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De formatione fractionum continuarum

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DE FORMATIONE
FRACTIONVM CONTINVARVM.

Auctore

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§. 1.

Principium vniuerſale ad fractiones continuas perducens reperitur in ſerie infinita quantitatum A, B, C etc.; quarum ternae ſibi ſuccedentes ſecundum certam legem, ſiue conſtantem ſiue vtcunque variabilem ita a ſe inuicem pendent, vt ſit

$$f\mathrm{A} = g\mathrm{B} + h\mathrm{C},\quad f'\mathrm{B} = g'\mathrm{C} + h'\mathrm{D},\quad f''\mathrm{C} = g'' + \mathrm{D} + h''\mathrm{E},$$
$$f'''\mathrm{D} = g'''\mathrm{E} + h'''\mathrm{F} \text{ etc.}$$

Hinc enim deducuntur ſequentes aequalitates:

$$\frac{f\mathrm{A}}{\mathrm{B}} = g + \frac{h\mathrm{C}}{\mathrm{B}} = g + \frac{f'h}{f'\mathrm{B:C}}$$
$$\frac{f'\mathrm{B}}{\mathrm{C}} = g' + \frac{h'\mathrm{D}}{\mathrm{C}} = g' + \frac{f''h'}{f''\mathrm{C:D}}$$
$$\frac{f''\mathrm{C}}{\mathrm{D}} = g'' + \frac{h''\mathrm{E}}{\mathrm{D}} = g'' + \frac{f'''h''}{f'''\mathrm{D:E}}$$
$$\frac{f'''\mathrm{D}}{\mathrm{E}} = g''' + \frac{h'''\mathrm{F}}{\mathrm{E}} = g''' + \frac{f''''h'''}{f''''\mathrm{E:F}}$$

etc. etc.

Quod si iam posteriores valores in prioribus continuo substituantur, sponte emerget sequens fractio continua:

$$\frac{f\,A}{B} = g + \cfrac{f'\,h}{g' + \cfrac{f''\,h'}{g'' + \cfrac{f'''\,h''}{g''' + \cfrac{f''''\,h'''}{g'''' + \text{etc.}}}}}$$

cuius ergo valor per solos duos primos terminos A & B seriei determinatur.

§. 2. Quoties igitur talis progressio quantitatum A, B, C, D, E etc. habetur, cuius lex ita fuerit comparata, vt terni quique eius termini sibi succedentes secundum legem quamcunque a se inuicem pendeant, toties inde deducitur fractio continua, cuius valor assignari potest. Quamobrem si formula quaecunque ita fuerit comparata, vt eius euolutio perducat ad huiusmodi seriem quantitatum A, B, C, D, E, etc. quarum quisque terminus per duos praecedentes determinatur, inde fractiones continuae deriuari poterunt, quod quomodo fiat, commodissime per aliquot exempla ostendi poterit.

I. Euolutio formulae.

$$s = x^n(\alpha - \beta x - \gamma x x).$$

§. 3. In hac formula exponens n indefinitus spectatur, successiue recipiens omnes valores 1, 2, 3, 4, 5, 6 etc., vnde, dummodo fuerit $n > 0$, haec formula euanescit, posito $x = 0$, tum vero etiam euanescit, sumto

$x = -$

$$x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma}.$$

His notatis differentietur ista formula, vt fiat

$$ds = n\alpha x^{n-1} dx - (n+1)\beta x^n dx - (n+2)\gamma x^{n+1} dx,$$

vnde per partes integrando et integrationem tantum indicando fiet

$$n\alpha\int x^{n-1} dx = (n+1)\beta\int x^n dx + (n+2)\gamma\int x^{n+1} dx + s,$$

Hinc, si post quamque integrationem, ita peractam, vt integrale euanescat posito $x = 0$, statuatur

$$x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma},$$

quippe quo casu fit $s = 0$, erit

$$n\alpha\int x^{n-1} dx = (n+1)\beta\int x^n dx + (n+2)\gamma\int x^{n+1} dx,$$

quae est eiusmodi relatio inter ternas formulas integrales sibi succedentes, qualem desideramus pro formatione fractionis continuae; quandoquidem hae formulae integrales, si loco n successiue scribantur numeri 1, 2, 3, 4, 5, 6 etc. nobis suppeditant quantitates A, B, C, D etc.

§. 4. Scribamus igitur loco n ordine numeros naturales 1, 2, 3, 4, etc. vt prodeant istae relationes:

$$\alpha\int dx = 2\beta\int x\,dx + 3\gamma\int xx\,dx$$
$$2\alpha\int x\,dx = 3\beta\int xx\,dx + 4\gamma\int x^3 dx$$
$$3\alpha\int xx\,dx = 4\beta\int x^3 dx + 3\gamma\int x^4 dx$$
$$4\alpha\int x^3 dx = 5\beta\int x^4 dx + 6\gamma\int x^5 dx$$

etc. etc.

Hinc igitur habebimus

$$A = \int dx = x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma},$$
$$B = \int x\,dx = \tfrac{1}{2}xx = \tfrac{1}{2}\left(\frac{-\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma}\right)^2,$$
$$C = \int xx\,dx = \tfrac{1}{3}x^3,\quad D = \int x^3 dx = \tfrac{1}{4}x^4$$

etc. etc.

