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TRIGONOMETRIA SPHAERICA VNIVERSA, EX PRIMIS PRINCIPIIS BREVITER ET DILVCIDE DERIVATA.

Auctore
L. EVLERO.

§. 1.

Tab. I. Fig. 7. Propofitum fit triangulum fphaericum quodcunque, cuius anguli litteris maiusculis A, B, C, latera autem minusculis *a*, *b*, *c*, in figura adfcriptis, defignentur, ita vt iisdem litteris maiusculis eaedem minusculae opponantur. Iam ex centro Sphaerae, cui literam O tribuamus, per fingulos angulos educantur rectae O C, O *a*, O *b*, quae in centro O angulum folidum conftituent, cuius anguli plani metientur latera trianguli, eorum autem inclinationes mutuae angulos trianguli.

Fig. 8. §. 2. His praemiffis capiatur O C ipfi radio Sphaerae aequalis $=1$, vnde ad O C in vtroque plano C O *a* et C O *b* normaliter ftatuantur rectae C *a* et C *b*; tum vero ex *b* ad C *a* demittatur perpendiculum *b p*, quod fimul ad

ad planum C O a erit normale; praeterea vero ex p ad O a normalis ducatur $p\,q$, ficque, ducta $b\,q$, ea etiam ad O a erit normalis. Hoc modo tota figura, qua indigemus, erit conftructa.

§. 3. Cum iam fit angulus C O a lateri b aequalis, erit

$$C\,a = \text{tang.}\, b \text{ et } O\,a = \text{fec.}\, b = \frac{1}{\text{cof.}\, b}.$$

Simili modo, ob angulum C O $b = a$, erit

$$C\,b = \text{tang.}\, a \text{ et } O\,b = \text{fec.}\, a = \frac{1}{\text{cof.}\, a}.$$

Porro autem, cum fit angulus

$$a\,O\,b = c \text{ et } O\,b = \frac{1}{\text{cof.}\, a} \text{ erit}$$

$$b\,q = \frac{\text{fin.}\, c}{\text{cof.}\, a} \text{ et } O\,q = \frac{\text{cof.}\, c}{\text{cof.}\, a}.$$

Hinc pro reliquis figurae lineis exprimendis, ob angulum

$$a\,C\,b = C, \text{ erit } b\,p = C\,b\, \text{fin.}\, C = \text{tang.}\, a\, \text{fin.}\, C \text{ et}$$

$$C\,p = C\,b\, \text{cof.}\, C = \text{tang.}\, a\, \text{cof.}\, C,$$

vnde porro colligitur

$$a\,p = C\,a - C\,p = \text{tang.}\, b - \text{tang.}\, a\, \text{cof.}\, C,$$

et quia angulus C a O $= 90° - b$, habebitur,

$$p\,q = a\,p\, \text{cof.}\, b = \text{fin.}\, b - \text{tang.}\, a\, \text{cof.}\, b\, \text{cof.}\, C \text{ et}$$

$$a\,q = a\,p\, \text{fin.}\, b = \frac{\text{fin.}\, b^2}{\text{cof.}\, b} - \text{tang.}\, a\, \text{fin.}\, b\, \text{cof.}\, C.$$

Quare, cum inuenerimus O $q = \frac{\text{cof.}\, c}{\text{cof.}\, a}$, fiet

$$O\,a = \frac{1}{\text{cof.}\, b} = \frac{\text{cof.}\, c}{\text{cof.}\, a} + \frac{\text{fin.}\, b^2}{\text{cof.}\, b} - \text{tang.}\, a\, \text{fin.}\, b\, \text{cof.}\, C,$$

ficque erit

$$\frac{\text{cof.}\, c}{\text{cof.}\, a} = \text{cof.}\, b + \text{tang.}\, a\, \text{fin.}\, b\, \text{cof.}\, C, \text{ fiue}$$

$$\text{cof.}\, c = \text{cof.}\, a\, \text{cof.}\, b + \text{fin.}\, a\, \text{fin.}\, b\, \text{cof.}\, C$$

§. 4. Cum iam angulus bqp praebeat inclinationem plani aOb ad aOC, erit iste angulus $bqp = A$, vnde ex triangulo bpq primo habebitur

$$\text{sin } A = \frac{bp}{bq} = \frac{\text{sin. } a \text{ sin. } C}{\text{sin. } c}, \text{ siue } \frac{\text{sin. } C}{\text{sin. } c} = \frac{\text{sin. } A}{\text{sin. } a};$$

vnde iam sequitur, sinus angulorum nostri trianguli proportionales esse sinubus laterum oppositorum. Deinde aequatio

$$\text{cos. } A = \frac{pq}{bq} = \frac{\text{cos. } a \text{ sin. } b - \text{sin. } a \text{ cos. } b \text{ cos. } C}{\text{sin. } a},$$

cum binis praecedentibus coniuncta totam Doctrinam sphaericam complectitur, quod autem vberiorem explicationem postulat, vnde singulas has tres aequationes magis euoluamus.

