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# Nova methodus fractiones quascumque rationales in fractiones simplices resolvendi

Leonhard Euler

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NOVA METHODVS  
 FRACTIONES QVASCVNQVE RATIONALES  
 IN FRACTIONES SIMPLICES  
 RESOLVENDI.

Auctore  
 L. EYLERO.

§. I.

Sit  $\frac{P}{Q}$  fractio quaecunque proposita, cuius tam numera-  
 tor  $P$  quam denominator  $Q$  sint functiones rationales  
 integrae quantitatis variabilis  $z$ , denominator autem  $Q$   
 sit productum ex quotcunque factoribus simplicibus for-  
 mae  $z \pm a$ , siue aequalibus siue inaequalibus inter se:  
 et notum est, istam fractionem semper resolui posse in  
 fractiones simplices, quarum denominatores singuli for-  
 mentur ex factoribus ipsius  $Q$ , numeratores vero sint quan-  
 titates constantes, siquidem variabilis  $z$  in numeratore  $P$  ad  
 pauciores dimensiones assurgat quam in denominatore  $Q$ ,  
 quoniam aliter praeter istas fractiones simplices insuper  
 partes integrae essent adijciendae. Qui casus cum nulla  
 laboret difficultate, propterea quod partes istae integrae  
 facile reperiuntur, dum numerator  $P$  per denominatorem  
 $Q$  actu diuiditur, sufficiet eiusmodi tantum fractiones con-  
 siderasse, in quarum denominatore  $Q$  variabilis  $z$  ad al-  
 tiores potestates ascendit quam in numeratore  $P$ . Tum  
 igitur,

igitur, quando pro singulis factoribus denominatoris  $Q$  inventae fuerint fractiones simplices ipsis respondentes, summa omnium harum fractionum aequabitur fractioni propositae  $\frac{P}{Q}$ . Primus quidem in Introductione mea ad Analysin infinitorum methodum tradidi satis simplicem, cuius ope omnes istae fractiones partiales pro singulis denominatoris factoribus reperiri queant, sine vilo respectu ad reliquas habito, quarum ratio antehac teneri debebat. Postmodum vero istam methodum magis excolui, et quemadmodum ope calculi differentialis facilius ad quosvis vsus accommodari possit, vberius ostendi. Nunc autem penitus noua Idea sese mihi obtulit eandem resolutionem perficiendi, quae plerumque negotium non mediocriter subleuare videtur. Imprimis autem ad functiones transcendentes mira facilitate accommodari potest, vnde operae pretium fore existimo, si istam nouam methodum accuratius expesiero.

§. 2. Sit igitur  $z - a$  factor simplex denominatoris  $Q$ , siue solitarius, siue quotcunque sibi aequales admittens. Ac priori casu inde fractio simplex oriunda erit  $\frac{\alpha}{z - a}$ . Sin autem denominator binos huiusmodi factores aequales inuoluat, scilicet  $(z - a)^2$ , tum resolutio binas dabit fractiones simplices  $\frac{\alpha}{(z - a)^2} + \frac{\beta}{z - a}$ ; at si factorem habeat cubicum  $(z - a)^3$ , fractiones simplices inde ortae erunt  $\frac{\alpha}{(z - a)^3} + \frac{\beta}{(z - a)^2} + \frac{\gamma}{z - a}$ , et ita porro pro altioribus potestatibus. Totum igitur negotium huc redit, vt pro singulis huiusmodi factoribus numeratores  $\alpha$ ,  $\beta$ ,  $\gamma$ , etc. definiantur, pro qua inuestigatione iam olim praecepta

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E      dedi.

dedi. Noua autem methodus, quam hic sum traditurus, huic innititur principio, quod posito  $z = a$  omnes istae fractiones partiales euadant infinitae, dum reliquae omnes manent finitae magnitudinis, ideoque prae illis quasi euanescant. Hinc si in fractione proposita  $\frac{P}{Q}$  statuatur  $z = a$ , ea utique etiam in infinitum excrescet, eiusque valor debite euolutus praebit ipsas illas fractiones simplices in infinitum abeuntes, id quod hic accuratius sum persecuturus.

