



1783

Supplementum calculi integralis pro integratione formularum irrationalium

Leonhard Euler

Follow this and additional works at: <https://scholarlycommons.pacific.edu/euler-works>

 Part of the [Mathematics Commons](#)

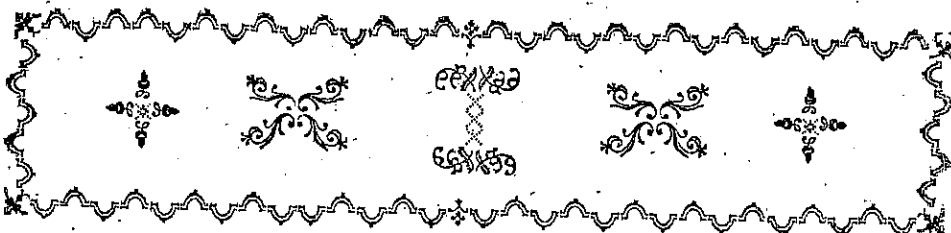
Record Created:

2018-09-25

Recommended Citation

Euler, Leonhard, "Supplementum calculi integralis pro integratione formularum irrationalium" (1783). *Euler Archive - All Works*. 539.
<https://scholarlycommons.pacific.edu/euler-works/539>

This Article is brought to you for free and open access by the Euler Archive at Scholarly Commons. It has been accepted for inclusion in Euler Archive - All Works by an authorized administrator of Scholarly Commons. For more information, please contact mgibney@pacific.edu.



SUPPLEMENTUM
CALCVLI INTEGRALIS

PRO
INTEGRATIONE FORMVLARVM
IRRATIONALIVM.

Auctore
L. E V L E R O.

Problema I.

§. 1.

Si functio X praeter ipsam variabilem x etiam formulam irrationalem $s = \sqrt{a + bx}$ inuoluat: ita tamen, ut X sit functio rationalis binarum quantitatum x et s, formulam differentialem $X dx$ ab irrationalitate liberare.

Solutio.

Cum irrationalitas tantum in formula $s = \sqrt{a + bx}$ insit: hanc tantum ita per idoneam substitutionem tolli

A 2

oportet,

portet, ut inde valor ipsius x non fiat irrationalis. Hoc autem praestabitur, ponendo $a + bx = zz$, ut fiat $s = z$ et $x = \frac{z^2 - a}{b}$, hincque $dx = \frac{2}{b} z dz$; quibus valoribus substitutis tota formula differentialis $X dx$, ad rationalem, novam variabilem z complectens, perducitur.

Exemplum 1.

§. 2. Si fuerit $dy = \frac{dx}{\sqrt{a+bx}}$, seu $dy = \frac{dx}{s}$,posito $\sqrt{a+bx} = z$, fiet $dy = \frac{2}{b} dz$, et integrando $y = \frac{2z}{b}$, vnde facta substitutione colligitur $y = \frac{2}{b} \sqrt{a+bx} + C$.

Exemplum 2.

§. 3. Si fuerit $dy = dx \sqrt{a+bx} = s dx$, sumto $\sqrt{a+bx} = z$ erit $dy = z dx = \frac{2}{b} z dz$, vnde integrando fit $y = \frac{2}{3b} z^3$, et facta substitutione prodit

$$y = \frac{2}{3b} (a+bx)^{\frac{3}{2}} + C.$$

Quod integrale si debeat evanescere facto $x = 0$, fiet $C = -\frac{2a\sqrt{a}}{3b}$, ideoque

$$y = \frac{2(a+bx)^{\frac{3}{2}} - 2a\sqrt{a}}{3b}.$$

Exemplum 3.

§. 4. Si fuerit $dy = \frac{x dx}{\sqrt{a+bx}}$, facta substitutione $\sqrt{a+bx} = z$, erit

$$dy = \frac{z(z^2 - a) dz}{bb} = \frac{z z dz - a dz}{bb},$$

vnde

vnde fit integrando $y = \frac{x}{\frac{2}{3} b b} z^{\frac{5}{3}} - \frac{\frac{2}{3} a}{\frac{2}{3} b} z + C$, et facta resti-
tutione

$$y = \frac{x}{\frac{2}{3} b b} (a + b x)^{\frac{5}{3}} - \frac{\frac{2}{3} a}{\frac{2}{3} b} \sqrt[3]{(a + b x)} + C$$

$$= \frac{\frac{2}{3} \sqrt[3]{(a + b x)}}{\frac{2}{3} b b} (\frac{5}{3} b x - \frac{2}{3} a) + C.$$

Exemplum 4.

§. 5. Si fuerit $dy = \frac{dx}{(a + bx)^{\frac{5}{3}}}$, facta substitutione

$\sqrt[3]{(a + bx)} = z$ erit $dy = \frac{dx}{z^{\frac{5}{3}}}$; quae formula porro ob
 $dx = \frac{z dz}{b}$ abit in $dy = \frac{z dz}{b z^{\frac{5}{3}}}$, quae integrata fit $y = -\frac{z}{b z}$,
seu facta restitutione, $y = \frac{-z}{b \sqrt[3]{(a + bx)}} + C$. Vbi notetur,
pro C sumi debere $\frac{x}{b \sqrt[3]{a}}$.

Problema 2.

§. 6. Si fuerit X functio quaecunque rationalis bi-
narum quantitatum x et s, existente $s = \sqrt[3]{(a + bx)}$, formu-
lam differentialem X dx ab irrationalitate liberare.

Solutio.

Ponatur $\sqrt[3]{(a + bx)} = z$, vt fit $s = z$, erit $a + bx = z^3$,
hincque $x = \frac{z^3 - a}{b}$ et $dx = \frac{3 z^2 dz}{b}$; quibus valoribus
substitutis tota formula fiet rationalis.

Exemplum 1.

§. 7. Si fuerit

$$dy = \frac{dx}{\sqrt[3]{(a + bx)}} = \frac{dx}{z}$$

$\sqrt[s]{a+bx} = z$ et substituto valore hinc nato $dx = \frac{z dz}{b}$,
erit $dy = \frac{s z dz}{b}$, vnde integrando fit

$$y = \frac{s}{2b} z^2 = \frac{s}{2b} \sqrt[s]{(a+bx)^2} + C.$$

Exemplum 2.

§. 8. Si fuerit

$$dy = \frac{dx}{\sqrt[s]{(a+bx)^2}} = \frac{dx}{s z^2}, \text{ posito}$$

$\sqrt[s]{a+bx} = z$ fiet $dy = \frac{z dz}{b}$, hinc integrando

$$y = \frac{s}{b} z = \frac{s}{b} \sqrt[s]{a+bx} + C.$$

Exemplum 3.

§. 9. Si fuerit $dy = dx \sqrt[s]{a+bx} = s dx$, fac-
ta substitutione fit $dy = \frac{s z^3 dz}{b}$, hinc integrando

$$y = \frac{s}{4b} z^4 = \frac{s}{4b} (a+bx) \sqrt[s]{a+bx} + C.$$

Problema 3.

