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De motu oscillatorio duorum corporum ex filo super trochleas traducto suspensorum

Leonhard Euler

DE MOTV OSCILLATORIO DVORVM CORPORVM EX FILO SVPER TROCHLEAS TRADVCTO SVSPENSORVM.

Auctore
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§. 1.

Filo A M N B, super duas trochleas M et N traducto, appensa sint duo corpora A et B. Per puncta M et N ducantur rectae verticales M P et N Q, ad easque horizontales A P et B Q, et elapso tempore t corpora teneant situm in figura repraesentatum. Tum pro situ corporum ponantur coordinatae $MP = x$ et $PA = y$, $NQ = x'$ et $QB = y'$, et quia longitudo fili manet inuariata, statuamus $MN = M$, $MA = a + z$ et $NB = b - z$, vt tota fili longitudo sit $= a + b + M$; tum vero ponamus angulos $AMP = \eta$ et $BNQ = \theta$ eritque Tab. III. Fig. 1.

$$x = (a + z) \text{ cof. } \eta = \text{ et } y = (a + z) \text{ fin. } \eta;$$

eodemque modo

$$x' = (b - z) \text{ cof. } \theta \text{ et } y' = (b - z) \text{ fin. } \theta.$$

§. 2. Ponatur nunc tensio fili $= T$, a qua quia ambo corpora sursum trahuntur, dum propria grauitate

 deor-

deorſum nituntur, principia motus nobis ſuppeditant quatuor ſequentes aequationes:

$$\text{I. } \frac{ddx}{2g\,dt^2} = \frac{A - T\cos.\eta}{A}; \quad \text{III. } \frac{ddx'}{2g\,dt^2} = \frac{B - T\sin.\theta}{B}$$

$$\text{II. } \frac{ddy}{2g\,dt^2} = -\frac{T\sin.\eta}{A}; \quad \text{IV. } \frac{ddy'}{2g\,dt^2} = -\frac{T\sin.\theta}{B},$$

ex quibus quatuor aequationibus 1°. tenſionem fili T: 2° quantitatem z; 3° et 4° angulos η et θ definiri oportet.

§. 3. At vero differentiando habebimus

$$dx = dz\cos.\eta - (a+z)\,d\eta\sin.\eta \text{ et } ddx = (ddz - (a+z)\,d\eta^2)\cos.\eta - (2dzd\eta + (a+z)dd\eta)\sin.\eta$$

$$dy = dz\sin.\eta + (a+z)\,d\eta\cos.\eta \text{ et } ddy = (ddz - (a+z)\,d\eta^2)\sin.\eta + (2dzd\eta + (a+z)dd\eta)\cos.\eta$$

Eodem modo reperietur

$$ddx' = -(ddz + (b-z)\,d\theta^2)\cos.\theta + (2\,dz\,d\theta - (b-z)\,dd\theta)\sin.\theta$$

$$ddy' = -(ddz + (b-z)\,d\theta^2)\sin.\theta - (2\,dz\,d\theta - (b-z)\,dd\theta)\cos.\theta$$

quibus valoribus ſubſtitutis, noſtrae aequationes erunt:

$$\text{I. } \frac{(ddz - (a+z)\,d\eta^2)\cos.\eta - (2\,dz\,d\eta + (a+z)\,dd\eta)\sin.\eta}{2g\,dt^2} = \frac{A - T\cos.\eta}{A}.$$

$$\text{II. } \frac{(ddz - (a+z)\,d\eta^2)\sin.\eta + (2\,dz\,d\eta + (a+z)\,dd\eta)\cos.\eta}{2g\,dt^2} = -\frac{T\sin.\eta}{A}.$$

$$\text{III. } -\frac{(ddz + (b-z)\,d\theta^2)\cos.\theta + (2\,dz\,d\theta - (b-z)\,dd\theta)\sin.\theta}{2g\,dt^2} = \frac{B - T\cos.\theta}{B}.$$

$$\text{IV. } -\frac{(ddz + (b-z)\,d\theta^2)\sin.\theta - (2\,dz\,d\theta - (b-z)\,dd\theta)\cos.\theta}{2g\,dt^2} = -\frac{T\sin.\theta}{B}.$$

