



1781

# Nova methodus motum planetarum determinandi

Leonhard Euler

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NOVA METHODVS  
MOTVM PLANETARVM DETERMINANDI

Auctore  
L. E V L E R O.

§. 1.

**A**gitur hic de celeberrimo illo Problemate *Kepleriano*, cuius plurimae solutiones passim sunt traditae, quae omnes in hoc conueniant, vt ex data anomalia vera Planetae eius anomalia media definiatur; cum tamen ad vsum astronomicum vicissim ex data anomalia media vera assignari deberet. Quanquam autem non difficile erat solutionem inuentam per conuersionem ad hunc scopum traducere, methodum directam, qua ex media anomalia inuestigari queat anomalia vera, vsu non esse carituram arbitror. Quamobrem hic constitui eam methodum, qua in determinando motu Lunae feliciter sum vsus, ad Planetas primarios etiam adcommodare, quo clarius natura et vis huius methodi ob oculos exponatur; quandoquidem in illa eiusmodi artificia occurrunt, quae propter multitudinem elementorum, quibus motus Lunae implicatur, non satis dilucide perspiciuntur.

M m 2

§. 2.

Tab. II.

Fig. 18.

§. 2. Referat igitur tabula planum, in quo Planeta moueatur, vbi punctum S sit centrum Solis, axis vero S V ad principium arietis dirigatur. Planeta autem elapso tempore  $\tau$ , quod in diebus exprimi assumo, pervenerit in P, vnde ad axem demisso perpendicularo P Q vocentur binæ coordinatae S Q =  $x$  et Q P =  $y$ ; ipsa autem distantia Planetæ a Sole sit S. P =  $v$ , ita ut  $v v = x x + y y$ . His positis constat principia motus sequentes dare binas aequationes:

$$\frac{d d x}{\Delta d \tau^2} = - \frac{x}{v^3} \text{ et } \frac{d d y}{\Delta d \tau^2} = - \frac{y}{v^3},$$

vbi elementum  $d \tau$  fumitur constans, et littera  $\Delta$  denotat certam quantitatem constantem, quam ex motu Terræ mox definiemus. Hic autem integrationi harum formularum non immoror, quam tum demum feliciori successum suscepturus, cum has aequationes ad usum commodiorem transformauero, vbi totum negotium multo facilius succedet.

§. 3. Quoniam hic ambæ coordinatae  $x$  et  $y$ , quarum valores ad quoduis tempus assignari oportet, maximis variationibus sunt obnoxiae, dum per totam Planetae orbitam modo fieri possunt positivæ modo negativæ, eas ante omnia ad alium axem transferri conveniet, vbi multo minores variationes sint subiturae. Hunc in finem statim motum medium eiusdem Planetæ in calculum introduco, quo scilicet singulae revolutiones motu aequabili circa solem in circulo peragantur. Ducatur igitur recta S M ad locum medium, quem Planeta eodem tempore est occupaturus; quippe qui locus ex tabulis mediorum motuum facillime innotescit. Vocemus igitur eius longitudi-

nem,

nem, seu angulum  $\angle S M = \zeta$ , qui ergo tempore est proportionalis, hancque rectam  $S M$  pro hoc tempore tanquam axem spectemus, ad quem locum Planetæ  $P$  per coordinatas orthogonales referamus, quæ sint

$S q = X$ ,  $q P = Y$ , vnde iterum fit  $o o = X X + Y Y$ ; ex his vero priores coordinatæ ita definiuntur, vt fit

$$x = X \cos. \zeta - Y \sin. \zeta \text{ et } y = X \sin. \zeta + Y \cos. \zeta.$$

Atque nunc iam istud commodum sumus affecti, vt, nisi Planeta enormem habuerit excentricitatem, hæe quantitates  $X$  et  $Y$  exiguas tantum mutationes sint passuræ, dum noua abscissa  $S q = X$  nunquam multum a distantia Planetæ media a Sole est discrepatura; applicata autem  $P q = Y$  nunquam certos limites, non adeo remotos, est transgressura. Si enim Planeta excentricitate penitus careret, perpetuo foret  $X$  quantitas constans et  $Y = 0$ , quandoquidem hoc casu motus Planetæ in motum medium recideret.

§. 4. Cum igitur fit  $x = X \cos. \zeta - Y \sin. \zeta$ , erit

$$d x = d X \cos. \zeta - d Y \sin. \zeta - d \zeta (X \sin. \zeta + Y \cos. \zeta)$$

et denuo differentiando, ob  $d \zeta$  constans, fiet

$$d d x = d d X \cos. \zeta - d d Y \sin. \zeta - 2 d \zeta (d X \sin. \zeta + d Y \cos. \zeta) - d \zeta^2 (X \cos. \zeta - Y \sin. \zeta).$$

Eodem modo, cum fit

$$y = X \sin. \zeta + Y \cos. \zeta, \text{ erit}$$

$$d y = d X \sin. \zeta + d Y \cos. \zeta + d \zeta (X \cos. \zeta - Y \sin. \zeta) \text{ et}$$

$$d d y = d d X \sin. \zeta + d d Y \cos. \zeta + 2 d \zeta (d X \cos. \zeta - d Y \sin. \zeta) - d \zeta^2 (X \sin. \zeta + Y \cos. \zeta).$$

Hinc igitur colligimus sequentes valores:

M m 3

$d d x$

$$\begin{aligned} d d x \cos. \zeta + d d y \sin. \zeta &= d d X - 2 d \zeta d Y - X d \zeta^2 \\ d d y \cos. \zeta - d d x \sin. \zeta &= d d Y + 2 d \zeta d X - Y d \zeta^2 \end{aligned}$$

§. 5. At vero ex aequationibus fundamentalibus sequitur fore

$$\begin{aligned} \frac{d d x \cos. \zeta + d d y \sin. \zeta}{\Delta d \tau^2} &= \frac{-x \cos. \zeta - y \sin. \zeta}{v^3} = \frac{-X}{v^3}, \\ \frac{d d y \cos. \zeta - d d x \sin. \zeta}{\Delta d \tau^2} &= \frac{-y \cos. \zeta + x \sin. \zeta}{v^3} = \frac{-Y}{v^3}, \end{aligned}$$

quae ergo aequationes, substitutis valoribus modo inuentis, nobis praebebunt sequentes formulas per X et Y expressas:

$$\begin{aligned} \frac{d d X - 2 d \zeta d Y - X d \zeta^2}{\Delta d \tau^2} &= \frac{-X}{v^3} \\ \frac{d d Y + 2 d \zeta d X - Y d \zeta^2}{\Delta d \tau^2} &= \frac{-Y}{v^3}, \end{aligned}$$

ex quibus ergo binas novas coordinatas X et Y definiri conuenit,

§. 6. Antequam autem hoc negotium suscipiamus, ex motu Terrae medio, quem pro penitus cognito assumere licet, quantitatem constantem  $\Delta$  determinemus. Hunc in finem statuamus distantiam mediam Terrae a Sole = 1, et iam loco Planetae P substituamus ipsam Terram, quasi motu suo medio in circulo, cuius radius = 1, circa Solem moueretur, fietque hoc casu X = 1 et Y = 0, hincque  $v = 1$ , vnde nostrae aequationes euadent:

$$-\frac{d \zeta^2}{\Delta d \tau^2} = -1 \text{ et } 0 = 0.$$

Hinc ergo fit

$$d \zeta^2 = \Delta d \tau^2, \text{ ideoque } \Delta = \frac{d \zeta^2}{d \tau^2},$$

vbi  $\zeta$  denotat longitudinem Terrae mediam, quae si pro tempore  $\tau$  dierum vocetur =  $t$ , erit  $\Delta = \frac{d t^2}{d \tau^2}$ ; sicque in nostris formulis loco  $\Delta d \tau^2$  scribi conueniet  $d t^2$ . Quamobrem,

