1783

De motu oscillatorio pendulorum ex filo tenso dependentium

Leonhard Euler

DE
MOTV OSCILLATORIO
PENDVLORVM EX FILO TENSO DEPENDENTIVM.

Auctore
L. EVLERO.

§. 1.

Concipiamus filum A E F B, in punctis A et B fixum, cui in punctis E et F appensa sint pendula E M et F N, quorum motum definiri oportet, postquam de statu aequilibrii vtcunque fuerint deturbata, cuiusmodi autem motus tanquam minimos spectamus, simulque a grauitate ipsius fili mentem abstrahimus, ita vt tantum massae duorum ponderum M et N in calculum ingrediantur. Tab. I. Fig. 1.

§. 2. Primum igitur consideremus statum aequilibrii, ac ponamus fili portiones $AE = a$, $EF = b$, $FB = c$; anguli autem, quibus hae portiones ad horizontalem A C inclinantur, sint α, β et γ; tum vero longitudinem pendulorum vocemus $EM = m$ et $FN = n$,

dum

dum eorum pondera literis M et N exprimuntur. Hinc igitur erit recta horizontalis

$$A\,C = a \operatorname{cos}. \alpha + b \operatorname{cos}. \beta + c \operatorname{cos}. \gamma$$

recta vero verticalis

$$C\,B = a \operatorname{sin}. \alpha + b \operatorname{sin}. \beta + c \operatorname{sin}. \gamma.$$

§. 3. Ponamus nunc statum aequilibrii ita esse turbatum, vt anguli α, β, γ, incrementa ceperint infinite parva ω, ω' et ω'', et quia illa intervalla A C et B C manent immutata, habebimus has duas aequationes:

$$a\,\omega \operatorname{sin}. \alpha + b\,\omega' \operatorname{sin}. \beta + c\,\omega'' \operatorname{sin}. \gamma = 0, \text{ et}$$

$$a\,\omega \operatorname{cos}. \alpha + b\,\omega' \operatorname{cos}. \beta + c\,\omega'' \operatorname{cos}. \gamma = 0.$$

sicque statim atque vnicus horum angulorum ω, ω', ω'' fuerit datus, bini reliqui simul determinantur.

§. 4. Sumamus nunc ambo pendula E M et F N de situ verticali declinare angulis ϕ et ϕ', itidem quam minimis, ac ductis inde ad verticalem horizontalibus $M\,m$ et $N\,n$, vocemus quasi-coordinatas

$$A\,m = x \text{ et } m\,M = y;\; A\,n = x' \text{ et } n\,N = y'$$

eritque

$$x = a \operatorname{sin}. (\alpha + \omega) + m \operatorname{cos}. \phi = a \operatorname{sin}. \alpha + a\,\omega \operatorname{cos}. \alpha + m \text{ et}$$

$$y = a \operatorname{cos}. (\alpha + \omega) + m \operatorname{sin}. \phi = a \operatorname{cos}. \alpha - a\,\omega \operatorname{sin}. \alpha + m\,\phi$$

deinde

$$x' = a \operatorname{sin}. (\alpha + \omega) + b \operatorname{sin}. (\beta + \omega') + n \operatorname{cos}. \phi'$$
$$= a \operatorname{sin}. \alpha + a\,\omega \operatorname{cos}. \alpha + b \operatorname{sin}. \beta + b\,\omega' \operatorname{cos}. \beta + n$$

$$y' = a \operatorname{cos}. (\alpha + \omega) + b \operatorname{cos}. (\beta + \omega') + n \operatorname{sin}. \phi'$$
$$= a \operatorname{cos}. \alpha - a\,\omega \operatorname{sin}. \alpha + b \operatorname{cos}. \beta - b\,\omega' \operatorname{sin}. \beta + n\,\phi'.$$

§. 5.

