1782

De motu oscillatorio mixto plurium pendulorum ex eodem corpore mobili suspensorum

Leonhard Euler

DE

MOTV OSCILLATORIO MIXTO

PLVRIVM PENDVLORVM

EX EODEM CORPORE MOBILI SVSPENSORVM.

Auctore

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§. 1.

Sit A A' A'' corpus quodcunque, quod libere gyrari queat circa punctum O, seu potius circa axem horizontalem, ad planum figurae, quod verticale est intelligendum, normalem, atque in quotcunque eius punctis A, A', A'', A''', etc. suspensa sint pendula A L, A' L', A'' L'' etc., quae ergo singula motum oscillatorium recipere possunt, dum ipsum corpus circa punctum O oscillationes peragit; quo posito quaeritur, qualis motus in hoc corpore et omnibus pendulis oriri debeat, postquam illis initio motus quicunque fuerit impressus. Per se autem manifestum est hic non nisi de oscillationibus minimis quaestionem esse posse. Tab. II. Fig. 1.

§. 2. Antequam autem in hos motus inquiram, considerari opportet statum aequilibrii, in quo corpus cum omnibus pendulis quiescere possit. Hunc in finem per O ducatur recta horizontalis E O F, et ductis ex O ad singula puncta A, A', A'' etc., ex quibus pendula suspenduntur, rectis O A, O A', O A'' etc. vocentur istae rectae $OA = a$, $OA' = a'$, $OA'' = a''$ etc.; tum vero ponantur anguli $EOA = \alpha$, $EOA' = \alpha'$, $EOA'' = \alpha''$ etc.; has scilicet distantias et angulos similibus literis designo, vt, quod de vno dicetur, simul ad omnia reliqua transferri possit. Porro huius corporis massa vocetur $= M$, eiusque momentum inertiae respectu puncti $O = Mkk$; eius vero centrum grauitatis reperiatur in puncto G, pro quo sit distantia $OG = e$ et angulus $EOG = \varepsilon$, qui quidem foret rectus in statu aequilibrii, si de solo hoc corpore esset sermo; verum hic simul consideramus omnia pendula A L, $A'L'$, $A''L''$ etc., quae singula situm teneant verticalem; ac primo eorum longitudines ita designemus: $AL = l$, $A'L' = l'$, $A''L'' = l''$ etc., quas tanquam fila grauitatis expertia spectamus, quibus appensa sint corpora L, L', L'' etc. Cum iam in hoc statu detur aequilibrium, necesse est vt summa omnium momentorum respectu puncti O nihilo fiat aequalis, vnde oritur sequens aequatio:

$$Me\cos.\varepsilon + La\cos.\alpha + L'a'\cos.\alpha' + L''a''\cos.\alpha'' + \text{etc.} = 0.$$

§. 3. Nunc autem, postquam hoc corpus cum pendulis suis de statu aequilibrii vtcunque fuerit deturbatum, elapso tempore $= t$ ipsum corpus declinet a situ aequilibrii angulo quam minimo $= \phi$, ita vt iam singuli anguli α, α', α'' etc. vna cum angulo ε idem acceperint augmentum ϕ; tum vero singula pendula declinent a directione verticali sinistrorsum angulis minimis ω, ω', ω'' etc., ac praeterea sint vires, quibus singula pendula tenduntur T, T', T'' etc.; qui-

quibus positis inuestigandum est, quantis viribus, tam ipsum corpus, quam singula pendula, ad motum concitentur.

§. 4. Quod igitur ad ipsum corpus M, quatenus circa punctum O est mobile, attinet, id primum a proprio pondere, in centro grauitatis G collecto, sollicitatur, cuius vis momentum est

$$M. OG. \text{cos.}(\varepsilon + \Phi) = M e \text{ cos.}(\varepsilon + \Phi).$$