§. 4. Hinc iam per idoneas combinationes formentur aequationes ſequentes ſimpliciores:

I. cof. η + II. ſin. η dat:

$$\text{I}^\circ.\ \frac{ddz - (a+z)\,d\eta^2}{2g\,dt^2} = \frac{A\cos.\eta - T}{A}; \text{ porro}$$

− I. ſin. η + II. cof. η dat:

II°.

II°. $\frac{2\,dz\,d\eta + (a+z)\,dd\eta}{2\,g\,a\,t^2} = -\sin.\eta.$

$-\,$III. $\cos.\theta$ $-$ IV. $\sin.\theta$ praebet:

III°. $\frac{ddz + (b-z)\,d\theta^2}{2\,g\,a\,t^2} = \frac{T - B\cos.\theta}{B}.$

$+\,$III. $\sin.\theta$ $-$ IV. $\cos.\theta$ producit:

IV°. $\frac{2\,dz\,d\theta - (b-z)\,dd\theta}{2\,g\,a\,t^2} = \sin.\theta.$

Sicque tantum in I^a et III^a. tenſio T occurrit, ſecunda autem et quarta tenſionis ſunt immunes.

§. 5. Ad tenſionem igitur eliminandam vtamur hac noua combinatione: I^a. A + III^a. B, quae praebet

$$\frac{(A+B)\,ddz - A(a+z)\,d\eta^2 + B(b-z)\,d\theta^2}{2\,g\,a\,t^2} = A\cos.\eta - B\cos.\theta$$

quae ergo aequatio cum ſuperiorum ſecunda et quarta totam ſolutionem problematis continet, vnde tam quantitatem z quam angulos η et θ definiri oportet.

§. 6. Antequam reſolutionem harum aequationum aggrediamur, quatuor aequationes primum inuentas alio modo tractemus, et quia eſt

$$dx = dz\cos.\eta - (a+z)\,d\eta\sin.\eta \text{ et } dy = dz\sin.\eta + (a+z)\,d\eta\cos.\eta$$

vtamur ſequentibus combinationibus:

I. $2\,dx$ + II. $2\,dy$, vnde fit

$$\frac{2\,dx\,ddx + 2\,dy\,ddy}{2\,g\,a\,t^2} = 2\,dx - \frac{2\,T\,dz}{A},$$

quae aequatio integrata dat

$$\frac{dx^2 + dy^2}{2\,g\,a\,t^2} = 2\,x - 2\int\frac{T\,dz}{A}.$$

Nunc vero haec combinatio: I. x + II. y praebet

$$\frac{x\,ddx + y\,ddy}{2\,g\,a\,t^2} = x - \frac{T(a+z)}{A},$$

Haec aequatio addatur ad priorem et prodibit

$$\frac{dx^2 + dy^2 + xddx + yddy}{2g\,dt^2} = 3x - 2\int \frac{T\,dz}{A} - \frac{T}{A}(a + z),$$

vbi quidem partis ad finiſtram integrale eſt $\frac{x\,dx + y\,dy}{2g\,dt^2}$; at ex membro ad dextram nihil concludi poſſet. Pari modo non ſuccederet haec combinatio: I. y — II. x, quae dat $\frac{yddx - xddy}{2g\,dt^2} = y$, vbi etiam membri ſiniſtri integrale eſt $\frac{y\,dx - x\,dy}{2g\,dt^2}$, ſed iterum membrum alterum nullam reductionem patitur.

