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De figura curvae elasticae contra obiectiones quasdam Illustris d'Alembert

Leonhard Euler

DE FIGVRA CVRVAE ELASTICAE CONTRA OBIECTIONES QVASDAM ILL. D'ALEMBERT.

Auctore

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§. 1.

Tab. I. Fig. 7. Consideretur virga elastica, in termino B muro firmiter infixa, cui in altero termino A appensum sit pondus Q, quo virgae inducatur figura incuruata BMA, quam ergo vtrum ex principio a *Iacobo Bernoulli* stabilito determinare liceat nec ne, videamus; siquidem *Ill. d'Alembert* in Tomo nouissimo Opusculorum contendit, hoc principium neutiquam sufficere, et curuam manere indeterminatam.

§. 2. Ponamus igitur totam virgae longitudinem $BMA = a$, et pro quouis puncto indefinito M vocetur arcus $BM = s$, abscissa $BP = x$, applicata $PM = y$ et inclinatio tangentis ad Horizontem $VTM = \Phi$. Praeterea ponatur pro altero termino A, abscissa $BF = f$, applicata $FA = g$ et inclinatio extremae tangentis $= \zeta$, quibus positis erit $dx = ds \operatorname{cos.} \Phi$ et $dy = ds \operatorname{sin.} \Phi$.

§. 3.

§. 3. Iam momentum ponderis Q respectu puncti M est Q. P F $= Q(f-x)$, quod sustineri debet ab elasticitate virgae in M, quae si vocetur $= E$, quia reciproce proportionalis est radio osculi $\frac{ds}{d\Phi}$, statim habebimus

$$Q(f-x) = E\frac{d\Phi}{ds}, \text{ vnde fit } \frac{d\Phi}{ds} = \frac{Q}{E}(f-x).$$

§. 4. Ponatur $\frac{d\Phi}{ds} = \frac{f-x}{as}$, vbi bb est quantitas ex statu quaestionis data; at quantitates f, g, ζ, ex inuenta demum Curua definiri poterunt. Hanc autem aequationem differentiando, posito ds constante, colligitur

$$\frac{dd\Phi}{ds} = -\frac{dx}{bb} = -\frac{ds \operatorname{cof.}\Phi}{bb}, \text{ vnde fit}$$

$$bb\,dd\Phi = -ds^2 \operatorname{cof.}\Phi.$$

Multiplicetur haec aequatio per $2d\Phi$ et integrando prodibit ista:

$$bb\,d\Phi^2 = -2ds^2 \operatorname{fin.}\Phi + C\,ds^2, \text{ vnde fit}$$

$ds^2 = \frac{bb\,d\Phi^2}{C - 2\operatorname{fin.}\Phi}$, et posito $C = 2\alpha$ integrando fiet

$$s = \frac{b}{\sqrt{2}}\int\frac{d\Phi}{\sqrt{(\alpha - \operatorname{fin.}\Phi)}}.$$

Tum vero, ob

$$f-x = \frac{bb\,d\Phi}{ds}, \text{ erit } f-x = b\sqrt{2(\alpha - \operatorname{fin.}\Phi)}.$$

§. 5. Hic primo patet, sumto $x = 0$ fieri debere $\Phi = 0$, quandoquidem tangens Curuae in B necessario manet horizontalis, ad quam conditionem *Ill. d'Alembert* non attendisse videtur: at hinc statim sequitur $f = b\sqrt{2\alpha}$. Deinde, posito $x = f$, fit $\Phi = \zeta$ et $\alpha = \operatorname{fin.}\zeta$ et $f = b\sqrt{2\operatorname{fin.}\zeta}$, hincque $x = b\sqrt{2\operatorname{fin.}\zeta} - b\sqrt{2(\alpha - \operatorname{fin.}\zeta)}$ atque

$$s = \frac{b}{\sqrt{2}}\int\frac{d\Phi}{\sqrt{(\operatorname{fin.}\zeta - \operatorname{fin.}\Phi)}},$$

quod integrale, a termino $\Phi = 0$ vsque ad $\Phi = \zeta$ extensum,

sum, dabit totam virgam $s = a$, ita vt

$$a = \frac{b}{\sqrt{2}} \int \frac{d\,\Phi}{\sqrt{(\sin.\zeta - \sin.\Phi)}} \left(\begin{smallmatrix} a & \Phi = 0 \\ ad & \Phi = \zeta \end{smallmatrix}\right),$$

