec series inuenta sit:

$$s = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.}$$

certi nunc sumus, in ea alias potestates ipsius x non occurrere, nisi quarum exponentes contineantur in hac formula generali: $\frac{3nn \pm n}{2}$, et quidem ita, vt si n fuerit numerus impar, bini termini inde nati habituri sint signum $-$, qui autem ex paribus oriuntur signum $+$.

Alia inuestigatio eiusdem seriei.

§. 11. Eadem series secundum potestates ipsius x procedens etiam sequenti modo inuestigari potest. Cum scilicet sit

G 2

s =

$$s = 1 - x - xx(1-x) - x^3(1-x)(1-xx) - x^4(1-x)(1-x^2)(1-x^3) \text{ etc.}$$

evoluatur statim secundum membrum $-xx(1-x)$, vt fiat

$$s = 1 - x - xx + x^3 - x^3(1-x)(1-xx) - x^4(1-x)(1-xx)(1-x^3) \text{ etc.}$$

ac statuatur $s = 1 - x - xx + A$ vt fit

$$A = x^3 - x^3(1-x)(1-xx) - x^4(1-x)(1-xx)(1-x^3) \text{ etc.}$$

cuius singula membra per evolutionem factoris $1-x$ in duas partes discerpantur, vt prodeat

$$A = x^3 - x^3(1-xx) - x^4(1-xx)(1-x^3) - x^6(1-x^2)(1-x^3)(1-x^4) + x^4(1-xx) + x^5(1-xx)(1-x^2) + x^6(1-x^2)(1-x^3)(1-x^4)$$

Hic iterum bina membra eadem potestate ipsius x affecta contracta praebebunt:

$$A = +x^3 + x^7(1-xx) + x^9(1-xx)(1-x^2) + x^{11}(1-x^2)(1-x^3)(1-x^4) \text{ etc.}$$

§. 12. Hic nunc iterum secundum membrum evoluatur, vt prodeat:

$$A = x^3 + x^7 - x^9 + x^9(1-xx)(1-x^3) + x^{11}(1-x^2)(1-x^3)(1-x^4) \text{ etc.}$$

Iam ponatur $A = x^5 + x^7 - B$, vt fit

$$B = x^9 - x^9(1-xx)(1-x^3) - x^{11}(1-x^2)(1-x^3)(1-x^4) \text{ etc.}$$

quare si vbique factor $1-x$ evoluatur, obtinebitur

$$B = x^9 - x^9(1-x^3) - x^{11}(1-x^3)(1-x^4) - x^{13}(1-x^3)(1-x^4)(1-x^5) + x^{11}(1-x^3) + x^{13}(1-x^3)(1-x^4) + x^{15}(1-x^3)(1-x^4)(1-x^5)$$

tum vero contrahendis binis membris orietur

$$B = x^{12} + x^{15}(1-x^3) + x^{18}(1-x^3)(1-x^4) + x^{21}(1-x^3)(1-x^4)(1-x^5) \text{ etc.}$$

§. 13. Evoluatur pariter secundum membrum ac statuatur $B = x^{12} + x^{15} - C$, eritque

$$C = x^{18} - x^{18}(1-x^3)(1-x^4) - x^{21}(1-x^3)(1-x^4)(1-x^5) - \text{etc.}$$

Nunc termini evoluantur secundum factorem $1 - x^3$, fietque

$$C = x^{18} - x^{18}(1-x^4) - x^{21}(1-x^4)(1-x^5) - x^{24}(1-x^4)(1-x^5)(1-x^6) \\ + x^{27}(1-x^4) + x^{30}(1-x^4)(1-x^5) + x^{33}(1-x^4)(1-x^5)(1-x^6)$$

Hinc binis membris contrahendis fiet

$$C = x^{22} + x^{26}(1-x^4) + x^{30}(1-x^4)(1-x^5) \\ + x^{34}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

§. 14. Evoluto nunc hic iterum secundo membro statuatur $C = x^{22} + x^{26} - D$, eritque

$$D = x^{30} - x^{30}(1-x^4)(1-x^5) - x^{34}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

vbi evolutio factoris $1 - x^4$ producet

$$D = x^{30} - x^{30}(1-x^5) - x^{34}(1-x^5)(1-x^6) - x^{38}(1-x^5)(1-x^6)(1-x^7) \\ + x^{34}(1-x^5) + x^{38}(1-x^5)(1-x^6) + x^{42}(1-x^5)(1-x^6)(1-x^7)$$

Hinc binis membris contractis fiet

$$D = x^{35} + x^{40}(1-x^5)(1-x^6) \text{ etc.}$$

§. 15. Evoluto secundo membro statuatur denuo

$$D = x^{35} + x^{40} - E, \text{ eritque}$$

$$E = x^{45} - x^{45}(1-x^5)(1-x^6) - x^{50}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

et evoluto factore secundo $1 - x^5$ fiet

$$E = x^{45} - x^{45}(1-x^6) - x^{50}(1-x^6)(1-x^7) - x^{55}(1-x^6)(1-x^7)(1-x^8) \\ + x^{50}(1-x^6) + x^{55}(1-x^6)(1-x^7) + x^{60}(1-x^6)(1-x^7)(1-x^8) \text{ etc.}$$

binisque terminis collectis elicitur

$$E = x^{51} + x^{57}(1-x^6) + x^{63}(1-x^6)(1-x^7) + x^{69}(1-x^6)(1-x^7)(1-x^8) \text{ etc.}$$

§. 16. Inuentis igitur his valoribus litterarum A, B, C, D, E, si singuli successiue substituantur, resultabit ista series:

$$1 - x - xx, + x^5 + x^7, - x^{12} - x^{15}, + x^{22} + x^{26}, - x^{35} - x^{40}, + \text{ etc.}$$

Hic autem ordo exponentium facilius perspicitur. Cum enim in valoribus litterarum A, B, C, D, primo constitutis primi termini simplices essent $x^5, x^9, x^{14}, x^{20}, x^{27}$, exponentes manifesto sunt numeri trigonales triplicati, vnde generatim pro numero n erit iste exponents $\frac{3n^2 + 3n}{2}$. Verum (hi termini sequuntur binas potestates ipsius x procedentes per eandem differentiam n , vnde numerum n ab hac formula bis subtrahendo orientur binæ potestates insequenter erunt n et $\frac{3n^2 - n}{2}$.

§. 17. Hinc igitur vicissim patet

$$s = 1 - x - xx + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - \text{ etc.}$$

in infinitum continuatam habere infinitos factores, qui scilicet erunt $(1-x), 1-xx, 1-x^5, 1-x^7, 1-x^{12}, \text{ etc.}$ ita vt si primo diuidatur per $1-x$, tum vero quotus per $1-xx$, iste quotus porro per $1-x^5$, hocque modo in infinitum diuisio continuetur, vltimum quotum resultantem vnitati aequalem esse oportebit.

§. 18. Quod si ergo, proposita fuerit ista aequatio in infinitum excurrentis:

$$1 - x - xx + x^3 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.} = 0,$$

eius omnes radices facile assignari possunt. Primum enim radix erit $x = 1$, deinde binae radices quadratae ex unitate, tum vero ternae radices cubicae ex unitate, porro quaternae radices biquadratae ex unitate, similique modo quinae radices potestatis quintae ex unitate, et ita porro, inter quas igitur ipsa unitas infinities occurrit; at vero -1 ibi reperietur, ubi radix potestatis parisi est extrahenda.