Quia nunc angulus ε incrementum cepit Φ, ob hunc angulum Φ infinite paruum, erit

$$\text{cos}(\varepsilon + \Phi) = \text{cos.}\varepsilon - \Phi \text{ fin.}\varepsilon$$

vnde hoc momentum erit $M e \text{ cos.}\varepsilon - M e \Phi \text{ fin }\varepsilon$, quod tendit dextrorsum ideoque ad inclinationem Φ augendam; si quidem positiuum habuerit valorem. Praeterea vero hoc corpus sollicitatur a singulis pendulis, quatenus scilicet fila A L, A' L', A'' L'' etc. viribus T, T', T'' tenduntur, vbi sufficiet vnicum considerasse. Igitur ex pendulo A L, si situm teneret verticalem, oriretur momentum

$$T. OA \text{ fin. } OAL = T a \text{ cos.}(\alpha + \Phi);$$

quia scilicet angulus α incrementum cepit $= \Phi$; quoniam autem hoc pendulum sinistrorsum a situ verticali declinat angulo ω, erit hoc momentum

$$T a \text{ cos.}(\alpha + \Phi + \omega) = T a \text{ cos.}\alpha - T a (\Phi + \omega) \text{ fin.}\alpha$$

quod pariter ad inclinationem Φ augendam tendit; vnde omnia momenta ex pendulis nata erunt

$$\begin{aligned} &T a \text{ cos.}\alpha + T' a' \text{ cos.}\alpha' + T'' a'' \text{ cos.}\alpha'' \\ &- T a(\Phi + \omega) \text{ fin.}\alpha - T' a'(\Phi + \omega') \text{ fin.}\alpha' - T'' a''(\Phi + \omega'') \text{ fin.}\alpha''. \end{aligned}$$

§. 5. Ponamus breuitatis gratia

$$T\,a\,\text{cof.}\,\alpha + T'\,a'\,\text{cof.}\,\alpha' + T''\,a''\,\text{cof.}\,\alpha'' + \text{etc.} = P,$$
$$T\,a\,\text{fin.}\,\alpha + T'\,a'\,\text{fin.}\,\alpha' + T''\,a''\,\text{fin.}\,\alpha'' + \text{etc.} = Q, \text{ et}$$
$$T\,a\,\omega\,\text{fin.}\,\alpha + T'\,a'\,\omega'\,\text{fin.}\,\alpha' + T''\,a''\,\omega''\,\text{fin.}\,\alpha'' + \text{etc.} = \Omega$$

ita vt totum momentum, motum gyratorium corporis A accelerans, fit

$$M\,e\,\text{cof.}\,\varepsilon - M\,e\,\Phi\,\text{fin.}\,\varepsilon + P - Q\,\Phi - \Omega;$$

vnde, cum momentum inertiae corporis M, refpectu puncti O, pofitum fit $= M\,k\,k$, principia motus hanc fuppeditant aequationem:

$$\frac{d\,d\,\Phi}{2\,g\,d\,t^2} = \frac{M\,e\,\text{cof.}\,\varepsilon - M\,e\,\Phi\,\text{fin.}\,\varepsilon + P - Q\,\Phi - \Omega}{M\,k\,k}.$$

Euidens autem eft, quia fingula pendula tantum infinite parum a fitu verticali declinant, tenfiones filorum ipfis ponderibus L, L', L'' fore aequales; vnde, cum fupra pro ftatu aequilibrii effet

$$M\,e\,\text{cof.}\,\varepsilon + L\,a\,\text{cof.}\,\alpha + L'\,a'\,\text{cof.}\,\alpha' + L''\,a''\,\text{cof.}\,\alpha'' + \text{etc.} = 0$$

erit hic $M\,e\,\text{cof}\,\varepsilon + P = 0$, ita vt iam habeamus

$$\frac{d\,d\,\Phi}{2\,g\,d\,t^2} = -\frac{M\,e\,\Phi\,\text{fin.}\,\varepsilon - Q\,\Phi - \Omega}{M\,k\,k},$$

exiftente

$$Q = L\,a\,\text{fin.}\,\alpha + L'\,a'\,\text{fin.}\,\alpha' + L''\,a''\,\text{fin.}\,\alpha'' + \text{etc. et}$$
$$\Omega = L\,a\,\omega\,\text{fin.}\,\alpha + L'\,a'\,\omega'\,\text{fin.}\,\alpha' + L''\,a''\,\omega''\,\text{fin.}\,\alpha'' + \text{etc.}$$

Tab. II. Fig. 2. §. 6. Porro pro motu fingulorum pendulorum definiendo vnicum confideraffe fufficiet. Hunc in finem ex puncto L in axem horizontalem O E ducatur verticalis L P, vt pro puncto L habeamus coordinatas $O\,P = x$ et $P\,L = y$; tum autem ducta etiam verticali $a\,A\,\alpha$ et horizontali $A\,b$, ob angulum $A\,O\,P = \alpha + \Phi$, erit