§. 5. Quia nunc pondera M et N primum verticaliter deorſum vrgentur, viribus M et N, fila autem EM et FN aequalibus viribus tenduntur, vires verticales ſe mutuo deſtruunt. Horizontaliter autem pondera ad axem AB pelluntur, viribus $M\Phi$ et $N\Phi'$, ſicque ex motus principiis erit

$$\frac{ddx}{2g\,dt^2} = 0,\quad \frac{ddy}{2g\,dt^2} = -\Phi,\quad \frac{ddx'}{2g\,dt^2} = 0 \text{ et } \frac{ddy'}{2g\,dt^2} = -\Phi',$$

vnde reſtitutis valoribus habebimus has quatuor aequationes:

$$\text{I. } + \frac{a\,dd\omega \cos.\alpha}{2g\,dt^2} = 0.$$

$$\text{II. } -\frac{a\,dd\omega \sin.\alpha + m\,dd\Phi}{2g\,dt^2} = -\Phi.$$

$$\text{III. } + \frac{a\,dd\omega \cos.\alpha + b\,dd\omega' \cos.\beta}{2g\,dt^2} = 0.$$

$$\text{IV. } -\frac{a\,dd\omega \sin.\alpha - b\,dd\omega' \sin.\beta + n\,dd\Phi'}{2g\,dt^2} = -\Phi'.$$

§. 6. Ex harum aequationum prima et tertia neutiquam concludere licet fore tam $\frac{ddx}{2g\,dt^2}$ quam $\frac{ddx'}{2g\,dt^2} = 0$; plus enim inde non ſequitur, quam tenſionem filorum EM et FN non exacte ponderibus M et N eſſe aequalem. Si enim ponamus tenſionem fili $EM = P$ et fili $FN = Q$, hae aequationes ita ſe habebunt:

$$\text{I. } \frac{a\,dd\omega \cos.\alpha}{2g\,dt^2} = 1 - \frac{P}{M} \text{ et}$$

$$\text{III. } \frac{a\,dd\omega \cos.\alpha + b\,dd\omega' \cos.\beta}{2g\,dt^2} = 1 - \frac{Q}{N},$$

quae formulae cum ſint quaſi infinite paruae, erit vtique proxime $P = M$ et $Q = N$; quibus valoribus in aequatione ſecunda et quarta ſine errore vti licet, quia ibi per quantitates minimas Φ et Φ' ſunt multiplicatae.

§. 7.

§. 7. Ad has ergo aequationes resoluendas vtamur methodo iam aliquoties adhibita, dum scilicet angulis variabilibus ω, ω', ω''; Φ et Φ', rationem constantem tribuimus, vt eorum mutationes ad motum penduli simplicis reuocentur, cuius longitudo sit $= r$; motus vero hac aequatione exprimatur: $\frac{d\,d\,z}{2\,g\,d\,t^2} = -\frac{z}{r}$, et iam ponamus

$$\omega = \mathfrak{A} z, \quad \omega' = \mathfrak{B} z, \quad \omega'' = \mathfrak{C} z, \quad \Phi = \mathfrak{M} z \text{ et } \Phi' = \mathfrak{N} z,$$

hincque ergo fiet:

$$\frac{d\,d\,\omega}{2\,g\,d\,t^2} = -\frac{\mathfrak{A} z}{r}; \quad \frac{d\,d\,\omega'}{2\,g\,d\,t^2} = -\frac{\mathfrak{B} z}{r};$$
$$\frac{d\,d\,\Phi}{2\,g\,d\,t^2} = -\frac{\mathfrak{M} z}{r}; \quad \frac{d\,d\,\Phi'}{2\,g\,d\,t^2} = -\frac{\mathfrak{N} z}{r}.$$

