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De mensura angulorum solidorum

Leonhard Euler

DE MENSURA ANGULORUM SOLIDORUM.

Auctore

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§. 1.

Quemadmodum anguli plani mensurantur per arcus circulares eos subtendentes, si scilicet vertex anguli in centro circuli collocatur: ita naturae rei consentaneum videtur, angulos solidos per portiones superficiei sphaericae metiri, quae eos quasi subtendant, si vertex anguli in centro sphaerae collocatur. Ita si angulus solidus ex tribus angulis, qui sint a, b, c, fuerit formatus, et circa verticem sphaera describatur, cuius radius vnitate exprimatur, mensura huius anguli solidi rite statuetur areae trianguli sphaerici aequalis, cuius latera sint illis angulis a, b et c aequalia; quandoquidem haec latera sunt mensurae istorum angulorum planorum. Eodem modo si angulus solidus ex quatuor vel pluribus angulis planis fuerit formatus, eius mensura erit area quadrilateri sphaerici, vel polygoni plurium laterum, cuius scilicet singula latera aequentur angulis planis, quibus angulus solidus componitur. Hac igitur ratione dimensio angulorum solidorum reducitur ad inuestigationem areae trianguli sphaerici, vel polygoni plurium late-

laterum, cuius latera fuerint data. Cum igitur area cuiusque trianguli sphaerici facillime ex eius angulis cognoscatur, quemadmodum iam dudum ab acutissimo Geometra *Alberto Girardo* est demonstratum, hanc ipsam demonstrationem, quoniam non inuulgus satis nota videtur, hic apponam.

Lemma.

Tab. II. Fig. 13. §. 2. *Area portionis sphaericae, inter duos meridianos, angulo α inuicem inclinatos, contenta, se habet ad superficiem totius sphaerae, vt angulus α ad* $360°$. Sint A C B et A D B duo semicirculi maximi in superficie sphaerica, se mutuo in polis oppositis A et B secantes, et inuicem inclinati angulo C A D vel C B D $= \alpha$, et euidens est, arcam huius sectoris sphaerici A C B D A toties contineri in superficie sphaerae tota, quoties angulus α continetur in 360 gradibus.

§. 3. Quod si ergo radius sphaerae ponatur $= r$, quia superficies totius sphaerae est $= 4\pi r r$, denotante π peripheriam circuli, cuius diameter $= 1$, erit area nostri sectoris sphaerici $= 4\pi r r . \frac{\alpha}{360°}$, si quidem angulus α in gradibus exprimatur; at si α detur in partibus radii, qui semper vnitate exprimatur, ob $360° = 2\pi$ erit area sectoris sphaerici $= 2\alpha r r$; vnde si radius sphaerae pariter vnitati aequalis statuatur, ista area erit $= 2\alpha$. Hoc igitur modo area istius sectoris per simplicem angulum repraesentari poterit, dum tota superficies est $= 4\pi$.

Theo-

Theorema

Alberti Girardi.

§. 4. *Area trianguli sphaerici semper aequalis est angulo, quo summa omnium trium angulorum trianguli sphaerici excedit duos angulos rectos.*

Demonstratio.

Sit A B C triangulum sphaericum propositum, cuius area quaeritur, eiusque anguli denotentur literis α, β, γ. Iam primo latera A B et A C in superficie sphaerica producantur, donec sibi mutuo iterum occurrant in polo a, ipsi angulo A opposito, et quia hi arcus A B a et A C a tanquam duo meridiani spectari possunt, a se innicem angulo α distantes, erit area istius sectoris A C a B $= 2\,\alpha$. Deinde eodem modo bina latera B A et B C continuentur vsque in b, quod punctum itidem erit polus, ipsi B oppositus, huiusque sectoris B A b C area erit $= 2\,\beta$. Denique producantur etiam latera C A et C B vsque in polum ipsi C oppositum in c, eritque istius sectoris C B c A area $= 2\,\gamma$. Hinc igitur si area trianguli A B C quaesita vocetur $=$ S, innotescent areae sequentium triangulorum: Tab. II. Fig. 14.

I°. a B C $= 2\,\alpha -$ S
II°. b A C $= 2\,\beta -$ S
III°. c A B $= 2\,\gamma -$ S.

§. 4. Quia nunc puncta a, b, c in superficie sphaerae punctis A, B et C e diametro sunt opposita, inter se etiam easdem tenebunt distantias, etiamsi in figura longe aliter videatur. Hinc ductis arcubus $a\,b$, $b\,c$, $c\,a$, erit $a\,b =$

$ab = AB$, $ac = AC$ et $bc = BC$; vnde et huius trianguli abc, in regione ſphaerae poſteriore ſiti, area quoque erit $= S$; ita vt iam tota ſuperficies ſphaerae contineat 1°. triangula $ABC = S$ et $abc = S$; 2°. triangula $aBC = 2\alpha - S$, $bAC = 2\beta - S$ et $cAB = 2\gamma - S$. Praeterea vero figura continet triangula abC, acB et bcA, quorum poſteriorum areae ex ſuperioribus innoteſcunt; namque pro triangulo abC primo eſt latus $ab = AB$, latus $aC = Ac$ et $bC = Bc$; vnde manifeſto hoc triangulum $abC = ABc = 2\gamma - S$. Eodem modo intelligitur fore triangulum $acB = ACb = 2\beta - S$; ac denique $bcA = BCa = 2\alpha - S$.

§. 5. Quare cum tota ſphaerae ſuperficies hic diſſecta ſit in octo triangula, quorum ſingulorum areas hic exhibuimus, earum ſumma aequalis eſſe debet toti ſuperficiei ſphaerae $= 4\pi$; ex qua aequalitate area quaeſita S definiri poterit. Singula igitur haec triangula cum ſuis areis conſpectui exponamus:

I. $ABC = S$	III. $aBC = 2\alpha - S$	VI. $Abc = 2\alpha - S$
II. $abc = S$	IV. $bAC = 2\beta - S$	VII. $Bac = 2\beta - S$
	V. $cAB = 2\gamma - S$	VIII. $Cab = 2\gamma - S$

$$\text{Summa} = 2S + 2(\alpha + \beta + \gamma) - 3S + 2(\alpha + \beta + \gamma) - 3S$$

vnde omnium octo triangulorum ſumma colligitur $= 4(\alpha + \beta + \gamma) - 4S$, quae ergo aequalis eſſe debet 4π, vnde per quatuor diuidendo oritur $\alpha + \beta + \gamma - S = \pi$, ideoque $S = \alpha + \beta + \gamma - \pi$, vbi $\alpha + \beta + \gamma$ eſt ſumma omnium angulorum trianguli propoſiti, et π eſt menſura duorum rectorum, ſiue 180°, ſicque area trianguli ſphaerici ci propositi reperitur, si a summa omnium angulorum $\alpha + \beta + \gamma$ duo recti seu 180°. subtrahantur, prorsus vti Theorema declarat.