O ω

$$O\,a = a \operatorname{cof.}(\alpha + \Phi) \text{ et } A\,a = a \operatorname{fin.}(\alpha + \Phi).$$

Deinde ob angulum $L\,A\,a = \omega$ erit

$$A\,b = l \operatorname{fin.} \omega \text{ et } L\,b = l \operatorname{cof.} \omega,$$

vnde colligitur

$$OP = x = a \operatorname{cof.}(\alpha + \Phi) + l \operatorname{fin.} \omega = a \operatorname{cof.} \alpha - a \Phi \operatorname{fin.} \alpha + l \omega, \text{ et}$$

$$PL = y = a \operatorname{fin.}(\alpha + \Phi) + l \operatorname{cof.} \omega = a \operatorname{fin.} \alpha + a \Phi \operatorname{cof.} \alpha + l.$$

§. 7. Vires autem corpus L vrgentes funt primo eius pondus $= L$, quod tendit ad quantitatem y augendam, tum vero, ob tenfionem fili T, furfum vrgetur vi $= T \operatorname{cof.} \omega = T$, dextrorfum autem vi $= T \operatorname{fin.} \omega = T\,\omega$; vnde ex principiis motus nancifcimur has aequationes:

$$\frac{d\,d\,x}{2\,g\,d\,t^2} = -\frac{T\,\omega}{L} \text{ et } \frac{d\,d\,y}{2\,g\,d\,t^2} = \frac{L - T}{L};$$

hinc, pro x et y valores fubftituendo, habebimus has aequationes:

$$-\frac{a\,d\,d\,\Phi \operatorname{fin.} \alpha + l\,d\,d\,\omega}{2\,g\,d\,t^2} = -\frac{T\,\omega}{L} \text{ et } \frac{a\,d\,d\,\Phi \operatorname{cof.} \alpha}{2\,g\,d\,t^2} = \frac{L - T}{L},$$

ex qua pofteriore intelligitur, tenfionem T infinite parum a pondere L difcrepare, quia membrum $\frac{a\,d\,d\,\Phi \operatorname{cof.} \alpha}{2\,g\,d\,t^2}$ pro infinite paruo eft habendum. Pofito ergo $T = L$ fola aequatio prior nobis relinquitur, quae eft

$$\frac{-a\,d\,d\,\Phi + l\,d\,d\,\omega}{2\,g\,d\,t^2} = -\omega.$$

§. 8. Similes plane aequationes pro fingulis reliquis pendulis reperiuntur. Verum quo omnes has aequationes concinniores reddamus, ponamus breuitatis gratia

$$a \operatorname{fin.} \alpha = b;\; a' \operatorname{fin.} \alpha' = b';\; a'' \operatorname{fin.} \alpha'' = b'';\; \text{etc.}$$

$$a \operatorname{cof.} \alpha = c;\; a' \operatorname{cof.} \alpha' = c';\; a'' \operatorname{cof.} \alpha'' = c'';\; \text{etc.}$$

ſic enim erit

$$Q = L\,b + L'\,b' + L''\,b'' + L'''\,b''' + \text{etc. et}$$

$$\Omega = L\,b\,\omega + L'\,b'\,\omega' + L''\,b''\,\omega'' + \text{etc.}$$

vnde erit primo pro motu ipſius corporis

$$\frac{d\,d\,\Phi}{2\,g\,d\,t^2} = -\frac{M\,e\,\Phi\,\text{ſin.}\,\varepsilon - Q\,\Phi - \Omega}{M\,k\,k};$$

pro ſingulis autem pendulis erunt aequationes

$$\text{I. } \frac{-b\,d\,d\,\Phi + l\,d\,d\,\omega}{2\,g\,d\,t^2} = -\,\omega;$$

$$\text{II. } \frac{-b'\,d\,d\,\Phi + l'\,d\,d\,\omega'}{2\,g\,d\,t^2} = -\,\omega';$$

$$\text{III. } \frac{-b''\,d\,d\,\Phi + l''\,d\,d\,\omega''}{2\,g\,d\,t^2} = -\,\omega'';$$

$$\text{IV. } \frac{-b'''\,d\,d\,\Phi + l'''\,d\,d\,\omega'''}{2\,g\,d\,t^2} = -\,\omega''';$$

quibus aequationibus totus motus determinatur. Pro ſtatu autem aequilibrii recordandum eſt eſſe

$$M\,e\,\text{coſ.}\,\varepsilon + L\,c + L'\,c' + L''\,c'' + L'''\,c''' + \text{etc.} = 0.$$