Euolutio primae formulae.

$$\frac{\text{sin. } C}{\text{sin. } c} = \frac{\text{sin. } A}{\text{sin. } a}.$$

§. 5. Cum tam litteras maiusculas A, B, C, quam minusculas a, b, c, inter se permutare liceat, si modo iisdem litteris maiusculis eaedem minusculae oppositae relinquantur, erit etiam $\frac{\text{sin. } C}{\text{sin. } c} = \frac{\text{sin. } B}{\text{sin. } b}$, sicque prodibit haec tergemina aequatio:

$$\frac{\text{sin. } A}{\text{sin. } B} = \frac{\text{sin. } B}{\text{sin. } b} = \frac{\text{sin. } C}{\text{sin. } c}.$$

Deinde etiam notasse iuuabit sequentes aequalitates:

$$\text{sin. } A \text{ sin. } b = \text{sin. } B \text{ sin. } a$$

$$\text{sin. } A \text{ sin. } c = \text{sin. } C \text{ sin. } a$$

$$\text{sin. } B \text{ sin. } c = \text{sin. } C \text{ sin. } b.$$

Euolu-

Euolutio formulae.

$$\text{cos. A sin. } c = \text{cos. } a \text{ sin. } b - \text{sin. } a \text{ cos. } b \text{ cos. C}$$

§. 6. Quia sin. A sin. c = sin. C sin. a, diuidatur huius aequationis membrum prius per sin. A sin. c, posterius vero per sin. C sin. a, atque obtinebitur

$$\text{cot. A} = \frac{\text{cos. } a \text{ sin. } b - \text{sin. } a \text{ cos. } b \text{ cos. C}}{\text{sin. } a \text{ sin. C}},$$

vnde iam ex datis binis lateribus a et b, cum angulo intercepto C, angulus A reperiri potest; similique modo ex iisdem datis colligetur angulus B per hanc formulam, ex illa, literas A, B, a, b permutando, deriuatam:

$$\text{cot. B} = \frac{\text{cos. } b \text{ sin. } a - \text{cos. } a \text{ sin. } b \text{ cos. C}}{\text{sin. } b \text{ sin. C}}.$$

§. 7. Si porro eiusdem, quam hic consideramus, formulae primum terminum per $\frac{\text{sin. C}}{\text{sin. } c}$, secundum per $\frac{\text{sin. B}}{\text{sin. } b}$, tertium vero per $\frac{\text{sin. A}}{\text{sin. } a}$ multiplicemus, orietur ista aequatio memorabilis:

$$\text{cos. A sin. C} = \text{cos. } a \text{ sin. B} - \text{cos. } b \text{ sin. A cos. C},$$

$$\text{cos. } a = \frac{\text{cos. A sin. C} + \text{sin. A cos. C cos. } b}{\text{sin. B}} \text{ et}$$

litteris B et C, item b et c inter se permutandis erit

$$\text{cos. } a = \frac{\text{cos. A sin. B} + \text{sin. A cos. B cos. } c}{\text{sin. C}}, \text{ siue}$$

$$\text{cos. } a \text{ sin. C} = \text{cos. A sin. B} + \text{sin. A cos. B cos. } c$$

quae a proposita aliter non discrepat, nisi quod literae maiusculae et minusculae inter se permutentur, insuper vero omnes cosinus negatiue accipiantur.

§. 8. Quod si iam huius postremae aequationis primum membrum per sin. a sin. C, posterius per sin. A sin. c

 diui-

diuidamus, orietur haec aequatio:

$$\cot. a = \frac{\cos. A \sin. B + \sin. A \cos. B \cos. c}{\sin. A \sin. c},$$

quae inferuit lateri a inueniendo, ex datis duobus angulis A, B, cum latere intercepto c; tum vero ex iisdem datis etiam latus b definietur hac aequatione:

$$\cot. b = \frac{\cos. B \sin. A + \sin. B. \cos. A \cos. c}{\sin. B \sin. c}.$$