§. 8. His igitur valoribus substitutis nanciscimur sequentes aequationes simplices:

$$\mathrm{I} - \frac{\mathfrak{A}\,a\,z\,\mathrm{cos}.\,\alpha}{r} = 1 - \frac{P}{M},$$
$$\mathrm{II} + \frac{\mathfrak{A}\,a\,z\,\mathrm{sin}.\,\alpha - \mathfrak{M}\,m\,z}{r} = -\mathfrak{M}\,z,$$
$$\mathrm{III} - \frac{\mathfrak{A}\,a\,z\,\mathrm{cos}.\,\alpha - \mathfrak{B}\,b\,z\,\mathrm{cos}.\,\beta}{r} = 1 - \frac{Q}{N},$$
$$\mathrm{IV} + \frac{\mathfrak{A}\,a\,z\,\mathrm{sin}.\,\alpha + \mathfrak{B}\,b\,z\,\mathrm{sin}.\,\beta - \mathfrak{N}\,n\,z}{r} = -\mathfrak{N}\,z,$$

ex quarum prima et tertia deducimus tensiones P et Q, quae sunt:

$$P = M + \frac{\mathfrak{A}\,a\,z\,M\,\mathrm{cos}.\,\alpha}{r}; \quad Q = N + \frac{Nz}{r}(\mathfrak{A}\,a\,\mathrm{cos}.\,\alpha + \mathfrak{B}\,b\,\mathrm{cos}.\,\beta);$$

ex secunda vero et quarta deducimus

$$\mathfrak{M} = \frac{\mathfrak{A}\,a\,\mathrm{sin}.\,\beta}{m - r} \text{ et } \mathfrak{N} = \frac{\mathfrak{A}\,a\,\mathrm{sin}.\,\alpha + \mathfrak{B}\,b\,\mathrm{sin}.\,\beta}{n - r}.$$

§. 9. Hos eosdem valores etiam substituamus in binis formulis, quae supra, ob puncta A et B fixa, sunt datae; vnde duas sequentes nanciscimur aequationes:

𝔄 a

$$\mathfrak{A}\, a \text{ fin. } \alpha + \mathfrak{B}\, b \text{ fin. } \beta + \mathfrak{C}\, c \text{ fin. } \gamma = 0 \text{ et}$$
$$\mathfrak{A}\, a \text{ cof. } \alpha + \mathfrak{B}\, b \text{ cof. } \beta + \mathfrak{C}\, c \text{ cof. } \gamma = 0$$

ex quibus, eliminando $\mathfrak{C}$, elicimus

$$\mathfrak{A}\, a \text{ fin. } (\alpha - \gamma) + \mathfrak{B}\, b \text{ fin. } (\beta - \gamma) = 0,$$

hincque porro

$$\mathfrak{B} = + \frac{\mathfrak{A}\, a \text{ fin. } (\alpha - \gamma)}{b \text{ fin. } (\gamma - \beta)} \text{ et } \mathfrak{C} = \frac{\mathfrak{A}\, a \text{ fin. } (\alpha - \beta)}{c \text{ fin. } (\beta - \gamma)},$$

vnde fequentes formamus valores concinniores:

$$\mathfrak{A}\, a = f \text{ fin. } (\beta - \gamma); \; \mathfrak{B}\, b = f \text{ fin. } (\gamma - \alpha) \text{ et } \mathfrak{C}\, c = f \text{ fin. } (\alpha - \beta).$$

§. 10. Quodfi hos valores in praecedentes introducamus, fiet

$$P = M + \frac{\mathfrak{M} f z}{r} \text{ fin. } (\beta - \gamma) \text{ cof. } \alpha \text{ et}$$
$$Q = N - \frac{\mathfrak{N} f z}{r} \text{ fin. } (\alpha - \beta) \text{ cof. } \gamma;$$

tum vero

$$\mathfrak{M} = \frac{f \text{ fin. } (\beta - \gamma) \text{ fin. } \alpha}{m - r} \text{ et } \mathfrak{N} = \frac{- f \text{ fin. } (\alpha - \beta) \text{ fin. } \gamma}{n - r}.$$