§. 10. Si fuerit X functio rationalis binarum quan-
titarum x et s , existente $s = \sqrt[n]{a+bx}$, formulam diffe-
rentialem $X dx$ ab irrationalitate liberare.

Solutio.

Ponatur $\sqrt[n]{a+bx} = z$, vt fit $s = z$, erit $a+bx = z^n$,
hinc $x = \frac{z^n - a}{b}$ et $dx = \frac{n z^{n-1} dz}{b}$; quibus valoribus

substi-

substitutis formula proposita $X dx$ certe fiet rationalis, si modo numerus exponentialis n fuerit integer.

Exemplum 1.

§. 11. Si fuerit

$$dy = \frac{dx}{\sqrt[n]{a+bx}} = \frac{dx}{s}, \text{ posito}$$

$$\sqrt[n]{a+bx} = z \text{ ob valorem inde natum } dx = \frac{nz^{n-1} dz}{b}$$

habebitur $dy = \frac{nz^{n-1} dz}{b}$; vnde integrando colligimus

$$y = \frac{n}{b(n-1)} z^{n-1} + C, \text{ siue restituis valoribus}$$

$$y = \frac{n}{b(n-1)} (a+bx)^{\frac{n-1}{n}} + C = \frac{n}{b(n-1)} \frac{a+bx}{\sqrt[n]{a+bx}} + C.$$

Exemplum 2.

§. 12. Si fuerit

$$dy = \frac{dx}{\sqrt[n]{(a+bx)^\lambda}} = \frac{dx}{s^\lambda}, \text{ posito}$$

$$\sqrt[n]{(a+bx)^\lambda} = z \text{ et substituto valore } dx = \frac{nz^{n-1} dz}{b}, \text{ fiet}$$

$$dy = \frac{nz^{n-1} dz}{bz^\lambda} = \frac{n}{b} z^{n-\lambda-1} dz,$$

cuius integrale dat

$$y = \frac{n}{b(n-\lambda)} (a+bx)^{\frac{n-\lambda}{n}} + C, \text{ siue}$$

$$y = \frac{n}{b(n-\lambda)} \frac{a+bx}{\sqrt[n]{(a+bx)^\lambda}}$$

Ex

Ex his autem exemplis iam apparet, integrationem non impediri, etiam si exponentes n et λ non fuerint numeri integri.

Problema 4.

§. 13. Si fuerit X functio rationalis binarum quantitatum x et s , existente $s = \sqrt{a + b \sqrt{f + gx}}$, quae formula ergo duplicem irrationalitatem inuoluit, formulam differentialem $X dx$ ab hac duplici irrationalitate liberare.

Solutio.

Ponatur iterum $\sqrt{a + b \sqrt{f + gx}} = z$, ut fit $s = z$, erit sumtis quadratis $a + b \sqrt{f + gx} = z z$, hinc $b \sqrt{f + gx} = z z - a$: ac sumtis denuo quadratis

$$b b (f + gx) = (z z - a)^2, \text{ unde colligitur}$$

$$x = \frac{(z z - a)^2 - f}{b b g}, \text{ hincque}$$

$$dx = \frac{2 z dz (z z - a)}{b b g}$$

Quibus valoribus substitutis tota formula reddetur rationalis.

Corollarium.

§. 14. Perspicuum est, eodem modo irrationalitatem tolli posse, si fuerit multo generalius.

$$s = \sqrt[n]{a + b \sqrt[m]{f + gx}}.$$

Posita enim hac formula $= z$, fiet

$$a + b \sqrt[m]{f + gx} = z^n \text{ et } b \sqrt[m]{f + gx} = z^n - a.$$

Porro $b^m (f + gx) = (z^n - a)^m$, et hinc colligitur

$$x =$$

$$x = \frac{(z^n - a)^m}{b^m g} - \frac{f}{g}, \text{ ideoque}$$

$$dx = \frac{mnz^{n-1} dz (z^n - a)^{m-1}}{b^m g}.$$

Sicque etiam hoc modo tota formula rationalis euadet.

Problema 5.

§. 15. Si fuerit X functio rationalis binarum quantitatum s et x , existente $s = \sqrt{\frac{a+bx}{f+gx}}$, formulam differentialem $X dx$ ab irrationalitate liberare.

Solutio.

Ponatur $\sqrt{\frac{a+bx}{f+gx}} = z$, et sumtis quadratis erit

$$\frac{a+bx}{f+gx} = z^2, \text{ hincque } x = \frac{fz^2 - a}{b - gz^2},$$

unde differentiando colligitur

$$dx = \frac{2bfz dz - 2agz dz}{(b-gz^2)^2}.$$

Hisque valoribus substitutis formula proposita $X dx$ ad rationalitatem erit perducta.

Exemplum I.

§. 16. Si fuerit $dy = \frac{dx}{s} = \frac{dx \sqrt{f+gx}}{\sqrt{a+bx}}$, posito

$$\sqrt{\frac{a+bx}{f+gx}} = z \text{ erit } dy = \frac{dx}{z},$$

et substituto loco dx valore supra inuento colligitur

$$dy = \frac{2(bf-az) dz}{(b-gz^2)^2};$$

quae formula, uti iam satis constat, reduci potest ad talem: $\int \frac{dz}{b-gz^2}$, cuius autem integratio vel per logarithmos vel per arcus circulares expediatur.

Exemplum 2.

§. 17. Sit specialius $dy = \frac{dx\sqrt{(1-x)}}{\sqrt{(1+x)}}$, vbi $f = 1$,
 $g = -1$, $a = 1$ et $b = 1$, ideoque

$$z = \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}} \text{ et } dx = \frac{2z dz}{(1+z^2)^2};$$

quibus valoribus substitutis fiet $dy = \frac{2 dz}{(1+z^2)^2}$. Statuatur
 ergo:

$$\int \frac{2 dz}{(1+z^2)^2} = \frac{A z}{1+z^2} + B \int \frac{dz}{1+z^2} = y,$$

vnde sumtis differentialibus fiet:

$$\frac{4}{(1+z^2)^2} = \frac{A - A z z}{(1+z^2)^2} + \frac{B}{1+z^2} = \frac{A + B + (B-A) z z}{(1+z^2)^2}.$$

Oportet igitur esse $A + B = 4$ et $B - A = 0$, ideoque
 $A = 2$ et $B = 2$; et quia $\int \frac{dz}{1+z^2} = A \text{ tang. } z$, adipiscimur
 $y = \frac{2z}{1+z^2} + 2 A \text{ tang. } z$; quocirca facta restitutione, ob
 $1 + z z = \frac{2}{1-x}$ obtinebitur

$$y = \sqrt{(1-x)} + 2 A \text{ tang. } \sqrt{\frac{1+x}{1-x}}.$$

Cum igitur huius arcus tangens sit $\sqrt{\frac{1+x}{1-x}}$, erit eius finus
 $= \sqrt{\frac{1+x}{2}}$ et cofinus $= \sqrt{\frac{1-x}{2}}$; anguli vero dupli finus erit
 $\sqrt{(1-x)}$ et cofinus $= -x$, vnde fiet

$$2 A \text{ tang. } \sqrt{\frac{1+x}{1-x}} = A \text{ cof. } -x = \frac{\pi}{2} + A \text{ fin. } x;$$

quocirca integrale quaesitum erit

$$y = \sqrt{(1-x)} + \frac{\pi}{2} + A \text{ fin. } x + C,$$

quod si ita capi debeat, vt evanescat posito $x = 0$, erit

$$C = -1 - \frac{\pi}{2}, \text{ ideoque } y = \sqrt{(1-x)} - 1 + A \text{ fin. } x.$$

Tum igitur, si sumatur $x = 1$, fiet $y = \frac{\pi}{2} - 1$, qui valor
 in fractionibus decimalibus dat 0,5707963.