§. 3. Ne igitur hic consideratio infiniti moram faceat, statuamus non  $z - a = 0$ , sed  $z - a = \omega$ , denotante  $\omega$  quantitatem infinite paruam atque adeo ipsam euanescentem, ac ponamus tam in numeratore  $P$  quam in denominatore  $Q$  ubique  $z = a + \omega$ , quo facto numerator  $P$  euoluatur in huiusmodi formam:

$$P = A + B\omega + C\omega\omega + D\omega^3 + \text{etc.}$$

denominator vero  $Q$ , quia per hypothesein euanescit posito  $z = a$ , talem induet formam:

$$Q = \mathfrak{A}\omega + \mathfrak{B}\omega\omega + \mathfrak{C}\omega^3 + \mathfrak{D}\omega^4 + \text{etc.}$$

vbi si factor  $z - a$  fuerit solitarius, primus terminus  $\mathfrak{A}\omega$  necessario aderit. Sin autem denominator  $Q$  factorem habeat  $(z - a)^2$ , erit  $\mathfrak{A} = 0$ , ac denominator a termino  $\mathfrak{B}\omega^2$  incipiet. Quodsi vero denominatoris factor fuerit  $(z - a)^3$ , primus terminus in denominatore erit  $\mathfrak{C}\omega^3$ , et ita porro, ita ut si in genere factor fuerit  $(z - a)^n$ , infima potestas in denominatore sit  $\mathfrak{N}\omega^n$ .

§. 4. Haec quidem substitutio, ponendo  $z = a + \omega$ , operationes tantum vulgares algebraicas postulat: interim tamen

tamen per nota principia differentialium mirifice subleuari potest. Nam si in genere loco  $z$  scribatur  $z + \omega$ , functio ipsius  $z$  quaecunque  $P$  accipiet istum valorem:

$$P + \frac{\omega dP}{1 \cdot dz} + \frac{\omega \omega d d P}{1 \cdot 2 d z^2} + \frac{\omega^3 d^3 P}{1 \cdot 2 \cdot 3 d z^3} + \frac{\omega^4 d^4 P}{1 \cdot 2 \cdot 3 \cdot 4 d z^4} + \text{etc.}$$

Hoc igitur modo statim forma tam numeratoris  $P$  quam denominatoris  $Q$  secundum potestates ipsius  $\omega$  disposita reperietur, tantumque opus est, vt in singulis terminis loco  $z$  vbique scribatur  $a$ . Quousque autem istas expressiones per potestates ipsius  $\omega$  continuari oporteat, ex primo denominatoris termino, seu infima potestate ipsius  $\omega$  facile diiudicabitur, vnde sequentes casus euoluamus.

### Casus I.

Quo denominatoris  $Q$  factor est  $z - a$ .

§. 5. Hoc igitur casu non erit  $\mathcal{A} = 0$ , vnde fractio nostra  $\frac{P}{Q}$ , facto  $z = a + \omega$ , induet hanc formam

$$\frac{A + B\omega + C\omega^2 + D\omega^3 + \text{etc.}}{\mathcal{A} + \mathcal{B}\omega + \mathcal{C}\omega^2 + \mathcal{D}\omega^3 + \text{etc.}}$$

vbi fractio ista per diuisionem euoluatur tantum vsque ad primam potestatem  $\omega$ , propterea quod in fractione praefixa  $\frac{1}{\omega}$  haec littera vnicam tantum habet dimensionem, vnde hoc casu tam numeratorem  $P$  quam denominatorem  $Q$  ad duos tantum terminos extendisse sufficit, ita vt sit

$\frac{A + B\omega}{\mathcal{A} + \mathcal{B}\omega}$ . Nunc igitur ex euolutione istius fractionis  $\frac{A + B\omega}{\mathcal{A} + \mathcal{B}\omega}$ , oriatur quotus  $\alpha + \beta\omega$ , eritque

$$\alpha = \frac{A}{\mathcal{A}}, \quad \beta = \frac{B}{\mathcal{A}} - \frac{A\mathcal{B}}{\mathcal{A}^2}.$$

His autem valoribus inuentis fractio nostra discernitur in has partes:  $\frac{\alpha}{\omega} + \alpha$ , quarum cum prima tantum fiat infini-

ta, si loco  $\omega$  restituamus valorem  $z - a$ , fractio simplex hinc resultans erit  $\frac{\alpha}{z-a}$ . Hoc igitur modo facillime fractiones simplices ex singulis denominatoris  $Q$  factoribus solitariis formae  $z - a$  obtinentur; neque adeo opus est, valorem ipsius  $\beta$  nosse, unde sufficere potuisset tantum primos terminos  $A$  et  $\mathcal{A}$  indagasse. Oritur autem  $A$  ex numeratore  $P$ , posito  $z = a$ ; at  $\mathcal{A}$  oritur ex formula  $\frac{dQ}{dz}$ , posito itidem  $z = a$ . Cum enim posito  $z = a$  fiat  $Q = 0$ , si loco  $z$  scribatur  $a + \omega$ , prodibit  $\mathcal{A} = \frac{dQ}{dz}$ .