§. 11. Supereft autem, vt etiam motus fili rationem in computum ducamus; in quo cum nullam inertiam admittamus, vires, quibus puncta E et F follicitantur, fe mutuo deftruere debent; haec autem puncta, praeter tenfionem filorum P et Q, potiffimum a tenfione fili follicitantur. Hunc in finem ponamus tenfionem fili $AE = A$, fili $EF = B$ et fili $FB = C$, quas vires fecundum directiones verticalem et horizontalem refoluere oportet. Hinc igitur punctum E horizontaliter finiftrorfum trahetur his viribus:

$$\text{I. } A \text{ cof. } (\alpha + \omega) - B \text{ cof. } (\beta + \omega') + P \text{ fin. } \Phi = 0,$$

verticaliter autem furfum trahitur his viribus:

$$\text{II.}\quad A \sin.(\alpha + \omega) - B \sin.(\beta + \omega') - P \cos.\Phi = 0.$$

Punctum autem F horizontaliter ſiniſtrorſum trahitur his viribus:

$$\text{III.}\quad B \cos.(\beta + \omega') - C \cos.(\gamma + \omega'') + Q \sin.\Phi' = 0,$$

verticaliter autem ſurſum, his:

$$\text{IV.}\quad B \sin.(\beta + \omega') - C \sin.(\gamma + \omega'') - Q \cos.\Phi' = 0.$$

§. 12. Reſoluamus iam iſtos angulos, ſimulque loco ω, ω', ω''; Φ et Φ' ſuos valores aſſumtos ſcribamus, et obtinebemus ſequentes formas:

$$\text{I.}\quad A \cos.\alpha - B \cos.\beta - A\mathfrak{A}z \sin.\alpha + B\mathfrak{B}z \sin.\beta + P\mathfrak{M}z = 0.$$

$$\text{II.}\quad A \sin.\alpha - B \sin.\beta + A\mathfrak{A}z \cos.\alpha - B\mathfrak{B}z \cos.\beta - P = 0.$$

$$\text{III.}\quad B \cos.\beta - C \cos.\gamma - B\mathfrak{B}z \sin.\beta + C\mathfrak{C}z \sin.\gamma + Q\mathfrak{N}z = 0.$$

$$\text{IV.}\quad B \sin.\beta - C \sin.\gamma + B\mathfrak{B}z \cos.\beta - C\mathfrak{C}z \cos.\gamma - Q = 0.$$

§. 13. Quia autem praecipuum negotium in hoc verſatur, vt tenſiones A, B, C ex calculo exterminemus, potius vtamur formis prioribus, vtpote ſimplicioribus; ac primo quidem ex prima et ſecunda literam A definiamus hoc modo:

$$A = \frac{B \cos.(\beta + \omega') - P\Phi}{\cos.(\alpha + \omega)} = \frac{B \sin.(\beta + \omega') - P}{\sin.(\alpha + \omega)},$$

ex quo duplici valore colligimus

$$B = \frac{P(\cos.(\alpha + \omega) + \Phi \sin.(\alpha + \omega))}{\sin.(\alpha - \beta + \omega - \omega')}.$$

Eodem modo ex tertia et quarta eliciamus valores literae C, qui ſunt

$$C = \frac{B \cos.(\beta + \omega') + Q\Phi'}{\cos.(\gamma + \omega'')} = \frac{B \sin.(\beta + \omega') - Q}{\sin.(\gamma + \omega'')},$$

vnde deducimus ſequentem valorem:

$$B = \frac{Q(\cos.(\gamma + \omega'') + \Phi' \sin.(\gamma + \omega''))}{\sin.(\beta - \gamma + \omega' - \omega'')}.$$

Quod ſi iam hi duo valores ipſius P inter ſe aequentur, obtine-

obtinebimus aequationem finalem, ex qua quantitatem r determinari oportebit.