§. 10. Quatenus hic  $yy$  est valde paruum prae  $(1+x)^2$ , formula nostra irrationalis sequenti modo in seriem euoluetur:

$$((1+x)^2 + yy)^{-\frac{1}{2}} = \frac{1}{(1+x)^2} - \frac{1}{2} \frac{yy}{(1+x)^3} + \frac{3}{8} \frac{y^2}{(1+x)^4} - \frac{5}{16} \frac{y^3}{(1+x)^5} + \text{etc.}$$

quo valore substituto nostrae aequationes euadent:

$$\begin{aligned} \frac{d^2 x}{dt^2} - 2n \frac{dy}{dt} - nn(1+x) &= -\frac{nn}{(1+x)^2} + \frac{1}{2} \frac{nn yy}{(1+x)^3} \\ &\quad - \frac{3}{8} \frac{nn y^2}{(1+x)^4} + \frac{5}{16} \frac{nn y^3}{(1+x)^5} \text{ etc.} \\ \frac{d^2 y}{dt^2} + 2n \frac{dx}{dt} - nn y &= -\frac{nn y}{(1+x)^3} + \frac{3}{2} \frac{nn y^3}{(1+x)^5} \\ &\quad - \frac{5}{8} \frac{nn y^5}{(1+x)^7} + \frac{35}{128} \frac{nn y^7}{(1+x)^9} \text{ etc.} \end{aligned}$$

§. 11. Quatenus autem quoque  $x$  prae unitate est valde paruum, denominatores nostrarum aequationum deuo in series conuertantur ope huius reductionis:

$$(1+x)^{-\lambda} = 1 - \lambda x + \frac{\lambda(\lambda+1)}{1 \cdot 2} x^2 - \frac{\lambda(\lambda+1)(\lambda+2)}{1 \cdot 2 \cdot 3} x^3 + \text{etc.}$$

vide fit

$$\begin{aligned} \frac{1}{(1+x)^2} &= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7 + \text{etc.} \\ \frac{1}{(1+x)^3} &= 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + 28x^6 - 36x^7 + \text{etc.} \\ \frac{1}{(1+x)^4} &= 1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + 84x^6 - 120x^7 + \text{etc.} \\ \frac{1}{(1+x)^5} &= 1 - 5x + 15x^2 - 35x^3 + 70x^4 - 126x^5 + 210x^6 - 330x^7 + \text{etc.} \\ \frac{1}{(1+x)^6} &= 1 - 6x + 21x^2 - 56x^3 + 126x^4 - 252x^5 + 462x^6 - 792x^7 + \text{etc.} \\ &\text{etc. etc.} \end{aligned}$$

§. 12. Substituamus nunc istos valores in nostris aequationibus, ac terminos ad partem dextram secundum dimensiones, quas literae  $x$  et  $y$  in iis obtinent, disponamus,

*Acta Acad. Imp. Sc. Tom. II. P. II.*      N n      quo

quo clarius conuergentia terminorum ob oculos ponatur, siquidem literas  $x$  et  $y$  vt quantitates valde paruas respicere licet. Erit igitur

$$\frac{d^2x}{dt^2} + \frac{2ndy}{dt} = 3nnx - 3nnxx + 4nnx^2 - 5nnx^3 + 6nnx^4 - 7nnx^5 \text{ etc.}$$

$$+ \frac{3}{2}nny - 6nnxy + 15nnxyy - 30nnx^2yy + \frac{105}{2}nnx^3yy$$

$$- \frac{15}{4}nny^2 + \frac{45}{2}nnxy^2 - \frac{315}{4}nnx^2y^2$$

$$+ \frac{35}{16}nny^3 \text{ etc.}$$

$$\frac{ddy}{dt^2} + \frac{2ndx}{dt} = +3nnxy - 6nnxxy + 10nnx^2y - 15nnx^3y + 21nnx^4y$$

$$+ \frac{3}{2}nny^2 - \frac{15}{2}nnxy^2 + \frac{45}{2}nnx^2y^2 - \frac{105}{2}nnx^3y^2$$

$$- \frac{15}{8}nny^3 + \frac{105}{4}nnxy^3 \text{ etc.}$$

§. 13. Diuidamus has aequationes per  $nn$ , et cum fit  $nt = \zeta$ , existente  $\zeta$  longitudine media Planetæ, siue angulo  $\sphericalangle$  S M, quem ex tabula motuum mediorum Planetæ ad quodvis tempus depromere licet, erit  $ndt = d\zeta$  et  $nn d^2t = d^2\zeta$ , vnde nostræ aequationes aliquanto fient simpliciores, scilicet:

$$\frac{d^2x}{d\zeta^2} - \frac{2dy}{d\zeta} = 3x - 3xx + 4x^2 - 5x^3 + 6x^4 - 7x^5 \text{ etc.}$$

$$+ \frac{3}{2}yy - 6xyy + 15xxyy - 30x^2yy + \frac{105}{2}x^3yy \text{ etc.}$$

$$- \frac{15}{4}y^2 + \frac{45}{2}xy^2 - \frac{315}{4}x^2y^2 \text{ etc.}$$

$$+ \frac{35}{16}y^3 \text{ etc.}$$

$$\frac{ddy}{d\zeta^2} + \frac{2dx}{d\zeta} = 3xy - 6xxy + 10x^2y - 15x^3y + 21x^4y \text{ etc.}$$

$$+ \frac{3}{2}y^2 - \frac{15}{2}xy^2 + \frac{45}{2}x^2y^2 - \frac{105}{2}x^3y^2 \text{ etc.}$$

$$- \frac{15}{8}y^3 + \frac{105}{4}xy^3 \text{ etc.}$$

§. 14. Iam obseruauimus, si orbita Planetæ excentricitate careret, perpetuo fore tam  $x = 0$  quam  $y = 0$ . Eatenus igitur hæc quantitates non euanescent, quatenus adest

adest excentricitas. Statuamus igitur excentricitatem esse  $= e$ , et cum ea tanquam satis exigua spectetur, facile intelligitur, ambas literas  $x$  et  $y$  per huiusmodi series convergentes exprimi posse;

$$x = eP + eeQ + e^3R + e^4S + e^5T \text{ etc.}$$

$$y = ep + eeq + e^3r + e^4s + e^5t + e^6u \text{ etc.}$$

ex quibus statim colligitur

$$dx = e dP + eedQ + e^3 dR + e^4 dS + e^5 dT + e^6 dU + \text{etc.}$$

$$dy = edp + eedq + e^3 dr + e^4 ds + e^5 dt + e^6 du + \text{etc.}$$

$$ddx = eddP + eed.dQ + e^3 ddR + e^4 ddS + e^5 ddT + e^6 ddU + \text{etc.}$$

$$ddy = ed.dp + eed.dq + e^3 d.dr + e^4 d.ds + e^5 d.dt + e^6 d.du + \text{etc.}$$

tum vero pro membris compositis habebimus has formulas usque ad potestatem sextam ipsius  $e$  continuatas:

$$xx = eePP + 2e^3PQ + 2e^4PR + 2e^5PS + 2e^6PT \\ + e^4QQ + 2e^5QR + 2e^6QS \\ + e^6RR$$

$$xy = eepP + e^3pQ + e^4pR + e^5pS + e^6pT \\ + e^3qP + e^4qQ + e^5qR + e^6qS \\ + e^4rP + e^5rQ + e^6rR \\ + e^5sP + e^6sQ \\ + e^6tP$$

$$yy + eepP + 2e^3pq + 2e^4pr + 2e^5ps + 2e^6pt \\ + e^4qq + 2e^5qr + 2e^6qs \\ + e^6rr$$

$$x^3 = e^3P^3 + 3e^4PPQ + 3e^5PPR + 3e^6PPS \\ + 3e^5PQQ + 6e^6PQR \\ + e^6Q^3$$