Problema 6.

§. 18. Si fuerit X functio rationalis binarum variabilium x et s , existente $s = \sqrt[n]{\frac{a+bx}{f+gx}}$, formulam differentialem $X dx$ ad rationalitatem perducere.

Solutio.

Pofito $s = \sqrt[n]{\frac{a+bx}{f+gx}} = z$, erit $\frac{a+bx}{f+gx} = z^n$, hincque $x = \frac{fz^n - a}{b - gz^n}$, confequenter $dx = \frac{n(bf - ag)z^{n-1} dz}{(b - gz^n)^2}$;

hisque valoribus fubftituis tota formula propofita $X dx$ ad rationalitatem erit perducta.

Problema 7.

§. 19. Si fuerit X functio binarum quantitatum x et s , existente $s = \sqrt{a + bxx}$, formulam differentialem $\frac{xdx}{x}$ ab irrationalitate liberare.

Solutio.

Ponamus $s = \sqrt{a + bxx} = z$, erit $a + bxx = zz$, hinc $xx = \frac{zz - a}{b}$, et quia in functione X tantum quadratum xx , eiusque ergo potestates pares occurrunt: hac fubftitutione iam functio X euadet rationalis. Sumtis vero logarithmis $2lx = l(zz - a) - lb$, differentiando fit

$$\frac{2dx}{x} = \frac{2zdz}{zz - a}, \text{ ideoque } \frac{dx}{x} = \frac{zdz}{zz - a}.$$

Hoc ergo modo formula propofita $X \cdot \frac{dx}{x}$ profus reddetur rationalis.

Exemplum 1.

§. 20. Si fuerit

$$dy = \frac{x dx}{\sqrt{(a + bxx)}}, \text{ erit } dy = \frac{dx}{x} \cdot \frac{xx}{\sqrt{(a + bxx)}} = \frac{xx}{s} \cdot \frac{dx}{x}.$$

Posito ergo $\sqrt{(a + bxx)} = z$ erit $dy = \frac{dz}{b}$, vnde colligitur integrando $y = \frac{z}{b} = \frac{\sqrt{(a + bxx)}}{b}$.

Exemplum 2.

§. 21. Si fuerit

$$dy = \frac{x^3 dx}{\sqrt{(a + bxx)}} = \frac{dx}{x} \cdot \frac{x^4}{s},$$

ponendo $\sqrt{(a + bxx)} = z$, vt fit

$$xx = \frac{zz - a}{b} \text{ et } \frac{dx}{x} = \frac{z dz}{zz - a},$$

erit $dy = \frac{z}{b} dz (zz - a)$, hincque integrando adipiscimur $y = \frac{z}{3bb} (zz - 3a)$; vnde facta restitutione prodibit integrale quaesitum $y = \frac{bxx - 3a}{3bb} \sqrt{(a + bxx)} + C$.

Exemplum 3.

§. 22. Si fuerit

$$dy = \frac{x^3 dx}{\sqrt{(a + bxx)^3}}, \text{ erit } dy = \frac{dx}{x} \cdot \frac{x^4}{s^3};$$

hinc posito

$$\sqrt{(a + bxx)} = s = z \text{ fiet } dy = \frac{dz}{b} \left(\frac{zz - a}{zz} \right),$$

vnde sumto integrali fiet $y = \frac{z}{bb} \left(\frac{zz + a}{z} \right)$, quocirca facta restitutione resultat $y = \frac{2a + bxx}{bb\sqrt{(a + bxx)}} + C$.

Problema 8.

§. 23. Si fuerit X functio rationalis binarum quantitatum x^n et s , existente $s = \sqrt[m]{(a + bx^n)}$, formulam differentialem $X \frac{dx}{x}$ ad rationalitatem perducere.

Solutio

Solutio.

Posito $s = \sqrt[m]{a + b x^n} = z$, fiet $a + b x^n = z^m$ et $x^n = \frac{z^m - a}{b}$. Quia igitur in functione X tantum pote-

stas x^n occurrit, ea rationalis reddetur, si hi valores substituuntur. Tum vero sumtis logarithmis habebitur

$$n \log x = \log(z^m - a) - \log b,$$

et differentiando

$$\frac{dx}{x} = \frac{m z^{m-1} dz}{n(z^m - a)},$$

ficque tota formula proposita fiet rationalis.

Exemplum.

§. 24. Sit

$$dy = \frac{x^{n-1} dx}{\sqrt[m]{a + b x^n}} = \frac{dx}{x} \cdot \frac{x^n}{s},$$

factaque substitutione orietur haec aequatio:

$$dy = \frac{m z^{m-2} dz}{b n},$$

qua integrata prodibit

$$y = \frac{m z^{m-1}}{n b (m-1)} = \frac{m}{n b (m-1)} \sqrt[m]{(a + b x^n)^{m-1}} + C, \text{ sive}$$

$$y = \frac{m}{n b (m-1)} \frac{a + b x^n}{\sqrt[m]{(a + b x^n)}} + C.$$

Problema.

§. 25. Si fuerit X functio rationalis quantitarum x et s , existente $s = \sqrt{\frac{a+bx}{j+gx}}$, formulam differentialem $X \frac{dx}{x}$ ab irrationalitate liberare.

Solutio.

Ponatur $s = \sqrt{\frac{a+bx}{j+gx}} = z$, eritque $\frac{a+bx}{j+gx} = z^2$, hinc $bx = \frac{fz^2 - a}{b - gz^2}$, vnde functio X penitus fit rationalis. Porro sumtis logarithmis.

$$2 \log x = \log(fz^2 - a) - \log(b - gz^2)$$

differentietur, vt prodeat

$$\frac{2 dx}{x} = \frac{2fz dz}{fz^2 - a} + \frac{2gz dz}{b - gz^2} = \frac{2(bf - ag)z dz}{(fz^2 - a)(b - gz^2)}$$

vnde fit

$$\frac{dx}{x} = \frac{(bf - ag)z dz}{(fz^2 - a)(b - gz^2)}$$

sicque tota formula differentialis fiet rationalis.

Exemplum.