§. 9. Praeterea vero ex eadem formula propofita cafus alias difficillimus, quo ex datis tribus angulis latera poftulantur, eruitur. Cum enim fit

$$\cos. A \sin. c = \cos. a \sin. b - \sin. a \cos. b \cos. C,$$

erit fimili modo, literis A et B permutatis,

$$\cos. B \sin. c = \cos. b \sin. a - \sin. b \cos. a \cos. C.$$

Si pofterior, ducta in cof. C, ad priorem addatur, prodibit ifta aequatio:

$$\sin. c (\cos. A + \cos. B \cos. C) = \cos. a \sin. b \sin. C^2;$$

at vero ob $\sin. b \sin. C = \sin. B \sin. c$, aequatio illa induet hanc formam:

$$\cos. A + \cos. B \cos. C = \cos. a \sin. B \sin. C;$$

fiue

$$\cos. A = - \cos. B \cos. C + \sin. C \cos. a.$$

Permutatis igitur literis A et C, manente B, fiet

$$\cos. C = - \cos. B \cos. A + \sin. B \sin. A \cos. c,$$

quae ex noftra tertia formula:

$$\cos. c = \cos. a \cos. b - \sin. a \sin. b \cos. C$$

nafcitur, fi litterae maiusculae et minusculae inter fe permutentur, omnes autem cofinus negatiue accipiantur.

Euolu-

Euolutio formulae.

$$\text{cof. } c = \text{cof. } a \text{ cof. } b + \text{fin. } a \text{ fin. } b \text{ cof. } C.$$

§. 10. Hic ſtatim euidens eſt, hanc formulam duplicem vſum praeſtare, alterum, quo ex datis lateribus a, b, c anguli ſunt definiendi, quod fit ope huius formulae

$$\text{cof. } C = \frac{\text{cof. } c - \text{cof. } a \text{ cof. } b}{\text{fin. } a \text{ fin. } b};$$

alterum vero, quando ex binis lateribus a et b, cum angulo intercepto C, tertium latus c quaeritur, quod fit ope huius formulae:

$$\text{cof. } c = \text{cof. } a \text{ cof. } b + \text{fin. } a \text{ fin. } b \text{ cof. } C.$$

§. 11. Nunc igitur hunc vſum etiam ad angulos transferre poterimus, quoniam modo iuuenimus

$$\text{cof. } C = -\text{cof. } A \text{ cof. } B + \text{fin. } A \text{ fin. } B \text{ cof. } c.$$

Hinc enim ſtatim, ſi dentur duo anguli A, B, cum latere intercepto c determinatur tertius angulus C. Deinde vero, ſi dentur omnes tres anguli trianguli ſphaerici, quodvis latus, veluti c, hoc modo definitur:

$$\text{cof. } c = \frac{\text{cof. } C + \text{cof. } A \text{ cof. } B}{\text{fin. } A \text{ fin. } B}.$$

§. 12. Cum igitur tota Trigonometria Sphaerica tribus aequationibus ſupra inuentis innitatur, permutatio angulorum et laterum generaliter locum habet, ſi modo omnes coſinus negatiue accipiantur. In prima enim formula:

$$\frac{\text{fin. } C}{\text{fin. } c} = \frac{\text{fin. } B}{\text{fin. } b} = \frac{\text{fin. } A}{\text{fin. } a},$$

haec permutabilitas per se est manifesta, quia nulli cosinus occurrunt, deinde ista permutabilitas pro ambabus reliquis formulis iam est euicta, vnde sequens Theorema insigne nascitur:

Theorema.

Proposito quocunque triangulo sphaerico, cuius anguli sint A, B, C, *et latera* a, b, c, *semper aliud triangulum analogum exhiberi potest, cuius anguli sint complementa laterum illius ad duos rectos, latera vero complementa angulorum ad duos rectos.* Hoc enim modo omnes sinus manent iidem, omnes vero cosinus euadunt negatiui, ideoque etiam tangentes et cotangentes. Constat autem tale triangulum formari ex Polis trium laterum trianguli propositi.

§. 13. Ad vsum ergo practicum omnia praecepta sub quatuor formis repraesentari possunt, quarum binae adeo ita arcte colligantur, vt altera ex altera formetur, dum litterae maiusculae et minusculae inter se permutantur, cosinibus negatiue sumtis, ita vt sufficiat duas tantum formas memoriae mandasse. Has igitur quatuor formas cum omnibus variationibus, quas transpositione litterarum recipere possunt, ante oculos exponamus.

Forma prima.

§ 14. Haec forma duos inuoluit casus, quorum altero ex datis tribus lateribus quidam angulus, altero vero

vero ex datis duobus lateribus, cum angulo intercepto, tertium latus inuenitur.