§. 7. Mirum autem non eſt, hunc motum, qualem in genere contemplamur, prorſus eſſe inextricabilem, quoniam ambo corpora A et B etiam inaequalia eſſe poſſent: hoc autem caſu grauius inter oſcillandum deſcenderet, leuius vero aſcenderet, ſicque motus prodiret nimis complicatus, quam vt per calculum determinari poſſet. Quamobrem neceſſe eſt noſtram inueſtigationem tantum ad corpora aequalia reſtringere, quia alioquin ſtatus aequilibrii locum habere non poſſet. Praeterea vero etiam neceſſe eſt diuagationes, ſeu angulos η et θ quam minimos aſſumere; vnde facile intelligitur, tenſionem fili hoc caſu ponderi cuiusque corporis fore aequalem, ita vt ſit $T = A = B$. Denique patet, niſi corporibus initio motus verticalis fuerit impreſſus, ambo corpora durante motu vix eſſe vel aſcenſura vel deſcenſura, ſicque etiam quantitas z quaſi vt infinite parua tractari poterit.

§. 8. Ponamus igitur ambo corpora A et B inter ſe aequalia, ac primo quidem remoueamus vtrumque motum oſcillatorium, ita vt ſit $\eta = 0$ et $\theta = 0$, ac remanebunt

bunt tantum aequationes prima et tertia

$$\frac{ddz}{2g\,dt^2} = 1 - \frac{T}{A};$$
$$\frac{d\cdot d\,z}{2g\,dt^2} = \frac{T}{A} - 1;$$

quae inuicem additae dant $\frac{2\,ddz}{2g\,dt^2} = 0$, et a se inuicem subtractae relinquunt $0 = -\frac{2T}{A} + 2$. Ex priore ergo sequitur $\frac{dz}{dt} = \alpha$, ideoque $z = \alpha\,t$; vnde patet, corpus A motu vniformi descendere celeritate $= \alpha$, alterum vero corpus B eadem celeritate ascendere. Ex posteriore vero fit $T = A$; tensio scilicet fili perpetuo erit eadem et aequalis ponderi vnius corporis.

§. 9. Nunc igitur tribuamus vtrique corpori quandam inclinationem η et θ, quasi infinite exiguam, et manifestum est, priorem motum inde non sensibiliter turbari, ita vt adhuc sit $z = \alpha\,t$ et $T = A$, nisi quatenus ob motum minimum corporum tensio aliquantillum immutetur; vnde literam B in calculo retineamus. Quatuor ergo aequationes nostrae erunt:

$$\mathrm{I}\ -\frac{(a+\alpha t)\,d\eta^2}{2g\,dt^2} = 1 - \frac{T}{A};\quad \mathrm{II}\ \frac{2\alpha\,dt\,d\eta + (a+\alpha t)\,dd\eta}{2g\,dt^2} = -\eta$$
$$\mathrm{III}\ \frac{(b-\alpha t)\,d\theta^2}{2g\,dt^2} = \frac{T}{A} - 1;\quad \mathrm{IV}\ \frac{2\alpha\,dt\,d\theta - (b-\alpha t)\,dd\theta}{2g\,dt^2} = \theta,$$

vbi ex secunda et quarta elici opportet ambos angulos η et θ. Prima vero ac tertia, quia inuoluunt quasi infinite-parua secundi ordinis, tantum inseruient correctionibus minimis, tam tensionis T quam quantitatis z, accuratius determinandis, quas igitur hic praetermittere licebit.

§. 10. Pro angulo igitur η inueniendo habemus hanc aequationem:

$$2\,\alpha\,dt\,d\eta + (a + \alpha t)\,dd\eta + 2\,g\,\eta\,dt^2 = 0,$$

quam quidem facile esset ad differentialia primi gradus reducere, quod autem nobis parum lucri esset allaturum. Ad ipsam aequationem autem commodius referendam ponamus $a = i\,\alpha$ et $\frac{2g}{\alpha} = n$, vt habeamus hanc aequationem:

$$2\,dt\,d\eta + (i + t)\,dd\eta + n\,\eta\,dt^2 = 0,$$

pro cuius integrali inueniendo fingamus hanc seriem:

$$\eta = A + Bt + Ctt + Dt^3 + Et^4 + Ft^5 + \text{etc.}$$

eritque

$$\frac{d\eta}{dt} = B + 2\,Ct + 3\,Dtt + 4\,Et^3 + 5\,Ft^4 + \text{etc. et}$$

$$\frac{dd\eta}{dt^2} = 1.\,2\,C + 2.\,3\,Dt + 3.\,4\,Ett + 4.\,5\,Ft^3\ \text{etc.}$$

qui valores substituantur vt sequitur:

$$\left.\begin{array}{lllllll}
\frac{i.\,dd\eta}{dt^2} & = 1.\,2\,iC & + 2.\,3\,iDt & + 3.\,4\,iFtt & + 4.\,5\,iFt^3 & + \text{etc.} \\
\frac{t.\,dd\eta}{dt^2} & = & + 1.\,2\,C & + 2.\,3\,D & + 3.\,4\,E & + \text{etc.} \\
\frac{2\,d\eta}{dt} & = 2\,B & + 4\,C & + 6\,D & + 8\,E & + \text{etc.} \\
n\,\eta & = n\,A & + n\,B & + n\,C & + n\,D & + \text{etc.}
\end{array}\right\} = 0$$

vnde deducuntur hae determinationes:

$$C = \frac{-2B - nA}{1.\,2.\,i};\quad D = \frac{-6C - nB}{2.\,3\,i};\quad E = \frac{-12D - nC}{3.\,4\,i};\ \text{etc.}$$

§. 11. Quia autem hic singuli coefficientes a binis praecedentibus pendent, huic incommodo medelam afferemus, ponendo $i + t = s$, vt habeamus hanc aequationem:

$$2\,ds\,d\eta + s\,dd\eta + n\,\eta\,ds^2 = 0,$$ et nunc ponamus

$$\eta = A + Bs + Css + Ds^3 + \text{etc.}$$

qua

qua ferie fubftituta fiet

$$\left.\begin{array}{l} \frac{s\,dd\eta}{ds^2} = 1.2\,C\,s + 2.3\,D\,ss + 3.4\,E\,s^3 + 4.5\,F\,s^4 + \text{etc.} \\ \frac{2\,d\eta}{ds} = 2\,B + 4\,C\,s + 6\,D\,ss + 8\,E\,s^3 + 10\,F\,s^4 + \text{etc.} \\ n\,\eta = n\,A + n\,B\,s + n\,C\,ss + n\,D\,s^3 + n\,E\,s^4 + \text{etc.} \end{array}\right\} = 0$$

Hinc fit

$$B = \frac{-n\,A}{2};\ C = \frac{-n\,B}{6};\ D = \frac{-n\,C}{12};\ E = \frac{-n\,D}{20};\ \text{etc.}$$

vnde feries affumta ita prodit expreffa:

$$\frac{\eta}{A} = 1 - \frac{n\,s}{2} + \frac{nn\,ss}{2.6} - \frac{n^3\,s^3}{2.6.12} + \frac{n^4\,s^4}{2.6.12.20} - \text{etc.}$$

quae, quia nullo cafu abrumpitur, nihil prodeft, nifi forte quamdiu tempus, ideoque et s, eft quantitas valde parua.

§. 12. Notatu etiam digna eft transformatio iftius aequationis, ftatuendo $\eta = \frac{y}{s}$; hinc enim erit

$$d\,\eta = \frac{dy}{s} - \frac{y\,ds}{ss} \text{ et } dd\,\eta = \frac{ddy}{s} - \frac{2\,dy\,ds}{ss} + \frac{2\,y\,ds^2}{s^3};$$

qui valores fubftituti producent hanc aequationem:

$$ddy + \frac{n\,y\,ds^2}{s} = 0.$$

Quod fi hic ponamus

$y = e^{\int u\,ds}$, vt fit $dy = u\,ds\,e^{\int u\,ds}$ et $ddy = (du\,ds + u^2\,ds^2)\,e^{\int u\,ds}$, fietque $du + uu\,ds + \frac{n\,ds}{s} = 0$, quae aequatio quia eft formae Riccatianae, quam nullo adhuc modo tractare licuit, omnis opera in ea euoluenda fruftra confumetur; ita vt determinationem huius motus ofcillatorii, quo corpora A et B ciebuntur, dum filum fuper trochleas vniformiter promouetur, pro cafu defperato declarare fimus coacti.