 Tunc

Tunc vero pro literis f, g, h habebuntur isti valores:

$$f = \alpha,\; f' = 2\,\alpha,\; f'' = 3\,\alpha,\; f''' = 4\,\alpha \text{ etc.}$$
$$g = 2\,\beta,\; g' = 3\,\beta,\; g'' = 4\,\beta,\; g''' = 5\,\beta \text{ etc.}$$
$$h = 3\,\gamma,\; h' = 4\,\gamma,\; h'' = 5\,\gamma,\; h''' = 6\,\gamma \text{ etc.}$$

ex quibus valoribus resultat sequens fractio continua:

$$\frac{\alpha A}{B} = 2\,\beta + \cfrac{6\,\alpha\gamma}{3\,\beta + \cfrac{12\,\alpha\gamma}{4\,\beta + \cfrac{20\,\alpha\gamma}{5\,\beta + \cfrac{30\,\alpha\gamma}{6\,\beta + \text{etc.}}}}}$$

cuius ergo valor est

$$\frac{4\,\alpha\,\gamma}{-\beta + \sqrt{(\beta\beta + 4\,\alpha\,\gamma)}} = \beta + \sqrt{(\beta\beta + 4\,\alpha\gamma)}.$$

§. 5. Quo haec fractio continua concinnior reddatur, loco $\alpha\gamma$ scribamus $\frac{1}{2}\delta$, et prodibit

$$\beta + \sqrt{(\beta\beta + 2\delta)} = 2\,\beta + \cfrac{3\,\delta}{3\,\beta + \cfrac{6\,\delta}{4\,\beta + \cfrac{10\,\delta}{5\,\beta + \cfrac{15\,\delta}{6\,\beta + \text{etc.}}}}}$$

Quoniam autem haec expressio capite truncata videtur, adiecto hoc capite ponamus

$$s = \beta + \cfrac{\delta}{2\,\beta + \cfrac{3\,\delta}{3\,\beta + \cfrac{6\,\delta}{4\,\beta + \cfrac{10\,\delta}{5\,\beta + \text{etc.}}}}},$$ eritque

$s = \beta$

$$s = \beta + \frac{\delta}{\beta + \sqrt{(\beta\beta + 2\delta)}}$$

quae expreſſio reducitur ad hanc:

$$s = \tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 2\delta)}.$$

§. 6. Haec autem fractio continua adhuc ad maiorem ſimplicitatem reduci poteſt, ſi loco δ ſcribamus 2ε, vt ſit

$$\tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 4\varepsilon)} = \beta + \cfrac{2\varepsilon}{2\beta + \cfrac{6\varepsilon}{3\beta + \cfrac{12\varepsilon}{4\beta + \cfrac{20\varepsilon}{5\beta + 20 \text{ etc.}}}}}$$

Quod ſi iam prima fractio deprimatur per 2, ſecunda per 3, tertia per 4, quarta per 5 etc. prodibit ſequens forma:

$$\tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 4\varepsilon)} = \beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \text{ etc.}}}}}$$

quae eſt ſimpliciſſima, cuius ſumma ſi tanquam incognita ſpectetur, ac vocetur $= z$, erit vtique $z = \beta + \frac{\varepsilon}{z}$, ideoque $zz = \beta z + \varepsilon$, vnde fit $z = \frac{\beta + \sqrt{(\beta\beta + 4\varepsilon)}}{2}$, quae eſt eadem.

§. 7. Verum iſta ſumma ſimpliciſſima immediate deduci poteſt ex ipſa formula initio aſſumta

$$s = x^{n}(\alpha - \beta x - \gamma x x),$$

quam

quam quoniam nihilo aequalem posuimus, erit vtique $\alpha = \beta x + \gamma x x$, eodemque modo

$$\alpha x = \beta x x + \gamma x^3, \; \alpha x x = \beta x^3 + \gamma x^4, \text{ etc.}$$

ita vt pro serie A, B, C, D, etc. habeamus hanc simplicem seriem potestatum: 1, x, x^2, x^3, x^4 etc., tum vero omnes literae, f, g, h etc. fiunt α, β, γ etc. vnde oritur ista fractio continua:

$$\frac{\alpha}{x} = \beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \text{etc.}}}}$$

vbi est $\frac{1}{x} = \frac{\beta + \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\alpha}$. Huius ergo fractionis valor est

$$\tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 4\alpha\gamma)}, \text{ vt ante, ob } \alpha\gamma = \varepsilon.$$

II. Euolutio formulae.

$$s = x^n (a - x).$$

§. 8. Haec igitur formula euanescit, ponendo $x = a$; hinc autem fit $ds = n a x^{n-1} dx - (n+1) x^n dx$, quae expressio cum duobus tantum constet terminis, reducatur ad fractionem, cuius denominator sit $\alpha + \beta x$, ita vt fiat

$$ds = \frac{n a \alpha x^{n-1} dx + (\beta n a - \alpha(n+1)) x^n dx - \beta(n+1) x^{n+1} dx}{\alpha + \beta x}.$$

His igitur membris seorsim integratis fiet

$$s = n a \alpha \int \frac{x^{n-1} dx}{\alpha + \beta x} + (n \beta a - (n+1)\alpha) \int \frac{x^n dx}{\alpha + \beta x} - \beta(n+1) \int \frac{x^{n+1} dx}{\alpha + \beta x}$$

quae

quare si post singulas integrationes statuamus $x = a$, vt fiat $s = 0$, habebimus hanc reductionem:

$$n a \alpha \int \frac{x^{n-1} dx}{\alpha + \beta x} = ((n+1)\alpha - n\beta a) \int \frac{x^n dx}{\alpha + \beta x} + (n+1)\beta \int \frac{x^{n+1} dx}{\alpha + \beta x}.$$