§. 9. Quia in his omnibus aequationibus incognitae Φ, ω, ω', ω'', ω''' etc. vbique vnicam tantum tenent dimenſionem, pro earum reſolutione methodo iam ſaepius adhibita vtamur, vnde primo quidem tantum ſolutionem ſpecialem reperimus, qua ſinguli motus inſtar pendulorum ſimplicium abſoluuntur. Pro angulo igitur Φ ſtatuamus, pendulum ſimplex, eius motui iſochronum, eſſe $= r$, ita vt ſit $\frac{d\,d\,\Phi}{2\,g\,d\,t^2} = -\frac{\Phi}{r}$; vnde deducitur integrando $\Phi = F\,\text{ſin}\,(f + t\sqrt{\tfrac{2g}{r}})$; vbi F et f ſunt conſtantes arbitrariae per integrationes ingreſſae; quantitas autem r etiamnunc eſt incognita ex reliquis motibus determinanda, pro qua cum plures erunt ſimus valores, iis coniungendis ſolutionem generaliſſimam completam adipiſcemur.

§. 10.

§. 10. In nostris igitur aequationibus vbique loco formulae $\frac{dd\Phi}{2g\,dt^2}$ scribamus eius valorem $-\frac{\Phi}{r}$; tum vero statuamus

$$\omega = b\Phi,\ \omega' = b'\Phi,\ \omega'' = b''\Phi, \text{ etc.}$$

vt sit

$$\frac{dd\omega}{2g\,dt^2} = -\frac{b\Phi}{r},\ \frac{dd\omega'}{2g\,dt^2} = -\frac{b'\Phi}{r},\ \frac{dd\omega''}{2g\,dt^2} = -\frac{b''\Phi}{r}, \text{ etc.}$$

quibus substitutis omnes nostras aequationes per angulum Φ diuidere licebit, hincque pro singulis pendulis sequentes orientur aequalitates:

$$\text{I. } \frac{b}{r} - \frac{bl}{r} = -b, \text{ vnde fit } b = \frac{b}{l-r};$$

$$\text{II. } \frac{b'}{r} - \frac{b'l'}{r} = -b' \text{ - - - } b' = \frac{b'}{l'-r};$$

$$\text{III. } \frac{b''}{r} - \frac{b''l''}{r} = -b'' \text{ - - } b'' = \frac{b''}{l''-r};$$

etc. etc. etc.

Aequatio autem pro motu ipsius corporis euadet

$$-\frac{\Phi}{r} = -\frac{\mathrm{M}e\Phi\,\mathit{fin.}\,\varepsilon - \mathrm{Q}\Phi - \Omega}{\mathrm{M}kk},$$

vbi cum sit

$$\mathrm{Q} = \mathrm{L}b + \mathrm{L}'b' + \mathrm{L}''b'' + \mathrm{L}'''b''' + \text{etc. et}$$

$$\Omega = \mathrm{L}bb\Phi + \mathrm{L}'b'b'\Phi + \mathrm{L}''b''b''\Phi + \text{ etc.}$$

his valoribus substitutis, si praeterea loco b, b', b'' etc. valores ante inuenti surrogentur, orietur sequens aequatio:

$$o = \frac{\mathrm{M}kk}{r} - \mathrm{M}e\,\text{fin.}\,\varepsilon - \mathrm{Q} - \frac{\mathrm{L}bb}{l-r} - \frac{\mathrm{L}'b'b'}{l'-r} - \frac{\mathrm{L}''b''b''}{l''-r} - \text{etc.}$$

ex qua aequatione quantitatem incognitam r erui oportet.