§. 6. Interim tamen bonum est, etiam valorem ipsius  $\beta$  nosse, quoniam inde quaestio non parum curiosa facile potest resolui. Cum enim ex factore  $z - a$  deducta sit fractio  $\frac{\alpha}{z-a}$ , si pro reliquis omnibus terminis scribamus litteram  $R$ , erit utique  $\frac{P}{Q} = \frac{\alpha}{z-a} + R$ . Quod si ergo desideretur summa omnium reliquorum terminorum  $R$ , casu quo ponitur  $z = a$ , siue  $z = a + \omega$ , quippe quae summa est finita, ex aequatione modo inuenta fiet  $R = \frac{P}{Q} - \frac{\alpha}{z-a}$ , ideoque posito  $z = a + \omega$ , erit

$$R = \frac{\alpha}{\omega} + \beta - \frac{\alpha}{\omega} = \beta;$$

sicque valor litterae  $\beta$ , quem inuenimus, exprimit summam omnium reliquorum terminorum pro casu  $z = a$ . Erat autem  $\beta = \frac{B}{\mathcal{A}} - \frac{A \mathcal{B}}{\mathcal{Q}^2}$ .

§. 7. Facile autem patet, hoc modo pro omnibus factoribus simplicibus denominatoris  $Q$  easdem prodire fractiones partiales, ad quas methodus antehac exposita deducit. Si enim ponamus  $\frac{P}{Q} = \frac{\alpha}{z-a} + R$ , ac per  $z - a$  multiplicemus, fiet.

$P(z)$

$$\frac{P(z-a)}{Q} = \alpha + R(z-a).$$

Quia nunc nouimus, numeratorem quaesitum  $\alpha$  esse constantem, pro eo semper idem valor prodire debet, quicquid pro  $z$  scribatur. Ponatur igitur  $z = a$ , ut ratio reliquorum terminorum  $R$  ex calculo egrediatur, fietque  $\alpha = \frac{P(z-a)}{Q}$ , posito scilicet  $z = a$ ; tum autem tam numerator quam denominator euanescit, unde si eorum loco sua differentialia ponantur, fiet

$$\alpha = \frac{(z-a) \frac{dP}{dz} + P \frac{dz}{dz}}{\frac{dQ}{dz}}$$

Posito igitur  $z = a$  erit  $\alpha = \frac{P \frac{dz}{dz}}{\frac{dQ}{dz}}$ . At vero supra assumimus, casu  $z = a$  fieri  $P = A$  et  $\frac{dQ}{dz} = Q$ , ita ut et hinc etiam prodeat  $\alpha = \frac{A}{Q}$ .

### Casus II.

Quo denominatoris  $Q$  factor est  $(z-a)^2$ .

§. 8. Hic igitur in forma, ad quam nostram fractionem  $\frac{P}{Q}$ , posito  $z = a + \omega$ , conuertimus, erit  $Q = 0$ , unde fractio pro hoc casu ita referri poterit:

$$\frac{1}{\omega^2} \frac{A + B\omega + C\omega\omega}{B + C\omega + D\omega\omega}$$

Hic scilicet potestates ipsius  $\omega$  non ultra secundam pro- tendimus. Nunc illa expressio in hanc formam:

$$\frac{1}{\omega^2} (\alpha + \beta\omega + \gamma\omega\omega),$$

reducatur, et reperietur calculo subducto

$$\alpha = \frac{A}{B}; \beta = \frac{B}{B} - \frac{AC}{B^2}; \gamma = \frac{C}{B} - \frac{BC}{B^2} = \frac{AD}{B^2} + \frac{AC^2}{B^3}.$$

His igitur valoribus inuentis nostra fractio discerpetur in has partes:  $\frac{\alpha}{\omega^2} + \frac{\beta}{\omega} + \gamma$ , quarum binæ priores, ob  $\omega = z - a$ ,

praebent istas fractiones partiales:  $\frac{\alpha}{(z-a)^2} + \frac{\beta}{z-a}$ : at quantitas  $\gamma$  aequabitur summae omnium reliquorum terminorum, siquidem in illis statuatur  $z = a$ .

### Casus III.

Quo denominatoris factor est  $(z-a)^3$ .