§. 9. Loco n substituamus nunc successiue numeros 1, 2, 3, 4 etc. atque comparatione cum formulis generalibus instituta habebimus

$$A = \int \frac{dx}{\alpha + \beta x}, \quad B = \int \frac{x dx}{\alpha + \beta x}, \quad C = \int \frac{xx dx}{\alpha + \beta x} \text{ etc.}$$

vbi quidem post integrationem fieri debet $x = a$. Praeterea vero habebimus

$$f = a\alpha, \; f' = 2a\alpha, \; f'' \, 3a\alpha, \; f''' \, 4a\alpha, \text{ etc.}$$

$$g = 2\alpha - \beta a, \; g' = 3\alpha - 2\beta a, \; g'' = 4\alpha - 3\beta a, \text{ etc.}$$

$$h = 2\beta, \; h' = 3\beta, \; h'' = 4\beta, \; h''' = 5\beta, \text{ etc.}$$

atque ex his oritur sequens fractio continua:

$$\frac{\alpha a A}{B} = (2\alpha - \beta a) + \cfrac{4 a\alpha\beta}{(3\alpha - 2\beta a) + \cfrac{9 a\alpha\beta}{(4\alpha - 3\beta a) + \cfrac{16 a\alpha\beta}{(5\alpha - 4\beta a) + \text{etc.}}}}$$

§. 10. Integratione autem instituta fit

$$\int \frac{dx}{\alpha + \beta x} = \frac{1}{\beta} l \frac{\alpha + \beta x}{\alpha},$$

quandoquidem integralia euanescere debent facto $x = 0$. Nunc igitur fiat $x = a$, eritque $A = \frac{1}{\beta} l \frac{\alpha + \beta a}{\alpha}$. Porro

$$\int \frac{x dx}{\alpha + \beta x} = \frac{1}{\beta}\left(x - \frac{\alpha}{\beta} l \frac{\alpha + \beta x}{\alpha}\right), \text{ factoque } x = a \text{ fiet}$$

$$B = \frac{a}{\beta} - \frac{\alpha}{\beta\beta} l \frac{\alpha + \beta a}{\alpha},$$

quamobrem valor noſtrae fractionis continuae erit

$$\frac{\alpha\, a\, \beta\, l\, \frac{\alpha + \beta\, a}{\alpha}}{a\, \beta - \alpha\, l\, \frac{\alpha + \beta\, a}{\alpha}}$$

euidens autem eſt, nihil de vniuerſalitate perire, etiamſi ſumatur $a = 1$; tum enim erit

$$\frac{\alpha\, \beta\, l\, \frac{\alpha + \beta}{\alpha}}{\beta - \alpha\, l\, \frac{\alpha + \beta}{\alpha}} = (2\,\alpha - \beta) + \cfrac{4\,\alpha\,\beta}{(3\,\alpha - 2\,\beta) + \cfrac{9\,\alpha\,\beta}{(4\,\alpha - 3\,\beta) + \text{etc.}}}$$

§. 11. Tota autem haec expreſſio manifeſto vnice pendet a ratione numerorum α et β; vnde ſumamus $\alpha = 1$ et $\beta = n$, atque orietur haec fractio continua:

$$\frac{n\, l\, (1 + n)}{n - l\, (1 + n)} = (2 - n) + \cfrac{4\, n}{(3 - 2\, n) + \cfrac{9\, n}{(4 - 3\, n) + \cfrac{16\, n}{(5 - 4\, n) + \text{etc.}}}}$$

cui ſi praefigamus ſecundum ordinis legem $1 + n$ et ſummam ſtatuamus $= s$, vt ſit

$$s = 1 + \cfrac{n}{(2 - n) + \cfrac{4\, n}{(3 - 2\, n) + \cfrac{9\, n}{(4 - 3\, n) + \cfrac{16\, n}{(5 - 4\, n) + \text{etc.}}}}}$$

erit

$$s = \frac{1 + n\, (n - l\, (1 + n))}{n\, l\, (1 + n)} = \frac{1 + n - l\, (1 + n)}{l\, (1 + n)} = \frac{n}{l\, (1 + n)}.$$

§. 12. Exempla aliquot percurramus, sitque primo $n = 1$, erit

$$\frac{1}{l\,2} = 1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}$$

Posito autem $n = 2$ erit

$$\frac{2}{l\,3} = 1 + \cfrac{2}{0 + \cfrac{8}{-1 + \cfrac{18}{-2 + \cfrac{32}{-3 + \cfrac{50}{-4 + \text{etc.}}}}}}$$

quae autem expressio, ob quantitates negatiuas, non satis est commoda; quod cum eueniat quando $n > 1$, operae pretium erit eos casus euoluere, quibus n vnitate minor accipitur.

§. 13. Quo hoc facilius fieri possit, reuertamur ad expressionem literas α et β continentes, atque capite, quod deerat suppleto, prodit ista forma:

$$\frac{\beta}{l\,\frac{\alpha+\beta}{\alpha}} = \alpha + \cfrac{\alpha\beta}{(2\alpha-\beta) + \cfrac{4\alpha\beta}{(3\alpha-2\beta) + \cfrac{9\alpha\beta}{(4\alpha-3\beta) + \text{etc.}}}}$$

Ponamus nunc $n = n - m$ et $\beta = 2m$, vt obtineamus se-

 quen-

quentem formam:

$$\frac{2m}{l\frac{n+m}{n-m}} = n-m+\cfrac{2m(n-m)}{2n-4m+\cfrac{8m(n-m)}{3n-7m+\cfrac{18m(n-m)}{4n-10m+\text{etc.}}}}$$

vnde sequentes casus speciales deducuntur.

Si $m=1$ et $n=3$ erit

$$\frac{2}{l2} = 2+\cfrac{4}{2+\cfrac{16}{2+\cfrac{36}{2+\cfrac{64}{2+\text{etc.}}}}}$$

quae fractio per 2 diuisa et reducta praebet istam:

$$\frac{1}{l2} = 1+\cfrac{1}{1+\cfrac{4}{1+\cfrac{9}{1+\cfrac{16}{1+\text{etc.}}}}}$$

quae iam supra est inuenta.