§. 6. Totum ergo negotium pro mensura angulorum solidorum huc reducitur: vt ex datis ternis lateribus trianguli sphaerici eius area definiatur; quamobrem sequens Problema resoluendum suscipiamus.

Problema generale.

Datis in triangulo sphaerico ternis lateribus $AB = c$, $AC = b$ *et* $BC = a$, *inuestigare aream huius trianguli sphaerici.* Tab. II. Fig. 15.

Solutio.

§. 7. Denotent literae A, B, C angulos huius trianguli, ponaturque eius area quam quaerimus $= S$, ac modo vidimus fore $S = A + B + C - 180^\circ$. Hinc ergo erit $\sin. S = -\sin. (A + B + C)$ et $\cos. S = -\cos. (A + B + C)$, hincque $\tang. S = + \tang. (A + B + C)$; sicque tantum opus est, vt loco angulorum A, B, C latera a, b, c in calculum introducantur. At vero per praecepta trigonometriae sphaericae anguli ex datis lateribus ita definiuntur, vt sit:

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}; \quad \cos. B = \frac{\cos. b - \cos. a \cos. c}{\sin. a \sin. c};$$
$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b};$$

vnde porro deducuntur sinus eorundem angulorum

$$\sin. A = \frac{\sqrt{(1 - \cos. a^2 - \cos. b^2 - \cos. c^2 + 2 \cos. a \cos. b \cos. c)}}{\sin. b \sin. c};$$
$$\sin. B = \frac{\sqrt{(1 - \cos. a^2 - \cos. b^2 - \cos. c^2 + 2 \cos. a \cos. b \cos. c)}}{\sin. a \sin. c};$$

 sin. C

$$\text{fin. } C = \frac{\surd(1 - \text{cof. } a^2 - \text{cof. } b^2 - \text{cof. } c^2 + 2 \text{ cof. } a \text{ cof. } b \text{ cof. } c)}{\text{fin. } a \text{ fin. } b},$$

vbi loco radicalis ponamus

$$\surd(1 - \text{cof. } a^2 - \text{cof. } b^2 - \text{cof. } c^2 + 2 \text{ cof. } a \text{ cof. } b \text{ cof. } c) = k$$

et ad calculum contrahendum pro numeratoribus ftatuamus

$$\text{cof. } a = \alpha, \ \text{cof. } b = \beta \text{ et cof. } c = \gamma,$$

vt fit

$$kk = 1 - \alpha\alpha - \beta\beta - \gamma\gamma + 2\alpha\beta\gamma.$$

Hoc facto erit

$$\text{cof. } A = \frac{\alpha - \beta\gamma}{\text{fin. } b \text{ fin. } c}; \ \text{cof. } B = \frac{\beta - \alpha\gamma}{\text{fin. } a \text{ fin. } c}; \ \text{cof. } C = \frac{\gamma - \alpha\beta}{\text{fin. } a \text{ fin. } b};$$

$$\text{fin. } A = \frac{k}{\text{fin. } b \text{ fin. } c}; \ \text{fin. } B = \frac{k}{\text{fin. } a \text{ fin. } c}; \ \text{fin. } C = \frac{k}{\text{fin. } a \text{ fin. } b}.$$

§. 8. Coniungamus nunc primo angulos A et B ac reperiemus

$$\text{fin. } (A+B) = \text{fin. } A \text{ cof. } B + \text{cof. } A \text{ fin. } B = \frac{k(1-\gamma)(\alpha+\beta)}{\text{fin. } a \text{ fin. } b \text{ fin. } c^2};$$

$$\text{cof. } (A+B) = \text{cof. } A \text{ cof. } B - \text{fin. } A \text{ fin. } B = \frac{(\alpha-\beta\gamma)(\beta-\alpha\gamma)-kk}{\text{fin. } a \text{ fin. } b \text{ fin. } c^2}.$$

Quod fi nunc tertium angulum C coniungamus, erit

$$\text{fin. } (A+B+C) = \text{fin. } A \text{ cof. } B \text{ cof. } C + \text{fin. } B \text{ cof. } A \text{ cof. } C$$
$$+ \text{fin. } C \text{ cof. } A \text{ cof. } B - \text{fin. } A \text{ fin. } B \text{ fin. } C;$$

$$\text{cof. } (A+B+C) = \text{cof. } A \text{ cof. } B \text{ cof. } C - \text{cof. } A \text{ fin. } B \text{ fin. } C$$
$$- \text{cof. } B \text{ fin. } A \text{ fin. } C - \text{cof. } C \text{ fin. } A \text{ fin. } B.$$

Tantum igitur fupereft, vt in his formulis loco literarum maiuscularum A, B, C, valores modo affignati fubftituantur.

Prima Inueftigatio,

pro fin. S.

§. 9. Cum fit fin. $S = -$ fin. $(A+B+C)$, erit

$$\text{fin. } S = \text{fin. } A \text{ fin. } B \text{ fin. } C - \text{fin. } A \text{ cof. } B \text{ cof. } C$$
$$- \text{fin. } B \text{ cof. } A \text{ cof. } C - \text{fin. } C \text{ cof. } A \text{ cof. } B,$$

quae

quae expreſſio cum conſtet quatuor membris, ſingula ſeorſim euoluamus. Erit igitur:

I. $\sin. A \sin. B \sin. C = \frac{k^3}{\sin. a^2 \sin. b^2 \sin. c^2} = \frac{k(1 - \alpha\alpha - \beta\beta - \gamma\gamma + 2\alpha\beta\gamma)}{\sin. a^2 \sin. b^2 \sin. c^2}$;

II. $\sin. A \cos. B \cos. C = \frac{k(\beta - \alpha\gamma)(\gamma - \alpha\beta)}{\sin. a^2 \sin. b^2 \sin. c^2} = \frac{k(\beta\gamma - \alpha\beta\beta - \alpha\gamma\gamma + \alpha\alpha\beta\gamma)}{\sin. a^2 \sin. b^2 \sin. c^2}$;

III. $\sin. B \cos. A \cos. C = \frac{k(\alpha - \beta\gamma)(\gamma - \alpha\beta)}{\sin. a^2 \sin. b^2 \sin. c^2} = \frac{k(\alpha\gamma - \beta\alpha\alpha - \beta\gamma\gamma + \beta\beta\alpha\gamma)}{\sin. a^2 \sin. b^2 \sin. c^2}$;

IV. $\sin. C \cos. A \cos. B = \frac{k(\alpha - \beta\gamma)(\beta - \alpha\gamma)}{\sin. a^2 \sin. b^2 \sin. c^2} = \frac{k(\alpha\beta - \gamma\alpha\alpha - \gamma\beta\beta + \gamma\gamma\alpha\beta)}{\sin. a^2 \sin. b^2 \sin. c^2}$.