§. 9. Hic igitur ob  $\mathcal{A} = 0$  et  $\mathcal{B} = 0$  fractio evolenda erit

$$\frac{z}{\omega^3} \cdot \frac{A + B\omega + C\omega\omega + D\omega^3}{\mathcal{E} + \mathcal{D}\omega + \mathcal{C}\omega\omega + \mathcal{F}\omega^3},$$

quae reducatur ad hanc formam:

$$\frac{z}{\omega^3} (\alpha + \beta\omega + \gamma\omega\omega + \delta\omega^3),$$

ope harum aequalitatum:

$$A = \alpha \mathcal{E}; \quad B = \alpha \mathcal{D} + \beta \mathcal{E}; \quad C = \alpha \mathcal{C} + \beta \mathcal{D} + \gamma \mathcal{E}; \\ D = \alpha \mathcal{F} + \beta \mathcal{C} + \gamma \mathcal{D} + \delta \mathcal{E},$$

quibus valoribus inuentis ex denominatoris  $Q$  factore cubico  $(z-a)^3$  obtinentur istae fractiones partiales:

$$\frac{\alpha}{(z-a)^3} + \frac{\beta}{(z-a)^2} + \frac{\gamma}{z-a}.$$

At vero  $\delta$  exhibet summam omnium reliquorum terminorum, si in ipsis vbique scribatur  $z = a$ . Facile autem intelligitur, hoc modo etiam ad altiores potestates procedi posse.

§. 10. Haec methodus etiam succedit, si factores denominatoris fuerint imaginarii, scilicet formae

$$z - a + b\sqrt{-1},$$

tum autem, quoniam etiam factor erit  $z - a - b\sqrt{-1}$ , binae fractiones partiales hinc ortae



$$\frac{\alpha}{z - a + b\sqrt{-1}} + \frac{\beta}{z - a - b\sqrt{-1}}$$

facile in factorem duplicatum realem contrahentur, cuius denominator erit  $(z - a)^2 + b^2$ . Hoc igitur sequenti exemplo ostendisse iuuabit.

### Exemplum.

*Si fractio proposita resoluenda fuerit  $\frac{P}{Q} = \frac{\sin. \Phi}{\text{tang. } \Phi - \text{cos. } \Phi}$  eam in fractiones simplices resolvere.*

### Solutio.

§. 11. Hic igitur primo omnes angulos  $\Phi$  quaeri oportet, quibus denominator  $\text{tang. } \Phi - \text{cos. } \Phi$  euanescit. Ponatur igitur

$$\text{tang. } \Phi - \text{cos. } \Phi = 0, \text{ siue } \sin. \Phi - \text{cos. } \Phi^2 = 0,$$

ita ut sit

$$\sin. \Phi^2 + \sin. \Phi = 1, \text{ vnde colligitur}$$

$$\sin. \Phi = -\frac{1 \pm \sqrt{5}}{2},$$

sicque pro  $\sin. \Phi$  duo habentur valores:

$$\sin. \Phi = -\frac{1 + \sqrt{5}}{2} \text{ et } \sin. \Phi = -\frac{1 - \sqrt{5}}{2},$$

quorum prior cum sit vnitatem minor, dabit valorem realem pro angulo  $\Phi$ . Cum enim sit proxime

$$\frac{\sqrt{5} - 1}{2} = 0,618034, \text{ erit } \Phi = 38^\circ. 10'. 22'',$$

quem angulum breuitatis gratia ponamus  $= \zeta$ , ita ut sit

$$\sin. \zeta = 0,618034 \text{ et } \text{cos. } \zeta = 0,786151$$

atque  $\text{tang. } \zeta = 0,786154$ , ideoque uti posuimus

$$\text{cos. } \zeta = \text{tang. } \zeta.$$

§. 12. Alter autem valor pro  $\sin. \Phi$  inuentus dat  $\sin. \Phi = -1,618034$ , qui cum sit unitate maior, monstrat hunc angulum esse imaginarium, ad quem definiendum notetur esse  $\cos. \theta \sqrt{-1} = \frac{e^{-\theta} + e^{+\theta}}{2}$ , qui valor cum manifesto maior sit unitate, et quidem positius utamur hac formula:

$$\cos. (\pi + \theta \sqrt{-1}) = \frac{-e^{\theta} - e^{-\theta}}{2}.$$

Quodsi ergo faciamus

$$\Phi = 90^{\circ} - \pi - \theta \sqrt{-1} = \theta \sqrt{-1} - \pi, \text{ erit}$$