§. 14. Quoniam vero initio angulos α, β, γ, ita assumsimus, vt, dum pondera M et N verticaliter dependent, status aequilibrii resultet, nostrum calculum ad istum statum aequilibrii facillime accommodabimus, si modo angulos ω, ω' et ω'', vna cum angulis Φ et Φ' nullos statuamus, tum vero loco P et Q ipsa pondera M et N scribamus, quo facto bini valores, qui pro B essent produturi, erunt

$$B = \frac{M\ \text{cos}.\ \alpha}{\text{sin}.\ (\alpha - \beta)} \text{ et } B = \frac{N\ \text{cos}.\ \gamma}{\text{sin}.\ (\beta - \gamma)},$$

quibus aequatis habebimus hanc aequationem pro statu aequilibrii

$$M\ \text{cos}.\ \alpha\ \text{sin}.\ (\beta - \gamma) = N\ \text{cos}.\ \gamma.\ \text{sin}.\ (\alpha - \beta)$$

quae reducitur ad hanc formam:

$$M\ (\text{sin}.\ (\alpha + \beta - \gamma) - \text{sin}.\ (\alpha - \beta + \gamma))$$
$$= N\ (\text{sin}.\ (\alpha - \beta + \gamma) + \text{sin}.\ (\alpha - \beta - \gamma)).$$

§. 15. Euidens nunc est, ex hac aequatione euoluta eam, quae ex superiori nostra aequatione oriretur, immediate formari posse, si loco angulorum α, β, γ, scribantur $(\alpha + \omega)$, $(\beta + \omega')$, $(\gamma + \omega'')$; deinde vero loco M scribi debebit $P + P\,\Phi$ tang. $(\alpha + \omega)$; at vero loco N scribere debebimus $Q + Q\,\Phi'$ tang. $(\gamma + \omega'')$. Quia vero supra inuenimus

$$P = M + \frac{M f z}{r}\ \text{sin}.\ (\beta - \gamma)\ \text{cos}.\ \alpha \text{ et}$$
$$Q = N - \frac{N f z}{r}\ \text{sin}.\ (\alpha - \beta)\ \text{cos}.\ \gamma,$$

nunc autem est $\Phi = M z$, loco tang. $(\alpha + \omega)$ scribere sufficit tang. α, ita vt nunc sufficiat loco M scribere formulam

M +

$$M + \frac{Mfz}{r}\,\text{fin.}\,(\beta - \gamma)\,\text{cof.}\,\alpha + M\,\mathfrak{M}\,z\,\text{tang.}\,\alpha.$$

Eodem modo loco N fcribi conueniet hanc formulam:

$$N - \frac{Nfz}{r}\,\text{fin.}\,(\alpha - \beta)\,\text{cof.}\,\gamma + N\,\mathfrak{N}\,z\,\text{tang.}\,\gamma.$$

§. 16. Facillime igitur noftram aequationem finalem impetrabimus, fi aequationem ex ftatu aequilibrii deductam, quae erat

$$M\,\text{cof.}\,\alpha\,\text{fin.}\,(\beta - \gamma) = N\,\text{cof.}\,\gamma\,\text{fin.}\,(\alpha - \beta),$$

differentiemus, omnibus literis M, N, α, β, γ, variabilibus fumtis, ita fcilicet vt fit

$$d\alpha = \omega = \mathfrak{A}\,z = \frac{fz\,\text{fin.}\,(\beta - \gamma)}{a},$$
$$d\beta = \omega' = \mathfrak{B}\,z = \frac{fz\,\text{fin.}\,(\gamma - \alpha)}{b},$$
$$d\gamma = \omega'' = \mathfrak{C}\,z = \frac{fz\,\text{fin.}\,(\alpha - \beta)}{c},$$
$$dM = \frac{Mfz}{r}\,\text{fin.}\,(\beta - \gamma)\,\text{cof.}\,\alpha + M\,\mathfrak{M}\,z\,\text{tang.}\,\alpha \text{ et}$$
$$dN = -\frac{Nfz}{r}\,\text{fin.}\,(\alpha - \beta)\,\text{cof.}\,\gamma + N\,\mathfrak{N}\,z\,\text{tang.}\,\gamma.$$