ex qua aequatione angulum ζ definitum iri, patet; sicque omnia per binas quantitates datas a et b determinantur, cum sit

$$x = b\sqrt{2}\,(\sqrt{\sin.\zeta} - \sqrt{(\sin.\zeta - \sin.\Phi)}) \text{ et}$$

$$y = \frac{b}{\sqrt{2}} \int \frac{d\,\Phi \sin.\Phi}{\sqrt{(\sin.\zeta - \sin.\Phi)}}.$$

§. 6. Cum igitur peruenerimus ad hanc aequationem

$$\frac{a\sqrt{2}}{b} = \int \frac{d\,\Phi}{\sqrt{(\sin.\zeta - \sin.\Phi)}} \left(\begin{smallmatrix} a & \Phi = 0 \\ ad & \Phi = \zeta \end{smallmatrix}\right),$$

videamus quomodo hoc integrale per seriem commodissime exprimi queat. Hunc in finem ponamus $\sin.\zeta = \alpha$ et $\sin.\Phi = z$, atque ob $d\,\Phi = \frac{dz}{\sqrt{(1 - zz)}}$ erit

$$\frac{a\sqrt{2}}{b} = \int \frac{dz}{\sqrt{(\alpha - z)}\sqrt{(1 - zz)}} \int \frac{dz}{\sqrt{(\alpha - z)}} \left(1 + \tfrac{1}{2} zz + \tfrac{1.3}{2.4} z^4 + \text{etc.}\right)$$

Ponatur

$$\int \frac{z^n\,dz}{\sqrt{(\alpha - z)}} = A\, z^n \sqrt{(\alpha - z)} + B \int \frac{z^{n-1}\,dz}{\sqrt{(\alpha - z)}}$$

et differentiando fiet

$$z^n = n\,A\,\alpha\, z^{n-1} - (n + \tfrac{1}{2})\,A\, z^n + B\, z^{n-1},$$

vnde colligitur

$$A = \frac{-2}{2n+1} \text{ et } B = \frac{2n\alpha}{2n+1}.$$

Pro terminis autem $z = 0$ et $z = \alpha$ membrum absolutum euanescit, vnde oritur ista reductio generalis:

$$\int \frac{z^n\,dz}{\sqrt{(\alpha - z)}} = \frac{2n\alpha}{2n+1} \int \frac{z^{n-1}\,dz}{\sqrt{(\alpha - z)}}.$$

§. 7.

§. 7. Quod ſi iam loco n ſucceſſiue ſcribamus numeros 1, 2, 3, 4, etc. habebimus valores ſequentes:

$$\int \frac{-dz}{\sqrt{(\alpha - z)}} = 2\sqrt{\alpha}; \qquad \int \frac{z\,dz}{\sqrt{(\alpha - z)}} = \tfrac{2}{3}\, 2\alpha\sqrt{\alpha}.$$

$$\int \frac{zz\,dz}{\sqrt{(\alpha - z)}} = \frac{2.\,4}{3.\,5}.\, 2\alpha\alpha\sqrt{\alpha}; \qquad \int \frac{z^3\,dz}{\sqrt{(\alpha - z)}} = \frac{2.\,4.\,6}{3.\,5.\,7}.\, 2\alpha^3\sqrt{\alpha}$$

$$\int \frac{z^4\,dz}{\sqrt{(\alpha - z)}} = \frac{2.\,4.\,6.\,8}{3.\,5.\,7.\,9}.\, 2\alpha^4\sqrt{\alpha}; \int \frac{z^5\,dz}{\sqrt{(\alpha - z)}} = \frac{2.\,4.\,6.\,8.\,10}{3.\,5.\,7.\,9.\,11}\, 2\alpha^5\sqrt{\alpha}$$

$$\int \frac{z^6\,dz}{\sqrt{(\alpha - z)}} = \frac{2.\,4.\,6.\,8.\,11.\,12}{3.\,5.\,7.\,9.\,11.\,13}.\, 2\alpha^6\sqrt{\alpha};\ \text{etc.}$$