§. 26. Si fuerit $dy = \frac{dx}{\sqrt{j+gx}}$, repraesentemus hanc formulam ita:

$$dy = \frac{dx}{x} \frac{x}{\sqrt{j+gx}} = \frac{dx}{x} \sqrt{\frac{xx}{j+gx}}$$

Hic ergo erit $a = 0$, $b = 1$, et

$$z = \frac{x}{\sqrt{j+gx}}$$
 ita vt $dy = \frac{z dz}{x}$;

erit autem

$$\frac{dx}{x} = \frac{dz}{z(1-gz^2)}$$
, vnde fit $dy = \frac{dz}{1-gz^2}$,

cuius formulae integratio per logarithmos expedietur, si fuerit g numerus positivus: sin autem fuerit negativus per arcus

arcus circulares absoluatur. Sit igitur 1°) $g = +hb$, erit

$$dy = \frac{dx}{1 - \frac{b^2 x^2}{f}}$$

$$y = \frac{1}{2b} l \frac{1 + \frac{bx}{f}}{1 - \frac{bx}{f}}$$

et restitutis valoribus supra indicatis erit

$$y = \frac{1}{2b} l \left(\frac{\sqrt{(f + hb^2 x^2) + bx}}{\sqrt{(f + hb^2 x^2) - bx}} \right) = \frac{1}{b} l \frac{\sqrt{(f + hb^2 x^2) + bx}}{\sqrt{f}}$$

Sit 2°) g quantitas negativa, puta $g = -hb$, erit

$$dy = \frac{dx}{1 + \frac{b^2 x^2}{f}} = \frac{1}{b} \frac{b dx}{1 + \frac{b^2 x^2}{f}}$$

vnde colligitur

$$y = \frac{1}{b} A \text{ tang. } bx = \frac{1}{b} A \text{ tang. } \frac{bx}{\sqrt{(f - b^2 x^2)}}$$

Vbi manifestum est, f esse debere quantitatem positivam, quia alioquin formula differentialis esset imaginaria.

Corollarium.

§. 27. Hinc ergo si proponatur formula

$$dy = \sqrt{(1 + x^2)}, \text{ ubi } f = 1 \text{ et } g = 1,$$

ex casu priore ob $b = +1$ erit

$$\int \frac{dx}{\sqrt{(1 + x^2)}} = l \left(\frac{\sqrt{(1 + x^2)} + x}{1} \right).$$

At si fuerit

$$dy = \frac{dx}{\sqrt{(1 - x^2)}}, \text{ ubi } f = 1 \text{ et } g = -1,$$

colligitur ex casu posteriore $y = A \text{ tang. } \frac{x}{\sqrt{(1 - x^2)}}$, vnde concluditur

$$\int \frac{dx}{\sqrt{(1 - x^2)}} = A \text{ fin. } x = A \text{ cos. } \sqrt{(1 - x^2)}.$$

Problema 10.

§. 28. Si fuerit X functio rationalis quantitatum

x^n

x^n et s , existente $s = \sqrt[n]{\frac{a + b x^n}{f + g x^n}}$, formulam differentialem $X \frac{dx}{x}$ rationalem efficere.

Solutio.

Ponatur $s = \sqrt[n]{\frac{a + b x^n}{f + g x^n}} = z$, eritque

$$\frac{a + b x^n}{f + g x^n} = z^n, \text{ hinc } x^n = \frac{f z^n - a}{b - g z^n},$$

tum autem sumtis logarithmis erit

$$n \log x = \log(f z^n - a) - \log(b - g z^n),$$

et differentiando

$$\frac{dx}{x} = \frac{f z^{n-1} dz}{f z^n - a} + \frac{g z^{n-1} dz}{b - g z^n} = \frac{(b f - a g) z^{n-1} dz}{(f z^n - a)(b - g z^n)},$$

quibus valoribus substitutis formula proposita fit rationalis.

Problema II.

§. 29. Si fuerit X functio rationalis binarum quantitatum x^n et s , existente $s = \sqrt[m]{\frac{a + b x^n}{f + g x^n}}$, formulam differentialem $X \frac{dx}{x}$ ab omni irrationalitate liberare.

Solutio.

Statuatur $s = \sqrt[m]{\frac{a + b x^n}{f + g x^n}} = z$, eritque

$$\frac{a + b x^n}{f + g x^n} = z^m, \text{ vnde fit } x^n = \frac{f z^m - a}{b - g z^m};$$

hinc

hinc sumtis logarithmis erit

$$n \log x = \log(fz^m - a) - \log(b - gz^m),$$

hinc differentiando

$$\frac{n dx}{x} = \frac{m(bf - ag)z^{m-1} dz}{(fz^m - a)(b - gz^m)}, \text{ ideoque}$$

$$\frac{dx}{x} = \frac{m(bf - ag)z^{m-1} dz}{n(fz^m - a)(b - gz^m)},$$

quibus valoribus substitutis irrationalitas formulae propositae penitus tollitur.

Problema 12.

§. 30. Si fuerit X functio rationalis quaecunque binarum quantitatum x et s , existente $s = \sqrt{a + \beta x + \gamma xx}$, formulam differentialem $X dx$ ad rationalitatem perducere.

Solutio.

Hic duos casus a se inuicem distingui conuenit, prout γ fuerit vel quantitas positua vel negatiua.

I. Sit γ quantitas positua, ac ponatur $\gamma = cc$ et $\beta = 2bc$, vt habeatur

$$s = \sqrt{a + 2bcx + ccxx} = \sqrt{a - bb + (b + cx)^2},$$

vbi loco $a - bb$ breuitatis ergo scribatur e , vt fit

$$s = \sqrt{e + (b + cx)^2}.$$

Iam statuatur $s = b + cx + z$, eritque

$$ss = e + (b + cx)^2 = (b + cx)^2 + 2(b + cx)z + zz,$$

vnde sequitur

$$e - z z = 2 z (b + c x), \text{ siue } b + c x = \frac{e - z z}{2 z};$$

hincque colligitur

$$x = \frac{e - z z}{2 c z} - \frac{b}{c}, \text{ seu } x = \frac{e - 2 b z - z z}{2 c z}.$$

Aequatio autem $b + c x = \frac{e - z z}{2 z}$ differentiata praebet

$$c d x = - \frac{e d z}{2 z z} - \frac{d z}{2} = - \frac{e d z - z z d z}{2 z z},$$

vnde deducitur

$$d x = - \frac{d z (e + z z)}{2 c z z}, \text{ at ob}$$

$$b + c x = \frac{e - z z}{2 z} \text{ fiet } s = \frac{e + z z}{2 z}.$$

His ergo valoribus substitutis formula nostra $X d x$ reddetur rationalis. Postquam igitur eius integrale fuerit inventum loco z valor ante inuentus $\sqrt{(e + (b + c x)^2) - b - c x}$ erit substituendus.