§. 13.

§. 13. Interim tamen, quia longitudo fili MA continuo crefcit, ita vt pendulum, corpus A fuftinens, continuo crefcat, evidens eft, ofcillationes continuo tardiores fieri debere; vnde fi pro tempore praefente quantitas s tanquam conftans fpectaretur, omiffo primo termino, vt haberemus:

$$d\,d\,y + \frac{n\,y\,d\,s^2}{s} = 0,$$ integrale foret

$$y = \mathfrak{A}\,\text{fin.}\,(\lambda + s\,\sqrt{\tfrac{n}{s}})$$ fiue

$$\eta\,s = \mathfrak{A}\,\text{fin.}\,(\lambda + \sqrt{n\,s})$$ fiue

$$\eta = \frac{\mathfrak{A}\,\alpha}{a + \alpha\,t}\,\text{fin.}\,(\lambda + \frac{\sqrt{2\,g\,(a + \alpha\,t)}}{\alpha});$$

quae expreffio non multum videtur a fcopo aberrare.

§. 14. Vt autem appareat, quantum ille valor

$$y = \mathfrak{A}\,\text{fin.}\,(\lambda + s\,\sqrt{\tfrac{n}{s}}) = \mathfrak{A}\,\text{fin.}\,(\lambda + \sqrt{n\,s})$$

a veritate difcrepet, eum in aequatione differentiali $s\,d\,d\,y + n\,y\,d\,s^2 = 0$ fubftituamus. Quia ergo eft

$$\frac{d\,y}{d\,s} = \frac{\mathfrak{A}\,\sqrt{n}}{2\,\sqrt{s}}\,\text{cof.}\,(\lambda + \sqrt{n\,s})$$ et

$$\frac{d\,d\,y}{d\,s^2} = -\frac{n\,\mathfrak{A}}{4\,s}\,\text{fin.}\,(\lambda + \sqrt{n\,s}) - \frac{n\,\mathfrak{A}}{4\,s\,\sqrt{n\,s}}\,\text{cof.}\,(\lambda + \sqrt{n\,s})$$

habebimus hanc aequationem:

$$-\frac{n\,\mathfrak{A}}{4}\,\text{fin.}\,(\lambda + \sqrt{n\,s}) - \frac{n\,\mathfrak{A}}{4\,\sqrt{n\,s}}\,\text{cof.}\,(\lambda + \sqrt{n\,s}) + n\,\mathfrak{A}\,\text{fin.}\,(\lambda + \sqrt{n\,s}) = 0$$

fiue

$$\frac{3\,n\,\mathfrak{A}}{4}\,\text{fin.}\,(\lambda + \sqrt{n\,s}) - \frac{\mathfrak{A}\,\sqrt{n}}{4\,\sqrt{s}}\,\text{cof.}\,(\lambda + \sqrt{n\,s}) = 0$$

vnde aberratio diiudicari debet.

§. 15. Quanquam igitur cafus, quo $B = A$, facillimus videbatur, tamen ftatim ac filum promouetur nihil plane

plane de motu corporum definire licet; quando autem filum quiescit, ita vt sit $\alpha = 0$, tum vtrumque corpus perinde oscillationes suas peraget, ac si firmiter esset suspensum. Tanto igitur minus erit sperandum, si corpora inter se inaequalia statuere velimus. Interim tamen occurrent certi quidam casus, quibus praeter omnem expectationem motum definire licebit, quos ergo vtique operae pretium erit accuratius euoluisse.