N n 2

xy



$$\begin{aligned}
 xxy &= e^3 pPP + 2e^4 pPQ + 2e^5 pPR + 2e^6 pPS \\
 &\quad + e^5 pQQ + 2e^6 pQR \\
 &\quad + e^4 qPP + 2e^5 qPQ + 2e^6 qPR \\
 &\quad \quad \quad + e^6 qQQ \\
 &\quad + e^5 rPP + 2e^6 rPQ \\
 &\quad \quad \quad + e^6 sPP
 \end{aligned}$$

$$\begin{aligned}
 xyx &= e^3 ppP + 2e^4 pqP + 2e^5 prP + 2e^6 psP \\
 &\quad + e^5 qqP + 2e^6 qrP \\
 &\quad + e^4 ppQ + 2e^5 pqQ + 2e^6 prQ \\
 &\quad \quad \quad + e^6 qqQ \\
 &\quad + e^5 ppR + 2e^6 pqR \\
 &\quad \quad \quad + e^6 ppS
 \end{aligned}$$

$$\begin{aligned}
 y^3 &= e^3 p^3 + 3e^4 ppq + 3e^5 ppr + 3e^6 pps \\
 &\quad + 3e^5 pqq + 6e^6 pqr \\
 &\quad \quad \quad + e^6 q^3
 \end{aligned}$$

$$\begin{aligned}
 x^4 &= e^4 P^4 + 4e^5 P^3 Q + 4e^6 P^2 R \\
 &\quad + 6e^6 PPQQ
 \end{aligned}$$

$$\begin{aligned}
 x^3 y &= e^4 pP^3 + 3e^5 pPPQ + 3e^6 pPPR \\
 &\quad + 3e^6 pPQQ \\
 &\quad + e^5 qP^3 + 3e^6 qPPQ \\
 &\quad \quad \quad + e^6 rP^3
 \end{aligned}$$

$$\begin{aligned}
 xxxyy &= e^4 pppP + 2e^5 ppPQ + 2e^6 ppPR \\
 &\quad + e^6 ppQQ \\
 &\quad + 2e^5 pqPP + 4e^6 pqPQ \\
 &\quad \quad \quad + 2e^6 prPP \\
 &\quad \quad \quad + e^6 qqPP
 \end{aligned}$$

$$\begin{aligned}
 xy^3 &= e^4 p^3 P + 3e^5 ppqP + 3e^6 pprP \\
 &\quad + 3e^6 pqqP \\
 &\quad + e^5 p^3 Q + 3e^6 ppqQ \\
 &\quad \quad \quad + e^6 p^3 R
 \end{aligned}$$

$$y^4 = e^4$$

$$y^4 = e^4 p^4 + 4 e^5 p^3 q + 4 e^6 p^2 r + 6 e^6 p p q q$$

$$x^5 = e^5 P^5 + 5 e^6 P^4 Q$$

$$x^4 y = e^5 p P^4 + 4 e^6 p P^3 Q + e^6 q P^4$$

$$x^3 y y = e^5 p p P^3 + 3 e^6 p p P P Q + 2 e^6 p q P^3$$

$$x x y^3 = e^5 p^3 P P + 3 e^6 p p q P P + 2 e^6 p^3 P Q$$

$$x y^4 = e^5 p^4 P + 4 e^6 p^3 q P + e^6 p^4 Q$$

$$y^5 = e^5 p^5 + 5 e^6 p^4 q$$

$$x^6 = e^6 P^6$$

$$x^5 y = e^6 p P^5$$

$$x^4 y y = e^6 p p P^4$$

$$x^3 y^3 = e^6 p^3 P^3$$

$$x x y^4 = e^6 p^4 P P$$

$$x y^5 = e^6 p^5 P$$

$$y^6 = e^6 p^6$$

§. 15. Concipiamus nunc omnes hos valores in nostris aequationibus substitui, et quia quantitates P, Q, R et p, q, r ab excentricitate e immunes esse debent, necesse est, ut in his aequationibus omnes termini, paribus potestibus ipsius e affecti, seorsim inter si aequentur, unde utraque aequatio in plures discerpatur, dum scilicet omnes termini simpliciter per e multiplicati inter se aequales statuuntur, deinde vero illi termini, qui per e e sunt affecti,

N n 3

affectedi, tum ii, qui per  $e^3$  sunt affecti etc. Hoc igitur modo plures adipiscimur ordines aequationum, ex quibus quantitates incognitas P, Q, R, S et  $p, q, r, s$  etc. determinari oportebit. Hos igitur ordines aequationum hic ob oculos ponamus.

Ordo primus,

continens partes sola litera  $e$  affectas.

I.  $\frac{d d P}{d \zeta^2} - \frac{2 d p}{d \zeta} = 3 P$

II.  $\frac{d d p}{d \zeta^2} + \frac{2 d P}{d \zeta} = 0.$

Ordo secundus,

continens partes per  $e e$  affectas.

I.  $\frac{d d Q}{d \zeta^2} - \frac{2 d q}{d \zeta} = 3 Q - 3 P P + \frac{3}{2} p p$

II.  $\frac{d d q}{d \zeta^2} + \frac{2 d Q}{d \zeta} = 3 P p$

Ordo tertius,

continens partes per  $e^3$  affectas.

I.  $\frac{d d R}{d \zeta^2} - \frac{2 d r}{d \zeta} = 3 R - 6 P Q + 4 P^3 + 3 p q - 6 p p P$

II.  $\frac{d d r}{d \zeta^2} + \frac{2 d R}{d \zeta} = 3 p Q + 3 q P - 6 p P P + \frac{3}{2} p^3.$

Ordo quartus,

continens partes per  $e^4$  affectas.

I.  $\frac{d d S}{d \zeta^2} - \frac{2 d s}{d \zeta} = 3 S - 6 P R - 3 Q Q + 3 p r + \frac{3}{2} q q - 6 p p Q + 12 P P Q - 12 p q P - 5 P^4 + 15 p p P P - \frac{15}{2} p^4$

II.  $\frac{d d s}{d \zeta^2} + \frac{2 d S}{d \zeta} = 3 p R + 3 q Q + 3 r P - 12 p P Q - 6 q P P + \frac{3}{2} p p q + 10 p P^3 - \frac{15}{2} p^3 P$

Ordo

Ordo quintus,

continens partes per  $e^5$  affectas.

$$\text{I. } \frac{ddT}{a^2} - \frac{2dt}{a^2} = 3T - 6PS - 6QR + 3ps + 3qr$$

$$+ 12PPR + 12PQQ - 12prP - 6qqP$$

$$- 20P^2Q + 30ppPQ + 30pqPP - \frac{15}{2}p^2q$$

$$+ 6P^3 - 30ppP^2 + \frac{45}{2}p^4P.$$

$$\text{II. } \frac{ddt}{a^2} + \frac{2dT}{a^2} = 3pS + 3qR + 3rQ + 3sP$$

$$- 12pPR - 6pQQ - 12qPQ - 6rPP$$

$$+ \frac{9}{2}ppr + \frac{9}{2}pqq + 30pPPQ + 10qP^2$$

$$- \frac{45}{2}ppqP - \frac{15}{2}p^2Q - 15pP^4 + \frac{45}{2}p^3PP$$

$$- \frac{15}{8}p^5$$

Ordo sextus,

continens partes per  $e^6$  affectas.