Sit $m=1$ et $n=4$ erit

$$\frac{2}{l\frac{5}{3}} = 3+\cfrac{6}{4+\cfrac{24}{5+\cfrac{54}{6+\cfrac{96}{7+\text{etc.}}}}}$$

$$= 3+\cfrac{6.1}{4+\cfrac{6.4}{5+\cfrac{6.9}{6+\cfrac{6.16}{7+\text{etc.}}}}}$$

Sit

Sit $m = 1$ et $n = 5$, erit

$$\frac{2}{l\frac{5}{4}} = 4 + \cfrac{8}{6 + \cfrac{32}{8 + \cfrac{72}{10 + \cfrac{128}{12 + \text{etc.}}}}}$$

siue

$$\frac{1}{l\frac{5}{4}} = 2 + \cfrac{2}{3 + \cfrac{8}{4 + \cfrac{18}{5 + \cfrac{32}{6 + \text{etc.}}}}}$$

$$= 2 + \cfrac{2.1}{3 + \cfrac{2.4}{4 + \cfrac{2.9}{5 + \cfrac{2.16}{6 + \text{etc.}}}}}$$

III. Euolutio formulae.

$$s = x^n (1 - x^2)$$

§. 14. Haec ergo formula euanescit casibus $x = 0$ et $x = 1$. Quoniam vero hinc fit

$$ds = n x^{n-1} dx - (n + 2) x^{n+1} dx,$$

reducatur hoc differentiale ad denominatorem $\alpha + \beta x x$, fietque

$$ds = \frac{n \alpha x^{n-1} dx + (n\beta - (n+2)\alpha) x^{n+1} dx - (n+2)\beta x^{n+3} dx.}{\alpha + \beta x x}$$

 Hinc

Hinc iam iterum integrando fit

$$s = n\alpha\int\frac{x^{n-1}\,dx}{\alpha+\beta xx} + (n\beta-(n+2)\alpha)\int\frac{x^{n+1}\,dx}{\alpha+\beta xx} - (n+2)\beta\int\frac{x^{n+3}\,dx.}{\alpha+\beta xx}$$

Quod ſi iam poſt integrationes ſtatuatur $x = 1$, prodibit haec integralium reductio:

$$n\alpha\int\frac{x^{n-1}\,dx}{\alpha+\beta xx} = ((n+2)\alpha - n\beta)\int\frac{x^{n+1}\,dx}{\alpha+\beta xx} + (n+2)\beta\int\frac{x^{n+3}\,dx}{\alpha+\beta xx}.$$

§. 15. Quoniam hic poteſtates ipſius x binario augentur, exponenti n ſucceſſiue tribuamus valores 1, 3, 5, 7, 9 etc. ac ſtatuatur:

$$A = \int\frac{dx}{\alpha+\beta xx},\quad B = \int\frac{xx\,dx}{\alpha+\beta xx},\quad C = \int\frac{x^4\,dx}{\alpha+\beta xx}\ \text{etc.}$$

Deinde vero literae f, g, h cum ſuis deriuatis erunt:

$$f = \alpha,\ f' = 3\alpha,\ f'' = 5\alpha,\ f''' = 7\alpha,\ \text{etc.}$$
$$g = 3\alpha-\beta,\ g' = 5\alpha-3\beta,\ g'' = 7\alpha-5\beta,\ \text{etc.}$$
$$h = 3\beta,\ h' = 5\beta,\ h'' = 7\beta,\ h''' = 9\beta,\ \text{etc.}$$

vnde naſcitur ſequens fractio continua:

$$\frac{\alpha A}{B} = 3\alpha-\beta + \cfrac{9\alpha\beta}{5\alpha-3\beta + \cfrac{25\alpha\beta}{7\alpha-5\beta + \cfrac{49\alpha\beta}{9\alpha-7\beta + \text{etc.}}}}$$

§. 16. Quia eſt $B = \int\frac{xx\,dx}{\alpha+\beta xx}$, erit

$$B = \tfrac{1}{\beta}\int dx - \tfrac{\alpha}{\beta}\int\frac{dx}{\alpha+\beta xx},\ \text{ideoque}\ B = \tfrac{1}{\beta} - \tfrac{\alpha}{\beta}A,$$

quo valore ſubſtituto habebimus

$\alpha\beta A$

$$\frac{\alpha\beta A}{1-\alpha A} = 3\alpha-\beta+\cfrac{9\alpha\beta}{5\alpha-3\beta+\cfrac{25\alpha\beta}{7\alpha-5\beta+\text{etc.}}}$$

cui, quia caput deest, praefigamus $\alpha+\beta+\alpha\beta$; tum autem erit summa $\beta+\frac{1}{A}$, ita vt habeamus

$$\beta+\frac{1}{A}=\alpha+\beta+\cfrac{\alpha\beta}{3\alpha-\beta+\cfrac{9\alpha\beta}{5\alpha-3\beta+\cfrac{25\alpha\beta}{7\alpha-5\beta+\text{etc.}}}}$$

existente $A=\int\frac{dx}{\alpha+\beta xx}$, integrali ita sumto, vt euanescat posito $x=0$, tum vero facto $x=1$.

§. 17. Euoluamus primo casum simplicissimum, quo $\alpha=1$ et $\beta=1$, vbi erit $A=\frac{\pi}{4}$, vnde habebimus

$$1+\frac{4}{\pi}=2+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

siue erit

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\text{etc.}}}}$$

quae est ipsa fractio continua olim a *Brounkero* primum producta, cuius inuestigatio, cum a *Wallisio* per calculos valde taediosos sit eruta, hic quasi sponte ex nostra formula sese prodidit.