§. 11. Quo hanc aequationem ad formam commodiorem redigamus, totam per $\mathrm{M}e\,\text{fin.}\,\varepsilon + \mathrm{Q}$ deuidamus, ponamusque breuitatis gratia

$$\frac{\mathrm{M}kk}{\mathrm{M}e\,\mathit{fin.}\,\varepsilon + \mathrm{Q}} = m,$$

tum

tum vero

$$\frac{L\,b\,b}{M\,e\,\sin.\varepsilon + Q} = n;\ \frac{L'\,b'\,b'}{M\,e\,\sin.\varepsilon + Q} = n';\ \frac{L''\,b''\,b''}{M\,e\,\sin.\varepsilon + Q} = n'';\ \text{etc.}$$

et impetrabimus hanc aequationem:

$$0 = \frac{m}{r} - 1 - \frac{n}{l-r} - \frac{n'}{l'-r} - \frac{n''}{l''-r} - \text{etc.}$$

quae aequatio, in ordinem redacta pro incognita r, afcendet ad gradum vnitate altiorem quam eft numerus pendulorum.

§. 12. Totum ergo negotium reductum eft ad refolutionem aequationis algebraicae, cuius radicem r erui opportet; vnde fi radix quaecunque r fuerit inuenta, ex ea folutio particularis noftri problematis deriuabitur, quae fequentibus formulis continebitur:

$$\Phi = F\,\text{fin.}\,(f + t\,\surd\tfrac{2g}{r}),\quad \omega = \frac{F\,b}{l-r}\,\text{fin.}\,(f + t\,\surd\tfrac{2g}{r}),$$

$$\omega' = \frac{F\,b'}{l'-r}\,\text{fin.}\,(f + t\,\surd\tfrac{2g}{r}),\quad \omega'' = \frac{F\,b''}{l''-r}\,\text{fin.}\,(f + t\,\surd\tfrac{2g}{r}),$$

$$\omega'' = \frac{F\,b'''}{l'''-r}\,\text{fin.}\,(f + t\,\surd\tfrac{2g}{r});\ \text{etc.}$$

Sicque hoc cafu omnes ofcillationes, tam ipfius corporis, quam fingulorum pendulorum, erunt regulares et inter fe ifochronae, dum omnes refpondent pendulo fimplici, cuius longitudo $= r$, vnde tempus vniuscuiusque ofcillationis erit $= \pi\,\surd\frac{r}{2g}$.

§. 13. Quotquot igitur quantitas r habuerit radices, ex fingulis talis motus regularis oriri poteft; finguli autem per fe tantum fpeciem motus fpecialiffimam complectuntur. At vero fi omnes iftae fpecies inter fe quomodocunque coniungantur, folutio inde refultabit generaliffima, quae omnes plane motus, quos hoc fyftema recipere poteft, in fe complectitur, ita vt, quomodocunque totum

tum systema initio fuerit agitatum et de statu aequilibrii deturbatum, verus motus, qui sequetur, assignari valeat.

§. 14. Vt hinc igitur istam solutionem maxime generalem adipiscamur, omnes valores ipsius r per sequentes characteres indicemus: $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ etc. pro coefficiente autem F scribamus successiue litteras Germanicas maiusculas: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc. atque formulae solutionem generalem praebentes erunt:

$$\Phi = \mathfrak{A}\,\text{sin.}\,(f + t\sqrt{\tfrac{2g}{\mathfrak{a}}}) + \mathfrak{B}\,\text{sin.}\,(f' + t\sqrt{\tfrac{2g}{\mathfrak{b}}}) + \mathfrak{C}\,\text{sin.}\,(f'' + t\sqrt{\tfrac{2g}{\mathfrak{c}}}) + \text{etc.}$$

$$\omega = \frac{\mathfrak{A}\,b}{l-\mathfrak{a}}\,\text{sin.}\,(f + t\sqrt{\tfrac{2g}{\mathfrak{a}}}) + \frac{\mathfrak{B}\,b}{l-\mathfrak{b}}\,\text{sin.}\,(f' + t\sqrt{\tfrac{2g}{\mathfrak{b}}}) + \frac{\mathfrak{C}\,b}{l-\mathfrak{c}}\,\text{sin.}\,(f'' + t\sqrt{\tfrac{2g}{\mathfrak{c}}}) + \text{etc.}$$

$$\omega' = \frac{\mathfrak{A}\,b'}{l'-\mathfrak{a}}\,\text{sin.}\,(f + t\sqrt{\tfrac{2g}{\mathfrak{a}}}) + \frac{\mathfrak{B}\,b'}{l'-\mathfrak{b}}\,\text{sin.}\,(f' + t\sqrt{\tfrac{2g}{\mathfrak{b}}}) + \frac{\mathfrak{C}\,b'}{l'-\mathfrak{c}}\,\text{sin.}\,(f'' + t\sqrt{\tfrac{2g}{\mathfrak{c}}}) + \text{etc.}$$