$$\text{I. } \frac{ddU}{a^2} - \frac{2du}{a^2} = 3U - 6PT - 6QS - 3RR + 3pr$$

$$+ 3qs + \frac{3}{2}rr + 12PPS + 24PQR + 4Q^2$$

$$- 12psP - 12qrP - 12prQ - 6qqQ$$

$$- 12pqR - 6ppS - 20P^2R - 30PPQQ$$

$$+ 30ppPR + 15ppQQ + 60pqPQ$$

$$+ 30prPP + 15qqPP - \frac{15}{2}p^2r - \frac{45}{2}p^2q^2$$

$$+ 30P^4Q - 90p^2P^2Q - 60pqP^3 + 45p^3qP$$

$$+ \frac{45}{2}p^4Q - 7P^6 + \frac{105}{2}ppP^4 - \frac{315}{8}p^4PP + \frac{315}{16}p^6.$$

$$\text{II. } \frac{ddu}{a^2} + \frac{2dU}{a^2} = 3pT + 3qS + 3rR + 3sQ + 3tP$$

$$- 12pPS - 12pQR - 12qPR - 6qQQ$$

$$- 12rPQ - 6sPP + \frac{9}{2}ppS + 9pqr + \frac{3}{2}q^2$$

$$+ 30pPPR + 30pPQQ + 30qPPQ$$

$$+ 10rP^2 - \frac{45}{2}pprP - \frac{45}{2}pqqP - \frac{45}{2}ppqQ$$

$$- \frac{15}{2}p^2R - 60pP^2Q - 15qP^4 + \frac{135}{2}ppqPP$$

$$+ 45p^3PQ - \frac{75}{8}p^4q + 21pP^5 - \frac{105}{2}p^3P^2 + \frac{105}{8}p^5P.$$

§. 16. In constitutione horum ordinum tota vis istius methodi potissimum continetur, quae adeo multo latius patet, cum etiam eius ope motus lunares expediri queant. Cum enim haecenus ne ullam quidem integrationem tentauerimus, nunc binae aequationes cuiusque ordinis facili negotio integrari poterunt, namque in ordine primo tantum occurrunt binae incognitae  $P$  et  $p$ , quarum valores per integrationem ad functiones temporis siue anguli  $\zeta$  reducuntur, quibus inuentis secundus ordo binas tantum incognitas  $Q$  et  $q$  complectitur, quas pari modo per tempus exprimere licebit; tum vero simili modo ex tertio ordine definiuntur literae incognitae  $R$  et  $r$ , et ita porro, vnde tandem veri valores pro binis incognitis principalibus  $x$  et  $y$  colligentur. Totum autem hoc integrationis negotium in sequenti Problemate generali ostendamus.

### Problema generale.

§. 17. Propositis duabus aequationibus differentialibus secundi gradus:

$$\text{I. } \frac{d^2 Z}{d\zeta^2} - 2 \frac{dZ}{d\zeta} = 3Z + M \text{ et}$$

$$\text{II. } \frac{d^2 z}{d\zeta^2} + 2 \frac{dz}{d\zeta} = N,$$

vbi  $M$  et  $N$  denotant functiones quascunque temporis siue anguli  $\zeta$ , inuestigare valores binarum quantitatum incognitarum  $Z$  et  $z$ .

### Solutio.

Incipiamus ab aequatione posteriore, quae ducta in  $d\zeta$  et integrata dat  $\frac{dz}{d\zeta} + 2Z = \int N d\zeta$ , vnde fit

$$\frac{dz}{d\zeta}$$

$$\frac{dz}{d\zeta} = fN d\zeta - 2Z,$$

qui valor in priore aequatione substitutus praebet

$$\frac{d^2 Z}{d\zeta^2} - 2fN d\zeta + Z = M, \text{ siue } \frac{d^2 Z}{d\zeta^2} + Z = M + 2fN d\zeta = L,$$

ponendo breuitatis ergo  $L = M + 2fN d\zeta$ . Constat autem, si esset  $L = 0$ , tum fore  $Z = a \cos. \zeta$ , qui valor, etiamsi tantum est integrale particulare, tamen sufficit ad integrale completum inuestigandum. Statuamus igitur pro nostro casu esse  $Z = v \cos. \zeta$ , eritque

$$\frac{dZ}{d\zeta} = -v \sin. \zeta + \frac{dv}{d\zeta} \cos. \zeta \text{ et}$$

$$\frac{d^2 Z}{d\zeta^2} = -v \cos. \zeta - \frac{dv}{d\zeta} \sin. \zeta + \frac{d^2 v}{d\zeta^2} \cos. \zeta,$$

sicque nostra aequatio euadet

$$\frac{d^2 v}{d\zeta^2} \cos. \zeta - \frac{dv}{d\zeta} \sin. \zeta = L$$

quae per  $d\zeta \cos. \zeta$  multiplicata praebet

$$\frac{d^2 v}{d\zeta^2} \cos. \zeta^2 - 2 dv \sin. \zeta \cos. \zeta = L d\zeta \cos. \zeta,$$

cuius integrale est

$$\frac{dv}{d\zeta} \cos. \zeta^2 = \int L d\zeta \cos. \zeta,$$

vnde colligitur

$$dv = \frac{d\zeta}{\cos. \zeta^2} \int L d\zeta \cos. \zeta,$$

quae aequatio denuo integrata dat

$$v = \int \frac{d\zeta}{\cos. \zeta^2} \int L d\zeta \cos. \zeta = \text{tang. } \zeta \int L d\zeta \cos. \zeta - \int L d\zeta \sin. \zeta,$$

quo valore inuento erit

$$Z = \sin. \zeta \int L d\zeta \cos. \zeta - \cos. \zeta \int L d\zeta \sin. \zeta,$$

vbi binae formulae integrales iam binas constantes per integrationes ingressas inuoluunt. Cum deinde sit

$$\frac{dz}{d\zeta} = fN d\zeta - 2Z, \text{ erit}$$

$$\frac{dz}{d\zeta} = fN d\zeta - 2 \sin. \zeta \int L d\zeta \cos. \zeta + 2 \cos. \zeta \int L d\zeta \sin. \zeta,$$

hincque integrando deducitur

$$z = \int d\zeta f N d\zeta - 2 \int d\zeta \sin. \zeta \int L d\zeta \cos. \zeta \\ + 2 \int d\zeta \cos. \zeta \int L d\zeta \sin. \zeta$$

quae aequatio, si posteriores formulae integrales duplicatae reducantur, sequentem induet formam:

$$z = \int d\zeta f N d\zeta + 2 \cos. \zeta \int L d\zeta \cos. \zeta + 2 \sin. \zeta \int L d\zeta \sin. \zeta \\ - 2 \int L d\zeta.$$

§. 18. Restituamus nunc loco L valorem assumptum  $M + 2 \int N d\zeta$ , ac pro valore ipsius Z reperiemus

$$\int L d\zeta \cos. \zeta = \int M d\zeta \cos. \zeta + 2 \sin. \zeta \int N d\zeta \\ - 2 \int N d\zeta \sin. \zeta \text{ et}$$

$$\int L d\zeta \sin. \zeta = \int M d\zeta \sin. \zeta - 2 \cos. \zeta \int N d\zeta \\ + 2 \int N d\zeta \cos. \zeta,$$

unde fit

$$Z = \sin. \zeta \int M d\zeta \cos. \zeta - \cos. \zeta \int M d\zeta \sin. \zeta \\ + 2 \int N d\zeta - 2 \sin. \zeta \int N d\zeta \sin. \zeta - 2 \cos. \zeta \int N d\zeta \cos. \zeta.$$