§. 18.

§. 18. Noſtra autem forma generalis infinitas alias ſimiles expreſſiones ſuppeditat, prouti literae α et β vario modo accipiuntur. Ac primo quidem, ſi α et β fuerint numeri poſitiui, valor literae A ſemper per arcum circularem exprimetur, contra vero per logarithmos. Sit igitur primo $\beta = 1$, eritque

$$A = \int \frac{dx}{\alpha + xx} = \frac{1}{\sqrt{\alpha}} \text{A tang.} \frac{x}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}} \text{A tang.} \frac{1}{\sqrt{\alpha}},$$

vnde naſcitur haec fractio continua:

$$1 + \frac{\sqrt{\alpha}}{\text{A tang.} \frac{1}{\sqrt{\alpha}}} = \alpha + 1 + \cfrac{\alpha}{3\alpha - 1 + \cfrac{9\alpha}{5\alpha - 3 + \cfrac{25\alpha}{7\alpha - 5 + \text{etc.}}}}$$

Hinc igitur ſi ſumatur $\alpha = 3$, quia $\text{A tang.} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$, habebimus

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3}{8 + \cfrac{27}{12 + \cfrac{75}{16 + \cfrac{147}{20 + \text{etc.}}}}}$$

ſiue

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3 \cdot 1}{8 + \cfrac{3 \cdot 9}{12 + \cfrac{3 \cdot 25}{16 + \cfrac{3 \cdot 49}{20 + \text{etc.}}}}}$$

§. 19.

§. 19. Sit nunc B numerus positiuus quicunque, et quia est

$$A = \int \frac{dx}{\alpha + \beta x x} = \frac{1}{\beta} \int \frac{dx}{\frac{\alpha}{\beta} + x x},$$

integrando fit $A = \frac{1}{\sqrt{\alpha\beta}} \text{A tang.} \sqrt{\frac{\beta}{\alpha}}$. Hinc igitur habebimus

$$\beta + \frac{\sqrt{\alpha\beta}}{\text{A tang.} \sqrt{\frac{\beta}{\alpha}}} = \alpha + \beta + \cfrac{\alpha\beta}{3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \text{etc.}}}$$

Faciamus igitur $\alpha + \beta = 2n$ et $\alpha - \beta = 2m$, vt fit $\alpha = m + n$ et $\beta = n - m$, quibus valoribus positis erit

$$n - m + \frac{\sqrt{(nn - mm)}}{\text{A tang.} \sqrt{\frac{n-m}{n+m}}} = 2n + \cfrac{nn - mm}{2n + 4m + \cfrac{9(nn - mm)}{2n + 8m + \text{etc.}}}$$

§. 20. Consideremus etiam casum, quo β est numerus negatiuus, et ponendo $\beta = -\gamma$, erit

$$A = \int \frac{dx}{\alpha - \gamma x x} = \frac{1}{\gamma} \int \frac{dx}{\frac{\alpha}{\gamma} - x x},$$

cuius integrale est

$$A = \frac{1}{2\sqrt{\alpha\gamma}} l \frac{\sqrt{\frac{\alpha}{\gamma}} + x}{\sqrt{\frac{\alpha}{\gamma}} - x};$$

facto ergo $x = 1$ erit

$$A = \frac{1}{2\sqrt{\alpha\gamma}} l \frac{\sqrt{\alpha} + \sqrt{\gamma}}{\sqrt{\alpha} - \sqrt{\gamma}},$$

 vnde

vnde nascitur ista fractio continua:

$$-\gamma+\frac{2\sqrt{\alpha\gamma}}{l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}}=\alpha-\gamma-\cfrac{\alpha\gamma}{3\alpha+\gamma-\cfrac{9\alpha\gamma}{5\alpha+3\gamma-\cfrac{25\alpha\gamma}{7\alpha+5\gamma-\text{etc.}}}}$$

hocque modo nacti sumus nouas fractiones continuas, quarum valores etiam per logarithmos exhibere licet, et quae prorsus discrepant ab illis, quas ante inuenimus.

§. 21. Hic casus prae reliquis notatu dignus se offert, quando $\gamma=\alpha$. Siue, quod eodem redit, $\alpha=1$ et $\gamma=1$; quia enim tum est $l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}=l\,\infty=\infty$, habebimus

$$-1=0-\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}}$$

siue mutatis signis

$$1=\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}}$$

hinc primus denominator

$$4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}\quad\text{debet esse}=1.$$

Erit ergo $0=3-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}$

siue

siue $1 = \cfrac{3}{8 - \cfrac{25}{12 - \text{etc.}}}$

vbi denominator debet esse $= 3$, vnde fit

$$0 = 5 - \cfrac{25}{12 - \text{etc.}}$$

cuius denominator debet esse $= 5$, vnde fit

$$0 = 7 - \cfrac{49}{16 - \cfrac{81}{20 - \text{etc.}}}$$

ex quo ordine facile veritas perspicitur.