Quia ergo vbique idem habetur denominator

$$\sin. a^2 \sin. b^2 \sin. c^2 = (1 - \alpha\alpha)(1 - \beta\beta)(1 - \gamma\gamma),$$

tria membra poſteriora, in vnam ſummam collecta, dabunt

$$\frac{k(\alpha\beta + \alpha\gamma + \beta\gamma - \alpha\beta\beta - \alpha\gamma\gamma - \beta\gamma\gamma - \alpha\alpha\beta - \alpha\alpha\gamma - \beta\beta\gamma + \alpha\beta\gamma(\alpha + \beta + \gamma))}{(1 - \alpha\alpha)(1 - \beta\beta)(1 - \gamma\gamma)}$$

§. 10. Ad has formulas tractabiliores reddendas ponamus breuitatis gratia:

$$\alpha + \beta + \gamma = p; \; \alpha\beta + \alpha\gamma + \beta\gamma = q \text{ et } \alpha\beta\gamma = r,$$

hincque erit

$$\alpha\alpha + \beta\beta + \gamma\gamma = pp - 2q,$$

vnde fit

$$kk = 1 - pp + 2q + 2r.$$

Deinde cum ſit

$$pq = \alpha\alpha\beta + \alpha\alpha\gamma + \beta\beta\alpha + \beta\beta\gamma + \gamma\gamma\alpha + \gamma\gamma\beta + 3\alpha\beta\gamma,$$

erit

$$\alpha\alpha(\beta + \gamma) + \beta\beta(\alpha + \gamma) + \gamma\gamma(\alpha + \beta) = pq - 3r,$$

quibus valoribus ſubſtitutis terna poſteriora membra iunctim praebent $\frac{k(q - pq + 3r + pr)}{(1 - \alpha\alpha)(1 - \beta\beta)(1 - \gamma\gamma)}$, quae ſumma, a primo membro $= \frac{k(1 - pp + 2q + 2r)}{(1 - \alpha\alpha)(1 - \beta\beta)(1 - \gamma\gamma)}$ ſubtracta, relinquit id quod quaerimus, ſcilicet:

$$\sin. S = \frac{k(1 + q - r - pp + pq - pr)}{(1 - \alpha\alpha)(1 - \beta\beta)(1 - \gamma\gamma)};$$

 vbi

vbi obſeruaſſe iuuabit, quia, poſito $\alpha = 1$, denominator euaneſcit, eodem caſu quoque numeratorem euaneſcere debere, quod idem quoque euenire debet caſibus $\beta = 1$ et $\gamma = 1$, ita vt numerator neceſſario habeat factores $1-\alpha$; $1-\beta$; $1-\gamma$, quorum productum cum ſit $1-p+q-r$, per hoc ſimul numerator erit diuiſibilis, et diuiſione facta quotus reperitur $= 1+p$; denominator vero, per eundem diuiſorem diuiſus, fit

$$(1+\alpha)(1+\beta)(1+\gamma) = 1+p+q+r,$$

ſicque reſultat iſta formula:

$$\text{ſin. } S = \frac{h(1+p)}{1+p+q+r},$$

ſiue valoribus reſtitutis

$$\text{ſin. } S = \frac{(1+\alpha+\beta+\gamma)\sqrt{(1-\alpha\alpha-\beta\beta-\gamma\gamma+2\alpha\beta\gamma)}}{(1+\alpha)(1+\beta)(1+\gamma)},$$

vbi denotat α, coſ. a; β, coſ. b; γ, coſ. c. Hancque formulam operae pretium erit aliquot exemplis illuſtrare.

§. 11. *Exemplum primum.* Sint latera b et c quadrantes, ita vt ſit $\beta = 0$ et $\gamma = 0$, eritque ſin. $S = \sqrt{(1-\alpha\alpha)}$, ideoque ſin. $S =$ ſin. a, conſequenter ipſa area $S = a$. Quando autem ambo latera A B et A C ſunt quadrantes et latus B C $= a$, tum ambo anguli B et C erunt recti, et ob coſ. A $= \alpha =$ coſ. a, erit angulus A $= \hat{a}$, hincque ſumma omnium angulorum $= 180° + a$, ideoque area quaeſita $S = a$.

§. 12. *Exemplum ſecundum.* Sit triangulum ſphaericum A B C ad A rectangulum, et cum ex ſphaericis ſit coſ. B C $=$ coſ. A B coſ. A C, erit coſ. $a =$ coſ. b coſ. c, ideoque $\alpha = \beta\gamma$; quo valore ſubſtituto prodibit:

ſin. S

$$\text{fin. } S = \frac{(1+\beta+\gamma+\beta\gamma)\sqrt{(1-\beta\beta-\gamma\gamma+\beta\beta\gamma\gamma)}}{(1+\beta)(1+\gamma)(1+\beta\gamma)} = \frac{\sqrt{(1-\beta\beta)(1-\gamma\gamma)}}{1+\beta\gamma}.$$

Cum igitur fit $\sqrt{(1-\beta\beta)} = \text{fin. } b$, et $\sqrt{(1-\gamma\gamma)} = \text{fin. } c$, erit pro area trianguli rectanguli

$$\text{fin. } S = \frac{\text{fin. } b \text{ fin. } c}{1+\text{cof. } b \text{ cof. } c} = \frac{\text{fin. } b \text{ fin. } c}{1+\text{cof. } a}.$$

§. 13. *Exemplum tertium.* Si triangulum fuerit aequilaterum, feu $\alpha = \beta = \gamma$, eius area ita exprimetur vt fit $\text{fin. } S = \frac{(1+3\alpha)\sqrt{(1-3\alpha^2+2\alpha^3)}}{(1+\alpha)^2}$, vbi formula radicalis factores habet $(1-\alpha)^2(1+2\alpha)$, vnde ergo fiet

$$\text{fin. } S = \frac{(1+3\alpha)(1-\alpha)\sqrt{(1+2\alpha)}}{1+\alpha^2}.$$

Hinc fi terna latera fuerint quadrantes, ideoque $\alpha = 0$, erit $\text{fin. } S = 1$, ideoque $S = \frac{\pi}{2}$.