$$\omega'' = \frac{\mathfrak{A}\,b''}{l''-\mathfrak{a}}\,\text{sin.}\,(f + t\sqrt{\tfrac{2g}{\mathfrak{a}}}) + \frac{\mathfrak{B}\,b''}{l''-\mathfrak{b}}\,\text{sin.}\,(f' + t\sqrt{\tfrac{2g}{\mathfrak{b}}}) + \frac{\mathfrak{C}\,b''}{l''-\mathfrak{c}}\,\text{sin.}\,(f'' + t\sqrt{\tfrac{2g}{\mathfrak{c}}}) + \text{etc.}$$

$$\omega''' = \frac{\mathfrak{A}\,b'''}{l'''-\mathfrak{a}}\,\text{sin.}\,(f + t\sqrt{\tfrac{2g}{\mathfrak{a}}}) + \frac{\mathfrak{B}\,b'''}{l'''-\mathfrak{b}}\,\text{sin.}\,(f' + t\sqrt{\tfrac{2g}{\mathfrak{b}}}) + \frac{\mathfrak{C}\,b'''}{l'''-\mathfrak{c}}\,\text{sin.}\,(f''' + t\sqrt{\tfrac{2g}{\mathfrak{c}}}) + \text{etc.}$$

etc. etc. etc.

Hic autem supponitur, omnes ipsius r valores esse inter se inaequales; si enim duo pluresue inter se essent aequales, solutio haec non amplius foret generalis, sed peculiari artificio opus foret, vt inde solutio generalis obtineatur.

§. 15. Quod quo clarius appareat, ponamus pendulorum numerum esse $= \lambda$, et cum peruenerimus ad

 hanc

hanc aequationem:

$$0 = \frac{m}{r} - 1 - \frac{n}{l-r} - \frac{n'}{l'-r} - \frac{n''}{l''-r} - \text{etc.}$$

ſi eam, ad fractiones tollendas, multiplicemus per productum omnium denominatorum $r(l-r)$ $(l'-r)$ $(l''-r)$ etc. prodibit aequatio ordinis $\lambda + 1$, ſi quidem omnes denominatores fuerint inaequales, quod euenit ſi omnium pendulorum longitudines l, l', l'' etc. fuerint inaequales. At ſi duae ſint inter ſe aequales, puta $l' = l$, aequatio illa factorem habebit $l - r$, vnde radix prodiret $r = l$, qui tamen valor in noſtris formulis locum habere nequit. Si enim eſſet $a = l$, valores angulorum ω, ω', ω'' fierent infiniti; quod incommodum multo magis turbaret, ſi plura quam duo pendula haberent eandem longitudinem; vnde his caſibus aliae radices pro r admitti nequeunt, niſi quae ex noſtra aequatione reſultant, poſtquam ea fuerit diuiſa per $l - r$, vel $(l-r)^2$, vel $(l-r)^3$ etc., prout plura pendula fuerint inter ſe aequalia; quare, ob diminutum numerum valorum ipſius r, non amplius tot conſtantes arbitrariae in calculum introducentur, quot requiruntur ad ſolutionem generalem reddendam.

§. 16. Quin etiam apparet, ſi eſſet $\lambda' = \lambda$, tum valores angulorum ω et ω' datam inter ſe habituros eſſe rationem, ſcilicet vt b ad b', ſiue valores fractionum $\frac{\omega}{b}$ et $\frac{\omega'}{b'}$ inter ſe futuros eſſe aequales; quod ſi vero inſuper $l'' = l' = l$, hi tres valores: $\frac{\omega}{b}$, $\frac{\omega'}{b'}$ et $\frac{\omega''}{b''}$ inter ſe forent aequales, ſicque haec pendula ſimilem motum oſcillatorium eſſent habitura. Neque ergo haec ſolutio amplius eſſet generalis, cum iam in ipſo initio his pendulis diuerſus motus imprimi poſſet; quocirca his caſibus formulae noſtrae

ſtrae inuentae quadam correctione indigebunt, qua tot nouae conſtantes introducantur, quot ad ſolutionem completam poſtulantur. Has igitur correctiones omnino neceſſe erit inueſtigare.