§. 22. Sumamus $\alpha = 4$ et $\gamma = 1$ et nanciscemur hanc fractionem:

$$-1 + \frac{4}{l\,3} = 3 - \cfrac{4.1}{13 - \cfrac{4.9}{23 - \cfrac{4.25}{33 - \cfrac{4.49}{43 - \text{etc.}}}}}$$

Sin autem accipiamus $\alpha = 9$ et $\gamma = 1$ erit

$$-1 + \frac{6}{l\,2} = 8 - \cfrac{9.1}{28 - \cfrac{9.9}{48 - \cfrac{9.25}{68 - \cfrac{9.49}{88 - \text{etc.}}}}}$$

IV. Euolutio formulae.

$$s = x^n e^{\alpha x} (1 - x)$$

§. 22. Hic e denotat numerum cuius logarithmus hyperbolicus est vnitas, ita vt $d.\, e^{\alpha x} = \alpha\, dx\, e^{\alpha x}$. Hinc ergo erit

$$ds = n x^{n-1}\, dx\, e^{\alpha x} + (\alpha - (n+1))\, x^n\, dx\, e^{\alpha x} - \alpha x^{n+1}\, dx\, e^{\alpha x},$$

vnde viciſſim integrando fit

$$s = n \int x^{n-1} dx\, e^{\alpha x} + (\alpha - (n+1)) \int x^n\, dx\, e^{\alpha x} - \alpha \int x^{n+1}\, dx\, e^{\alpha x}.$$

Quod ſi ergo poſt integrationem ſtatuatur $x = 1$, erit

$$n \int x^{n-1} dx\, e^{\alpha x} = (n + 1 - \alpha) \int x^n\, dx\, e^{\alpha x} + \alpha \int x^{n+1}\, dx\, e^{\alpha x}.$$

§. 23. Quodſi iam loco n ſucceſſiue ſcribamus numeros 1, 2, 3, 4, ac faciamus

$$A \int e^{\alpha x} dx = \tfrac{1}{\alpha}(e^{\alpha} - 1) \text{ et } B = \int x\, dx\, e^{\alpha x} = \tfrac{\alpha - 1}{\alpha\alpha} e^{\alpha} + \tfrac{1}{\alpha\alpha}$$

$$f = 1,\ f' = 2,\ f'' = 3,\ f''' = 4, \text{ etc.}$$

$$g = 2 - \alpha,\ g' = 3 - \alpha,\ g'' = 4 - \alpha, \text{ etc.}$$

$$h = \alpha,\ h' = \alpha,\ h'' = \alpha,\ h''' = \alpha, \text{ etc.}$$

prodibit iſta fractio continua:

$$\frac{A}{B} = 2 - \alpha + \cfrac{2\alpha}{3 - \alpha + \cfrac{3\alpha}{4 - \alpha + \cfrac{4\alpha}{5 - \alpha + \text{etc.}}}}$$

Adiungamus adhuc ſuperne $1 - \alpha + \alpha$, erit eius valor

$$1 - \alpha + \frac{(\alpha - 1)\, e^{\alpha} + 1}{e^{\alpha} - 1} = \frac{\alpha}{e^{\alpha} - 1},$$

vnde

vnde habebitur haec fractio continua satis concinna:

$$\frac{\alpha}{e^{\alpha}-1}=1-\alpha+\cfrac{\alpha}{2-\alpha+\cfrac{2\alpha}{3-\alpha+\cfrac{3\alpha}{4-\alpha+\text{etc.}}}}$$

vnde patet, si fuerit $\alpha=0$, ob $e^{\alpha}-1=\alpha$, fore vtique $1=1$.

§. 24. Consideremus nonnullos casus speciales; ac primo, si sit $\alpha=1$, erit

$$\frac{1}{e-1}=0+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\text{etc.}}}}}$$

quae fractio facile transfunditur in hanc:

$$\frac{1}{e-1}=\cfrac{1}{1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{4}}{1+\text{etc.}}}}}}$$

vnde fit

$$e-1=1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\text{etc.}}}}$$

Haec autem porro a fractionibus partialibus liberata dat

$$e - 1 = + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \text{etc.}}}}}}$$

vnde sequitur

$$\frac{1}{e - 2} = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{4 + \text{etc.}}}}}$$

quae formae ob simplicitatem maxime sunt notatu dignae. Ex penultima, qua fit

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \text{etc.}}}}}$$

sumendo successiue 1, 2, 3, pluraue membra, orientur sequentes approximationes:

$$e = 2,0000$$
$$e = 3.0000$$
$$e = 2,6666$$
$$e = 2,7272$$
$$e = 2,7169$$

qui valores, alternatim maiores et minores, satis prompte ad veritatem conuergunt.

§. 25.

§. 25. Sumamus $\alpha = 2$ erit

$$\frac{2}{ee-1} = -1 + \cfrac{2}{0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}}$$

Ex hac fractione porro deducitur iſta:

$$\frac{2(ee-1)}{ee+1} = 0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}$$

ſimilique modo, ſi pro α maiores numeri accipiantur, reductio fieri poterit.

§. 26. Poſſunt etiam pro α numeri negativi accipi. Ita ſi fuerit $\alpha = -1$ fiet

$$\frac{e}{e-1} = 2 - \cfrac{1}{3 - \cfrac{2}{4 - \cfrac{3}{5 - \cfrac{4}{6 - \text{etc.}}}}}$$

quae reducitur ad hanc formam:

$$\frac{e}{e-1} = 2 + \cfrac{1}{-3 + \cfrac{2}{4 + \cfrac{3}{-5 + \cfrac{4}{6 + \text{etc.}}}}}$$

ſimilique modo maiores valores expediri poſſunt.