Deinde vero pro altero valore z, ubi eadem formulae iam reductae occurrunt, habebimus

$$z = 2 \cos. \zeta \int M d\zeta \cos. \zeta + 2 \sin. \zeta \int M d\zeta \sin. \zeta \\ - 2 \int M d\zeta - 4 \cos. \zeta \int N d\zeta \sin. \zeta \\ + 4 \sin. \zeta \int N d\zeta \cos. \zeta - 3 \int d\zeta f N d\zeta.$$

§. 19. Quod si ergo M et N fuerint functiones quaecunque ipsius  $\zeta$ , valores integrales pro Z et z inuenti non parum euadunt perplexi: Verum in resolutione nostrorum ordinum commode vsu venit, vt quantitates M et N perpetuo per sinus et cosinus huiusmodi angulorum:  $n\zeta + a$ , exprimantur, quemadmodum mox videbimus. Quando autem literae M et N huiusmodi induunt valores, tum integralia

tegralia quaesita satis succincte et concinne assignare licebit, id quod in sequenti Problemate speciali clarius ob oculos ponamus.

### Problema speciale.

§. 20. Si binae aequationes differentiales secundi gradus huiusmodi habeant formam:

$$I. \frac{d^2 Z}{d\zeta^2} - \frac{z dz}{d\zeta} - 3 Z = C \cos. (n \zeta + \alpha),$$

$$II. \frac{d^2 z}{d\zeta^2} + \frac{z dZ}{d\zeta} = c \sin. (n \zeta + \alpha),$$

inuenire valores integrales pro binis quantitibus incognitis Z et z.

### Solutio.

Haud difficulter quidem Solutio huius Problematis ex praecedenti deduci posset, siquidem valores integrales completi desiderarentur; verum quia pro nostro instituto integralia particularia sufficere possunt, quemadmodum in sequentibus manifesto patebit, ista integralia multo facilius inuestigare poterimus, quandoquidem forma integralium haud difficulter perspicitur. Hunc in finem statuamus  $Z = F \cos. (n \zeta + \alpha)$  et  $z = f \sin. (n \zeta + \alpha)$ , vbi tantum coefficientes F et f definiri oportet. Substituamus igitur istos valores assumtos in binis nostris aequationibus propositis, et cum sit

$$\frac{dZ}{d\zeta} = -n F \sin. (n \zeta + \alpha) \text{ et } \frac{d^2 Z}{d\zeta^2} = -n n F \cos. (n \zeta + \alpha)$$

$$\frac{dz}{d\zeta} = +n f \cos. (n \zeta + \alpha) \text{ et } \frac{d^2 z}{d\zeta^2} = -n n f \sin. (n \zeta + \alpha)$$

aequatio prior in omnibus terminis continebit  $\cos. (n \zeta + \alpha)$  posterior vero  $\sin. (n \zeta + \alpha)$ , quibus factoribus omissis eae induent has formas:



$$-nnF - 2nf - 3F = C \text{ et } -nnf - 2nF = 0$$

ex quibus inuestigari oportet literas F et f. Ex posteriore quidem statim colligitur  $f = -\frac{zF}{n} - \frac{c}{nn}$ , qui valor in priore substitutus praebet  $-nnF + F + \frac{zc}{n} = C$ , vnde fit

$$F = \frac{zc - nC}{n(nn - 1)} \text{ et } f = \frac{znC - c(z + nn)}{nn(nn - 1)}$$

§. 21. Pro hoc ergo casu proposito valores integrales quaesiti erunt

$$Z = \frac{zc - nC}{n(nn - 1)} \text{ cof. } (n\zeta + \alpha) \text{ et } z = \frac{znC - (z + nn)c}{nn(nn - 1)} \text{ fin. } (n\zeta + \alpha)$$

atque hinc intelligitur, si quantitates M et N plures contineant huiusmodi terminos, veluti si aequationes nostrae essent

$$\frac{d^2 Z}{d\zeta^2} - \frac{z d^2 z}{d\zeta^2} - 3Z = C \text{ cof. } (n\zeta + \alpha) + C' \text{ cof. } (n'\zeta + \alpha') + C'' \text{ cof. } (n''\zeta + \alpha'') + \text{etc.}$$

$$\frac{d^2 z}{d\zeta^2} + \frac{z d^2 Z}{d\zeta^2} = c \text{ fin. } (n\zeta + \alpha) + c' \text{ fin. } (n'\zeta + \alpha') + c'' \text{ fin. } (n''\zeta + \alpha'') + \text{etc.}$$

tum, quia Z et z vbique vnicam tantum habent dimensionem, fore

$$Z = \frac{zc - nC}{n(nn - 1)} \text{ cof. } (n\zeta + \alpha) + \frac{z c' - n' C'}{n'(n' n' - 1)} \text{ cof. } (n'\zeta + \alpha') + \frac{z c'' - n'' C''}{n''(n'' n'' - 1)} \text{ cof. } (n''\zeta + \alpha'') + \text{etc.}$$

$$z = \frac{znC - (z + nn)c}{nn(nn - 1)} \text{ fin. } (n\zeta + \alpha) + \frac{zn' C' - (z + n' n') c'}{n' n' (n' n' - 1)} \text{ fin. } (n'\zeta + \alpha') + \text{etc.}$$

His igitur praemissis singulos nostros ordines percurramus.

### Resolutio aequationum primi ordinis.

§. 22. Quoniam hic binae aequationes propositae sunt:

ddP

$$\frac{d d P}{d \zeta^2} - \frac{2 d p}{d \zeta} - 3 P = 0,$$

$$\frac{d d p}{d \zeta^2} + \frac{2 d P}{d \zeta} = 0$$

earum integratio ita in genere instituat. Primo posterior integrata dat  $\frac{d p}{d \zeta} + 2 P = \alpha$ , unde fit  $\frac{d p}{d \zeta} = \alpha - 2 P$ , qui valor in prima substitutus praebet:  $\frac{d d P}{d \zeta^2} + P = 2 \alpha$ . Ponatur hic, ut in solutione generali,  $P = v \cos. \zeta$ , fietque

$$\frac{d d v}{d \zeta^2} \cos. \zeta - \frac{2 d v}{d \zeta} \sin. \zeta = 2 \alpha$$

quae aequatio, in  $d \zeta \cos. \zeta$  ducta et integrata, praebet

$$\frac{d v}{d \zeta} \cos. \zeta^2 = 2 \alpha \sin. \zeta + \beta,$$

hincque fit

$$d v = \frac{2 \alpha d \zeta \sin. \zeta}{\cos. \zeta^2} + \frac{\beta d \zeta}{\cos. \zeta^2},$$

unde integrando oritur

$$v = \frac{2 \alpha}{\cos. \zeta} + \beta \text{ tang. } \zeta + \gamma,$$

quocirca habebimus

$$P = 2 \alpha + \beta \sin. \zeta + \gamma \cos. \zeta.$$

Tum vero porro erit

$$\frac{d p}{d \zeta} = -3 \alpha - 2 \beta \sin. \zeta - 2 \gamma \cos. \zeta, \text{ unde fit}$$

$$p = -3 \alpha \zeta + 2 \beta \cos. \zeta - 2 \gamma \sin. \zeta + \delta$$

ficque quatuor in calculum ingressae sunt quantitates constantes  $\alpha, \beta, \gamma, \delta$ , quemadmodum integratio binarum aequationum differentialium secundi gradus postulat, ut integralia completa reperiantur.