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \quad \Big| \quad \cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A$$

$$\cos. B = \frac{\cos. b - \cos. a \cos. c}{\sin. a \sin. c} \quad \Big| \quad \cos. b = \cos. a \cos. c + \sin. a \sin. c \cos. B$$

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b} \quad \Big| \quad \cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C.$$

Forma ſecunda.

§. 15. Haec forma etiam duos caſus continet, quorum altero ex datis tribus angulis aliquod latus, altero vero ex datis duobus angulis, cum latere intercepto, tertius angulus quaeritur:

$$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C} \quad \Big| \quad \cos. A = -\cos. B \cos. C + \sin. B \sin. C \cos. a$$

$$\cos. b = \frac{\cos. B + \cos. A \cos. C}{\sin. A \sin. C} \quad \Big| \quad \cos. B = -\cos. A \cos. C + \sin. A \sin. C \cos. b$$

$$\cos. c = \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B} \quad \Big| \quad \cos. C = -\cos. A \cos. B + \sin. A \sin. B \cos. c.$$

Forma tertia.

§. 16. Haec forma eum caſum complectitur, quo ex duobus lateribus, cum angulo intercepto, duo reliqui anguli determinantur, quae formulae cum ſuis variationibus ita ſe habebunt:

$$\cot. A = \frac{\cos. a \sin. b - \sin. a \cos. b \cos. C}{\sin. a \sin. C} \quad \Big| \quad \cot. B = \frac{\sin. a \cos. b - \cos. a \sin. b \cos. C}{\sin. b \sin. C}$$

$$\cot. B = \frac{\cos. b \sin. c - \sin. b \cos. c \cos. A}{\sin. b \sin. A} \quad \Big| \quad \cot. C = \frac{\sin. b \cos. c - \cos. b \sin. c \cos. A}{\sin. c \sin. A}$$

$$\cot. C = \frac{\cos. c \sin. a - \sin. c \cos. a \cos. B}{\sin. c \sin. B} \quad \Big| \quad \cot. A = \frac{\sin. c \cos. a - \cos. c \sin. a \cos. B}{\sin. a \sin. B}$$

Forma

Forma quarta.

§. 17. Haec forma reſpicit caſum, quo ex duobus angulis, cum latere intercepto, bina reliqua latera definiuntur, quae formulae cum variationibus ita ſe habent:

$$\cot. a = \frac{\cos. A \sin. B + \sin. A \cos. B \cos. c}{\sin. A \sin. c} \quad \Big| \quad \cot. b = \frac{\sin. A \cos. B + \cos. A \sin. B \cos. c}{\sin. B \sin. c}$$

$$\cot. b = \frac{\cos. B \sin. C + \sin. B \cos. C \cos. a}{\sin. B \sin. a} \quad \Big| \quad \cot. c = \frac{\sin. B \cos. C + \cos. B \sin. C \cos. a}{\sin. C \sin. a}$$

$$\cot. c = \frac{\cos. C \sin. A + \sin. C \cos. A \cos. b}{\sin. C \sin. b} \quad \Big| \quad \cot. a = \frac{\sin. C \cos. A + \cos. C \sin. A \cos. b}{\sin. A \sin. b}$$

§. 18. Haec ſimplicitas eo magis eſt notatu digna, quod reſolutio triangulorum rectangulorum adeo ſex formulas a ſe inuicem prorſus diuerſas requirat. Quod ſi enim angulus C fuerit rectus, ideoque c hypothenuſa et a et b ambo catheti, ſex formulae requiſitae ſunt ſequentes:

$$\cos. c = \cos. a \cos. b$$
$$\cos. c = \cot. A \cot. B$$
$$\sin. a = \sin. c \sin. A \quad \text{ſiue} \quad \sin. b = \sin. c \sin. B$$
$$\tang. b = \tang. c \cos. A \; - - \; \tang. a = \tang. c \cos. B$$
$$\tang. a = \tang. A \sin. b \; - - \; \tang. b = \tang. B \sin. a$$
$$\cos. A = \cos. a \sin. B \; - - \; \cos. B = \cos. b \sin. A$$

quae formulae ex ſuperioribus ſponte deriuantur, poſito
$\cos. C = 0$ et $\sin. C = 1$.

§. 19. Quo autem logarithmi in vſum vocari queant, ex formis ſuperioribus aliae eius indolis ſunt deriuandae, quae ex factoribus conſtent, id quod per certas transformationes obtineri poteſt, quibus ad ſemiſſes tam angulo-

angulorum quam laterum deducimur. Has autem transformationes sequentibus modis succincte instituere licet.