§. 14. *Exemplum quartum.* Sint omnia latera trianguli, a, b, c quam minima, quo cafu triangulum fphaericum abit in triangulum planum, et cum fit

$$\alpha = \text{cof. } a = 1 - \tfrac{1}{2}aa + \tfrac{1}{24}a^4 - \text{etc.},$$

fimilique modo

$$\beta = 1 - \tfrac{1}{2}bb + \tfrac{1}{24}b^4 - \text{etc. et } \gamma = 1 - \tfrac{1}{2}cc + \tfrac{1}{24}c^4 - \text{etc.},$$

factor rationalis noftrae formulae fiet $= \frac{4}{2^3} = \frac{1}{2}$, neglectis fcilicet partibus minimis. At in formula irrationali non folum partes finitae fe mutuo deftruunt, fed etiam termini, vbi a, b, c habent duas dimenfiones; quamobrem fingulas partes vsque ad quatuor dimenfiones euolui oportet. Habebimus ergo vt fequitur:

$\alpha\alpha = 1 - aa + \frac{1}{3}a^4$	$\alpha\beta = 1 - \frac{1}{2}aa - \frac{1}{2}bb + \frac{1}{24}a^4 + \frac{1}{24}b^4 + \frac{1}{4}aabb$, ideoque
$\beta\beta = 1 - bb + \frac{1}{3}b^4$	$\alpha\beta\gamma = 1 - \frac{1}{2}aa - \frac{1}{2}bb - \frac{1}{2}cc + \frac{1}{24}a^4 + \frac{1}{24}b^4 + \frac{1}{24}c^4$
$\gamma\gamma = 1 - cc + \frac{1}{3}c^4$	$+ \frac{1}{4}aabb + \frac{1}{4}aacc + \frac{1}{4}bbcc.$

Hinc

Hinc igitur colligitur quantitas poſt ſignum radicale vt ſequitur

$$-2+aa+bb+cc-\tfrac{1}{3}a^4-\tfrac{1}{3}b^4-\tfrac{1}{3}c^4$$
$$+2-aa-bb-cc+\tfrac{1}{12}a^4+\tfrac{1}{12}b^4+\tfrac{1}{12}c^4$$
$$+\tfrac{1}{2}aabb+\tfrac{1}{2}aacc+\tfrac{1}{2}bbcc,$$

quae, deletis terminis ſe deſtruentibus, reducitur ad hanc:

$$\tfrac{1}{2}aabb+\tfrac{1}{2}aacc+\tfrac{1}{2}bbcc-\tfrac{1}{4}a^4-\tfrac{1}{4}b^4-\tfrac{1}{4}c^4.$$

Quare cum etiam area S ſit quam minima, ideoque ſin. S $=$ S, habebimus aream quaeſitam:

$$S=\tfrac{1}{2}\sqrt{(\tfrac{1}{2}aabb+\tfrac{1}{2}aacc+\tfrac{1}{2}bbcc-\tfrac{1}{4}a^4-\tfrac{1}{4}b^4-\tfrac{1}{4}c^4)},$$

ſiue

$$S=\tfrac{1}{4}\sqrt{(2aabb+2aacc+2bbcc-a^4-b^4-c^4)},$$

quae eſt formula notiſſima pro area trianguli plani.

Inueſtigatio ſecunda, pro coſinu S.

§. 15. Cum ſit coſ. S $=-$ coſ. (A $+$ B $+$ C), erit

$$\text{coſ. } S=\text{coſ. } A \text{ ſin. } B \text{ ſin. } C+\text{coſ. } B \text{ ſin. } A \text{ ſin. } C$$
$$+\text{coſ. } C \text{ ſin. } A \text{ ſin. } B-\text{coſ. } A \text{ coſ. } B \text{ coſ. } C,$$

quae quatuor membra ſeorſim euoluta dabunt:

$$\text{I. coſ. } A \text{ ſin. } B \text{ ſin. } C=\frac{kk(\alpha-\beta\gamma)}{\sin. a^2 \sin. b^2 \sin. c^2};$$
$$\text{II. coſ. } B \text{ ſin. } A \text{ ſin. } C=\frac{kk(\beta-\alpha\gamma)}{\sin. a^2 \sin. b^2 \sin. c^2};$$
$$\text{III. coſ. } C \text{ ſin. } A \text{ ſin. } B=\frac{kk(\gamma-\alpha\beta)}{\sin. a^2 \sin. b^2 \sin. c^2}.$$

Pro termino poſtremo erit primo

$$\text{coſ. } A \text{ coſ. } B=\frac{(\alpha-\beta\gamma)(\beta-\alpha\gamma)}{\sin. a \sin. b \sin. c^2}=\frac{\alpha\beta-\alpha\alpha\gamma-\beta\beta\gamma+\alpha\beta\gamma\gamma}{\sin. a \sin. b \sin. c^2},$$

hincque

$$\text{coſ.A coſ.B coſ.C}=\frac{\alpha\beta\gamma-\alpha\alpha\beta\beta-\alpha\alpha\gamma\gamma-\beta\beta\gamma\gamma+\alpha\beta\gamma^3+\alpha\gamma\beta^3+\beta\gamma\alpha^3-\alpha^2\beta^2\gamma^2}{\sin. a^2 \sin. b^2 \sin. c^2}.$$

§. 16.

§. 16. Quod si iam iterum ponamus $\alpha+\beta+\gamma=p$; $\alpha\beta+\alpha\gamma+\beta\gamma=q$ et $\alpha\beta\gamma=r$, tria membra priora, in vnam summam collecta, dabunt $\frac{kk(p-q)}{\sin. a^2 \sin. b^2 \sin. c^2}$; vltimum autem membrum, si hoc modo repraesentetur:

$$\frac{\alpha\beta\gamma-\alpha\alpha\beta\beta-\alpha\alpha\gamma\gamma-\beta\beta\gamma\gamma+\alpha\beta\gamma(\alpha\alpha+\beta\beta+\gamma\gamma)-\alpha\alpha\beta\beta\gamma\gamma}{\sin. a^2 \sin. b^2 \sin. c^2},$$

ob $\alpha\alpha+\beta\beta+\gamma\gamma=pp-2q$ et

$$\alpha\alpha\beta\beta+\alpha\alpha\gamma\gamma+\beta\beta\gamma\gamma=qq-2pr,$$

induet hanc formam:

$$\frac{r-qq+2pr+ppr-2qr-rr}{\sin. a^2 \sin. b^2 \sin. c^2}.$$

Quare cum sit $kk=1-pp+2q+2r$, omnibus membris collectis habebimus:

$$\cos. S=\frac{(p-q)(1-pp+2q+2r)-r+qq-2pr-ppr+2qr+rr}{\sin. a^2 \sin. b^2 \sin. c^2},$$

quae formula euoluta fit

$$\cos. S=\frac{p-q-r+2pq+ppq-ppr-qq+rr-p^3}{\sin. a^2 \sin. b^2 \sin. c^2}.$$

§. 17. Quia hic iterum denominator euanescit casibus quibus $\alpha=1$, $\beta=1$ et $\gamma=1$, necesse est vt iisdem casibus etiam numerator euanescat, ideoque istum factorem habeat:

$$(1-\alpha)(1-\beta)(1-\gamma)=1-p+q-r.$$

Facta igitur hac diuisione pro numeratore nanciscemur hunc quotum: $p-q-r+pp$; pro denominatore autem quotus erit:

$$(1+\alpha)(1+\beta)(1+\gamma)=1+p+q+r,$$

sicque nacti sumus istam expressionem:

$$\cos. S=\frac{p(1+p)-q-r}{1+p+q+r};$$

ac, pro literis p, q et r restitutis valoribus, erit

 cos.