Supra vero iam inuenimus

$$\mathfrak{M} = \frac{f\,\text{fin.}\,(\beta - \gamma)\,\text{fin.}\,\alpha}{m - r} \text{ et } \mathfrak{N} = -\frac{f\,\text{fin.}\,(\alpha - \beta)\,\text{fin.}\,\gamma}{n - r}.$$

Hi igitur valores in aequatione illa differentiata fubftitui debent, quae ita fe habet

$$\begin{aligned} &dM\,\text{cof.}\,\alpha\,\text{fin.}\,(\beta - \gamma) - M\,d\alpha\,\text{fin.}\,\alpha\,\text{fin.}\,(\beta - \gamma) \\ &\quad + M\,(d\beta - d\gamma)\,\text{cof.}\,\alpha\,\text{cof.}\,(\beta - \gamma) = \\ &dN\,\text{cof.}\,\gamma\,\text{fin.}\,(\alpha - \beta) - N\,d\gamma\,\text{fin.}\,\gamma\,\text{fin.}\,(\alpha - \beta) \\ &\quad + N\,(d\alpha - d\beta)\,\text{cof.}\,\gamma\,\text{cof.}\,(\alpha - \beta). \end{aligned}$$

Calculus autem nimis fieret prolixus, fi hos valores actu fubftituere vellemus. Sufficiat autem hic obferuaffe, quantitatem r ex ifta aequatione determinari poffe, quandoquidem, facta euolutione, totum negotium ad aequationem cubicam reducetur.

Appli-

Applicatio ad exemplum.

Tab. I. Fig. 2.

§. 17. Sumamus primo puncta A et B, in quibus filum est fixum, in eadem recta horizontali esse sita; deinde sint portiones fili AE, EF, FB, inter se aequales, ita vt sit $b = a$ et $c = a$; tertio sit in statu aequilibrii portio media EF horizontalis, ita vt sit $\beta = 0$; hinc ergo manente angulo $BAE = \alpha$, fiet angulus $\gamma = -\alpha$, vnde sequitur interuallum $AB = 2\,a \operatorname{cos.} \alpha + a$. His positis aequatio pro statu aequilibrii dabit

$$M \operatorname{cos.} \alpha \operatorname{sin.} \alpha = N \operatorname{cos.} \alpha \operatorname{sin.} \alpha$$

vnde patet, pondera M et N inter se esse debere aequalia; pendulorum autem longitudines m et n maneant adhuc indeterminatae.

§. 18. His ergo constitutis, valores differentialium sequenti modo definientur:

$$d\alpha = \frac{f\,z \operatorname{sin.} \alpha}{a};\ d\beta = -\frac{f\,z \operatorname{sin.} 2\alpha}{a};\ d\gamma = \frac{f\,z \operatorname{sin.} \alpha}{a},$$

$$dM = \frac{M f z}{r} \operatorname{sin.} \alpha \operatorname{cos.} \alpha + \frac{M f z \operatorname{sin.} \alpha^2 \operatorname{tang.} \alpha}{m - r},$$

$$dN = -\frac{M f z}{r} \operatorname{sin.} \alpha \operatorname{cos.} \alpha - \frac{M f z \operatorname{sin.} \alpha^2 \operatorname{tang.} \alpha}{n - r}.$$