II. Sin autem γ fuerit quantitas negativa, ponatur $\gamma = -c c$ et $\beta = -2 b c$, vt habeatur

$$s = \sqrt{(a - 2 b c x - c c x x)} = \sqrt{(a + b b - (b + c x)^2)},$$

vbi evidens est, quantitatem $a + b b$ necessario esse debere positivam, quia alioquin s euaderet imaginarium. Quamobrem ponamus breuitatis gratia $a + b b = a a$, vt fiat $s = \sqrt{(a a - (b + c x)^2)}$, ad quam formam rationalem efficiendam statuamus

$$\sqrt{(a a - (b + c x)^2)} = a - (b + c x) z,$$

vnde sumtis quadratis erit

$$a a - (b + c x)^2 = a a - 2 a z (b + c x) + (b + c x)^2 z z$$

quae aequatio reducitur ad hanc:

$$-(b + c x) = -2 a z + (b + c x) z z,$$

vnde reperitur

$b +$

$$b + cx = \frac{az}{1+zz}, \text{ ideoque}$$

$$x = \frac{az - b - bzz}{c(1+zz)}.$$

Illa autem aequatio differentiatia dat

$$cdx = \frac{adz(1+zz) - azzdz}{(1+zz)^2} = \frac{adz(1-zz)}{(1+zz)^2};$$

unde fit

$$dx = \frac{adz(1-zz)}{c(1+zz)^2}.$$

Porro autem, cum fit

$$s = a - (b + cx)z, \text{ ob } b + cx = \frac{az}{1+zz}$$

erit $s = \frac{a(1-zz)}{1+zz}$, quocirca, si loco x , s et dx inveni

hi valores substituantur, formula proposita differentialis $X dx$ euadet rationalis, et per variabilem z exprimetur, cuius integrale postquam fuerit inuentum, loco z vbique eius restituatur valor assumtus

$$z = a - \sqrt{aa - (b + cx)^2},$$

et integrale obtinebitur per solam variabilem x expressum.

Exemplum I.

§. 31. Si fuerit

$$dy = \frac{dx}{\sqrt{(e + (b + cx)^2)}},$$

quae formula ad casum priorem pertinet, erit

$$dy = \frac{dx}{s} = -\frac{dz}{cz}, \text{ ob } dx = -\frac{dz(e+zz)}{2ezz} \text{ et } s = \frac{e+zz}{2z},$$

cuius integrale est $y = -\frac{1}{c} l z$; restituto, ergo valore

$$z = \sqrt{(e + (b + cx)^2)} - b - cx, \text{ erit}$$

$$y = -\frac{1}{c} l (\sqrt{(e + (b + cx)^2)} - b - cx) + C,$$

quod integrale si euanescere debeat posito $x = 0$, fiet

$$C = \frac{1}{c} l (\sqrt{(e + bb)} - b).$$

C 2

Corol

Corollarium.

§. 32. Si ponatur $b = 0$ et $c = 1$, siue

$$dy = \frac{dx}{\sqrt{(e + xx)}}, \text{ erit integrale}$$

$$y = -l(\sqrt{(e + xx)} - x) + l\sqrt{e} = l \frac{\sqrt{e}}{\sqrt{(e + xx)} - x}$$

quae formula reducitur ad hanc:

$$y = l \frac{\sqrt{(e + xx)} + x}{\sqrt{e}}$$

Cum vero porro fit

$$d: \sqrt{(e + xx)} = \frac{x dx}{\sqrt{(e + xx)}} \text{ erit}$$

$$\int \frac{x dx}{\sqrt{(e + xx)}} = \sqrt{(e + xx)}$$

Si igitur hae duae formulae combinentur, habebitur ista integratio notatu digna:

$$\int \frac{A dx + B x dx}{\sqrt{(e + xx)}} = A l \frac{\sqrt{(e + xx)} + x}{\sqrt{e}} + B \sqrt{(e + xx)}$$

Exemplum II.

§. 33. Sit $dy = \frac{dx}{\sqrt{(aa - (b + cx)^2)}}$, quae formula ad casum secundum est referenda, ita ut fit $dy = \frac{dx}{s}$.

Cum igitur fit

$$dx = \frac{2adz(1 - zz)}{c(1 + zz)^2} \text{ et } s = \frac{a(1 - zz)}{1 + zz}, \text{ erit}$$

$$y = \frac{dx}{s} = \frac{2}{c} \cdot \frac{dz}{1 + zz}$$

vnde fit integrando $y = \frac{2}{c} A \text{ tang. } z$.

Quia igitur est

$$z = \frac{a - \sqrt{(aa - (b + cx)^2)}}{b + cx}, \text{ erit}$$

$$y = \frac{2}{c} A \text{ tang. } \frac{a - \sqrt{(aa - (b + cx)^2)}}{b + cx} + C.$$

Corollarium.

§. 34. Sit igitur iterum $b = 0$ et $c = 1$, seu formu-

formula differentialis proposita $dy = \frac{dx}{\sqrt{(aa - xx)}}$, reperietur
que

$$y = 2A \operatorname{tang.} \frac{a - \sqrt{(aa - xx)}}{x} + C.$$

Quia igitur tangens huius arcus est $\frac{a - \sqrt{(aa - xx)}}{x}$, tan-
gens dupli arcus erit $\frac{x}{\sqrt{(aa - xx)}}$, ita ut sit

$$y = A \operatorname{tang.} \frac{x}{\sqrt{(aa - xx)}};$$

huius autem arcus sinus erit $\frac{x}{a}$; sicque integrale quaesi-
tum

$$\int \frac{dx}{\sqrt{(aa - xx)}} = A \operatorname{fin.} \frac{x}{a}.$$

Quia porro

$$d. \sqrt{(aa - xx)} = - \frac{x dx}{\sqrt{(aa - xx)}}, \text{ erit}$$

$$\int \frac{x dx}{\sqrt{(aa - xx)}} = - \sqrt{(aa - xx)},$$

quocirca ista generalior conficitur integratio:

$$\int \frac{A dx + B x^2}{\sqrt{(aa - xx)}} = A \operatorname{tang.} \frac{x}{a} - B \sqrt{(aa - xx)}.$$

Problema 13.

§. 35. Si fuerit V functio rationalis binarum quan-
tatum v^n et s , existente $s = \sqrt{(a + \beta v^n + \gamma v^{2n})}$, for-
mulam differentialem $V v^{n-1} dv$ ab irrationalitate liberare.

Solutio.

Ponatur $v^n = x$, erit

$$s = \sqrt{(a + \beta x + \gamma xx)} \text{ et } v^{n-1} dv = \frac{dx}{n};$$

hic ergo iam erit V functio rationalis binarum quantita-
tum x et s , existente

$$s = \sqrt{(a + \beta x + \gamma xx)}$$

C 3

et

et formula ab irrationalitate liberanda erit $\frac{\sqrt{dx}}{n}$; qui casus profus convenit cum problemate praecedente, ideoque eandem habebit solutionem.

Scholion.