§. 16. Primum igitur iterum faciamus $\eta = 0$ et $\theta = 0$, et aequationes nostrae erunt:

I. $\frac{ddz}{2g\,dt^2} = 1 - \frac{T}{A}$.

III. $\frac{ddz}{2g\,dt^2} = \frac{T}{B} - 1$; vnde fit $T = \frac{2AB}{A+B}$, hincque

$$\frac{ddz}{2g\,dt^2} = \frac{A-B}{A+B} = \frac{1}{n}, \text{ vt fit } n = \frac{A+B}{A-B}.$$

Hinc iam erit

$$\frac{dz}{2g\,dt} = \frac{t}{n}, \text{ ideoque } dz = \frac{2gt\,dt}{n} \text{ et } z = \frac{gtt}{n},$$

vbi constantes non addimus, quia hinc multo magis quam supra in aequationes inextricabiles illaberemus; sic igitur assecuti sumus has duas aequationes: $T = \frac{2AB}{A+B}$ et $z = \frac{gtt}{n}$, existente $n = \frac{A+B}{A-B}$, ita vt n sit numerus positiuus, si $A > B$, contra vero negatiuus.

§. 17. Nunc etiam vtrique corpori minimas tribuamus inclinationes, a quibus cum praecedentes valores non immutari sint censendi, tantum secunda et quarta aequationum nostrarum in computum erunt ducendae, quae ob $dz = \frac{2gt\,dt}{n}$ erunt:

$$\frac{4gt\,dt\,d\eta + (an + gtt)\,dd\eta}{2gn\,dt^2} = -\eta \text{ et}$$

$$\frac{4gt\,dt\,d\theta - (bn - gtt)\,dd\theta}{2gn\,dt^2} = +\theta;$$

quae cum inter se sint similes, tractasse solam primam sufficiet, quae quo commodior reddatur, faciamus $n a = i g$, vt sit $i = \frac{n a}{g}$, et aequatio resoluenda erit:

$$4 t\, d t\, d \eta + (i + t t)\, d d \eta + 2 n \eta\, d t^2 = 0$$

quam etiam, vt supra per series integrare tentemus.

§. 18. Fingamus igitur sequentem seriem:

$$\eta = A + B t + C t t + D t^3 + E t^4 + F t^5 + G t^6 + \text{etc. erit}$$

$$\frac{d \eta}{d t} = B + 2 C t + 3 D t t + 4 E t^3 + 5 F t^4 + 6 G t^5 + \text{etc. et}$$

$$\frac{d d \eta}{d t^2} = 1.2 C + 2.3 D t + 3.4 E t t + 4.5 F t^3 + 5.6 G t^4 + \text{etc.}$$

quibus substitutis fiet:

$$\left.\begin{array}{lllllll}
\frac{i\, d d \eta}{d t^2} = & 1.2 i C & + 2.3 i D t & + 3.4 i E t t & + 4.5 i F t^3 & + 5.6 i G t^4 & + \text{etc.} \\
\frac{t t\, d d \eta}{d t^2} = & & & + 1.2 C & + 2.3 D & + 3.4 E & + \text{etc.} \\
\frac{4 t\, d \eta}{d t} = & & 4 B & + 8 C & + 12 D & + 16 E & + \text{etc.} \\
2 n \eta = & 2 A n + & 2 B n & + 2 C n & + 2 D n & + 2 E n & + \text{etc.}
\end{array}\right\} = 0$$

vnde sequuntur sequentes denominationes:

$$C = -\frac{2 A n}{1.2 i};\; D = -\frac{B(2 n + 4)}{2.3 i};\; E = -\frac{C(2 n + 10)}{3.4 i};$$

$$F = -\frac{D(2 n + 18)}{4.5 i};\; G = -\frac{E(2 n + 28)}{5.6 i};\; H = -\frac{F(2 n + 40)}{5.6 i} \text{ etc.}$$

sicque bini primi coefficientes A et B manent indeterminati.