$$\text{cos. } S = \frac{(\alpha+\beta+\gamma)(1+\alpha+\beta+\gamma)-\alpha\beta-\alpha\gamma-\beta\gamma-\alpha\beta\gamma}{(1+\alpha)(1+\beta)(1+\gamma)};$$

siue etiam

$$\text{cos. } S = \frac{\alpha+\beta+\gamma+\alpha\alpha+\beta\beta+\gamma\gamma+\alpha\beta+\alpha\gamma+\beta\gamma-\alpha\beta\gamma}{(1+\alpha)(1+\beta)(1+\gamma)}.$$

§. 18. *Exemplum primum.* Sint duo latera b et c quadrantes, ideoque $\beta = 0$ et $\gamma = 0$, quo ergo casu prodibit $\text{cos. } S = \frac{\alpha(1+\alpha)}{1+\alpha} = \alpha = \text{cos. } a$; consequenter erit iterum vt supra $S = a$.

§. 19. *Exemplum secundum.* Sit triangulum sphaericum rectangulum, existente angulo A recto, eritque, vti supra vidimus, $\text{cos. } a = \text{cos. } b \text{ cos. } c$, siue $\alpha = \beta\gamma$, quo valore substituto reperitur:

$$\text{cos. } S = \frac{\beta+\gamma+2\beta\gamma+\beta\beta+\gamma\gamma+\beta\beta\gamma+\beta\gamma\gamma}{(1+\beta)(1+\gamma)(1+\beta\gamma)}, \text{ siue}$$

$$\text{cos. } S = \frac{\beta+\gamma(1+\beta+\gamma+\beta\gamma)}{(1+\beta)(1+\gamma)(1+\beta\gamma)} = \frac{\beta+\gamma}{1+\beta\gamma}.$$

Pro eodem vero casu supra inuenimus $S = \frac{\sqrt{(1-\beta\beta)(1-\gamma\gamma)}}{1+\beta\gamma}$, quod egregie congruit, cum hinc fiat

$$\text{sin. } S^2 + \text{cos. } S^2 = \frac{1+2\beta\gamma+\beta\beta\gamma\gamma}{(1+\beta\gamma)^2} = 1.$$

§. 20. *Exemplum tertium.* Sit triangulum aequilaterum, siue $\alpha = \beta = \gamma$, eritque $\text{cos. } S = \frac{3\alpha+6\alpha\alpha-\alpha^3}{(1+\alpha)^3}$. Supra autem inuenimus pro hoc casu

$$\text{sin. } S = \frac{(1+3\alpha)(1-\alpha)\sqrt{(1+2\alpha)}}{1+\alpha^3};$$

ad quarum expressionum consensum ostendendum sumamus vtriusque formulae quadratum, ac prodibit:

$$\text{cos. } S^2 = \frac{9\alpha\alpha+36\alpha^3+30\alpha^4-12\alpha^5+\alpha^6}{(1+\alpha^6)} \text{ et}$$

$$\text{sin. } S^2 = \frac{(1+6\alpha+9\alpha\alpha)(1-3\alpha\alpha+2\alpha^3)}{(1+\alpha)^6} = \frac{1+6\alpha+6\alpha\alpha-16\alpha^3-15\alpha^4+18\alpha^5}{(1+\alpha)^6};$$

qua-

quarum fractionum summa praebet

$$\frac{1+6\alpha+15\alpha\alpha+20\alpha^3+15\alpha^4+6\alpha^5+\alpha^6}{(1+\alpha)^6}=1.$$

§. 21. *Exemplum quartum.* Sint latera trianguli quam minima, et quia etiam area quasi sit euanescens, erit cos. $S = 1 - \frac{1}{2}SS$; hinc ex formula, per literas p, q, r expressa, erit $1 - \frac{1}{2}SS = \frac{p(1+p)-q-r}{1+p+q+r}$, vnde colligitur.

$$SS = \frac{2+4q+4r-2pp}{1+p+q+r}.$$

et restitutis pro p, q, r valoribus fiet

$$SS = \frac{2kk}{(1+\alpha)(1+\beta)(1+\gamma)};$$

vbi in denominatore pro literis α, β, γ sufficit scribere vnitatem, quo facto denominator erit 8. Supra vero vidimus, pro numeratore fieri $k = \sqrt{(1-\alpha\alpha-\beta\beta-\gamma\gamma+2\alpha\beta\gamma)}$

$$= \sqrt{(\tfrac{1}{2}aabb+\tfrac{1}{2}aacc+\tfrac{1}{2}bbcc-\tfrac{1}{4}a^4-\tfrac{1}{4}b^4-\tfrac{1}{4}c^4)},$$

quo valore posito reperitur

$$SS = \frac{2aabb+2aacc+2bbcc-a^4-b^4-c^4}{16},$$

vnde fit vtique

$$S = \tfrac{1}{4}\sqrt{(2aabb+2aacc+2bbcc-a^4-b^4-c^4.)}$$

Tertia inuestigatio,

pro tang. S et tang. $\frac{1}{2}$ S.

§. 22. Postquam pro area nostri trianguli sphaerici tam sin. S quam cos. S inuenimus, sponte se prodit tangens istius areae, scilicet:

$$\text{tang. } S = \frac{(1+\alpha+\beta+\gamma)\sqrt{(1-\alpha\alpha-\beta\beta-\gamma\gamma+2\alpha\beta\gamma)}}{\alpha+\beta+\gamma+\alpha\alpha+\beta\beta+\gamma\gamma+\alpha\beta+\alpha\gamma+\beta\gamma-\alpha\beta\gamma},$$

quam formulam succinctius in genere exprimere non licet.