§. 36. Praecepta hactenus tradita ad omnes fere formulas differentiales, quae quidem adhuc tractari potuerunt, extenduntur. Interim tamen eiusmodi casus occurrere possunt, quibus idonea substitutio, ad irrationalitatem tollendam necessaria, non tam facile perspicitur: sed acri iudicio demum inuestigare licet, in quo negotio cum praeepta generalia tradere nondum liceat, exempla quaedam particularia speciminis loco in medium afferamus.

Exemplum I.

§. 37. Si proposita fuerit haec formula irrationalis: $dP = \frac{dx(1+xx)}{(1-xx)\sqrt{1+xx}}$, eius integrale P inuestigare.

Si quis hic eiusmodi vti vellet substitutione, qua formula $\sqrt{1+xx}$ ad rationalitatem perduceretur, oleum et operam effret perditurus, interim tamen singulari artificio sequens substitutio negotium conficere poterit. Statuatur $\frac{x\sqrt{2}}{1-xx} = p$, eritque $1+pp = \frac{1+xx}{(1-xx)^2}$, hinc

$$\sqrt{1+pp} = \frac{\sqrt{1+xx}}{1-xx};$$

tum vero erit differentiando

$$dp = \frac{dx\sqrt{2}(1+xx)}{(1-xx)^2},$$

ex quibus valoribus colligitur

$$\frac{dp}{\sqrt{1+pp}} = \frac{dx\sqrt{2}(1+xx)}{(1-xx)\sqrt{1+xx}},$$

quae

quae feliciter cum formula ipsa proposita conuenit, ita
vt fit

$$\frac{d p}{\sqrt{(1+p p)}} = d P \sqrt{2}, \text{ siue } d P = \frac{1}{\sqrt{2}} \frac{d p}{\sqrt{1+p p}},$$

vnde colligitur integrando

$$P = \frac{1}{\sqrt{2}} l(\sqrt{(1+p p)} + p).$$

Quare si loco p et $\sqrt{(1+p p)}$ valores dati substituantur,
haec obtinetur integratio satis memorabilis:

$$P = \int \frac{d x (1+x x)}{(1+x x) \sqrt{(1+x x)}} = \frac{1}{\sqrt{2}} l \frac{\sqrt{(1+x x)} + \sqrt{2}}{1+x x}.$$

Exemplum 2.

§. 38. Si proposita fuerit haec formula irrationalis:

$$\frac{d x (1-x x)}{(1+x x) \sqrt{(1+x x)}}, \text{ eius integrale } Q \text{ inuestigare.}$$

Ad hoc praestandum fiat $\frac{x \sqrt{2}}{1+x x} = q$, eritque

$$\sqrt{(1-q q)} = \frac{\sqrt{(1+x x)}}{1+x x};$$

tum vero erit $d q = \frac{d x (1-x x) \sqrt{2}}{(1+x x)^2}$, atque hinc colligitur

$$\frac{d q}{\sqrt{(1-q q)}} = \frac{d x (1-x x) \sqrt{2}}{(1+x x) \sqrt{(1+x x)}} = d Q \sqrt{2};$$

vnde fit

$$Q = \frac{1}{\sqrt{2}} \int \frac{d q}{\sqrt{(1-q q)}} = \frac{1}{\sqrt{2}} A \text{ fin. } q.$$

Restituto ergo pro q valore assumpto ista obtinebitur inte-
gratio:

$$Q = \int \frac{d x (1-x x)}{(1+x x) \sqrt{(1+x x)}} = \frac{1}{\sqrt{2}} A \text{ fin. } \frac{x \sqrt{2}}{1+x x}.$$

Scholion.

§. 39. Cum istae duae formulae:

$$\frac{d x (1+x x) \sqrt{2}}{(1-x x) \sqrt{(1+x x)}} \text{ et } \frac{d x (1-x x) \sqrt{2}}{(1+x x) \sqrt{(1+x x)}}$$

per-

perductae sint ad has simplices:

$$\frac{dp}{\sqrt{(1+p^2)}} \text{ et } \frac{dq}{\sqrt{(1-q^2)}};$$

quarum vtraque facile ab irrationalitate liberatur, istae ipsae formulae propositae ope idoneae substitutionis ab irrationalitate liberari possunt; unde mirum non est, earum integralia siue per logarithmum siue per arcum circulem exhiberi potuisse. Satis enim iam est ostensum: omnium formularum differentialium rationalium integralia semper vel per logarithmos et arcus circulares, vel adeo algebraice exhiberi posse; quod igitur etiam de illis formulis irrationalibus est tenendum, quas certae substitutionis ope ad rationalitatem perducere licet. Unde vicissim plures Geometrae concluderunt: si quae formula differentialis nullo plane modo ab irrationalitate liberari queat, tum eius integrale etiam neque per logarithmos nec arcus circulares, multo minus algebraice exprimi posse, sed ad aliud genus quantitatum transcendentium referri oportere. Ceterum combinatio duorum praecedentium exemplorum manuducit ab solutionem sequentium.

Exemplum 3.

§. 40. Si proposita fuerit haec formula differentialis: $dy = \frac{dx \sqrt{(1+x^2)}}{1-x^2}$, eius integrale inuenire.

Hanc formulam per neutram substitutionem ante vsurpatam rationalem reddere licet: vtraque tamen iuncta negotium confici poterit; namque eius integrale per logarithmos et arcus circulares sequenti artificio expedietur: Formula enim proposita in binas sequentes partes discerpi potest, quae sunt

dy

$$dy = \frac{\frac{1}{2} dx (1 + x x)}{(1 - x x) \sqrt{(1 + x^4)}} + \frac{\frac{1}{2} dx}{(1 + x x) \sqrt{(1 + x^4)}}$$

quippe quarum summa ipsam formulam nostram proposi-
tam producit; prodit enim:

$$dy = \frac{\frac{1}{2} dx (1 + x x)^2 + \frac{1}{2} dx (1 - x x)^2}{(1 - x^4) \sqrt{(1 + x^4)}} \\ = \frac{dx (1 + x^4)}{(1 - x^4) \sqrt{(1 + x^4)}} = \frac{dx \sqrt{(1 + x^4)}}{1 - x^4}$$

Quod si ergo duo praecedentia exempla in subsidium vo-
centur, manifesto fiet $dy = \frac{1}{2} dP + \frac{1}{2} dQ$, consequenter
integrale quaesitum erit $y = \frac{1}{2} P + \frac{1}{2} Q$, quod sequenti
modo exprimere licebit:

$$\int \frac{dx \sqrt{(1 + x^4)}}{1 - x^4} = \frac{1}{2\sqrt{2}} \int \frac{\sqrt{(1 + x^4)} + x\sqrt{2}}{1 - x x} + \frac{1}{2\sqrt{2}} A \text{ fin. } \frac{x\sqrt{2}}{1 + x x}$$

Exemplum. 4.