Quod si igitur hi valores in aequatione differentiali supra exhibita substituantur, in singulis terminis occurret factor Mfz, quem ergo omittamus, vnde ista aequatio sequentem induet formam:

$$\left(\frac{\operatorname{sin.} \alpha \operatorname{cos.} \alpha}{r} + \frac{\operatorname{sin.} \alpha^2 \operatorname{tang.} \alpha}{m - r}\right) \operatorname{sin.} \alpha \operatorname{cos.} \alpha - \frac{\operatorname{sin.} \alpha^3}{a} - \frac{\operatorname{cos.} \alpha^2 (\operatorname{sin.} 2\alpha + \operatorname{sin.} \alpha)}{a}$$

$$= -\left(\frac{\operatorname{sin.} \alpha \operatorname{cos.} \alpha}{r} + \frac{\operatorname{sin.} \alpha^2 \operatorname{tang.} \alpha}{n - r}\right) \operatorname{sin.} \alpha \operatorname{cos.} \alpha + \frac{\operatorname{sin.} \alpha^3}{a}$$

$$+ \frac{\operatorname{cos.} \alpha^2 (\operatorname{sin.} 2\alpha + \operatorname{sin.} \alpha)}{a},$$

quae euoluta abit in formam sequentem:

$$\frac{2\,\text{fin.}\,\alpha^2\,\text{cof.}\,\alpha^2}{r}+\frac{\text{fin.}\,\alpha^4}{m-r}+\frac{\text{fin.}\,\alpha^4}{n-r}-\frac{2\,\text{fin.}\,2\,\alpha\,\text{cof.}\,\alpha^2}{a}-\frac{2\,\text{fin.}\,\alpha\,\text{cof.}\,\alpha^2}{a}=0,$$

ex qua aequatione noftram incognitam r quaeri oportet.

§. 19. Quod fi iam vtrique pendulo eandem tribuamus longitudinem, vt fit $n=m$, aequatio noftra fiet

$$\frac{\text{fin.}\,\alpha^2\,\text{cof.}\,\alpha^2}{r}+\frac{\text{fin.}\,\alpha^4}{m-r}-\frac{\text{fin.}\,2\,\alpha\,\text{cof.}\,\alpha^2}{a}-\frac{\text{fin.}\,\alpha\,\text{cof.}\,\alpha^2}{a}=0,$$

quae reducitur ad hanc formam:

$$\frac{m\,\text{fin.}\,\alpha^2\,\text{cof.}\,\alpha^2-r\,\text{fin.}\,\alpha^2\,\text{cof.}\,2\,\alpha}{r\,(m-r)}=\frac{\text{fin.}\,2\,\alpha\,\text{cof.}\,\alpha^2+\text{fin.}\,\alpha\,\text{cof.}\,\alpha^2}{a}$$
$$=\frac{\text{cof.}\,\alpha^2\,(\text{fin.}\,2\,\alpha+\text{fin.}\,\alpha)}{a},$$

hincque porro fit

$$a\,m\,\text{fin.}\,\alpha^2\,\text{cof.}\,\alpha^2-a\,r\,\text{fin.}\,\alpha^2\,\text{cof.}\,2\,\alpha$$
$$=r\,(m-r)\,\text{cof.}\,\alpha^2\,(\text{fin.}\,2\,\alpha+\text{fin.}\,\alpha),$$

quae aequatio iam tantum eft quadratica, cum praecedens ad tertium gradum afcendiffet.

§. 20. Cum igitur cafu, quo ambo pendula longitudinem habent inaequalem, perueniatur ad aequationem cubicam, ponamus eius tres radices effe 1°) $r=k$; 2°) $r=k'$; 3°) $r=k''$; ex quibus triplex motus regularis oritur, quorumque finguli folutionem fpecialem noftri problematis exhibent, quarum vnam euoluiffe fufficiet, ex radice $r=k$ oriundam: Pro binis reliquis enim tantum opus eft loco k vel k' vel k'' fcribere. Cum igitur ex aequatione differentio-differentiali affumta fit $z=\text{fin.}\,(\zeta+t\sqrt{\tfrac{2g}{r}})$, denotante ζ angulum conftantem quemcunque, erit ad quoduis tempus, tam pro fitu fili quam amborum corporum M et N, vt fiquitur:

$$\omega=\frac{f\,z\,\text{fin.}\,\alpha}{a}=\frac{f}{a}\,\text{fin.}\,\alpha\,\text{fin.}\,(\zeta+t\sqrt{\tfrac{2g}{k}});$$