§. 23. Verum tangens dimidiae areae, siue tang. $\frac{1}{2}$S, multo concinnius exprimi poterit. Cum enim sit

$$\text{tang. } \tfrac{1}{2} S = \frac{\text{sin. } S}{1 + \text{cos. } S},$$

retineamus initio literas p, q et r, ita vt pro numeratore habeamus

$$\text{sin. } S = \frac{(1 + p)\sqrt{(1 - pp + 2q + 2r)}}{(1 + p + q + r)},$$

at vero pro denominatore, ob

$$\text{cos. } S = \frac{p(1 + p) - q - r}{1 + p + q + r}, \text{ erit}$$

$$1 + \text{cos. } S = \frac{(1 + p)^2}{1 + p + q + a};$$

quare his valoribus substitutis reperitur

$$\text{tang. } \tfrac{1}{2} S = \frac{\sqrt{(1 - pp + 2q + 2r)}}{1 + p},$$

et restitutis valoribus,

$$\text{tang. } \tfrac{1}{2} S = \frac{\sqrt{(1 - \alpha\alpha - \beta\beta - \gamma\gamma + 2\alpha\beta\gamma)}}{1 + \alpha + \beta + \gamma};$$

quae formula ad vsum vtique est aptissima.

§. 24. *Exemplum primum.* Si bina latera b et c fuerint quadrantes, ideoque $\beta = 0$ et $\gamma = 0$, erit

$$\text{tang. } \tfrac{1}{2} S = \frac{\sqrt{(1 - \alpha\alpha)}}{1 + \alpha} = \frac{\text{sin. } \alpha}{1 + \text{cos. } \alpha},$$

vnde manifestum est fore tang. $\frac{1}{2}$S $=$ tang. $\frac{1}{2}a$, ideoque $S = a$, vti iam supra inuenimus.

§. 25. *Exemplum secundum.* Sit triangulum sphaericum ad A rectangulum, ideoque cos. $a =$ cos. b cos. c et $\alpha = \beta\gamma$; hoc autem valore substituto reperitur

$$\text{tang. } \tfrac{1}{2} S = \frac{\sqrt{(1 - \beta\beta - \gamma\gamma + \beta\beta\gamma\gamma)}}{1 + \beta + \gamma + \beta\gamma} = \frac{\sqrt{((1 - \beta\beta)(1 - \gamma\gamma))}}{(1 + \beta)(1 + \gamma)},$$

quae fractio, supra et infra diuidendo per $\sqrt{(1 + \beta)(1 + \gamma)}$, reducitur ad hanc:

$$\text{tang.}\ \tfrac{1}{2} S = \sqrt{\frac{(1-\beta)(1-\gamma)}{(1+\beta)(1+\gamma)}}.$$

Eſt vero

$$\sqrt{\frac{1-\beta}{1+\beta}} = \sqrt{\frac{1-\cos. b}{1+\cos. b}} = \text{tang.}\ \tfrac{1}{2} b,$$

ſimilique modo $\sqrt{\frac{1-\gamma}{1+\gamma}} = \text{tang.}\ \tfrac{1}{2} c$; quocirca reſultat ſequens formula maxime memorabilis:

$$\text{tang.}\ \tfrac{1}{2} S = \text{tang.}\ \tfrac{1}{2} b.\ \text{tang.}\ \tfrac{1}{2} c,$$

cuius conſenſus cum ſupra inuentis haud difficulter oſtenditur.

§. 26. *Exemplum tertium.* Si triangulum fuerit aequilaterum, ſiue $\alpha = \beta = \gamma$, erit

$$\text{tang.}\ \tfrac{1}{2} S = \frac{\sqrt{(1-3\alpha\alpha+2\alpha^3)}}{1+3\alpha} = \frac{(1-\alpha)\sqrt{(1+2\alpha)}}{1+3\alpha},$$

vnde caſu, quo ſingula latera ſunt quadrantes, ideoque $\alpha = 0$, erit tang. $\frac{1}{2} S = 1$, ideoque $\frac{1}{2} S = 45°$ et $S = \frac{\pi}{2}$.

§. 27. *Exemplum quartum.* Sint denique tria latera a, b, c, quam minima, et quia

$$\text{tang.}\ \tfrac{1}{2} S = \tfrac{1}{2} S,\ \text{erit}\ S = \frac{2\sqrt{(1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma)}}{1+\alpha+\beta+\gamma}.$$

Nunc igitur pro denominatore ſufficit ſumi $\alpha = 1$, $\beta = 1$, $\gamma = 1$, ita vt Coefficiens formulae radicalis ſit $= \frac{1}{2}$; ipſam autem formulam radicalem iam ſupra aliquoties vidimus eſſe

$$\sqrt{(\tfrac{1}{2} aabb + \tfrac{1}{2} aacc + \tfrac{1}{2} bbcc - \tfrac{1}{4} a^4 - \tfrac{1}{4} b^4 - \tfrac{1}{4} c^4)},$$

vnde area prorſus vt ante exprimitur.

Problema.

§. 28. *Propoſito angulo ſolido $AOBC$, ex tribus angulis planis $BOC = a$, $AOC = b$ et $AOB = c$ formatum, eius veram menſuram aſſignare.* Tab. II. Fig. 16.

 Solutio

Solutio.

Quoniam huius anguli ſolidi menſura ſtatui poteſt aequalis areae trianguli ſphaerici, cuius latera ſint a, b, c, radio ſpaerae exiſtente $= 1$, ex praecedentibus intelligitur, angulos ſolidos, perinde ac planos, ſiue per gradus et minuta, ſiue per arcus circulares exprimi poſſe. Ponamus igitur S exprimi menſuram anguli ſolidi propoſiti, ac poſito breuitatis gratia.

$$\text{coſ.}\, a = \alpha,\ \text{coſ.}\, b = \beta,\ \text{coſ.}\, c = \gamma,$$

triplici modo iſta menſura S aſſignari poterit; primo enim erit per ſinus:

$$\text{ſin. S} = \frac{1+\alpha+\beta+\gamma}{(1+\alpha)(1+\beta)(1+\gamma)} \sqrt{(1-\alpha\alpha-\beta\beta-\gamma\gamma+2\alpha\beta\gamma)};$$

deinde per coſinus:

$$\text{coſ. S} = \frac{\alpha+\beta+\gamma+\alpha\alpha+\beta\beta+\gamma\gamma+\alpha\beta+\alpha\gamma+\beta\gamma-\alpha\beta\gamma}{(1+\alpha)(1+\beta)(1+\gamma)};$$

tertio vero commodiſſime per tangentem ſemiſſis:

$$\text{tang.}\, \tfrac{1}{2}\text{S} = \frac{\sqrt{(1-\alpha\alpha-\beta\beta-\gamma\gamma+2\alpha\beta\gamma)}}{1+\alpha+\beta+\gamma}.$$

Vbi imprimis notaſſe iuuabit, ſi omnes tres anguli a, b, c fuerint recti, tum menſura anguli ſolidi prodire $= 90°$; id quod mirifice conuenit cum communi loquendi more, dum huiusmodi anguli ſolidi etiam ab opificibus anguli recti vocari ſolent; ex quo ſimul intelligere licet, quinam anguli ſiue maiores ſiue minores angulo recto ſint reputandi.