Transformatio prima.

§. 20. Haec transformatio ex primae formae hac formula:

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c},$$

commodissime deriuatur. Hinc enim primo sequitur:

$$1 - \cos. A = \frac{\cos. (b - c) - \cos. a}{\sin. b \sin. c}$$

$$1 + \cos. A = \frac{\cos. a - \cos. (b + c)}{\sin. b \sin. c};$$

Hinc cum sit

$$\frac{1 - \cos. A}{1 + \cos. A} = \text{tang.} \tfrac{1}{2} A^2, \text{ erit}$$

$$\text{tang.} \tfrac{1}{2} A^2 = \frac{\cos. (b - c) - \cos. a}{\cos. a - \cos. (b + c)};$$

constat autem esse

$$\cos. p - \cos. q = 2 \sin. \tfrac{q-p}{2} \sin. \tfrac{p+q}{2},$$

vnde habebimus

$$\text{tang.} \tfrac{1}{2} A = \sqrt{\frac{\sin. \frac{a-b+c}{2} \cdot \sin. \frac{a+b-c}{2}}{\sin. \frac{b+c-a}{2} \cdot \sin. \frac{a+b+c}{2}}}.$$

Transformatio secunda.

§. 21. Haec petitur ex formae prioris formula

$$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C},$$

vnde deducitur

$$1 - \cos. a = \frac{-\cos. (B + C) - \cos. A}{\sin. B \sin. C}$$

$$1 + \cos. a = \frac{\cos. A + \cos. (B - C)}{\sin. B \sin. C},$$

 sicque

ſicque erit

$$\text{tang.}\ \tfrac{1}{2} a^2 = -\frac{\text{coſ.}(B+C)+\text{coſ.}A}{\text{coſ.}(B-C)+\text{coſ.}A}.$$

Cum iam ſit

$$\text{coſ.}\ p + \text{coſ.}\ q = 2\ \text{coſ.}\ \tfrac{p-q}{2}\ \text{coſ.}\ \tfrac{p+q}{2},\ \text{erit}$$

$$\text{tang.}\ \tfrac{1}{2} a = \sqrt{\frac{-\text{coſ.}\ \frac{B+C-A}{2}\ \text{coſ.}\ \frac{B+C+A}{2}}{\text{coſ.}\ \frac{B+A-C}{2}\ \text{coſ.}\ \frac{A+C-B}{2}}}.$$

Transformatio tertia.

§. 22. Hanc transformationem etiam ex prima forma expedire licet, combinandis his duabus formulis:

$$\text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c = \text{ſin.}\ b\ \text{ſin.}\ c\ \text{coſ.}\ A,$$

$$\text{coſ.}\ b - \text{coſ.}\ a\ \text{coſ.}\ c = \text{ſin.}\ a\ \text{ſin.}\ c\ \text{coſ.}\ B;$$

quarum illa per hanc diuiſa praebet,

$$\frac{\text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c}{\text{coſ.}\ b - \text{coſ.}\ a\ \text{coſ.}\ c} = \frac{\text{ſin.}\ b\ \text{coſ.}\ A}{\text{ſin.}\ a\ \text{coſ.}\ B} = \frac{\text{ſin.}\ B\ \text{coſ.}\ A}{\text{ſin.}\ A\ \text{coſ.}\ B}.$$

Addatur vtrinque vnitas, fietque

$$(\text{coſ.}\ a + \text{coſ.}\ b)(1 - \text{coſ.}\ c) = \frac{\text{ſin.}(A+B)}{\text{ſin.}\ A\ \text{coſ.}\ B},$$

ſubtrahatur vtrinque vnitas, prodibit

$$(\text{coſ.}\ a - \text{coſ.}\ b)(1 + \text{coſ.}\ c) = \frac{\text{ſin.}(B-A)}{\text{ſin.}\ A\ \text{coſ.}\ B},$$

quae aequatio per priorem diuiſa dat

$$\frac{\text{coſ.}\ a - \text{coſ.}\ b}{\text{coſ.}\ a + \text{coſ.}\ b}.\ \text{cot.}\ \tfrac{1}{2} c^2 = \frac{\text{ſin.}(B-A)}{\text{ſin.}(B+A)}.$$