$$\sin. \Phi = -\frac{e^{-\theta} - e^{+\theta}}{2},$$

quamobrem debet esse

$$\frac{e^{-\theta} + e^{\theta}}{2} = +1,618034 = \frac{1 + \sqrt{5}}{2}$$

pro quo numero breuitatis gratia scribamus  $\varepsilon$ , ut sit  $e^{\theta} + e^{-\theta} = 2\varepsilon$ , vnde colligitur

$$e^{\theta} = \varepsilon + \sqrt{(\varepsilon\varepsilon - 1)} = \frac{\varepsilon + \sqrt{5}}{2} + \sqrt{\frac{\varepsilon - 1}{2}} = \varepsilon + \sqrt{\varepsilon},$$

et substituto valore fit  $e^{\theta} = 2,0581710$ . Hinc igitur fiet  $\theta = L. 2,0581710$ , sumendo scilicet logarithmum hyperbolicum, qui reperitur si logarithmus vulgaris multiplicetur per  $2,3025851$ . Cum igitur logarithmus vulgaris fit  $0,3134816$  erit

$$\theta = 0,3134816 \cdot 2,3025851 = 0,7218177.$$

§. 13. Inventis igitur his valoribus pro  $\zeta$  et  $\theta$ , ex priore  $\Phi = \zeta$  omnes anguli  $\Phi$ , quibus noster denominator

nator tang.  $\Phi - \cos. \Phi$  evanescit, sunt in genere

$$2i\pi + \zeta \text{ et } (2i + 1)\pi - \zeta,$$

quippe qui anguli omnes eundem habent sinum, vnde colliguntur omnes factores simplices reales. Pro imaginariis autem tantum loco  $\zeta$  scribi oportet  $\theta\sqrt{-1} - \frac{1}{2}\pi$ , sicque simul obtinentur omnes factores imaginarii.

§. 14. Primo igitur denominatoris nostri factor sit  $\Phi - 2i\pi - \zeta$ , ponaturque hic factor  $= \omega$ , ita ut sit,

$$\begin{aligned} \Phi &= 2i\pi + \zeta + \omega, \text{ eritque sin. } \Phi = \text{sin. } (\zeta + \omega) \\ &= \text{sin. } \zeta \cos. \omega + \cos. \zeta \text{ sin. } \omega = \text{sin. } \zeta + \omega \cos. \zeta, \end{aligned}$$

quoniam non ultra primam dimensionem ipsius  $\omega$  progredi necesse est. Deinde vero erit

$$\cos. \Phi = \cos. (\zeta + \omega) = \cos. \zeta - \omega \text{ sin. } \zeta,$$

denique

$$\text{tang. } \Phi = \text{tang. } (\zeta + \omega) = \text{tang. } \zeta + \frac{\omega}{\cos. \zeta^2}$$

Hinc igitur denominator erit

$$\text{tang. } \zeta - \cos. \zeta + \omega \left( \text{sin. } \zeta + \frac{1}{\cos. \zeta^2} \right);$$

at vero per hypothesin est tang.  $\zeta - \cos. \zeta = 0$ , vnde denominator iste erit  $\omega \left( \text{sin. } \zeta + \frac{1}{\cos. \zeta^2} \right)$ . Vbi notetur, si accuratius procedere voluissemus, in denominatorem insuper terminum  $\omega^3$  ingressurum fuisse, quem autem hic negligere licet, quia nullus factor bis occurrit. Hinc ergo nascitur valor infinitus nostrae fractionis

$$\frac{\text{sin. } \zeta}{\omega \left( \text{sin. } \zeta + \frac{1}{\cos. \zeta^2} \right)} = \frac{\text{sin. } \zeta \cos. \zeta^2}{\omega \left( \text{sin. } \zeta \cos. \zeta^2 + 1 \right)}$$

ex quo haec fractio partialis deducitur:

$$\frac{\sin. \zeta \cos. \zeta^2}{\sin. \zeta \cos. \zeta^2 + 1} \cdot \Phi - \frac{1}{2i\pi - \zeta}$$

quare si pro  $i$  omnes numeros tam positivos quam negativos statuamus, prodibit ista series fractionum:

$$\frac{\sin. \zeta \cos. \zeta^2}{1 + \sin. \zeta \cos. \zeta^2} \left( \Phi - \frac{1}{\zeta} + \Phi - \frac{1}{2\pi - \zeta} + \Phi - \frac{1}{2\pi + \zeta} + \Phi - \frac{1}{4\pi - \zeta} + \Phi - \frac{1}{4\pi + \zeta} + \text{etc.} \right)$$