$\omega'=$

$$\omega' = -\frac{f}{a}\,\text{sin.}\,2\alpha\;\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right);$$
$$\omega'' = \frac{f}{a}\,\text{sin.}\,\alpha\;\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right);$$
$$\Phi = \frac{f}{m-k}\,\text{sin.}\,\alpha^2\;\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right) \text{ et}$$
$$\Phi' = \frac{f}{n-k}\,\text{sin.}\,\alpha^2\;\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right),$$

quibus formulis totus motus, tam fili quam pendulorum appensorum, pro radice $r = k$ definitur.

§. 21. Similes igitur formulae etiam pro reliquis radicibus $r = k'$ et $r = k''$ prodibunt, in quibus quidem loco anguli constantis ζ scribi conueniet ζ' et ζ'', similique modo loco coefficientis f scribatur f' et f'', quo facto solutio generalis et completa, quae omnes plane motus in se complectatur, sequenti modo exhiberi poterit:

$$\text{I.}\quad \omega = \frac{f}{a}\,\text{sin.}\,\alpha\,\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right) + \frac{f'}{a}\,\text{sin.}\,\alpha\,\text{sin.}\left(\zeta' + t\sqrt{\tfrac{2g}{k'}}\right) + \frac{f''}{a}\,\text{sin.}\,\alpha\,\text{sin.}\left(\zeta'' + t\sqrt{\tfrac{2g}{k''}}\right),$$

$$\text{II.}\quad \omega' = -\frac{f}{a}\,\text{sin.}\,2\alpha\,\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right) - \frac{f'}{a}\,\text{sin.}\,2\alpha\,\text{sin.}\left(\zeta' + t\sqrt{\tfrac{2g}{k'}}\right) - \frac{f''}{a}\,\text{sin.}\,2\alpha\,\text{sin.}\left(\zeta'' + t\sqrt{\tfrac{2g}{k''}}\right),$$

$$\text{III.}\quad \omega'' = \frac{f}{a}\,\text{sin.}\,\alpha\,\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right) + \frac{f'}{a}\,\text{sin.}\,\alpha\,\text{sin.}\left(\zeta' + t\sqrt{\tfrac{2g}{k'}}\right) + \frac{f''}{a}\,\text{sin.}\,\alpha\,\text{sin.}\left(\zeta'' + t\sqrt{\tfrac{2g}{k''}}\right),$$

$$\text{IV.}\quad \Phi = \frac{f}{m-k}\,\text{sin.}\,\alpha^2\,\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right) + \frac{f'}{m-k'}\,\text{sin.}\,\alpha^2\,\text{sin.}\left(\zeta' + t\sqrt{\tfrac{2g}{k'}}\right) + \frac{f''}{m-k''}\,\text{sin.}\,\alpha^2\,\text{sin.}\left(\zeta'' + t\sqrt{\tfrac{2g}{k''}}\right),$$

$$\text{V.}\quad \Phi' = \frac{f}{n-k}\,\text{sin.}\,\alpha^2\,\text{sin.}\left(\zeta + t\sqrt{\tfrac{2g}{k}}\right) + \frac{f'}{n-k'}\,\text{sin.}\,\alpha^2\,\text{sin.}\left(\zeta' + t\sqrt{\tfrac{2g}{k'}}\right) + \frac{f''}{n-k''}\,\text{sin.}\,\alpha^2\,\text{sin.}\left(\zeta'' + t\sqrt{\tfrac{2g}{k''}}\right),$$

in quas formulas ingrediuntur sex constantes arbitrariae, scilicet f, f', f'' et ζ, ζ', ζ'', quemadmodum ratio omnium motuum possibilium postulat.