Scholion I.

§. 29. Egregium foret, ſi iſta angulorum ſolidorum menſura etiam ad eiusmodi eximias proprietates perduceret, quales pro figuris planis locum habent; veluti: quod ſumma angulorum planorum aequalis eſt duobus rectis. Interim tamen talis proprietas in figuris solidis neutiquam occurrit, ratione nostrae mensurae. Neque enim in omnibus Tetraëdis, quae quatuor constant angulis solidis, summa omnium angulorum solidorum eandem quantitatem constituit, sed prouti Tetraëdra magis minusue obliqua construuntur, summa quatuor angulorum solidorum modo maior modo minor fieri potest. Si enim Tetraëdron regulare examini subiiciamus, cuius singuli anguli solidi ex ternis angulis planis sexaginta graduum formantur, habebimus $\alpha = \beta = \gamma = \frac{1}{3}$; vnde cuiusque anguli solidi mensura ita reperitur, vt sit tang. $\frac{1}{2} S = \frac{\sqrt{2}}{5}$, vnde ex tabulis colligitur

$$\tfrac{1}{2} S = 15^\circ.\ 48', \text{ siue } S = 31^\circ.\ 36',$$

ideoque summa omnium quatuor angulorum huius Tetraëdri erit $126^\circ.\ 24'$. Nunc consideremus Pyramidem triangularem, cuius basis itidem sit triangulum aequilaterum, vertex autem desinat in cuspidem acutissimam, cuius itaque mensura euanescat; pro ternis autem angulis solidis ad basin vnus angulus erit $a = 60^\circ$, bini vero reliqui $b = c = 90^\circ$, ita vt sit

$$\alpha = \tfrac{1}{3} \text{ et } \beta = \gamma = 0; \text{ vnde prodit}$$

$$\text{tang. } \tfrac{1}{2} S = \tfrac{1}{\sqrt{3}} = \text{tang. } 30^\circ, \text{ ita vt sit } S = 60;$$

vnde huius Pyramidis summa omnium angulorum solidorum erit 180°, cum ante pro Tetraëdro fuisset tantum 126°. Quanquam autem in summa angulorum solidorum cuiusque solidi nulla insignis proprietas elucet, in aliis fortasse relationibus ista mensura proprietates haud contemnendas patefacere poterit.

Scholion II.

§. 30. Quae hactenus sunt tradita ad mensuram eorum angulorum solidorum spectant, qui ex tribus tantum

tum angulis planis sunt compositi. At si angulus solidus ex quatuor pluribusue angulis planis fuerit formatus, eius mensura erit area quadrilateri sphaerici, vel polygoni plurium laterum, cuius singula latera aequentur angulis planis solidum constituentibus. Tum igitur nihil aliud opus est, nisi vt tale Polygonum in triangula sphaerica resoluatur, et singulorum areae inuestigentur, quippe quorum summa dabit mensuram anguli solidi. His autem casibus non sufficit singulos angulos planos tantum nosse, sed insuper necesse est, vt inclinatio mutua binorum pluriumue sit cognita. Haec cum satis sint manifesta, hic tantum adiungam dimensionem angulorum solidorum regularium, qui ex quotcunque angulis, planis inter se aequalibus et pariter inclinatis, formentur.

Problema.

§. 31. *Si angulus solidus componatur ex* n *angulis planis inter se aequalibus, qui singuli sint* = a, *et aequaliter inter se inclinentur, inuenire mensuram huius anguli solidi.*

Solutio.

Si huic angulo solido sphaera concipiatur circumscripta, cuius radius = 1, eius mensura erit Polygonum regulare sphaericum, cuius omnia latera erunt = a, eorumque numerus = n; et quia etiam omnes anguli inter se erunt aequales, Polygonum erit regulare, ideoque in eius medio dabitur eius centrum, quod sit in O; vnum vero quodque latus Poligoni sit latus AB = a, ex cuius terminis ad O ducantur arcus AO et BO, qui erunt inter se aequales, vt habeatur triangulum AOB. Quia igitur nume-

Tab. II. Fig. 17.

numerus talium triangulorum est $= n$, erit

angulus $A O B = \frac{2\pi}{n}$;

at si area totius Polygoni statuatur $= S$, quae simul erit mensura anguli propositi, area istius trianguli $A O B$ erit $= \frac{S}{n}$. Iam ex O in latus $A B$ ducatur normalis $O P$, latus $A B$ bisceans, eritque $A P = \frac{1}{2} a$, et

angulus $A O P = \frac{\pi}{n}$.

Vocetur iam angulus $O A B = \Phi$, eritque ex sphaericis

$$\text{sin.}\, \Phi = \frac{\text{cos.}\, \frac{\pi}{n}}{\text{cos.}\, \frac{1}{2} a}.$$

Quia igitur huic angulo Φ etiam aequalis est angulus $O B A$, summa angulorum trianguli $A O B$ erit $= 2\Phi + \frac{2\pi}{n}$, vnde ablatis duobus rectis obtinebitur area trianguli $A O B$

$$\frac{S}{n} = 2\Phi + \frac{2\pi}{n} - \pi,$$

hincque area totius Polygoni

$$S = 2n\Phi + 2\pi - n\pi = 2n\Phi - (n-2)\pi,$$

quae ergo erit mensura anguli solidi regularis propositi.

Corollarium I.

§. 32. Si igitur angulus solidus constet ex tribus angulis planis aequalibus $= a$, ob $n = 3$, erit

$$\text{sin.}\, \Phi = \frac{\text{cos.}\, 60^\circ}{\text{cos.}\, \frac{1}{2} a};$$

quo angulo inuento erit mensura anguli solidi

$$S = 6\Phi - \pi = 6\Phi - 180^\circ.$$

Corollarium II.

§. 33. Si angulus solidus ex quatuor constet angulis planis inter se aequalibus $= a$, ob $n = 4$ quaeratur

 angu-

angulus Φ, vt fit fin. $\Phi = \frac{\text{cof. } 45^\circ}{\text{cof. } \frac{1}{2} a}$; atque hinc reperietur menfura anguli folidi $S = 8\Phi - 2\pi = 8\Phi - 360^\circ$.

Corollarium III.

§. 34. Si angulus folidus conftet ex quinque angulis planis inter fe aequalibus $= a$, ob $n = 5$ quaeratur angulus Φ, vt fit fin. $\Phi = \frac{\text{cof. } 36^\circ}{\text{cof. } \frac{1}{2} a}$; hinc vero menfura iftius anguli folidi erit $S = 10\Phi - 3\pi = 10\Phi - 540^\circ$.