At si pro  $\zeta$  scribamus  $\theta \sqrt{-1} - \frac{1}{2}\pi$ , series fractionum imaginariarum erit

$$\frac{\sin. \zeta \cos. \zeta^2}{1 + \sin. \zeta \cos. \zeta^2} \left( \frac{1}{\Phi + \frac{1}{2}\pi - \theta \sqrt{-1}} + \frac{1}{\Phi - \frac{1}{2}\pi - \theta \sqrt{-1}} + \frac{1}{\Phi + \frac{3}{2}\pi - \theta \sqrt{-1}} + \frac{1}{\Phi - \frac{3}{2}\pi - \theta \sqrt{-1}} + \text{etc.} \right)$$

§. 15. Pro altero casu, quo in genere erat factor

$$\Phi - (2i + 1)\pi + \zeta = \omega, \text{ erit}$$

$$\Phi = (2i + 1)\pi - \zeta + \omega$$

hincque

$$\sin. \Phi = \sin. (\zeta - \omega) = \sin. \zeta - \omega \cos. \zeta,$$

tum vero

$$\cos. \Phi = -\cos. (\zeta - \omega) = -\cos. \zeta - \omega \sin. \zeta \text{ et}$$

$$\text{tang. } \Phi = -\text{tang. } (\zeta - \omega) = -\text{tang. } \zeta + \frac{\omega}{\cos. \zeta^2},$$

quare totus denominator erit

$$-\text{tang. } \zeta + \cos. \zeta + \omega \left( \frac{1}{\cos. \zeta^2} + \sin. \zeta \right)$$

$$= \omega \left( \sin. \zeta + \frac{1}{\cos. \zeta^2} \right) \text{ ob } -\text{tang. } \zeta + \cos. \zeta = 0,$$

vnde pars infinita nostrae fractionis erit

$$\frac{\sin. \zeta}{\omega \left( \sin. \zeta + \frac{1}{\cos. \zeta^2} \right)} = \frac{\sin. \zeta \cos. \zeta^2}{\omega (1 + \sin. \zeta \cos. \zeta^2)}$$

Nunc

Nunc igitur, si loco  $\omega$  scribamus valorem assumtum

$$\Phi - (2i + 1)\pi + \zeta,$$

oriatur forma generalis fractionum simplicium

$$\frac{\sin. \zeta \cos. \zeta^2}{(1 + \sin. \zeta \cos. \zeta^2) \cdot \Phi - (2i + 1)\pi + \zeta}.$$

Loco  $i$  ergo successively scribamus omnes valores

$$0, \pm 1, \pm 2, \pm 3, \pm 4 \text{ etc.}$$

et colligetur sequens series fractionum:

$$\frac{\sin. \zeta \cos. \zeta^2}{1 + \sin. \zeta \cos. \zeta^2} \left( \frac{1}{\Phi - \pi + \zeta} + \frac{1}{\Phi + \pi + \zeta} + \frac{1}{\Phi - 3\pi + \zeta} \right. \\ \left. + \frac{1}{\Phi + 3\pi + \zeta} + \frac{1}{\Phi - 5\pi + \zeta} + \text{etc.} \right)$$

Quodsi iam hic pro  $\zeta$  scribamus  $\theta\sqrt{-1} - \frac{1}{2}\pi$ , orientur fractiones imaginariae, quae erunt

$$\frac{\sin. \zeta \cos. \zeta^2}{1 + \sin. \zeta \cos. \zeta^2} \left( \frac{1}{\Phi + \frac{1}{2}\pi + \theta\sqrt{-1}} + \frac{1}{\Phi - \frac{3}{2}\pi + \theta\sqrt{-1}} \right. \\ \left. + \frac{1}{\Phi + \frac{5}{2}\pi + \theta\sqrt{-1}} + \frac{1}{\Phi - \frac{7}{2}\pi + \theta\sqrt{-1}} + \text{etc.} \right)$$