§. 41. Si proposita fuerit haec formula differentialis:

$$dy = \frac{x x dx}{(1 - x^4) \sqrt{(1 + x^4)}}, \text{ eius integrale inuestigare.}$$

Haec formula simili modo ac praecedens tractari
potest; discernatur enim in sequentes duas partes:

$$\frac{\frac{1}{2} dx (1 + x x)}{(1 - x x) \sqrt{(1 + x^4)}} - \frac{\frac{1}{2} dx (1 - x x)}{(1 + x x) (\sqrt{1 + x^4})^2}$$

quippe quae coniunctae producunt

$$dy = \frac{\frac{1}{2} dx (1 + x x)^2 - \frac{1}{2} dx (1 - x x)^2}{(1 - x^4) \sqrt{(1 + x^4)}} \\ = \frac{\frac{1}{2} dx \cdot 4 x x}{(1 - x^4) \sqrt{(1 + x^4)}} = \frac{x x dx}{(1 - x^4) \sqrt{(1 + x^4)}}$$

quae cum sit ipsa formula proposita, erit ex praecedenti-
 bus exemplis: $dy = \frac{1}{2} dP - \frac{1}{2} dQ$, consequenter $y = \frac{1}{2} P - \frac{1}{2} Q$,
 hinc integrale quaesitum ita reperietur expressum:

$$\int \frac{x x dx}{(1-x^2) \sqrt{(1+x^2)}} = \frac{1}{2\sqrt{2}} \left[\frac{\sqrt{(1+x^2)} + x\sqrt{2}}{1-x^2} - \frac{1}{x\sqrt{2}} \right] \text{ A fin. } \frac{x\sqrt{2}}{1+x^2}$$

Scholion.

§. 42. Haec duo postrema exempla si nullo plane modo ope cuiuspiam substitutionis ad rationalitatem per-
 duci possent, insigne praebere documentum, quod con-
 clusio supra memorata quandoque fallere possit: Re autem
 attentius perpensa inveni, omnia haec quatuor exempla
 ope vnica substitutione immediate ad rationalitatem per-
 duci ideoque integrari posse; id quod ostendisse utique
 operae erit pretium.

Alia resolutio

quatuor postremorum exemplorum:

§. 43. Statuatur pro primo exemplo

$$v = \frac{x\sqrt{2}}{\sqrt{(1+x^2)}}, \text{ eritque } \sqrt{(1+vv)} = \frac{1+x^2}{\sqrt{(1+x^2)^2}}$$

$$\text{tum vero } \sqrt{(1-uv)} = \frac{1-x^2}{\sqrt{(1+x^2)}}, \text{ vnde fit}$$

$$\sqrt{\frac{1+vv}{1-uv}} = \frac{1+x^2}{1-x^2} \text{ et } \sqrt{(1-uv)^2} = \frac{1-x^4}{1+x^2}$$

At differentiando adipiscimur

$$dv = \frac{dx (1-x^2) \sqrt{2}}{(1+x^2) \sqrt{(1+x^2)^2}}$$

Cum nunc sit $\frac{1-x^2}{1+x^2} = \sqrt{(1-uv)}$; erit

$$dv = \frac{dx \sqrt{2} \sqrt{(1-uv)}}{\sqrt{(1+x^2)}}, \text{ siue } \frac{dv}{\sqrt{(1-uv)}} = \frac{dx \sqrt{2}}{\sqrt{(1+x^2)}};$$

quae aequalitas maxime est notatu digna. Quod si iam haec

haec aequatio multiplicetur per $\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx}$, nascetur
 haec aequatio:

$$\frac{dv}{1-vv} = \frac{dx(1+xx)\sqrt{2}}{(1-xx)(1+x^2)^2}$$

ficque erit

$$\int \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^2)}} = \frac{1}{\sqrt{2}} \int \frac{dv}{1-vv} = \frac{1}{2\sqrt{2}} \int \frac{1+v}{1-v}$$

Deinde aequatio

$$\frac{1}{\sqrt{2}} \frac{dv}{\sqrt{(1-v^2)}} = \frac{dx}{\sqrt{(1+x^2)}}$$

multiplicetur per

$$\sqrt{\frac{1-vv}{1+vv}} = \frac{1-xx}{1+xx}$$

ac prodibit formula exempli secundi

$$\int \frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^2)}} = \frac{1}{\sqrt{2}} \int \frac{dv}{1+vv} = \frac{1}{\sqrt{2}} \text{A tang. } v.$$

Porro eadem aequatio

$$\frac{1}{\sqrt{2}} \frac{dv}{\sqrt{(1-v^2)}} = \frac{dx}{\sqrt{(1+x^2)}}$$

diuidatur per

$$\sqrt{(1-v^2)} = \frac{1-x^2}{1+x^2} \text{ et prodibit}$$

$$\frac{1}{\sqrt{2}} \frac{dv}{1-v^2} = \frac{dx\sqrt{(1+x^2)}}{1-x^2}$$

quae est ipsa formula exempli tertii, ita vt iam fit

$$\int \frac{dx\sqrt{(1+x^2)}}{1-x^2} = \frac{1}{\sqrt{2}} \int \frac{dv}{1-v^2} = \frac{1}{2\sqrt{2}} \int \frac{dv}{1+vv} + \frac{1}{2\sqrt{2}} \int \frac{dv}{1-vv}$$

quod integrale cum ante inuento egregie conuenit. Tandem postrema aequatio hic inuenta:

$$\frac{1}{\sqrt{2}} \frac{dv}{1-v^2} = \frac{dx\sqrt{(1+x^2)}}{1-x^2}$$

ducatur in $vv = \frac{xx}{1+x^2}$, vt prodeat:

$$\frac{1}{\sqrt{2}} \frac{vv dv}{1-v^2} = \frac{2xx dx \sqrt{(1+x^2)}}{(1-x^2)(1+x^2)} = \frac{2xx dx}{(1-x^2)\sqrt{(1+x^2)}}$$

vnde pro exemplo quarto colligitur

$$\int \frac{xx dx}{(1-x^2)\sqrt{(1+x^2)}} = \frac{1}{2\sqrt{2}} \int \frac{vv dv}{1-v^2} = -\frac{1}{4\sqrt{2}} \int \frac{dv}{1+vv} + \frac{1}{4\sqrt{2}} \int \frac{dv}{1-vv},$$

vnde cum fit $v = \frac{x\sqrt{z}}{\sqrt{(1+x^2)}}$, erit

$$\int \frac{dv}{1-v^2} = \frac{1}{2} \int \frac{1+v}{1-v} = \frac{1}{2} \int \frac{\sqrt{(1+x^2)} + x\sqrt{z}}{\sqrt{(1+x^2)} - x\sqrt{z}}$$

$$= \frac{1}{2} \int \frac{(\sqrt{(1+x^2)} + x\sqrt{z})^2}{(1-xx)^2} = \int \frac{\sqrt{(1+x^2)} + x\sqrt{z}}{1-xx}$$

Deinde vero est

$$\int \frac{dv}{1+v^2} = A \text{ tang. } v = A \text{ fin. } \frac{v}{\sqrt{(1+v^2)}} = A \text{ fin. } \frac{x\sqrt{z}}{1+xx}$$

Scholion.