§. 17. Ponamus igitur duo pendula longitudine inter ſe eſſe aequalia, ſiue eſſe $l' = l$; hinc autem duae aequationes differentio differentiales primae erunt:

$$-\frac{b\,dd\Phi + l\,dd\omega}{2g\,dt^2} = -\omega, \text{ et}$$
$$-\frac{b'\,dd\Phi + l\,dd\omega'}{2g\,dt^2} = -\omega';$$

vnde, ſi poſterior per b' diuiſa a priore per b diuiſa ſubtrahatur, remanebit:

$$\frac{l}{2g\,dt^2}\left(\frac{dd\omega}{b} - \frac{dd\omega'}{b'}\right) = -\frac{\omega}{b} + \frac{\omega'}{b'}.$$

Hinc ſi ponamus

$$\frac{\omega}{b} + \frac{\omega'}{b'} = 2p \text{ et } \frac{\omega}{b} - \frac{\omega'}{b'} = 2q, \text{ vt fiat}$$
$$\frac{\omega}{b} = p + q \text{ et } \frac{\omega'}{b'} = p - q$$

litera p manifeſto exprimit valores illos aequales, qui ex ſuperiori ſolutione pro $\frac{\omega}{b}$ et $\frac{\omega'}{b'}$ prodierunt, ita vt nunc ad alterum quantitas q addi, ab altero vero ſubtrahi debeat. Aequatio autem inuenta nunc induet hanc formam:

$$\frac{l}{2g\,dt^2}\,2\,ddq = -2q, \text{ ſiue } \frac{ddq}{2g\,dt^2} = -\frac{q}{l},$$

qua motus penduli ſimplicis longitudinis $= l$ exprimitur, ita vt ſit $q = \mathfrak{I}\,\text{ſin.}\,(i + t\sqrt{\frac{2g}{l}})$. Hinc ergo quaeſita correctio pro caſu, quo $l' = l$, in hoc conſiſtit, vt, ſi p denotet valores ſupra exhibitos pro $\frac{\omega}{b}$ et $\frac{\omega'}{b'}$, nunc reuera ſit

$$\frac{\omega}{b} = p + \mathfrak{I}\,\text{ſin.}\,(i + t\sqrt{\tfrac{2g}{l}}) \text{ et}$$
$$\frac{\omega'}{b'} = p - \mathfrak{I}\,\text{ſin.}\,(i + t\sqrt{\tfrac{2g}{l}});$$

 ſicque

ſicque duae nouae conſtantes $\mathfrak{J}$ et i in calculum ſunt introductae. Reliquorum autem pendulorum pariter atque ipſius corporis motus iidem manent ac ſupra ſunt inuenti.

§. 18. Sint nunc tria pendula l, l' et l'' inter ſe aequalia, tum praeter duas ſuperiores aequationes accedit nunc inſuper tertia

$$-\frac{b'' d d \Phi + l d d \omega''}{2 g d t^2} = -\omega'';$$

Nunc igitur ex prima et tertia colligitur:

$$\frac{l}{2 g d t^2}\left(\frac{d d \omega}{b} - \frac{d d \omega''}{b''}\right) = -\frac{\omega}{b} + \frac{\omega''}{b''};$$

at vero ex ſecunda ac tertia

$$\frac{l}{2 g d t^2}\left(\frac{d d \omega'}{b'} - \frac{d d \omega''}{b''}\right) = -\frac{\omega'}{b'} + \frac{\omega''}{b''}.$$

Hinc, ſi vt ſupra operemur, reperiemus ſimili modo

$$\frac{\omega}{b} - \frac{\omega'}{b'} = \mathfrak{J} \text{ ſin.} \left(i + t \sqrt{\frac{2g}{l}}\right)$$

$$\frac{\omega'}{b'} - \frac{\omega''}{b''} = \mathfrak{J}' \text{ ſin.} \left(i' + t \sqrt{\frac{2g}{l}}\right) \text{ et}$$

$$\frac{\omega''}{b''} - \frac{\omega}{b} = \mathfrak{J}'' \text{ ſin.} \left(i'' + t \sqrt{\frac{2g}{l}}\right),$$

quarum ergo trium formularum ſumma nihilo debet eſſe aequalis, vnde ſequitur fore