§. 16. Colligamus nunc primo seorsim omnes fractiones reales, et cum omnes habeant eundem coefficientem constantem  $\frac{\sin. \zeta \cos. \zeta^2}{1 + \sin. \zeta \cos. \zeta^2}$ , ante omnia in eius valorem numericum inquiramus. Primo igitur cum fuisset

$$\sin. \zeta - \cos. \zeta^2 = 0, \text{ erit } \cos. \zeta^2 = \sin. \zeta,$$

ideoque iste coefficientis  $= \frac{\sin. \zeta^2}{1 + \sin. \zeta^2}$ ; porro vero erat

$$\sin. \zeta^2 + \sin. \zeta = 1, \text{ ideoque } \sin. \zeta^2 = 1 - \sin. \zeta$$

vnde fit coefficientis  $\frac{1 - \sin. \zeta}{1 + \sin. \zeta}$ . Denique vero pro factoribus realibus eruimus  $\sin. \zeta = \frac{\sqrt{5} - 1}{2}$ , vnde noster coefficientis evadet  $\frac{3 - \sqrt{5}}{5 - \sqrt{5}} = \frac{5 - \sqrt{5}}{10}$ , cuius ergo valor erit 0, 2763932,

pro quo brevitatis gratia scribamus  $\alpha$ , et omnes fractiones simplices reales in ordinem redactae erunt

$$\frac{\alpha}{\Phi - \zeta} + \frac{\alpha}{\Phi - \zeta - 2\pi} + \frac{\alpha}{\Phi - \zeta + 2\pi} + \frac{\alpha}{\Phi - \zeta - 4\pi} + \frac{\alpha}{\Phi - \zeta + 4\pi} + \text{etc.}$$

$$\frac{\alpha}{\Phi + \zeta - \pi} + \frac{\alpha}{\Phi + \zeta + \pi} + \frac{\alpha}{\Phi + \zeta - 3\pi} + \frac{\alpha}{\Phi + \zeta + 3\pi} + \frac{\alpha}{\Phi + \zeta - 5\pi} + \text{etc.}$$

§. 17. Pro partibus autem imaginariis idem coefficientis communis

$\frac{\sin. \zeta \cos. \zeta^2}{1 + \sin. \zeta \cos. \zeta^2}$ , ob  $\cos. \zeta^2 = \sin. \zeta$  et  $\sin. \zeta^2 = 1 - \sin. \zeta$ , transmutatur ut ante in hanc formam:  $\frac{1 - \sin. \zeta}{2 - \sin. \zeta}$ . At vero

pro partibus imaginariis invenimus  $\sin. \zeta = -\frac{1 - \sqrt{5}}{2}$ , in quo iam inuoluitur substitutio ante memorata  $\zeta = \theta \sqrt{-1 - \frac{1}{2}\pi}$ . Hoc ergo valore substituto coefficientis communis erit

$$\frac{2 + \sqrt{5}}{5 + \sqrt{5}} = \frac{5 + \sqrt{5}}{10}$$

ideoque in numeris 0,7236068, pro quo numero scribamus  $\beta$ , ita ut sit  $\alpha + \beta = 1$ , hanc ob rem bini ordines fractionum imaginariarum erunt

$$\frac{\beta}{\Phi - \theta \sqrt{-1 + \frac{1}{2}\pi}} + \frac{\beta}{\Phi - \theta \sqrt{-1 - \frac{3}{2}\pi}} + \frac{\beta}{\Phi - \theta \sqrt{-1 + \frac{5}{2}\pi}}$$

$$+ \frac{\beta}{\Phi - \theta \sqrt{-1 - \frac{7}{2}\pi}} \text{ etc.}$$

$$\frac{\beta}{\Phi + \theta \sqrt{-1 + \frac{1}{2}\pi}} + \frac{\beta}{\Phi + \theta \sqrt{-1 - \frac{3}{2}\pi}} + \frac{\beta}{\Phi + \theta \sqrt{-1 + \frac{5}{2}\pi}}$$

$$+ \frac{\beta}{\Phi + \theta \sqrt{-1 - \frac{7}{2}\pi}} \text{ etc.}$$

Hinc ergo si binae harum fractionum in vnam summam colligantur, imaginaria se mutuo destruent, ac prodibit sequens series:

$$\frac{\beta(2\Phi + \pi)}{(\Phi + \frac{1}{2}\pi)^2 + \theta\theta} + \frac{\beta(2\Phi - 3\pi)}{(\Phi - \frac{3}{2}\pi)^2 + \theta\theta} + \frac{\beta(2\Phi + 5\pi)}{(\Phi + \frac{5}{2}\pi)^2 + \theta\theta} + \frac{\beta(2\Phi - 7\pi)}{(\Phi - \frac{7}{2}\pi)^2 + \theta\theta}, \text{ etc.}$$

vbi notetur esse  $\theta\theta = 0,5210210$ .