Conſtat autem eſſe

$$\frac{\text{coſ.}\ p - \text{coſ.}\ q}{\text{coſ.}\ p + \text{coſ.}\ q} = \text{tang.}\ \tfrac{q+p}{2}\ \text{tang.}\ \tfrac{q-p}{2},$$

vnde colligitur:

$$\text{tang.}\ \tfrac{b-a}{2}.\ \text{tang.}\ \tfrac{b+a}{2}.\ \text{cot.}\ \tfrac{1}{2} c^2 = \frac{\text{ſin.}(B-A)}{\text{ſin.}(B+A)}.$$

§. 23. Iam in subsidium vocemus ex proprietate primaria hanc formulam:

$\frac{\sin. b}{\sin. a} = \frac{\sin. B}{\sin. A}$, vnde deducimus

$\frac{\sin. b - \sin. a}{\sin. b + \sin. a} = \frac{\sin. B - \sin. A}{\sin. B + \sin. A}$,

quae reducitur ad hanc formam:

$$\text{tang.} \frac{b-a}{2} \cot. \frac{b+a}{2} = \text{tang.} \frac{B-A}{2} \cot. \frac{B+A}{2}.$$

Quod si iam aequationem ante inuentam per hanc multiplicemus, prodibit ista:

$$(\text{tang.} \frac{b-a}{2})^2 \cot. \tfrac{1}{2} c^2 = \frac{(\sin. \frac{B-A}{2})^2}{(\sin. \frac{B+A}{2})^2},$$

siue extracta radice

$$\text{tang.} \frac{b-a}{2} \cot. \tfrac{1}{2} c = \frac{\sin. \frac{B-A}{2}}{\sin. \frac{B+A}{2}}.$$

At vero prior formula per posteriorem diuisa dat

$$\text{tang.} \frac{b+a}{2} \cot. \tfrac{1}{2} c = \frac{\text{cos.} \frac{B-A}{2}}{\text{cos.} \frac{B+A}{2}}.$$

His igitur formulis resoluitur casus, quo dantur duo anguli A et B cum latere intercepto c, et quaeruntur ambo latera a et b, quod fit ope harum formularum:

$$\text{tang.} \frac{b-a}{2} = \text{tang.} \tfrac{1}{2} c. \frac{\sin. \frac{B-A}{2}}{\sin. \frac{B+A}{2}}$$

$$\text{tang.} \frac{b+a}{2} = \text{tang.} \tfrac{1}{2} c. \frac{\text{cos.} \frac{B-A}{2}}{\text{cos.} \frac{B+A}{2}}.$$

Transformatio quarta.

§. 24. Haec ſimili modo deducitur ex his formulis:

$$\text{coſ. A} + \text{coſ. B coſ. C} = \text{ſin. B ſin. C coſ. } a$$

$$\text{coſ. B} + \text{coſ. A coſ. C} = \text{ſin. A ſin. C coſ. } b$$

quarum illa per hanc diuiſa praebet

$$\frac{\text{coſ. A} + \text{coſ. B coſ. C}}{\text{coſ. B} + \text{coſ. A coſ. C}} = \frac{\text{ſin. B coſ. } a}{\text{ſin. A coſ. } b} = \frac{\text{ſin. } b \text{ coſ. } a}{\text{ſin. } a \text{ coſ. } b}.$$

Vnde vnitatem tam addendo quam ſubtrahendo ſequentes nouae deriuantur aequationes:

$$(\text{coſ. A} + \text{coſ. B})(1 + \text{coſ. C}) = \frac{\text{ſin. }(a+b)}{\text{ſin. } a \text{ coſ. } b}.$$

$$(\text{coſ. A} - \text{coſ. B})(1 - \text{coſ. C}) = \frac{\text{ſin. }(b-a)}{\text{ſin. } a \text{ coſ. } b},$$

et diuidendo illam per hanc nanciſcimur:

$$\frac{\text{coſ. A} + \text{coſ. B}}{\text{coſ. A} - \text{coſ. B}} \text{ cot. } \tfrac{1}{2} \text{C}^2 = \frac{\text{ſin. }(a+b)}{\text{ſin. }(b-a)}, \text{ ſiue}$$

$$\text{tang. } \frac{\text{B}-\text{A}}{2} . \text{ tang. } \frac{\text{B}+\text{A}}{2} = \text{cot. } \tfrac{1}{2} \text{C}^2 . \frac{\text{ſin. }(b-a)}{\text{ſin. }(b+a)},$$

quae aequatio, multiplicata et diuiſa per iſtam:

$$\text{tang. } \frac{\text{B}-\text{A}}{2} . \text{ cot. } \frac{\text{B}+\text{A}}{2} = \text{tang. } \frac{b-a}{2} \text{ cot. } \frac{b+a}{2},$$