$$\mathfrak{J} \text{ ſin. } i + \mathfrak{J}' \text{ ſin. } i' + \mathfrak{J}'' \text{ ſin. } i'' = 0 \text{ et}$$

$$\mathfrak{J} \text{ coſ. } i + \mathfrak{J}' \text{ coſ. } i' + \mathfrak{J}'' \text{ coſ. } i'' = 0,$$

vnde numerus conſtantium ad quatuor reducitur. His notatis, ſi p denotet quantitatem, quae ſupra pro formulis $\frac{\omega}{b}$, $\frac{\omega'}{b'}$, $\frac{\omega''}{b''}$ fuit inuenta, nunc, correctione adiecta, habebimus

$$\frac{\omega}{b} = p + \mathfrak{J} \text{ ſin.} \left(i + t \sqrt{\frac{2g}{l}}\right)$$

$$\frac{\omega'}{b'} = p + \mathfrak{J}' \text{ ſin.} \left(i' + t \sqrt{\frac{2g}{l}}\right) \text{ et}$$

$$\frac{\omega''}{b''} = p + \mathfrak{J}'' \text{ ſin.} \left(i'' + t \sqrt{\frac{2g}{l}}\right),$$

vbi

vbi constantes ita comparatas esse oportet, vt summa trium harum formularum fiat aequalis $3p$.

§. 19. Eodem modo ratiocinium erit instituendum, si plura pendula inter se fuerint aequalia; ita vt nunc super hoc problemate nihil amplius sit desiderandum, quotcunque etiam pondera fuerint appensa. Atque hinc etiam clarissime elucet summus vsus faecundissimi principii Illustr. *Dan. Bernoullii*, quo omnes huiusmodi motus oscillatorios semper ex aliquot motibus pendulorum simplicium compositos esse statuit. Imprimis vero etiam hinc patet, quam egregie istud principium cum primis Mechanicae principiis conspiret, atque adeo ex iis immediate deduci queat. Pulcherrime scilicet hoc principium connexum est cum ea conditione, qua in omnibus huiusmodi motibus definiendis peruenitur ad eiusmodi aequationes differentiales secundi gradus, in quibus omnes incognitae vnicam vbique habent dimensionem, ita vt semper dentur eiusmodi solutiones speciales, in quibus omnes incognitae constantes inter se teneant rationes, quo ipso oscillationes regulares ac simplices innuuntur. Tum vero ex omnibus his solutionibus specialibus per ipsam naturam huiusmodi aequationum solutio generalis et completa formari potest.

§. 20. Quanquam autem haec solutio maxime est generalis, et omnes plane casus in se complectitur, siue omnia pendula sint longitudine aequalia siue inaequalia, si quidem pro aequalibus correctio exposita adhibeatur: tamen dantur insuper casus, qui peculiarem resolutionem requirunt, qui sunt: quando pendula ab ipso axe horizontali E O F ſuſpenduntur, veluti ſi punctum A cadat in iſtum axem, ideoque angulus A O E $= \alpha$ euaneſcat; tum enim diſtantia $b = a$ ſin. α euaneſcet, vnde expreſſio noſtra, pro angulo ω inuenta, nullum plane motum huius penduli indicabit, cum tamen vtique motum recipere queat. Hoc autem caſu aequatio differentialis pro hoc pendulo non amplius inuoluet angulum ϕ, ſed erit ſimpliciter

$$\frac{l\,dd\omega}{2g\,dt^2} = -\omega;$$

vnde patet, iſtud pendulum motum oſcillatorium regularem recipere, perinde ac ſi ex puncto fixo eſſet ſuſpenſum, ita vt eius motus neque ab ipſo corpore M, neque a reliquis pendulis afficiatur. Ac viciſſim quoque hoc pendulum nihil plane conferet ad motum corporis M; quia enim $b = 0$, in quantitatem Q plane non ingreditur; ſimulque etiam quantitas n euaneſcit; vnde patet ab hoc pendulo motum corporis M nullo modo perturbari. Eodem modo res ſe habebit, ſi plura pendula ex ipſo axe E O F fuerint ſuſpenſa, tum enim ſingula libere ſuas oſcillationes peragent, neque vllo modo in motum reliquorum pendulorum, neque ipſius corporis M effectum exerent, quorum igitur motus perinde ſe habebit, ac ſi illa pendula prorſus abeſſent.

INVESTI-