§. 19. Hinc igitur perspicitur, hanc seriem abrumpi sequentibus casibus: scil. $n = 0$; $n = -2$; $n = -5$; $n = -9$; $n = -14$; $n = -20$, hincque in genere si $n = -\frac{i(i+3)}{2}$; vbi quidem alternatim vel A vel B nihilo aequale sumi debet, ita vt his casibus motum desideratum assignare valeamus. Praecipuos igitur euoluamus:

I. Si

I. Si $n = -2$ erit $\eta = Bt$

II. Si $n = -5$ erit $\eta = A + \frac{10 A tt}{1 \cdot 2 i}$;

III. Si $n = -9$ erit $\eta = Bt + \frac{14 B t^3}{2 \cdot 3 i}$;

IV. Si $n = -14$ erit $\eta = A + \frac{28 A tt}{1 \cdot 2 i} + \frac{28 \cdot 18 A}{1 \cdot 2 \cdot 3 \cdot 4 ii}$;

V. Si $n = -20$ erit $\eta = Bt + \frac{36 B t^3}{2 \cdot 3 i} + \frac{36 \cdot 2 B t^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 ii}$;

VI. Si $n = -27$ erit $\eta = A + \frac{54 A tt}{1 \cdot 2 i} + \frac{54 \cdot 44 A t^4}{1 \cdot 2 \cdot 3 \cdot 4 ii} + \frac{54 \cdot 44 \cdot 26 A t^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 i^3}$;

etc. etc. etc. etc.

§. 20. Euoluamus igitur casum $n = -2$, vnde pro nostris corporibus prodit $B = 3A$, ita vt corpus A sit ascensurum; tum igitur erit $\eta = Bt$, quod autem est integrale particulare, vnde ante omnia integrale completum inuestigari debet. Hunc in finem ponamus $\eta = tv$, ita vt sit $d\eta = t dv + v dt$ et $dd\eta = t ddv + 2 dt dv$, et prodibit

$$4 tt dt dv + (i + tt) t ddv + 2(i + tt) dt dv = 0, \text{ siue}$$

$$(i + tt) t ddv + 2(i + 3tt) dt dv = 0, \text{ vnde fit}$$

$$ddv = -\frac{2(i + 3tt) dt dv}{t(i + tt)}, \text{ hinc}$$

$$\frac{ddv}{dv} = -\frac{2(i + 3tt) dt}{t(i + tt)} = -\frac{2 dt}{t} - \frac{4 t dt}{i + tt},$$

vnde fit integrando

$$l \frac{dv}{dt} = -2 lt - 2 l(i + tt) + l C, \text{ consequenter}$$

$$\frac{dv}{dt} = \frac{C}{tt(i + tt)^2}, \quad \text{ideoque} \quad dv = \frac{C dt}{tt(i + tt)^2},$$

quae ita resoluitur:

$$dv = \frac{C dt}{ii tt} - \frac{C dt}{i(i + tt)^2} - \frac{C dt}{ii(i + tt)}, \text{ vnde fit}$$

$$v = -\frac{C}{iit} - \frac{C}{i} \int \frac{dt}{(i + tt)^2} - \frac{C}{ii} \int \frac{dt}{i + tt}.$$

§. 21. Erat autem $i = \frac{n a}{g} = -\frac{2 a}{g}$, ideoque i numerus negatiuus. Ponamus igitur $i = -cc$ et habebimus

$$v = -\frac{C}{c^4 t} + \frac{C}{cc}\int\frac{dt}{(tt-cc)^2} - \frac{C}{c^4}\int\frac{dt}{tt-cc}.$$