§. 23. Quoniam nostrum Problema utique est determinatum, ex conditionibus quas innoluit valores singularum harum constantium determinari oportet. Ac primo quidem quia angulus  $\zeta$  denotat longitudinem mediam Planetae

netae, a qua locus verus nunquam ultra datos limites discrepare potest, manifestum est esse debere  $\alpha = 0$ . Nisi enim esset  $\alpha = 0$ , quantitas  $p$ , ideoque et  $x$ , crescente  $\zeta$ , tandem in infinitum excrecere posset, id quod indicium esset, motum medium non rite esse constitutum, ex quo necesse est fieri  $\alpha = 0$ . Secundo ob eandem rationem quoque quarta constans  $\delta$  nihilo aequari debet, ut locus medius cum vero in certis orbitae locis conveniat. Tertio pro binis reliquis constantibus faciamus  $\beta = k \sin. \epsilon$  et  $\gamma = k \cos. \epsilon$ , ut obtineamus

$$P = k \sin. \epsilon \sin. \zeta + k \cos. \epsilon \cos. \zeta = k \cos. (\zeta - \epsilon) \text{ et}$$

$$p = 2 k \sin. \epsilon \cos. \zeta - 2 k \cos. \epsilon \sin. \zeta = -2 k \sin. (\zeta - \epsilon)$$

ex quo posteriore valore patet, si fuerit  $\zeta = \epsilon$ , tum fore  $p = 0$ , ideoque etiam  $y = 0$ , quatenus scilicet a  $p$  pendet, ita ut hoc casu locus verus cum medio conveniat. Quamobrem si assumamus cum Astronomis, hoc evenire in ipso Aphelio, constans  $\epsilon$  exhibebit longitudinem Aphelii. Tum vero pro quovis alio situ angulus  $\zeta - \epsilon$  exhibet anomaliam mediam, quae cum sit praecipuum elementum in motu Planetarum, statuamus brevitatis gratia  $\zeta - \epsilon = \theta$ , eritque  $P = k \cos. \theta$  et  $p = -2 k \sin. \theta$ ; unde ex hoc saltem ordine erit  $x = e k \cos. \theta$ , ubi  $e k$  iam denotat excentricitatem, quandoquidem, sumpta anomalia media  $\theta = 0$ , foret distantia Planetae a Sole  $a(1 + e k)$ ; at vero sumto  $\theta = 180^\circ$  prodiret distantia Perihelii  $= a(1 - e k)$ . Quare cum excentricitas supponatur  $= e$ , necesse est ut fiat  $k = 1$ , consequenter constantibus nostris rite determinatis resolutio ordinis primi ita se habebit;  $P = \cos. \theta$  et  $p = -2 \sin. \theta$ ; ubi  $\theta$  exprimit anomaliam mediam, et cum sit  $\theta = \zeta - \epsilon$ , erit utique  $d\theta = d\zeta$ , ideoque in ordinibus sequentibus lo-

eo  $d\zeta$  scribere licebit  $d\theta$ , ita vt sola anomalia media in calculum fit ingressura.

Resolutio aequationum secundi ordinis.

§. 24. In hoc ordine continentur quantitates incognitae  $Q$  et  $q$ , his aequationibus expressae:

$$I. \frac{d^2 Q}{d\theta^2} - \frac{2}{d\theta} \frac{dQ}{d\theta} - 3Q = -3P.P + \frac{3}{2} p.p,$$

$$II. \frac{d^2 q}{d\theta^2} + \frac{2}{d\theta} \frac{dq}{d\theta} = 3P.p;$$

vbi cum inuenerimus  $P = \cos. \theta$  et  $p = -2 \sin. \theta$ , erit

$$P^2 = \frac{1}{2} + \frac{1}{2} \cos. 2\theta, P.p = -\sin. 2\theta \text{ et } p.p = 2 - 2 \cos. 2\theta,$$

hincque nostrae aequationes erunt

$$\frac{d^2 Q}{d\theta^2} - \frac{2}{d\theta} \frac{dQ}{d\theta} - 3Q = +\frac{3}{2} - \frac{3}{2} \cos. 2\theta = M,$$

$$\frac{d^2 q}{d\theta^2} + \frac{2}{d\theta} \frac{dq}{d\theta} = -3 \sin. 2\theta = N$$

Quo nunc hic solutione speciali (§. 20.) vti queamus, ambae literae  $M$  et  $N$  in duas partes discriptae concipiantur, quarum priores contineant angulum  $0\theta$ , alterae vero angulum  $2\theta$ , quandoquidem has literas ita repraesentari licet:  $M = \frac{3}{2} \cos. 0\theta - \frac{3}{2} \cos. 2\theta$  et  $N = \sin. 0\theta - 3 \sin. 2\theta$ . Quia autem pro prioribus partibus fieret  $n = 0$ , formulae supra inuentae euadent incongruae, vnde hunc casum seorsim euolui conueniet. Sit ergo in genere

$$\frac{d^2 Z}{d\theta^2} - \frac{2}{d\theta} \frac{dZ}{d\theta} - 3Z = C \text{ et } \frac{d^2 z}{d\theta^2} + \frac{2}{d\theta} \frac{dz}{d\theta} = 0,$$

vbi est  $C = \frac{3}{2}$ . Ex posteriore aequatione fit  $\frac{dz}{d\theta} = \alpha - 2Z$ , vnde prior euadit  $\frac{d^2 z}{d\theta^2} + Z = 2\alpha + C = D$ . Posito nunc  $Z = v \cos. \theta$

erit  $\frac{d^2 v}{d\theta^2} \cos. \theta - \frac{2}{d\theta} \frac{dv}{d\theta} \sin. \theta = D$ ; quae aequatio ducta in  $d\theta \cos. \theta$  praebet integrale

$$\frac{dv}{d\theta} \cos. \theta^2 = D \sin. \theta + \beta, \text{ hincque}$$

$$dv =$$

$$dv = \frac{D d \theta \sin. \theta}{\cos. \theta^2} + \frac{\beta d \theta}{\cos. \theta^2}$$

vnde integrando colligitur

$$v = \frac{D}{\cos. \theta} + \beta \text{ tang. } \theta + \gamma; \text{ erit ergo}$$

$$Z = D + \beta \sin. \theta + \gamma \cos. \theta, \text{ consequenter}$$

$$\frac{dz}{d\theta} = \alpha - 2D - 2\beta \sin. \theta - 2\gamma \cos. \theta$$

et integrando

$$z = \alpha \theta - 2D \theta + 2\beta \cos. \theta - 2\gamma \sin. \theta + \delta,$$

vbi ante omnia observandum est, constantem  $\alpha$  ita accipi debere, vt fiat  $\alpha - 2D = 0$ , quia alioquin motus medius non rite esset constitutus. Erit ergo

$$\alpha = 2D = 4\alpha + 2C, \text{ vnde fit } \alpha = -\frac{2}{3}C;$$

tum vero ob rationes supra allegatas etiam esse oportet  $\delta = 0$ . Quod autem ad constantes  $\beta$  et  $\gamma$  attinet, quia angulus  $\theta$  in primo ordine iam rite constitutus supponitur, evidens est poni debere  $\beta = 0$  et  $\gamma = 0$ ; sicque habebimus  $Z = -\frac{1}{3}C$  et  $z = 0$ , qui valores adeo ex ipsis aequationibus differentialibus concludi potuissent, dum scilicet ambae vt constantes essent spectatae.