§. 44. Quanquam autem haec quatuor exempla ad rationalitatem reducere licuit: tamen conclusio supra memorata, quod omnes formulae integrales, quae nullo modo rationales effici queant, ad aliud pertineant transcendentium genus, neque per solos logarithmos et arcus circulares expediri possint, non solum manet suspecta, sed etiam falsitas eius euidenter ob oculos poni potest. Sit enim functio

$$X = \frac{a}{\sqrt{(1+xx)}} + \frac{b}{\sqrt[3]{(1+x^3)}} + \frac{c}{\sqrt[4]{(1+x^4)}};$$

tum certe formula differentialis $X dx$ nullo modo ad rationalitatem perducitur; interim tamen singulos eius partes

$$\frac{a dx}{\sqrt{(1+xx)}}, \frac{b dx}{\sqrt[3]{(1+x^3)}} \text{ et } \frac{c dx}{\sqrt[4]{(1+x^4)}}$$

seorsim rationales effici et integralia per logarithmos et arcus circulares exhiberi possunt. Coronidis loco hic sequens problema notatu dignum adiungamus.

Pro-

Problema.

§. 45. Formularum integralium $\int \frac{dx}{\sqrt{1+x^2}}$ et $\int \frac{dv}{\sqrt{1-v^2}}$ valores per series inuestigare; pro casibus, quibus ponitur tam $v = 1$ quam $x = 1$.

Solutio.

Cum posito $v = \frac{x\sqrt{2}}{\sqrt{1+x^2}}$, vt supra fecimus, eu-
dens sit, sumto $x = 0$ fore etiam $v = 0$, et sumto $x = 1$
fore $v = 1$, ita vt hae duae quantitates x et v simul eua-
nescant et simul unitati aequentur: hinc deducimus istam
aequationem differentialem attentione dignissimam:

$$\frac{x}{\sqrt{1+x^2}} \cdot \frac{dv}{\sqrt{1-v^2}} = \frac{dx}{\sqrt{1+x^2}}$$

quas ergo ambas formulas in series conuertii oportet; erit
autem

$$\frac{x}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \text{etc. et}$$

$$\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \text{etc.}$$

Illae iam per dv multiplicatae et integratae praebet:

$$\int \frac{dv}{\sqrt{1-v^2}} = v + \frac{1}{2 \cdot 5}v^5 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9}v^9 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 13}v^{13} + \text{etc.}$$

unde posito $v = 1$ valor huius integralis erit:

$$1 + \frac{1}{2 \cdot 5} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 13} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 17} + \text{etc.}$$

quam seriem littera A indicemus. Simili modo altera se-
ries in dx ducta et integrata producit

$$\int \frac{dx}{\sqrt{1+x^2}} = x - \frac{1}{2 \cdot 5}x^5 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9}x^9 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 13}x^{13} + \text{etc.}$$

D 3

cuius

cuius valor factò $x = 1$ erit

$$1 - \frac{1}{2.5} + \frac{1.3}{2.4.9} - \frac{1.3.5}{2.4.6.12} + \frac{1.3.5.7}{2.4.6.8.17} - \text{etc.}$$

quèm littera B designemus, ita ut sit $B = \frac{A}{\sqrt{2}}$, siue $A = B\sqrt{2}$,
vnde patet, priorem seriem se habere ad posteriorem ut
 $\sqrt{2} : 1$.

Scholion.

§. 46. Valor formulæ integralis $\int \frac{x^2 v}{\sqrt{(1-v^2)}}$ etiam
hoc modo per seriem inuestigari potest. Cum sit

$$\frac{x}{\sqrt{(1-v^2)}} = \frac{(1+vw)^{-\frac{1}{2}}}{\sqrt{(1-v^2)}} \text{ et}$$

$$(1+vw)^{-\frac{1}{2}} = 1 - \frac{1}{2}vw + \frac{1.3}{2.4}v^2 - \frac{1.3.5}{2.4.6}v^3 + \text{etc.}$$

notetur esse $\int \frac{dvw}{\sqrt{(1-v^2)}} = \frac{\pi}{2}$. Deinde pro integration reli-
quorum terminorum ponatur

$$\int \frac{v^{n+2} dv}{\sqrt{(1-v^2)}} = A v^{n+1} \sqrt{(1-v^2)} + B \int \frac{v^n dv}{\sqrt{(1-v^2)}}$$

quæ æquatio differentiata dat

$$\frac{v^{n+2}}{\sqrt{(1-v^2)}} = (n+1) A v^n \sqrt{1-v^2} - \frac{A v^{n+2}}{\sqrt{(1-v^2)}} + \frac{B v^n}{\sqrt{(1-v^2)}}$$

vnde per $\sqrt{(1-v^2)}$ multiplicando prodit

$$v^{n+2} = (n+1) A v^n - (n+1) A v^{n+2} - A v^{n+2} + B v^n.$$

Hinc termini in quibus inest v^{n+2} inter se æquati præ-
bent $1 = -(n+2) A$, ideoque $A = -\frac{1}{n+2}$, termini vero
 v^n continentés præbent $0 = (n+1) A + B$, vnde fit
 $B = \frac{n+1}{n+2}$, ita ut in genere fit

$$\int \frac{v^{n+2} dv}{\sqrt{(1-v^2)}} = -\frac{1}{n+2} v^{n+2} \sqrt{(1-v^2)} + \frac{n+1}{n+2} \int \frac{v^n dv}{\sqrt{(1-v^2)}},$$

quod

quod integrale vti requiritur evanescit posito $v = 0$. Ponatur nunc $v = 1$, eritque:

$$\int \frac{v^{n+2} dv}{\sqrt{(1-vv)}} = \frac{n+1}{n+2} \int \frac{v^n dv}{\sqrt{(1-vv)}}$$

hinc ergo pro n scribendo successively valores 0, 2, 4, 6, 8, etc. erit:

I. $\int \frac{v dv}{\sqrt{(1-vv)}} = \frac{1}{2} \cdot \frac{\pi}{2}$

II. $\int \frac{v^3 dv}{\sqrt{(1-vv)}} = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

III. $\int \frac{v^5 dv}{\sqrt{(1-vv)}} = \frac{5}{8} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

etc. etc.

quibus valoribus adhibitis erit casu $v = 1$

$$\int \frac{dv}{\sqrt{(1-v^4)}} = \frac{\pi}{2} - \frac{1}{2^2} \cdot \frac{\pi}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{\pi}{2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{\pi}{2} + \text{etc.}$$

$$= \frac{\pi}{2} \left(1 - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1^2 \cdot 3 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \text{etc.} \right)$$

ita vt fit ex problemate praecedente.

$$I = \frac{1}{2 \cdot 5} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 9 \cdot 13} - \text{etc.}$$

$$= \frac{\pi}{2} \left(\frac{1}{2} - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{etc.} \right)$$

unde fit

$$\frac{\pi}{2} = \frac{I - \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 9 \cdot 13} + \text{etc.}}{I - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{etc.}}$$