quibus rite ſubſtitutis colligitur:

$$\frac{a\sqrt{2}}{b} = 2\sqrt{\alpha}\left(1 + \frac{1}{2}.\frac{2.\,4}{2.\,5}\alpha^2 + \frac{1.\,3}{2.\,4}.\frac{2.\,4.\,6.\,8}{3.\,5.\,7.\,9}\alpha^4 + \text{etc.}\right)$$

quae expreſſio facile ad hanc concinniorem reducitur:

$$\frac{a}{b\sqrt{2}} = \sqrt{\alpha}\left(1 + \frac{2.\,2}{3.\,5}\alpha^2 + \frac{2.\,2.\,6.\,6}{3.\,5.\,7.\,9}\alpha^4 + \frac{2.\,2.\,6.\,6.\,10.\,10}{3.\,5.\,7.\,9.\,11.\,13}\alpha^6 + \text{etc.}\right)$$

ſiue

$$a = f\left(1 + \frac{2.\,2}{3.\,5}\alpha^2 + \frac{2.\,2.\,6.\,6}{3.\,5.\,7.\,9}\alpha^4 + \text{etc.}\right),\ \text{ob}\ f = b\sqrt{2\alpha}.$$

§. 8. Cum porro ſit

$$y = \frac{b}{\sqrt{2}}\int \frac{z\,dz}{\sqrt{(\alpha - z)}\sqrt{(1 - zz)}}$$

erit per ſeriem

$$y = \frac{b}{\sqrt{2}}\int \frac{z\,dz}{\sqrt{(\alpha - z)}}\left(1 + \frac{1}{2}zz + \frac{1.\,3}{2.\,4}z^4 + \frac{1.\,3.\,5}{2.\,4.\,6}z^6 + \text{etc.}\right)$$

vnde, integrando a termino $z = 0$ ad terminum $z = \alpha$, ob $y = FA = g$ erit

$$\frac{g\sqrt{2}}{b} = 2\alpha\sqrt{\alpha}\left(\frac{2}{3} + \frac{1}{2}.\frac{2.\,4.\,6}{3.\,5.\,7}\alpha^2 + \frac{1.\,3}{2.\,4}.\frac{2.\,4.\,6.\,8.\,10}{3.\,5.\,7.\,9.\,11}\alpha^4 + \text{etc.}\right),$$

ſiue

$$\frac{g}{b\sqrt{2}} = \tfrac{2}{3}\alpha\sqrt{\alpha}\left(1 + \frac{2.\,6}{5.\,7}\alpha^2 + \frac{2.\,6.\,6.\,10}{5.\,7.\,9.\,11}\alpha^4 + \frac{2.\,6.\,6.\,10.\,10.\,14}{5.\,7.\,9.\,11.\,13.\,15}\alpha^6 + \text{etc.}\right)$$

§. 9.

§. 9. Quod fi iam angulus ζ valde paruus ftatuatur, ita vt fufficiat terminos primos adhibuiffe, ob α valde paruum, ergo $\frac{a}{b\sqrt{2}} = \sqrt{\alpha}$, ideoque

$$\alpha = \text{fin.}\, \zeta, = \frac{aa}{2bb}, \text{ erit}$$

$$\frac{g}{b\sqrt{2}} = \tfrac{2}{3}\alpha\sqrt{\alpha} = \tfrac{2}{3}\frac{a^3}{2b^3\sqrt{2}}, \text{ ideoque } g = \frac{a^3}{3bb} \text{ et}$$

$$f = \frac{a}{1 + \frac{a^4}{15b^4}} = a\left(1 - \frac{a^4}{15b^4}\right),$$

figura hinc curuae fatis exacte cognofcitur. Erit enim

$$y = \frac{xx(3a - x)}{6bb}$$

et radius ofculi in puncto $M = \frac{bb}{a - x}$; vnde patet radium ofculi in B fore $\frac{bb}{a}$ et in $A = \infty$. Quod fi axis horizontalis ex A capiatur, ac ponatur $AV = t$ et $VM = u$, reperietur $u = \frac{3aat - t^3}{6bb}$; vnde patet Curuam cis et vltra A binas portiones aequales habere atque vtrinque in infinitum porrigi.

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