§. 25. Priores igitur partes pro litteris nostris  $Q$  et  $q$  praebent  $Q = -\frac{1}{3}C = -\frac{1}{2}$  et  $q = 0$ ; pro partibus autem posterioribus, quia est  $n = 2$ , formulae supra (§. 20) exhibitae nulla laborant ambiguitate, et cum pro hoc casu fit  $C = -\frac{2}{3}$  et  $c = -3$ , colligitur  $F = \frac{1}{2}$ , et  $f = \frac{1}{2}$ , ita vt nunc sit  $Z = \frac{1}{3} \cos. 2\theta$  et  $z = \frac{1}{2} \sin. 2\theta$ . Vtrosque ergo valores coniungendo habebimus pro ordine secundo has determinaciones:

$$Q = -\frac{1}{2} + \frac{1}{2} \cos. 2\theta \text{ et } q = \frac{1}{2} \sin. 2\theta.$$

Resolu.

Resolutio aequationum tertii ordinis.

§. 26. Binae aequationes huius ordinis ita se habent:

$$\text{I. } \frac{d^2 R}{d\theta^2} - \frac{2}{d\theta} R - 3R = -6PQ + 4P^2 + 3pq - 6ppP = M$$

$$\text{II. } \frac{d^2 r}{d\theta^2} + \frac{2}{d\theta} R = 3pQ + 3qP - 6ppP + \frac{3}{2}p^2 = N.$$

Cum igitur iam inuenerimus

$$P = \cos. \theta, \quad p = -2 \sin. \theta, \quad Q = -\frac{1}{2} + \frac{1}{2} \cos. 2\theta \text{ et}$$

$$q = \frac{1}{4} \sin. 2\theta,$$

per notas reductiones angulorum, quibus est

$$\cos. \alpha \cos. \beta = \frac{1}{2} \cos. (\alpha - \beta) + \frac{1}{2} \cos. (\alpha + \beta)$$

$$\sin. \alpha \cos. \beta = \frac{1}{2} \sin. (\alpha - \beta) + \frac{1}{2} \sin. (\alpha + \beta)$$

$$\sin. \alpha \sin. \beta = \frac{1}{2} \cos. (\alpha - \beta) - \frac{1}{2} \cos. (\alpha + \beta)$$

$$\cos. \alpha \sin. \beta = \frac{1}{2} \sin. (\alpha + \beta) - \frac{1}{2} \sin. (\alpha - \beta)$$

colligimus pro priore aequatione

$$PQ = -\frac{1}{4} \cos. \theta + \frac{1}{4} \cos. 3\theta, \quad pq = -\frac{1}{4} \cos. \theta + \frac{1}{4} \cos. 3\theta,$$

$$P^2 = \frac{1}{4} \cos. \theta + \frac{1}{4} \cos. 3\theta, \quad ppP = \cos. \theta - \cos. 3\theta,$$

hincque colligitur

$$M = -\frac{2}{4} \cos. \theta + \frac{25}{4} \cos. 3\theta.$$

Simili modo pro altera aequatione erit

$$pQ = \frac{3}{2} \sin. \theta - \frac{1}{2} \sin. 3\theta, \quad qP = +\frac{1}{2} \sin. \theta + \frac{1}{2} \sin. 3\theta,$$

$$ppP = -\frac{1}{2} \sin. \theta - \frac{1}{2} \sin. 3\theta, \quad p^2 = -6 \sin. \theta + 2 \sin. 3\theta,$$

unde fit

$$N = -\frac{2}{2} \sin. \theta + \frac{20}{2} \sin. 3\theta.$$

§. 27. Hic literae M et N iterum ex duabus constant partibus, pro quarum prioribus est  $n = 1$ , pro posterioribus autem  $n = 3$ . Priore autem casu formulae supra datae sunt incongruae, unde hunc casum seorsim euolui conueniet. Sit igitur

$$\frac{d^2 z}{d\theta^2} - 2 \frac{dz}{d\theta} - 3Z = C \operatorname{cof.} \theta \text{ et } \frac{d^2 z}{d\theta^2} + 2 \frac{dz}{d\theta} = e \operatorname{fin.} \theta,$$

ita ut fit  $C = -\frac{e}{2}$  et  $c = -\frac{e}{2}$ . Iam ex posteriore fit,

$$\frac{dz}{d\theta} = -c \operatorname{cof.} \theta - 2Z + \alpha,$$

hincque prior aequatio prodibit

$$\frac{d^2 z}{d\theta^2} + Z = (C - 2c) \operatorname{cof.} \theta + 2\alpha.$$

Cum autem nostro casu fit  $C - 2c = 0$ , habebimus

$$\frac{d^2 z}{d\theta^2} + Z = 2\alpha.$$

Hinc si ut hactenus ponatur  $Z = v \operatorname{cof.} \theta$ , erit

$$\frac{d^2 v}{d\theta^2} \operatorname{cof.} \theta - 2 \frac{dv}{d\theta} \operatorname{fin.} \theta = 2\alpha,$$

et integrando

$$\frac{dv}{d\theta} \operatorname{cof.} \theta^2 = 2\alpha \operatorname{fin.} \theta + \beta, \text{ unde colligitur}$$

$$dv = \frac{2\alpha d\theta \operatorname{fin.} \theta}{\operatorname{cof.} \theta^2} + \frac{\beta d\theta}{\operatorname{cof.} \theta^2},$$

hincque porro

$$v = \frac{2\alpha}{\operatorname{cof.} \theta} + \beta \operatorname{tang.} \theta + \gamma,$$

consequenter

$$Z = 2\alpha + \beta \operatorname{fin.} \theta + \gamma \operatorname{cof.} \theta, \text{ ex quo porro fit}$$

$$\frac{dz}{d\theta} = -c \operatorname{cof.} \theta - 3\alpha - 2\beta \operatorname{fin.} \theta + 2\gamma \operatorname{cof.} \theta,$$

unde reperitur

$$z = -c \operatorname{fin.} \theta - 3\alpha \theta + 2\beta \operatorname{cof.} \theta - 2\gamma \operatorname{fin.} \theta + \delta.$$

§. 28. Cum hic motum medium rite constitutum esse assumamus, necesse est ut fit  $\alpha = 0$ , tum vero etiam debet esse  $\delta = 0$ . Quod autem ad constantes  $\beta$  et  $\gamma$  attinet, quia etiam Aphelium rite constitutum assumimus, debet etiam

etiam esse  $\beta = 0$  et  $\gamma = 0$ , sicque valores nostri quaesiti erunt

$$Z = 0 \text{ et } z = -c \sin. \theta = +\frac{2}{3} \sin. \theta.$$

Hic autem probe observasse iuvabit, si non fuisset  $C - 2c = 0$ , ad terminos fuisse peruentum, qui ipsum angulum  $\theta$  continuisent, quos nullo modo per constantes tollere licuisset. Hoc scilicet casu motus Planetæ verus non ad medium reuocari potuisset; vnde ista conditio  $C - 2c = 0$  necessario in natura rei est fundata, atque etiam in sequentibus ordinibus, quoties Sinus et Cosinus anguli simplicis  $\theta$  occurrunt, semper necessario euadere debet  $C = 2c$ , indeque semper erit  $Z = 0$  et  $z = -c \sin. \theta$ .