producit

$$\text{tang. } \frac{\text{B}-\text{A}}{2} = \text{cot. } \tfrac{1}{2} \text{C} . \frac{\text{ſin. } \frac{b-a}{2}}{\text{ſin. } \frac{b+a}{2}}$$

$$\text{tang. } \frac{\text{B}+\text{A}}{2} = \text{cot. } \tfrac{1}{2} \text{C} . \frac{\text{coſ. } \frac{b-a}{2}}{\text{coſ. } \frac{b+a}{2}}$$

quae formulae valent pro caſu, quo dantur duo latera cum angulo intercepto.

§ 25. Quoniam praecedentium formarum omnes variationes appoſuimus, etiam hos quatuor caſus cum omnibus

nibus variationibus conspectui exponamus.

$$\text{tang.}\ \tfrac{1}{2} A = \sqrt{\frac{\text{sin.}\ \frac{a+b-c}{2}\ \text{sin.}\ \frac{a+c-b}{2}}{\text{sin.}\ \frac{b+c-a}{2}\ \text{sin.}\ \frac{a+b+c}{2}}}$$

$$\text{tang.}\ \tfrac{1}{2} B = \sqrt{\frac{\text{sin.}\ \frac{b+c-a}{2}\ \text{sin.}\ \frac{a+b-c}{2}}{\text{sin.}\ \frac{a+c-b}{2}\ \text{sin.}\ \frac{a+b+c}{2}}}$$

$$\text{tang.}\ \tfrac{1}{2} C = \sqrt{\frac{\text{sin.}\ \frac{a+c-b}{2}\ \text{sin.}\ \frac{b+c-a}{2}}{\text{sin.}\ \frac{a+b-c}{2}\ \text{sin.}\ \frac{a+b+c}{2}}}$$

$$\text{tang.}\ \tfrac{1}{2} a = \sqrt{\frac{-\text{cos.}\ \frac{B+C-A}{2}\ \text{cos.}\ \frac{A+B+C}{2}}{\text{cos.}\ \frac{A+B-C}{2}\ \text{cos.}\ \frac{A+C-B}{2}}}$$

$$\text{tang.}\ \tfrac{1}{2} b = \sqrt{\frac{-\text{cos.}\ \frac{A+C-B}{2}\ \text{cos.}\ \frac{A+B+C}{2}}{\text{cos.}\ \frac{B+C-A}{2}\ \text{cos.}\ \frac{A+B-C}{2}}}$$

$$\text{tang.}\ \tfrac{1}{2} c = \sqrt{\frac{-\text{cos.}\ \frac{A+B-C}{2}\ \text{cos.}\ \frac{A+B+C}{2}}{\text{cos.}\ \frac{A+C-B}{2}\ \text{cos.}\ \frac{B+C-A}{2}}}$$

$$\text{tang.}\ \frac{b-a}{2} = \text{tang.}\ \tfrac{1}{2} c\ \frac{\text{sin.}\ \frac{B-A}{2}}{\text{sin.}\ \frac{B+A}{2}} \qquad \text{tang.}\ \frac{b+a}{2} = \text{tang.}\ \tfrac{1}{2} c\ \frac{\text{cos.}\ \frac{B-A}{2}}{\text{cos.}\ \frac{B+A}{2}}$$

$$\text{tang.}\ \frac{c-b}{2} = \text{tang.}\ \tfrac{1}{2} a\ \frac{\text{sin.}\ \frac{C-B}{2}}{\text{sin.}\ \frac{C+B}{2}} \qquad \text{tang.}\ \frac{c+b}{2} = \text{tang.}\ \tfrac{1}{2} a\ \frac{\text{cos.}\ \frac{C-B}{2}}{\text{cos.}\ \frac{C+B}{2}}$$

$$\text{tang.}\ \frac{a-c}{2} = \text{tang.}\ \tfrac{1}{2} b\ \frac{\text{sin.}\ \frac{A-C}{2}}{\text{sin.}\ \frac{A+C}{2}} \qquad \text{tang.}\ \frac{a+c}{2} = \text{tang.}\ \tfrac{1}{2} b\ \frac{\text{cos.}\ \frac{A-C}{2}}{\text{cos.}\ \frac{A+C}{2}}$$