Corollarium IV.

§. 35. Si angulus folidus ex fex conftet angulis planis inter fe aequalibus $= a$, ob $n = 6$ quaeratur angulus Φ, vt fit fin. $\Phi = \frac{\text{cof. } 30^\circ}{\text{cof. } \frac{1}{2} a}$; tum vero menfura huius anguli folidi erit $S = 12\Phi - 4\pi = 12\Phi - 720^\circ$.

Scholion.

§. 36. Secundum haec praecepta computemus angulos folidos quinque corporum regularium, quo facilius eos cum angulo recto, qui in folidis pariter eft 90 graduum, comparare valeamus; vbi quidem conueniet angulos folidos minores quam 90° nomine acutorum, qui autem excedunt 90° nomine obtuforum infignire.

Menfura angulorum folidorum
Tetraëdri.

§. 37. Cum hic terni anguli plani 60 graduum concurrant ad angulos folidos conftituendos, erit $\frac{1}{2} a = 30^\circ$, et $n = 3$;

$n = 3$; vnde fecundum corrollarium 1. calculus per logarithmos ita inftituetur:

$$l \text{ cof. } 60^\circ = 9,6989700$$
$$l \text{ cof. } 30^\circ = 9,9375306$$
$$l \text{ fin. } \Phi = 9,7614394$$

hincque $\Phi = 35^\circ.\ 15'.\ 52''$

ergo $6\,\Phi = 211^\circ.\ 35'.\ 12''$

vnde quisque angulus folidus Tetraëdri reperietur

$S = 31^\circ.\ 35'.\ 12''$;

ficque hic angulus vix fuperat trientem anguli recti.

Menfura angulorum folidorum Octaëdri.

§. 38. Cum quilibet angulus componatur ex quaternis angulis planis 60 graduum, erit $\frac{1}{2}a = 30^\circ$, et $n = 4$; vnde fecundum praecepta corollarii II calculus per logarithmos inftituatur, vti fequitur:

$$l \text{ cof. } 45^\circ = 9,8494850$$
$$l \text{ cof. } 30^\circ = 9,9375306$$
$$l \text{ fin. } \Phi = 9,9119544$$

hincque erit $\Phi = 54^\circ.\ 44'.\ 8''$, ergo $8\,\Phi = 437^\circ.\ 53'.\ 4''$; vnde anguli folidi Octaëdri menfura erit $S = 77^\circ.\ 53'.\ 4''$, qui ergo angulus non multum a recto deficit. Caeterum hic angulus Φ eft complementum praecedentis ad 90°.

Menſura angulorum ſolidorum Icoſaëdri.

§. 39. Cum hic angulus ſolidus ex quinis angulis planis $a = 60^\circ$ componatur, erit $\frac{1}{2}a = 30^\circ$ et $n = 5$; vnde ex coroll. 3 calculum ita inſtitui opportet:

$$l\,\text{cof.}\ 36^\circ = 9,9079576$$
$$l\,\text{cof.}\ 30 = 9,9375306$$
$$l\,\text{fin.}\ \Phi = 9,9704270$$

vnde colligitur $\Phi = 69^\circ.\ 5'.\ 41\frac{1}{2}''$, ergo $10\,\Phi = 690^\circ.\ 56'.\ 55''$; hinc anguli ſolidi Icoſaëdri menſura erit $S = 150^\circ.\ 56'.\ 55''$, qui ergo angulus iam valde eſt obtuſus.

Menſura angulorum ſolidorum Hexaëdri.

§. 40. Cum hic ſinguli anguli ſolidi conſteut ternis angulis planis rectis, erit $a = 90^\circ$, $\frac{1}{2}a = 45^\circ$, et $n = 3$; hinc ex Coroll. 1. calculus iſta inſtituatur:

$$l\,\text{cof.}\ 60^\circ = 9,6989700$$
$$l\,\text{cof.}\ 45^\circ = 9,8494850$$
$$l\,\text{fin.}\ \Phi = 9,8494850$$

ideoque fit $\Phi = 45^\circ$, ergo $6\,\Phi = 270^\circ$; vnde menſura anguli ſolidi Hexaëdri erit 90°, ſcilicet hic angulus ipſe eſt rectus.

Menſura angulorum ſolidorum Dodecaëdri.

§. 41. Cum hic quilibet angulus conſtet ex ternis

nis planis, quorum singuli continent $108°$, erit $\frac{1}{2} a = 54°$, et $n = 3$; vnde calculus secundum coroll. 1. ita institui debet:

$$l \cos. 60° = 9,6989700$$
$$l \cos. 54° = 9,7692187$$
$$l \sin. \Phi = 9,9297513$$

hincque erit ipse angulus

$$\Phi = 58°.\ 16'.\ 57'', \text{ ergo } 6\Phi = 349°.\ 41'.\ 425''.$$

Mensura igitur anguli solidi Dodecaëdri erit $169°.\ 41'.\ 42''$; sicque hic angulus Dodecaëdri inter omnia corpora regularia est maximus.

Scholion.

§. 42. Quodsi angulus solidus formetur ex sex angulis planis $a = 60°$, vt sit $\frac{1}{2} a = 30°$ et $n = 6$, corpus regulare inde ortum est ipsa sphaera, in cuius superficie omnes anguli solidi in planum sunt depressi, sicque aequivalebunt quatuor angulis rectis; id quod etiam calculus secundum Coroll. 4. institutus declarat:

$$l \cos. 30° = 9,9375306$$
$$l \cos. 30 = 9,9375306$$
$$l \sin. \Phi = 10,000000$$

hincque angulus

$$\Phi = 90° \text{ et } 12\Phi = 1080°,$$

vnde fit angulus solidus $S = 360°$. Idem euenit si angulus solidus ex quatuor planis rectis componatur, vt sit $\frac{1}{2} a = 45$ et $n = 4$; tum enim erit

$$\text{fin.}\ \Phi = \frac{cof.\ 45^\circ}{cof.\ 45^\circ} = 1,\ \text{ideoque}\ \Phi = 90^\circ,$$

et angulus folidus $S = (8 - 4)\ 90 = 360^\circ$. Denique fi angulus folidus conftet ex tribus planis, ita vt fit

$$a = 120^\circ,\ \text{erit}\ \tfrac{1}{2} a = 60^\circ\ \text{et}\ n = 3;$$

vnde iterum fit

$$\text{fin.}\ \Phi = \frac{cof.\ 60^\circ}{cof.\ 60^\circ} = 1,\ \text{ideoque}\ \Phi = 90^\circ,$$

et angulus folidus $S = (6 - 2)\ 90 = 360^\circ$.

AD