Ponamus porro $\int\frac{dt}{(tt-cc)^2} = \frac{\alpha t}{tt-cc} + \int\frac{\beta\, dt}{tt-cc}$, eritque differentiando et per dt diuidendo

$$\frac{1}{(tt-cc)^2} = \frac{\alpha}{tt-cc} - \frac{2\alpha tt}{(tt-cc)^2} + \frac{\beta}{tt-cc},$$

vnde colligimus $\alpha = \beta = -\frac{1}{2cc}$, ita vt nunc fit

$$v = -\frac{C}{c^4 t} - \frac{Ct}{2c^4(tt-cc)} - \frac{3C}{2c^4}\int\frac{dt}{tt-cc}.$$

Est vero

$$\int\frac{dt}{tt-cc} = -\int\frac{dt}{cc-tt} = -\frac{1}{2c}\, l\frac{c+t}{c-t}$$

consequenter

$$v = -\frac{C}{c^4 t} - \frac{Ct}{2c^4(cc-tt)} + \frac{3C}{4c^5}\, l\frac{c+t}{c-t},$$

vel, posito breuitatis gratia $C = -Dc^5$, fiet

$$v = \frac{Dc}{t} + \frac{Dct}{2(cc-tt)} - \frac{3D}{4}\, l\frac{c+t}{c-t} + E.$$

§. 23. Inuento hoc valore erit angulus noster

$$\eta = Dc + \frac{Dctt}{2(cc-tt)} - \frac{3}{4}Dt\, l\frac{c+t}{c-t} + Et,$$

vbi notetur esse $cc = -i = +\frac{2a}{g}$. Posito igitur $t = 0$ fiet $\eta = Dc$; sicque Dc exprimit inclinationem initialem. Hinc crescente t hoc pendulum MA ascendet, et angulus etiam crescit, nisi forte constans D fuerit negatiua; verum tempus t non vltra c augeri potest, quia alioquin expressio pro η adeo in infinitum excresceret. Quod quo clarius appareat, consideremus etiam celeritatem angularem $\frac{d\eta}{dt} = \frac{Dc^3 t}{(cc-tt)^2} - \frac{3}{4}D\, l\frac{c+t}{c-t} - \frac{3}{2}\cdot\frac{Dct}{cc-tt} + E$. Nunc igitur ponamus initio, quo $t = 0$, fuisse $\eta = \alpha$ et $\frac{d\eta}{dt} = 0$, erit-

eritque $\alpha = D\,c$ et $E = 0$, ideoque et $D = \frac{\alpha}{c}$, ficque erit

$$\eta = \alpha + \frac{\alpha tt}{2(cc - tt)} - \frac{3\alpha t}{4c}\, l\, \frac{c+t}{c-t} \text{ et}$$

$$\frac{d\eta}{dt} = \frac{\alpha cct}{(cc-tt)^2} - \frac{3\alpha}{4c}\, l\, \frac{c+t}{c-t} - \frac{3\alpha t}{2(cc-tt)}, \text{ fiue}$$

$$\frac{d\eta}{dt} = \frac{\alpha t(3tt - cc)}{2(cc-tt)^2} - \frac{3\alpha}{4c}\, l\, \frac{c+t}{c-t},$$

ficque hinc ad quoduis tempus t tam angulum η quam celeritatem angularem $\frac{d\eta}{dt}$ definire licet.

§. 24. Ex indole harum formularum perfpicuum eft, tempus t non vltra terminum C augeri poffe, quippe quo tempore longitudo fili $MA = a + z$ ad nihilum redigitur, et tam angulus η, quam celeritas $\frac{d\eta}{dt}$ in infinitum excrefcunt, quod quidem cum pendulo infinite breui facile conciliari poteft. Verum iam multo ante, quam hoc euenit, angulus η tam fit magnus, vt non amplius pro tam paruo haberi poffit, qualem natura noftri calculi fupponit; ficque etiam iftius motus determinatio mox erronea euadet. Quod autem ad ofcillationes alterius corporis maioris B attinet, earum motus ob defectum Analyfeos nullo plane cafu definire licet, quoniam omnes valores numeri n, quibus integratio fuccedit, funt negatiui, ideoque tantum in pendulo afcendente locum habere poffunt.