$$\text{tang.}\ \frac{B-A}{2} = \text{cot.}\ \tfrac{1}{2} C\ \frac{\text{sin.}\ \frac{b-a}{2}}{\text{sin.}\ \frac{b+a}{2}} \qquad \text{tang.}\ \frac{B+A}{2} = \text{cot.}\ \tfrac{1}{2} C\ \frac{\text{cos.}\ \frac{b-a}{2}}{\text{cos.}\ \frac{b+a}{2}}$$

$$\text{tang.}\ \frac{C-B}{2} = \text{cot.}\ \tfrac{1}{2} A\ \frac{\text{sin.}\ \frac{c-b}{2}}{\text{sin.}\ \frac{c+b}{2}} \qquad \text{tang.}\ \frac{C+B}{2} = \text{cot.}\ \tfrac{1}{2} A\ \frac{\text{cos.}\ \frac{c-b}{2}}{\text{cos.}\ \frac{c+b}{2}}$$

$$\text{tang.}\ \frac{A-C}{2} = \text{cot.}\ \tfrac{1}{2} B\ \frac{\text{sin.}\ \frac{a-c}{2}}{\text{sin.}\ \frac{a-c}{2}} \qquad \text{tang.}\ \frac{A+C}{2} = \text{cot.}\ \tfrac{1}{2} B\ \frac{\text{cos.}\ \frac{a-c}{2}}{\text{cos.}\ \frac{a+c}{2}}$$

§. 26.

§. 26. Ex his postremis formulis iam facile expeditur casus, quem nondum attigimus, quo dantur duo latera cum angulis oppositis, et vel tertium latus vel tertius angulus quaeritur, quorum vtrumque duplici modo fieri potest. Has ergo formulas cum variationibus apponamus.

$$\text{tang.}\,\tfrac{1}{2}c = \text{tang.}\,\frac{b-a}{2}\,\frac{\text{sin.}\,\frac{B+A}{2}}{\text{sin.}\,\frac{B-A}{2}} \qquad \text{tang.}\,\tfrac{1}{2}c = \text{tang.}\,\frac{b+a}{2}\,\frac{\text{cos.}\,\frac{B+A}{2}}{\text{cos.}\,\frac{B-A}{2}}$$

$$\text{tang.}\,\tfrac{1}{2}a = \text{tang.}\,\frac{c-b}{2}\,\frac{\text{sin.}\,\frac{C+B}{2}}{\text{sin.}\,\frac{C-B}{2}} \qquad \text{tang.}\,\tfrac{1}{2}a = \text{tang.}\,\frac{c+b}{2}\,\frac{\text{cos.}\,\frac{C+B}{2}}{\text{cos.}\,\frac{C-B}{2}}$$

$$\text{tang.}\,\tfrac{1}{2}b = \text{tang.}\,\frac{a-c}{2}\,\frac{\text{sin.}\,\frac{A+C}{2}}{\text{sin.}\,\frac{A-C}{2}} \qquad \text{tang.}\,\tfrac{1}{2}b = \text{tang.}\,\frac{a+c}{2}\,\frac{\text{cos.}\,\frac{A+C}{2}}{\text{cos.}\,\frac{A-C}{2}}$$

$$\text{cot.}\,\tfrac{1}{2}C = \text{tang.}\,\frac{B-A}{2}\,\frac{\text{sin.}\,\frac{b+a}{2}}{\text{sin.}\,\frac{b-a}{2}} \qquad \text{cot.}\,\tfrac{1}{2}C = \text{tang.}\,\frac{B+A}{2}\,\frac{\text{cos.}\,\frac{b+a}{2}}{\text{cos.}\,\frac{b-a}{2}}$$

$$\text{cot.}\,\tfrac{1}{2}A = \text{tang.}\,\frac{C-B}{2}\,\frac{\text{sin.}\,\frac{c+b}{2}}{\text{sin.}\,\frac{c-b}{2}} \qquad \text{cot.}\,\tfrac{1}{2}A = \text{tang.}\,\frac{C+B}{2}\,\frac{\text{cos.}\,\frac{c+b}{2}}{\text{cos.}\,\frac{c-b}{2}}$$

$$\text{cot.}\,\tfrac{1}{2}B = \text{tang.}\,\frac{A-C}{2}\,\frac{\text{sin.}\,\frac{a+c}{2}}{\text{sin.}\,\frac{a-c}{2}} \qquad \text{cot.}\,\tfrac{1}{2}B = \text{tang.}\,\frac{A+C}{2}\,\frac{\text{cos.}\,\frac{a+c}{2}}{\text{cos.}\,\frac{a-c}{2}}$$

Hoc igitur modo praesens tractatio tanquam systema completum totius Trigonometriae sphaericae spectari potest.

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