§. 27. Statuamus etiam $\alpha = \frac{1}{2}$, ac reperietur ista expressio:

$$\frac{1}{2(\sqrt{e}-1)} = \frac{1}{2} + \cfrac{\frac{1}{2}}{\frac{3}{2} + \cfrac{1}{\frac{5}{2} + \cfrac{\frac{3}{2}}{\frac{7}{2} + \cfrac{\frac{5}{2}}{\frac{9}{2} + \text{etc.}}}}}$$

quae liberata a fractionibus partialibus euadit

$$\frac{1}{-1+\sqrt{e}} = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \cfrac{8}{9 + \text{etc.}}}}}$$

Simili modo si sumus $\alpha = \frac{1}{3}$ erit

$$\frac{1}{3(\sqrt[3]{e}-1)} = 2:3 + \cfrac{1:3}{5:3 + \cfrac{2:3}{8:3 + \cfrac{3:3}{11:3 + \cfrac{4:3}{14:3 + \text{etc.}}}}}$$

quae a fractionibus partialibus liberata dat

$$\frac{1}{-1+\sqrt[3]{e}} = 2 + \cfrac{3}{5 + \cfrac{6}{8 + \cfrac{9}{11 + \cfrac{12}{14 + \text{etc.}}}}}$$

At

At fi ponatur $\alpha = \frac{2}{3}$, prodit haec fractio continua:

$$\frac{2}{3(\sqrt[3]{(ee-1)})} = 1:3 + \cfrac{2:3}{4:3 + \cfrac{4:3}{7:3 + \cfrac{6:3}{10:3 + \cfrac{8:3}{13:3 + \text{etc.}}}}}$$

quae a fractionibus partialibus liberata fit

$$\frac{2}{\sqrt[3]{(ee-1)}} = 1 + \cfrac{6}{4 + \cfrac{12}{7 + \cfrac{18}{10 + \cfrac{24}{13 + \text{etc.}}}}}$$

§. 28. His formulis tanquam principalibus ac fimplicioribus euolutis, fimili modo alias multo generaliores tractare licebit, quae ad fractiones continuas multo magis abfconditas perducent, vti ex cafibus qui fequuntur patebit.

V. Euolutio formulae.

$$s = x^n (a - b\,x^\theta - c\,x^{2\theta})^\lambda.$$

§. 29. Hinc igitur erit

$$ds = (a - b\,x^\theta - c\,x^{2\theta})^{\lambda-1} (n\,a\,x^{n-1}\,dx - b(n+\lambda\,\theta)\,x^{n+\theta-1}\,dx$$
$$- c(n+2\lambda\,\theta)\,x^{n+2\theta-1}\,dx),$$

vnde per partes integrando, tum vero ftatuendo $a - b\,x^\theta - c\,x^{2\theta} = 0$, (quod fit fi fuerit $x^\theta = -\frac{b \pm \sqrt{(bb+4ac)}}{2c}$) habebitur ifta reductio generalis:

$$n a \int x^{n-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda - 1}$$
$$= (n + \lambda\theta) b \int x^{n+\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1}$$
$$+ (n + 2\lambda\theta) c \int x^{n+2\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1}.$$

§. 30. Quodsi iam hanc formam cum nostra generali initio tradita comparare velimus, valores pro litera n successiue assumendi per differentiam θ augeri debent. Deinde non necesse est vt primus valor ipsius n, vt hactenus fecimus, sumatur $= 1$; statuamus igitur eius primum valorem $= \alpha$, et quaeramus valores binarum sequentium formularum integralium, scilicet:

$$A = \int x^{\alpha-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1} \text{ et}$$
$$B = \int x^{\alpha+\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1},$$

quae integralia ita sunt capienda, vt euanescant posito $x = 0$, quo facto ipsi x ille valor tribui debet, qui reddat formulam $a - b x^{\theta} - c x^{2\theta} = 0$. Quoniam autem hoc in genere exsequi non licet, istos valores per literas A et B indicare contenti simus, quos ergo tanquam cognitos spectemus.

§. 31. Praeterea vero literae f, g, h, cum suis deriuatis sequentes induent valores:

$f = \alpha a$, $f' = (\alpha + \theta) a$, $f'' = (\alpha + 2\theta) a$, $f''' = (\alpha + 3\theta) a$, etc.
$g = (\alpha + \lambda\theta) b$, $g' = (\alpha + \theta + \lambda\theta) b$, $g'' = (\alpha + 2\theta + \lambda\theta) b$, etc.
$h = (\alpha + 2\lambda\theta) c$, $h' = (\alpha + \theta + 2\lambda\theta) c$, $h'' = (\alpha + 2\theta + 2\lambda\theta) c$; etc.

Ex his igitur formabitur sequens fractio continua:

$$\frac{\alpha a A}{B} = (\alpha+\lambda\theta)b + \cfrac{(\alpha+\theta)(\alpha+\lambda\theta)ac}{(\alpha+\theta+\lambda\theta)b + \cfrac{(\alpha+2\theta)(\alpha+\theta+2\lambda\theta)ac}{(\alpha+2\theta+\lambda\theta)b + \cfrac{(\alpha+3\theta)(\alpha+2\theta+2\lambda\theta)ac}{(\alpha+3\theta+\lambda\theta)b \text{ etc.}}}}$$

quae

quae forma vtique eſt maxime generalis, cuius autem vlteriori euolutioni non immoramur.

VI. Euolutio formulae.

$$s = x^n (1 - x^\theta)^\lambda$$

§. 32. Hinc ergo fit

$$ds = n x^{n-1} dx (1 - x^\theta)^\lambda - \lambda \theta x^{n+\theta-1} dx (1 - x^\theta)^{\lambda-1},$$

vnde tantum duae formulae integrales orirentur; quamobrem huic differentiali denominatorem arbitrarium tribuamus $a + b x^\theta$, vt habeamus:

$$ds = \frac{(1 - x^\theta)^{\lambda-1}}{a + b x^\theta} (n a x^{n-1} dx - (a(n + \lambda\theta) - bn) x^{n+\theta-1} dx$$
$$- b(n + \lambda\theta) x^{n+2\theta-1} dx.$$