§. 18. Quoniam partes imaginariae commode se contrahi sunt passae, vt similis contractio in partibus realibus succedat statuamus  $\zeta = \frac{1}{2}\pi + \eta$  et ambae series ita se habebunt:

$$\frac{\alpha}{\Phi - \frac{1}{2}\pi - \eta} + \frac{\alpha}{\Phi - \frac{5}{2}\pi - \eta} + \frac{\alpha}{\Phi + \frac{3}{2}\pi - \eta} + \frac{\alpha}{\Phi - \frac{7}{2}\pi - \eta} + \frac{\alpha}{\Phi + \frac{9}{2}\pi - \eta} + \text{etc.}$$

$$\frac{\alpha}{\Phi - \frac{1}{2}\pi + \eta} + \frac{\alpha}{\Phi + \frac{3}{2}\pi + \eta} + \frac{\alpha}{\Phi - \frac{5}{2}\pi + \eta} + \frac{\alpha}{\Phi + \frac{7}{2}\pi + \eta} + \frac{\alpha}{\Phi - \frac{9}{2}\pi + \eta} + \text{etc.}$$

Hic ergo cuilibet termino conuenit quasi focus, binisque contractis orietur sequens series:

$$\frac{\alpha(2\Phi - \pi)}{(\Phi - \frac{1}{2}\pi)^2 - \eta\eta} + \frac{\alpha(2\Phi + 3\pi)}{(\Phi + \frac{3}{2}\pi)^2 - \eta\eta} + \frac{\alpha(2\Phi - 5\pi)}{(\Phi - \frac{5}{2}\pi)^2 - \eta\eta} + \frac{\alpha(2\Phi + 7\pi)}{(\Phi + \frac{7}{2}\pi)^2 - \eta\eta} + \text{etc.}$$

vbi notetur, cum sit  $\eta = \zeta - \frac{1}{2}\pi$ , quoniam in partibus radii est  $\zeta = 0,6662405$  fore  $\eta = -0,9045558$ , ideoque  $\eta\eta = 0,8182214$ , cum ante fuisset  $\theta\theta = 0,5210210$ .

§. 19. Quae igitur hactenus sunt allata huc redeunt, ut fractio proposita  $\frac{\sin. \Phi}{\tan g. \Phi - \cos. \Phi}$ , aequetur binis sequentibus seriebus iunctim sumtis:

$$\begin{aligned} & \frac{\alpha(2\Phi - \pi)}{(\Phi - \frac{1}{2}\pi)^2 - \eta\eta} + \frac{\alpha(2\Phi + 3\pi)}{(\Phi + \frac{3}{2}\pi)^2 - \eta\eta} \\ & + \frac{\alpha(2\Phi - 5\pi)}{(\Phi - \frac{5}{2}\pi)^2 - \eta\eta} + \frac{\alpha(2\Phi + 7\pi)}{(\Phi + \frac{7}{2}\pi)^2 - \eta\eta} + \text{etc.} \\ & \frac{\beta(2\Phi + \pi)}{(\Phi + \frac{1}{2}\pi)^2 + \theta\theta} + \frac{\beta(2\Phi - 3\pi)}{(\Phi - \frac{3}{2}\pi)^2 + \theta\theta} \\ & + \frac{\beta(2\Phi + 5\pi)}{(\Phi + \frac{5}{2}\pi)^2 + \theta\theta} + \frac{\beta(2\Phi - 7\pi)}{(\Phi - \frac{7}{2}\pi)^2 + \theta\theta} + \text{etc.} \end{aligned}$$

Hinc igitur sequitur, si sumatur  $\Phi = 0$ , quo casu fractio ipsa in nihilum abit, fore

$$0 = \left\{ \begin{aligned} & - \frac{4.1\alpha\pi}{\pi\pi - 4\eta\eta} + \frac{4.3.\alpha\pi}{9\pi\pi - 4\eta\eta} - \frac{4.5\alpha\pi}{25\pi\pi - 4\eta\eta} + \frac{4.7\alpha\pi}{49\pi\pi - 4\eta\eta} - \text{etc.} \\ & + \frac{4.1\beta\pi}{\pi\pi + 4\theta\theta} - \frac{4.3.\beta\pi}{9\pi\pi + 4\theta\theta} + \frac{4.5\beta\pi}{25\pi\pi + 4\theta\theta} - \frac{4.7\beta\pi}{49\pi\pi + 4\theta\theta} + \text{etc.} \end{aligned} \right.$$

Ceterum hoc exemplum perquam idoneum est visum, quo applicatio ad factores imaginarios illustraretur.