§. 29. Pro terminis autem posterioribus, angulum  $3\theta$  inuoluentibus, formulæ supra datae tuto adhiberi possunt: erit enim

$n = 3$ ,  $C = \frac{25}{4}$  et  $c = \frac{39}{8}$ , vnde fit  $F = -\frac{7}{8}$  et  $f = -\frac{7}{24}$ , quocirca ex hoc ordine nanciscimur sequentes determinationes:

$$R = -\frac{7}{8} \cos. 3\theta \text{ et } r = +\frac{2}{3} \sin. \theta - \frac{7}{24} \sin. 3\theta.$$

Resolutio aequationum quarti ordinis,

§. 30. Binae huius ordinis aequationes ita se habent:

$$\text{I. } \left. \begin{aligned} \frac{d^2 ds}{d\theta^2} - \frac{2 ds}{d\theta} - 3S &= -6PR - 3QQ + 3pr + 3qq \\ + 6ppQ + 12PPQ - 12pqP \\ - 5P^2 + 15ppPP - \frac{15}{8}p^4 \end{aligned} \right\} = M$$

P p 2

II.



$$\text{II. } \frac{d ds}{d \theta^2} + \frac{z ds}{d \theta} = \left. \begin{aligned} &3 p R + 3 q Q + 3 r P \\ &- 12 p P Q - 6 q P P + \frac{9}{2} p p q \\ &+ 10 p P^2 - \frac{15}{2} p^2 P \end{aligned} \right\} = N.$$

Hic singuli termini, ut hactenus factum est, euoluantur, ac pro priore quidem aequatione reperietur:

$$\begin{aligned} p R &= -\frac{3}{16} \cos. 2 \theta - \frac{3}{16} \cos. 4 \theta, & q Q &= \frac{3}{8} - \frac{1}{8} \cos. 2 \theta + \frac{1}{8} \cos. 4 \theta, \\ p r &= -\frac{9}{8} + \frac{17}{12} \cos. 2 \theta - \frac{7}{24} \cos. 4 \theta, & q q &= \frac{1}{32} - \frac{1}{32} \cos. 4 \theta, \\ p p Q &= -\frac{3}{8} + 2 \cos. 2 \theta - \frac{1}{8} \cos. 4 \theta, & P P Q &= -\frac{1}{8} + \frac{1}{8} \cos. 4 \theta, \\ p q P &= -\frac{1}{8} + \frac{1}{8} \cos. 4 \theta, & P^2 &= \frac{3}{8} + \frac{1}{8} \cos. 2 \theta + \frac{1}{8} \cos. 4 \theta, \\ p p P P &= \frac{1}{2} - \frac{1}{2} \cos. 4 \theta, & p^2 &= 6 - 8 \cos. 2 \theta + 2 \cos. 4 \theta, \end{aligned}$$

vnde colligitur:

$$M = -\frac{1221}{8} + \frac{251}{9} \cos. 2 \theta - \frac{967}{64} \cos. 4 \theta.$$

Pro altera vero aequatione erit

$$\begin{aligned} p R &= -\frac{3}{8} \sin. 2 \theta + \frac{3}{8} \sin. 4 \theta, & q Q &= -\frac{1}{8} \sin. 2 \theta + \frac{1}{16} \sin. 4 \theta, \\ r P &= +\frac{5}{12} \sin. 2 \theta - \frac{7}{24} \sin. 4 \theta, & p P Q &= \frac{1}{8} \sin. 2 \theta - \frac{1}{4} \sin. 4 \theta, \\ q P P &= \frac{1}{8} \sin. 2 \theta + \frac{1}{16} \sin. 4 \theta, & p p q &= \frac{1}{8} \sin. 2 \theta - \frac{1}{4} \sin. 4 \theta, \\ p P^2 &= -\frac{1}{2} \sin. 2 \theta - \frac{1}{4} \sin. 4 \theta, & p^2 P &= -2 \sin. 2 \theta + \sin. 4 \theta, \end{aligned}$$

vnde colligitur

$$N = \frac{21}{4} \sin. 2 \theta - \frac{61}{8} \sin. 4 \theta.$$

§. 30. Hic igitur occurrunt tres partes, quarum prima est constans, secunda vero angulo  $2 \theta$  et tertia angulo  $4 \theta$  affecta. Pro prima igitur parte erit

$$C = -\frac{1221}{8}, c = 0 \text{ et } n = 0,$$

vnde, quemadmodum in ordine secundo iam vidimus, colligitur

$$Z = -\frac{1}{8} C = +\frac{407}{8} \text{ et } z = 0.$$

Porro

Porro pro secunda parte est

$$n = 2, C = \frac{251}{9} \text{ et } c = \frac{21}{4}, \text{ vnde fit}$$

$$F = \frac{2c - 2C}{6} = \frac{c - C}{3} = -\frac{209}{24} \text{ et } f = -F - \frac{c}{4} = \frac{255}{24}.$$

Pro tertia tandem parte, vbi

$$n = 4, C = -\frac{965}{24} \text{ et } c = -\frac{61}{2}, \text{ erit}$$

$$F = \frac{2c - 4C}{60} = \frac{c - 2C}{30} = +\frac{719}{960}, \text{ hincque}$$

$$f = -\frac{F}{2} - \frac{c}{16} = \frac{49}{480},$$

quibus valoribus substitutis habebimus

$$S = \frac{407}{9} - \frac{209}{24} \cos. 2\theta + \frac{719}{960} \cos. 4\theta,$$

$$s = \frac{255}{48} \sin. 2\theta + \frac{49}{480} \sin. 4\theta.$$

§. 31. Nimis longum foret huiusmodi calculos pro ordinibus superioribus exsequi, praecipue cum motus Planetarum primariorum iam aliunde satis sit cognitus, atque hic institutum nostrum in eo tantum versetur, vt specimen huius nouae methodi tradamus. Omissis igitur sequentibus ordinibus pro binis nostris incognitis  $x$  et  $y$  sequentes consecuti sumus valores:

$$x = e \cos \theta + e e \left( \frac{1}{2} - \frac{1}{2} \cos. 2\theta \right) - \frac{5}{9} e^2 \cos. 3\theta \\ + e^4 \left( \frac{407}{9} - \frac{209}{24} \cos. 2\theta + \frac{719}{960} \cos. 4\theta \right)$$

$$y = -2e \sin. \theta + \frac{1}{4} e e \sin. 2\theta + e^2 \left( \frac{2}{9} \sin. \theta - \frac{7}{24} \sin. 3\theta \right) \\ + e^4 \left( \frac{255}{48} \sin. 2\theta + \frac{49}{480} \sin. 4\theta \right),$$

§. 31. Inuentis autem his valoribus  $x$  et  $y$  innotescet angulus  $MSP$ , qui vocatur aequatio centri Planetae, quae si ponatur  $= \omega$ , erit longitudo Planetae vera

$$= \angle SP = \zeta + \omega;$$

vbi  $\zeta$  denotat longitudinem Planetae median.

Pro aequatio-

quatione autem centri  $\omega$ , ob

$$S q = a(1+x) \text{ et } P \hat{q} = ay, \text{ erit } \text{tang. } \omega = \frac{y}{1+x};$$

denique ex ipsa hac aequatione  $\omega$  colligetur distantia Planetæ a Sole

$$SP = \frac{a(1+x)}{\cos. \omega} = a(1+x) \sec. \omega,$$

vbi  $a$  designat distantiam Planetæ mediam a Sole.

§. 32. Pro Planetis quidem primariis neūtiqum consultum foret eiusmodi tabulas construere, quæ pro singulis anomalis mediis exhiberent valores litterarum  $x$  et  $y$ , cum tabularum consuetarum ope totum negotium multo facilius expediri queat. Verum si perturbationes, quas Planetæ ob actionem mutuam sibi inferunt, inuestigare voluerimus, tum istam methodum, etsi per se non parum molestam, pari successu applicare licebit, dum contra aliae methodi ob summam integrationum difficultatem vix in vsum vocari possunt. Huius igitur methodi vis potissimum in eo consistit, quod in singulis ordinibus, quos constituimus, negotium integrationis mira facilitate expediri potest, id quod in hac dissertatione imprimis ostendere mihi erat propositum, quo facilius noua theoria mea motuum lunarium diiudicari possit, quandoquidem ea tota isti artificio inuititur.