Nunc igitur, ponendo poſt integrationem $x = 1$, deducimus hanc reductionem:

$$na \int \frac{x^{n-1} dx (1 - x^\theta)^{\lambda-1}}{a + b x^\theta} = (a(n + \lambda\theta) - bn) \int \frac{x^{n+\theta-1} dx (1 - x^\theta)^{\lambda-1}}{a + b x^\theta}$$
$$+ b(n + \lambda\theta) \int \frac{x^{n+2\theta-1} dx (1 - x^\theta)^{\lambda-1}}{a + b x^\theta}.$$

§. 33. Hic iterum euidens eſt valores ipſius n per differentiam θ creſcere debere. Statuatur autem primus valor ipſius $n = \alpha$, et quaerantur pro quouis caſu oblato binae ſequentes formulae integrales:

$$A = \int \frac{x^{\alpha-1} dx (1 - x^\theta)^{\lambda-1}}{a + b x^\theta} \text{ et } B = \int \frac{x^{\alpha+\theta-1} dx (1 - x^\theta)^{\lambda-1}}{a + b x^\theta},$$

vbi ſcilicet poſt integrationem poſitum ſit $x = 1$. Quibus

bus inuentis, cum hinc fiat

$$f = \alpha a,\ f' = (\alpha + \theta) a,\ f'' = (\alpha + 2\theta) a,\ f''' = (\alpha + 3\theta) a, \text{ etc.}$$
$$g = (\alpha + \lambda\theta) a - \alpha b,\ g' - (\alpha + \theta + \lambda\theta) a - (\alpha + \theta) b,$$
$$g'' = (\alpha + 2\theta + \lambda\theta) a - (\alpha + 2\theta) b, \text{ etc.}$$
$$h = (\alpha + \lambda\theta) b,\ h = (\alpha + \theta + \lambda\theta) b,\ h'' = (\alpha + \theta + 2\lambda\theta) b, \text{ etc.}$$

inde formabitur fequens fractio continua:

$$\frac{\alpha a A}{B} = (\alpha+\lambda\theta)a - \alpha b + \cfrac{(\alpha+\theta)(\alpha+\lambda\theta)ab}{(\alpha+\theta+\lambda\theta)a - (\alpha+\theta)b + \cfrac{(\alpha+2\theta)(\alpha+\theta+\lambda\theta)ab}{(\alpha+2\theta+\lambda\theta)a - (\alpha+2\theta)b + (\alpha+3\theta)(\alpha+2\theta+\lambda\theta)ab \text{ etc.}}}$$

cuius formae vberiore euolutione fuperfedemus.

VII. Euolutio formulae.

$$s = x^n (e^{\alpha x} (1 - x)^\lambda)$$

§. 34. Hinc ergo fit

$$ds = (1-x)^{\lambda-1} (n x^{n-1} dx - (n+\lambda-\alpha) x^n dx - \alpha x^n dx),$$

hinc igitur fi poft integrationem vbique ftatuatur $x = 1$, quippe quo cafu fit $s = 0$, habebimus hanc reductionem:

$$n \int x^{n-1} dx\, e^{\alpha x} (1-x)^{\lambda-1} = (n+\lambda-\alpha) \int x^n dx\, e^{\alpha x} (1-x)^{\lambda-1}$$
$$+ \alpha \int x^{n+1} dx\, e^{\alpha x} (1-x)^{\lambda-1}.$$

§. 35. In his ergo formulis exponenti n valores vnitate crefcentes tribui debebunt, tum vero hic minimum eius valorem fumamus $n = \delta$, atque valores literarum A et B ex his formulis erui oportebit, ponendo poft integrationem $x = 1$,

$$A = \int x^{\delta-1} dx\, e^{\alpha x} (1-x)^{\lambda-1},\quad B = \int x^{\delta} dx\, e^{\alpha x} (1-x)^{\lambda-1}$$

deinde vero ob hos valores:

$$f = \delta,$$

$f = \delta$, $f' = \delta + 1$, $f'' = \delta + 2$, $f''' = \delta + 3$, etc.

$g = \delta + \lambda - \alpha$, $g' = \delta + 1 + \lambda - a$, $g'' = \delta + 2 + \lambda - \alpha$, etc.

$h = \alpha$, $h' = \alpha$, $h'' = \alpha$, etc.

fequitur ifta fractio continua:

$$\frac{\delta A}{B} = \delta + \lambda - \alpha + \cfrac{(\delta + 1)\alpha}{\delta + 1 + \lambda - \alpha + \cfrac{(\delta + 2)\alpha}{\delta + 2 + \lambda - \alpha + \cfrac{(\delta + 3)\alpha}{\delta + 3 + \lambda - \alpha + \text{etc.}}}}$$

Vbi imprimis notari oportet, exponentes λ et δ neceffario nihilo maiores accipi debere, quia alioquin formula principalis $x^n e^{\alpha x} (1 - x)^\lambda$ cafibus $x = 1$ non euanefceret.

§. 36. Si literis δ et λ tribuatur valor $= 1$, prodibit cafus iam fupra tractatus; ac fi his literis numeri integri affignentur, eiusmodi fractiones continuae orientur, quas per certas operationes ad priores reducere licebit. Verum fi his literis δ et λ, vel alterutri, vel vtrique, fractiones affignemus, tum formae orientur ad priores prorfus irreductibiles, quarumque valor haud aliter quam per quantitates maxime tranfcendentes exprimere liceat. Velvti fi fuerit $\delta = \frac{1}{2}$ et $\lambda = \frac{1}{2}$, valor literae A quaeri debebit ex hac formula integrali: $A = \frac{e^{\alpha x}\, dx}{\sqrt{(x - xx)}}$, cuius integratio ad quantitates maxime tranfcendentes perducit, ita vt valor talium fractionum continuarum prodeat maxime abftrufus.